

DIFFERENTIAL OPERATORS ON G/U AND THE AFFINE GRASSMANNIAN

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Abstract We describe the equivariant cohomology of cofibers of spherical perverse sheaves on the affine Grassmannian of a reductive algebraic group in terms of the geometry of the Langlands dual group. In fact we give two equivalent descriptions: one in terms of \mathcal{D} -modules of the basic affine space, and one in terms of intertwining operators for universal Verma modules. We also construct natural collections of isomorphisms parameterized by the Weyl group in these three contexts, and prove that they are compatible with our isomorphisms. As applications we reprove some results of the first author and of Braverman and Finkelberg.

1. Introduction

1.1.

The geometric Satake equivalence relates perverse sheaves (with complex coefficients in our case) on the affine Grassmannian \mathbf{Gr} of a complex connected reductive algebraic group \check{G} and representations of the Langlands dual (complex) reductive group G . The underlying vector space of the representation $\mathbb{S}(\mathcal{F})$ attached to a perverse sheaf \mathcal{F} is given by its total cohomology $\mathbf{H}^*(\mathbf{Gr}, \mathcal{F})$. It turns out that various *equivariant* cohomology groups attached to \mathcal{F} also carry information on the representation $\mathbb{S}(\mathcal{F})$; see e.g. [9, 22, 42]. In this paper, if \check{T} is a maximal torus of \check{G} , we describe, in terms of G , the equivariant cohomology of cofibers of \mathcal{F} at a \check{T} -fixed point, with respect to the action of \check{T} or of $\check{T} \times \mathbb{C}^\times$, where \mathbb{C}^\times acts on \mathbf{Gr} by loop rotation. In fact these groups can be described in two equivalent ways, either in terms of \mathcal{D} -modules on the basic affine space or in terms of intertwining operators for universal Verma modules. We also describe the Weyl group action on this collection of spaces induced by the action of $N_{\check{G}}(\check{T})$ on \mathbf{Gr} .

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1.2.

To state our results more precisely, choose some Borel subgroup $\check{B} \subset \check{G}$ containing \check{T} , and let T, B be the maximal torus and the Borel subgroup of G provided by the geometric Satake equivalence. Note that the Tannakian construction of G also provides no zero vectors in each simple root subspace of $\mathfrak{g} := \text{Lie}(G)$. In this paper we study three families of graded modules over a polynomial algebra, attached to G or \check{G} , and endowed with symmetries parameterized by their common Weyl group W .

Let $\mathfrak{t} := \text{Lie}(T)$ and $S_{\hbar} := S(\mathfrak{t}[\hbar])$, considered as a graded algebra where \hbar and the vectors in \mathfrak{t} are in degree 2. (Here, $S(\mathfrak{t})$ is the symmetric algebra of the vector space \mathfrak{t} .) Let also $\mathbb{X} := X^*(T)$ be the character lattice. Let $\text{Rep}(G)$ be the category of finite dimensional algebraic G -modules.

Our first family of graded modules over S_{\hbar} is of “geometric” nature. Let U be the unipotent radical of B , and let $\mathcal{X} := G/U$ be the basic affine space. Consider the algebra $\mathcal{D}_{\hbar}(\mathcal{X})$ of (global) asymptotic differential operators on \mathcal{X} , i.e. the Rees algebra of the algebra $\Gamma(\mathcal{X}, \mathcal{D}_{\mathcal{X}})$ of differential operators on \mathcal{X} , endowed with the order filtration (see § 2.4 for details). This algebra is naturally graded, and endowed with an action of T induced by *right* multiplication on \mathcal{X} . We denote by $\mathcal{D}_{\hbar}(\mathcal{X})_{\lambda}$ the weight space associated with $\lambda \in \mathbb{X}$. Then we set

$$\mathcal{M}_{V,\lambda}^{\text{geom}} := (V \otimes^{(\lambda)} \mathcal{D}_{\hbar}(\mathcal{X})_{\lambda})^G.$$

(Here the twist functor $^{(\lambda)}(\cdot)$ will be defined in § 2.4.)

Our second family of graded S_{\hbar} -modules is of “algebraic” nature. Let $U_{\hbar}(\mathfrak{g})$ be the asymptotic enveloping algebra of \mathfrak{g} (i.e. the Rees algebra of the algebra $U(\mathfrak{g})$ endowed with the Poincaré–Birkhoff–Witt filtration; see § 2.1 for details). For $\lambda \in \mathbb{X}$ we let $\mathbf{M}(\lambda)$ be the asymptotic universal Verma module associated with λ , a graded $(S_{\hbar}, U_{\hbar}(\mathfrak{g}))$ -bimodule whose precise definition is recalled in § 2.1. Then we set

$$\mathcal{M}_{V,\lambda}^{\text{alg}} := \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))$$

where we consider morphisms in the category of $(S_{\hbar}, U_{\hbar}(\mathfrak{g}))$ -bimodules.

We will also construct a third family of graded modules, of “topological” nature, which is associated with the “Langlands dual data”. Let $\check{\mathfrak{t}} := \text{Lie}(\check{T})$. We have canonical identifications $\mathbb{X} \cong X_*(\check{T})$ and $S_{\hbar} \cong S(\check{\mathfrak{t}}^*)[\hbar]$. Consider the category $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ of $\check{G}(\mathbf{O})$ -equivariant perverse sheaves on the affine Grassmannian \mathbf{Gr} of \check{G} . Then for any $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ we set

$$\mathcal{M}_{\mathcal{F},\lambda}^{\text{top}} := H_{\check{T} \times \mathbb{C}^{\times}}^{\bullet+\lambda(2\check{\rho})}(i_{\lambda}^! \mathcal{F}).$$

Here i_{λ} is the inclusion of the point of \mathbf{Gr} naturally associated with λ , \mathbb{C}^{\times} acts on \mathbf{Gr} by loop rotation, and $\check{\rho}$ is the half-sum of positive coroots of G . Then $\mathcal{M}_{\mathcal{F},\lambda}^{\text{top}}$ is in a natural way a graded S_{\hbar} -module.

1.3.

Each of these families is endowed with a kind of “symmetry” governed by the Weyl group W of (G, T) or (\check{G}, \check{T}) . (Note that these Weyl groups can be canonically identified.)

Namely, we have isomorphisms of graded S_{\hbar} -modules

$$A_{V,\lambda,w} : \mathcal{M}_{V,\lambda} \xrightarrow{\sim} {}^w\mathcal{M}_{V,w\lambda} \quad \text{or} \quad A_{\mathcal{F},\lambda,w} : \mathcal{M}_{\mathcal{F},\lambda} \xrightarrow{\sim} {}^w\mathcal{M}_{\mathcal{F},w\lambda}$$

for all $w \in W$. (Here the twist functor ${}^w(\cdot)$ will be defined in § 2.4.)

In the “geometric” case, the isomorphisms $A_{V,\lambda,w}^{\text{geom}}$ are constructed using a W -action on $\mathcal{D}_{\hbar}(\mathcal{X})$ given by partial Fourier transforms due to Gelfand and Graev and studied in particular by Bezrukavnikov, Braverman and Positselskii in [8]. These operators depend on a choice of (non-zero) simple root vectors in \mathfrak{g} , which we choose to be those provided by the geometric Satake equivalence.

In the “algebraic” setting, the isomorphisms $A_{V,\lambda,w}^{\text{alg}}$ are constructed using properties of intertwining operators, between a Verma module and a tensor product of a G -module and a Verma module. Our constructions are “renormalized” variants of classical constructions appearing in the definition of the dynamical Weyl group (see [19, 39]) but, as opposed to those considered in *loc. cit.*, our isomorphisms do not have poles. Again, the operators $A_{V,\lambda,w}^{\text{alg}}$ depend on a choice of simple root vectors in \mathfrak{g} , which we choose as above.

In the “topological” setting, the isomorphisms $A_{\mathcal{F},\lambda,w}^{\text{top}}$ are induced by the action of $N_{\check{G}}(\check{T})$ on Gr by left multiplication.

In each setting, the collection of operators is compatible with the product in W in the sense that

$${}^y(A_{V,y\lambda,x}) \circ A_{V,\lambda,y} = A_{V,\lambda,xy} \quad \text{or} \quad {}^y(A_{\mathcal{F},y\lambda,x}) \circ A_{\mathcal{F},\lambda,y} = A_{\mathcal{F},\lambda,xy}$$

for any λ, V, \mathcal{F} as above and $x, y \in W$.

1.4.

In addition, these families of graded modules are endowed with morphisms

$$\begin{aligned} \text{Conv}_{V,V',\lambda,\mu} &: \mathcal{M}_{V,\lambda} \otimes_{S_{\hbar}} {}^{(\lambda)}\mathcal{M}_{V',\mu} \rightarrow \mathcal{M}_{V \otimes V', \lambda + \mu}, \\ \text{Conv}_{\mathcal{F},\mathcal{F}',\lambda,\mu} &: \mathcal{M}_{\mathcal{F},\lambda} \otimes_{S_{\hbar}} {}^{(\lambda)}\mathcal{M}_{\mathcal{F}',\mu} \rightarrow \mathcal{M}_{\mathcal{F} \star \mathcal{F}', \lambda + \mu} \end{aligned}$$

related to the monoidal structure on the category $\text{Rep}(G)$ (denoted as \otimes) or $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$ (denoted as \star).

In the “geometric” setting, morphisms $\text{Conv}_{V,V',\lambda,\mu}^{\text{geom}}$ are induced by the product in the algebra $\mathcal{D}_{\hbar}(\mathcal{X})$. In the “algebraic” setting, morphisms $\text{Conv}_{V,V',\lambda,\mu}^{\text{alg}}$ are induced by composition of morphisms of bimodules. In the “topological” setting, morphisms $\text{Conv}_{\mathcal{F},\mathcal{F}',\lambda,\mu}^{\text{top}}$ are defined using a standard construction considered in particular in [3].

1.5.

Our main result might be stated as follows (see Corollary 2.4.2, Theorem 2.5.5, and Proposition 8.1.5).

Theorem. *For \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$ and $\lambda \in \mathbb{X}$ there exist canonical isomorphisms*

$$\mathcal{M}_{\mathcal{F},\lambda}^{\text{top}} \cong \mathcal{M}_{S(\mathcal{F}),\lambda}^{\text{geom}} \cong \mathcal{M}_{S(\mathcal{F}),\lambda}^{\text{alg}},$$

where $\mathbb{S} : \text{Perv}_{\check{G}(\mathfrak{O})}(\text{Gr}) \xrightarrow{\sim} \text{Rep}(G)$ is the geometric Satake equivalence. These families of isomorphisms are compatible with operators \mathbf{A} and with morphisms Conv .

The proof of this theorem is based on another crucial property of the modules $\mathcal{M}_{V,\lambda}^{\text{geom}}$, $\mathcal{M}_{V,\lambda}^{\text{alg}}$ and $\mathcal{M}_{\mathcal{F},\lambda}^{\text{top}}$: they are all compatible with restriction to a Levi subgroup in the appropriate sense. This property is used to reduce the proof of our claims to the case G and \check{G} have semisimple rank 1, in which case they can be checked by explicit computation. This strategy is rather classical in this context; see e.g. [1, 9, 10, 12].

1.6.

The present paper is closely related to, and motivated by, results of [3] and [9]. In fact, in a follow-up paper the results of the present article will be used to obtain a common generalization of the equivalences of categories established in these papers. A similar generalization can also be obtained using recent results of Dodd [18], but our approach is different and, we believe, more explicit. We will follow the strategy of [3] and a key technical step in our approach is the following algebra isomorphism, which is a “quantum” analogue of [3, Theorem 8.5.2] and which follows from the theorem stated in § 1.5:

$$\bigoplus_{\lambda \in \mathbb{X}^+} \text{Ext}_{\check{T} \times \mathbb{C}^\times}^\bullet(\mathcal{R}_G, \mathcal{W}^\lambda \star \mathcal{R}_G) \cong U_{\hbar}(\mathfrak{g}) \times \left(\bigoplus_{\lambda \in \mathbb{X}^+} \mathcal{D}_{\hbar}(\mathcal{X})_\lambda \right).$$

Here, \mathcal{W}^λ is the Wakimoto sheaf associated with λ , \mathcal{R}_G is an ind-perverse sheaf on Gr corresponding to the regular representation of G , and we refer the reader to [3, § 8] for this and other unexplained notation.

1.7.

We will also consider “classical analogues” of the above constructions, by which we mean specializing \hbar to 0, and hence replacing \mathbf{S}_{\hbar} by $\mathbf{S}(\mathfrak{t})$ or $\mathbf{S}(\check{\mathfrak{t}}^*)$. The classical analogues of \mathcal{M}^{top} are easy to define: we simply set

$$\overline{\mathcal{M}}_{\mathcal{F},\lambda}^{\text{top}} := \mathbf{H}_{\check{T}}^{\bullet+\lambda(2\check{\rho})}(i_{\lambda}^! \mathcal{F}).$$

We also have morphisms $\overline{\mathbf{A}}^{\text{top}}$ and $\overline{\text{Conv}}^{\text{top}}$ given by the same constructions as for \mathbf{A}^{top} and Conv^{top} .

There is no interesting classical analogue of \mathcal{M}^{alg} . The classical analogues of $\mathcal{M}^{\text{geom}}$ are defined using the geometry of the Grothendieck–Springer resolution $\check{\mathfrak{g}}$. More precisely, we set

$$\overline{\mathcal{M}}_{V,\lambda}^{\text{geom}} := (V \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda)))^G$$

where $\mathcal{O}_{\check{\mathfrak{g}}}(\lambda)$ is the G -equivariant line bundle on $\check{\mathfrak{g}}$ associated with λ . The operators $\overline{\text{Conv}}^{\text{geom}}$ are induced by the natural morphisms

$$\Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda)) \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\mu)) \rightarrow \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda + \mu)).$$

Finally, the operators $\overline{\mathbf{A}}^{\text{geom}}$ are defined using the W -action on the regular part of $\check{\mathfrak{g}}$. Again, these operators depend on a choice of simple root vectors in \mathfrak{g} . This construction seems to be new, and has interesting consequences (see § 5.5).

Then we prove the following (see Corollary 2.4.5, Theorem 2.5.7 and Remark 8.1.6(2)).

Theorem. For \mathcal{F} in $\text{Perv}_{G(\mathbf{O})}(\text{Gr})$ and $\lambda \in \mathbb{X}$, there exist canonical isomorphisms

$$\overline{\mathcal{M}}_{\mathcal{F},\lambda}^{\text{top}} \cong \overline{\mathcal{M}}_{\mathbb{S}(\mathcal{F}),\lambda}^{\text{geom}}.$$

This family of isomorphisms is compatible with operators $\overline{\mathbf{A}}$ and with morphisms $\overline{\text{Conv}}$.

The modules appearing in the theorem (and the corresponding morphisms) are related to those appearing in the theorem of § 1.5 by the functor $\mathbb{C} \otimes_{\mathbb{C}[\hbar]} (-)$ (where \hbar acts by zero on \mathbb{C}). For \mathcal{M}^{top} and $\overline{\mathcal{M}}^{\text{top}}$, this easily follows from the parity vanishing of $\mathbf{H}^*(i_\lambda^! \mathcal{F})$; see Lemma 6.2.4. For $\mathcal{M}^{\text{geom}}$ and $\overline{\mathcal{M}}^{\text{geom}}$, this requires a more subtle argument; see § 3.5. In particular, our results establish a relation between the automorphisms of $\mathcal{D}_\hbar(\mathcal{X})$ induced by partial Fourier transforms and the W -action on the regular part of $\tilde{\mathfrak{g}}$, which seems to be new.

1.8.

As applications of our constructions we give new proofs of two results: a geometric description of the Brylinski–Kostant filtration due to the first author (see [22]), and a geometric construction of the dynamical Weyl group due to Braverman and Finkelberg (see [12]). We also observe that some of our technical preliminary results have interesting applications: they allow us to give simpler proofs of results on the structure of the algebra $\mathcal{D}(\mathcal{X})$ of differential operators on \mathcal{X} (see § 3.6) and to construct an action of W on the regular part of $T^*\mathcal{X}$ which “lifts” the action on the regular part of $\tilde{\mathfrak{g}}$; see § 5.5.

One important tool in the first proof of the geometric Satake equivalence in [22] was the specialized equivariant cohomology of cofibers (see in particular [*loc. cit.*, § 3.5]), while in [35] the authors replaced this tool by the cohomology of corestrictions to semi-infinite orbits \mathfrak{T}_λ . Our descriptions of $\mathbf{H}_T^*(i_\lambda^! \mathcal{F})$, $\mathbf{H}_T^*(t_\lambda^! \mathcal{F})$ (where t_λ denotes the inclusion of \mathfrak{T}_λ) and the natural morphism between them (see Theorem 2.3.1) shed some light on the precise relation between these points of view.

1.9. Description of the paper

In Section 2 we define our main players, and state our main results. In Section 3 we study the modules $\mathcal{M}_{V,\lambda}^{\text{geom}}$ and define their symmetries. In Section 4 we study the modules $\mathcal{M}_{V,\lambda}^{\text{alg}}$, define their symmetries, and relate this algebraic family to the geometric one. In Section 5 we study the modules $\overline{\mathcal{M}}_{V,\lambda}^{\text{geom}}$, define their symmetries, and relate them to the modules $\mathcal{M}_{V,\lambda}^{\text{geom}}$. In Section 6 we recall the construction of the geometric Satake equivalence and its main properties. In Section 7 we prove our main results. In Section 8 we give some complements and applications of these results. Finally, the paper finishes with two appendices: Appendix A collects computations in semisimple rank 1 that are needed in our proofs, and Appendix B is a reminder on partial Fourier transforms for (asymptotic) \mathcal{D} -modules.

1.10. Conventions

Throughout, we will work over the ground field \mathbb{C} of complex numbers and write $\otimes = \otimes_{\mathbb{C}}$. If $M = \bigoplus_{n \in \mathbb{Z}} M_n$ is a graded vector space and $m \in \mathbb{Z}$, we define the graded vector space $M\langle m \rangle$ by the following rule: $(M\langle m \rangle)_n = M_{n-m}$. Note that $\langle 1 \rangle$ is a “homological” shift, i.e. it shifts graded vector spaces to the *right*. We will always consider $\mathbb{C}[\hbar]$ as a graded algebra where \hbar has degree 2. If A and B are $\mathbb{C}[\hbar]$ -algebras, by an (A, B) -bimodule we mean an $(A \otimes_{\mathbb{C}[\hbar]} B^{\text{op}})$ -module. If A is an algebra, we write $\text{Hom}_{-A}(-, -)$ for $\text{Hom}_{A^{\text{op}}}(-, -)$.

2. Statement of the main results

2.1. Asymptotic Verma modules

Given a filtered \mathbb{C} -algebra $A = \bigcup_{i \in \mathbb{Z}_{\geq 0}} F_i A$, we let A_{\hbar} be the Rees algebra (sometimes referred to as the “graded” or “asymptotic” version) of the filtered algebra A . It can be defined as the following subalgebra of $A[\hbar]$:

$$A_{\hbar} := \bigoplus_{i \in \mathbb{Z}} A_{\hbar}^i \quad \text{with } A_{\hbar}^i = \begin{cases} 0 & \text{if } i \text{ is odd;} \\ \hbar^i \cdot F_{i/2} A & \text{if } i \text{ is even.} \end{cases} \tag{2.1.1}$$

Thus, A_{\hbar} is a graded $\mathbb{C}[\hbar]$ -algebra, where the indeterminate \hbar has grade degree 2. (The reason for our convention will become clear later.) Moreover, one has a natural isomorphism

$$A_{\hbar}/\hbar \cdot A_{\hbar} \cong \text{gr}^F A$$

(where degrees are doubled on the left-hand side).

If \mathfrak{k} is a Lie algebra, the enveloping algebra $U(\mathfrak{k})$ comes equipped with a natural ascending filtration, the Poincaré–Birkhoff–Witt filtration, such that $\text{gr } U(\mathfrak{k}) = S(\mathfrak{k})$. The corresponding asymptotic enveloping algebra $U_{\hbar}(\mathfrak{k}) := U(\mathfrak{k})_{\hbar}$ has an alternative (equivalent) definition as the $\mathbb{C}[\hbar]$ -algebra generated by \mathfrak{k} , with relations $xy - yx = \hbar[x, y]$ for $x, y \in \mathfrak{k}$. Here elements of \mathfrak{k} have degree 2. We will use this description of $U_{\hbar}(\mathfrak{k})$, and still denote by x the image of an element $x \in \mathfrak{k}$. (If we were using the description (2.1.1), this element should rather be denoted as $\hbar x$.)

Let G be a connected reductive group over \mathbb{C} with Lie algebra \mathfrak{g} . We fix a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$, so $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$ is a Borel subalgebra. Let T be the maximal torus and $B = T \cdot U$ the Borel subgroup corresponding to the Lie algebras \mathfrak{t} and \mathfrak{b} , respectively. We will denote by R the set of roots of G (relative to T), by R^+ the positive roots (i.e. the roots of \mathfrak{u}), and by W the Weyl group of (G, T) . Let $\rho \in \mathfrak{t}^*$ (resp. $\check{\rho} \in \mathfrak{t}$) be the half-sum of positive roots (resp. coroots). We also let \mathbb{X} be the lattice of characters of T , and \mathbb{X}^+ (resp. \mathbb{X}^-) be the subsemigroup of dominant (resp. antidominant) weights. We will frequently consider elements of \mathfrak{t}^* (resp. of \mathbb{X}) as linear forms on \mathfrak{b} (resp. characters of B) which are trivial on \mathfrak{u} (resp. on U). Also, as usual, when convenient we identify \mathbb{X} with a subset of \mathfrak{t}^* via the differential.

We consider asymptotic $\mathbb{C}[\hbar]$ -algebras $U_{\hbar}(\mathfrak{g})$ and $U_{\hbar}(\mathfrak{t})$. The latter algebra is a commutative graded algebra which is clearly isomorphic to $S_{\hbar} := S(\mathfrak{t})[\hbar]$ where $S(\mathfrak{t})$, the symmetric algebra of \mathfrak{t} , is equipped with its natural grading. Let $Z(\mathfrak{g})$ be the center of the algebra $U(\mathfrak{g})$. The Poincaré–Birkhoff–Witt filtration on $U(\mathfrak{g})$ induces, by restriction,

a filtration on $Z(\mathfrak{g})$. The corresponding asymptotic algebra $Z_{\hbar}(\mathfrak{g})$ is the center of the algebra $U_{\hbar}(\mathfrak{g})$. One has a Harish-Chandra isomorphism $Z_{\hbar}(\mathfrak{g}) \cong S_{\hbar}^W = S(\mathfrak{t})^W[\hbar]$, a graded $\mathbb{C}[\hbar]$ -algebra isomorphism induced by the composition

$$Z_{\hbar}(\mathfrak{g}) \hookrightarrow U_{\hbar}(\mathfrak{g}) \twoheadrightarrow U_{\hbar}(\mathfrak{g})/(\mathfrak{u} \cdot U_{\hbar}(\mathfrak{g}) + U_{\hbar}(\mathfrak{g}) \cdot \mathfrak{u}^-) \xleftarrow{\sim} S_{\hbar} \xrightarrow{\sim} S_{\hbar}$$

where the isomorphism on the right-hand side sends $t \in \mathfrak{t}$ to $t + \hbar\rho(t)$. Using this isomorphism, we may (and will) identify the algebra $Z_{\hbar}(\mathfrak{g})$ with a subalgebra of S_{\hbar} .

For any $\lambda \in \mathfrak{t}^*$, let $S_{\hbar}\langle\lambda\rangle$ be the $U_{\hbar}(\mathfrak{b})$ -bimodule defined as follows. As a $\mathbb{C}[\hbar]$ -module, it is isomorphic to S_{\hbar} . The left $U_{\hbar}(\mathfrak{b})$ -module structure is given by the natural isomorphism $S_{\hbar} \cong U_{\hbar}(\mathfrak{b})/U_{\hbar}(\mathfrak{b}) \cdot \mathfrak{n}$. Then in the right $U_{\hbar}(\mathfrak{b})$ -module structure, the Lie ideal $\mathfrak{u} \subset \mathfrak{b}$ acts by 0, and $t \in \mathfrak{t}$ acts by multiplication by $t + \hbar\lambda(t)$.

We define a graded $(U_{\hbar}(\mathfrak{b}), U_{\hbar}(\mathfrak{g}))$ -bimodule, a certain asymptotic version of the universal Verma module, as follows:

$$\mathbf{M}(\lambda) = S_{\hbar}\langle\lambda + \rho\rangle \otimes_{U_{\hbar}(\mathfrak{b})} U_{\hbar}(\mathfrak{g}).$$

(We will mainly only consider $\mathbf{M}(\lambda)$ as an $(S_{\hbar}, U_{\hbar}(\mathfrak{g}))$ -bimodule.) The action of $U_{\hbar}(\mathfrak{g})$ on the vector $\mathbf{v}_{\lambda} := 1 \otimes 1 \in \mathbf{M}(\lambda)$ induces an isomorphism of right $U_{\hbar}(\mathfrak{g})$ -modules

$$\mathbf{M}(\lambda) \cong U_{\hbar}(\mathfrak{g})/(\mathfrak{u} \cdot U_{\hbar}(\mathfrak{g})). \tag{2.1.2}$$

Under this isomorphism, the right S_{\hbar} -module structure is such that the action of $t \in \mathfrak{t}$ is induced by right multiplication by $t - \hbar \cdot (\lambda + \rho)(t)$ on $U_{\hbar}(\mathfrak{g})$.

It is immediate from the definitions that there is well-defined ‘adjoint’ action $b : m \mapsto \text{ad } b(m)$ of the Lie algebra \mathfrak{b} on $\mathbf{M}(\lambda)$, which is related to the bimodule structure by the equation

$$\hbar \cdot \text{ad } b(m) = (b + \hbar\rho(b)) \cdot m - m \cdot b \quad \forall b \in \mathfrak{b}.$$

The adjoint action of the subalgebra $\mathfrak{t} \subset \mathfrak{b}$ is semisimple. Therefore, one has a weight decomposition $\mathbf{M}(\lambda) = \bigoplus_{\mu \in \mathfrak{t}^*} \mathbf{M}(\lambda)_{\mu}$. In particular, we have $\mathbf{M}(\lambda)_{-\lambda} = \mathbb{C}[\hbar] \cdot \mathbf{v}_{\lambda}$, and $\mathbf{M}(\lambda)_{\mu} = 0$ unless $\mu \in -\lambda - \mathbb{Z}_{\geq 0} R^+$. For $\lambda \in \mathbb{X}$, the adjoint action on $\mathbf{M}(\lambda)$ can be exponentiated to an algebraic B -action.

Let $\text{Rep}(G)$ be the tensor category of finite dimensional rational G -modules. For $V \in \text{Rep}(G)$ and $\lambda \in \mathbb{X}$, let V_{λ} denote the T -weight space of V of weight λ .

The assignment $\hbar \mapsto 1 \otimes \hbar$, $x \mapsto -x \otimes \hbar + 1 \otimes x$ has a unique extension to an algebra homomorphism $U_{\hbar}(\mathfrak{g}) \rightarrow U(\mathfrak{g})^{\text{op}} \otimes U_{\hbar}(\mathfrak{g})$. Via this homomorphism, for any right $U_{\hbar}(\mathfrak{g})$ -module M and $V \in \text{Rep}(G)$, the vector space $V \otimes M$ acquires the structure of a right $U_{\hbar}(\mathfrak{g})$ -module. This gives an $(U_{\hbar}(\mathfrak{b}), U_{\hbar}(\mathfrak{g}))$ -bimodule structure on $V \otimes \mathbf{M}(\lambda)$, where the left action of the algebra $U_{\hbar}(\mathfrak{b})$ on $V \otimes \mathbf{M}(\lambda)$ comes from its action on $\mathbf{M}(\lambda)$ on the left. If $\lambda \in \mathbb{X}$, the differential of the diagonal B -action on $V \otimes \mathbf{M}(\lambda)$ and the bimodule structure are related as follows: if $b \in \mathfrak{b} \subset U(\mathfrak{b})$, \hbar times the action of b is given by the assignment $n \mapsto (b + \hbar\rho(b)) \cdot n - n \cdot b$.

One has a natural morphism of left S_{\hbar} -modules $p_{\lambda} : \mathbf{M}(\lambda) \rightarrow S_{\hbar}$, induced by the projection $U_{\hbar}(\mathfrak{g}) \rightarrow U_{\hbar}(\mathfrak{b})$ orthogonal to $U_{\hbar}(\mathfrak{g}) \cdot \mathfrak{u}^-$. If $\lambda \in \mathbb{X}$, then p_{λ} is also a morphism of T -modules $\mathbf{M}(\lambda) \rightarrow S_{\hbar} \otimes \mathbb{C}_{-\lambda}$. An important role will be played below by the morphism of

graded S_{\hbar} -modules $\kappa_{V,\lambda}^{\text{alg}}$ defined (for V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$) as the following composition:

$$(V \otimes \mathbf{M}(\lambda))^B \hookrightarrow (V \otimes \mathbf{M}(\lambda))^T \xrightarrow{\text{id}_V \otimes p_\lambda} (V \otimes S_{\hbar} \otimes \mathbb{C}_{-\lambda})^T = V_\lambda \otimes S_{\hbar}. \tag{2.1.3}$$

Note that $(V \otimes \mathbf{M}(\lambda))^B$ has a natural structure of a $Z_{\hbar}(\mathfrak{g})$ -module, induced by the (right) action of $Z_{\hbar}(\mathfrak{g}) \subset U_{\hbar}(\mathfrak{g})$ on $\mathbf{M}(\lambda)$. With this definition, $\kappa_{V,\lambda}^{\text{alg}}$ is also a morphism of $Z_{\hbar}(\mathfrak{g})$ -modules, where $Z_{\hbar}(\mathfrak{g}) = S_{\hbar}^W$ acts on S_{\hbar} via the restriction of the right action of S_{\hbar} on $S_{\hbar}\langle\langle\lambda\rangle\rangle$.

2.2. The affine Grassmannian: equivariant cohomology of cofibers

Write \mathbb{G}_m for the multiplicative group. Let \check{G} be the Langlands dual group of G . The group \check{G} comes equipped with the maximal torus $\check{T} \subset \check{G}$, with opposite Borel subgroups $\check{B} = \check{T} \cdot \check{U}$ and $\check{B}^- = \check{T} \cdot \check{U}^-$, and with a canonical isomorphism $\mathbb{X} = \text{Hom}(\mathbb{G}_m, \check{T})$, the cocharacter lattice of \check{T} . (To be completely precise, one should first choose $\check{G}, \check{B}, \check{T}$, and then use the affine Grassmannian of \check{G} to define G, B, T by the Tannakian formalism; see § 6.1 for details.)

Let $\mathbf{K} = \mathbb{C}\langle\langle z \rangle\rangle$ (resp. $\mathbf{O} = \mathbb{C}\llbracket z \rrbracket$). Let $\text{Gr}_{\check{G}} := \check{G}(\mathbf{K})/\check{G}(\mathbf{O})$ (resp. $\text{Gr}_{\check{T}} := \check{T}(\mathbf{K})/\check{T}(\mathbf{O})$) be the affine Grassmannian associated with the group \check{G} (resp. \check{T}). (We will consider the *reduced* ind-scheme structure on these affine Grassmannians.) Thus, one has $\mathbb{X} = \text{Gr}_{\check{T}}$ and there is a natural embedding $\mathbb{X} = \text{Gr}_{\check{T}} \hookrightarrow \text{Gr}_{\check{G}}$. For $\lambda \in \mathbb{X}$, we let λ be the image of λ and let $i_\lambda : \{\lambda\} \hookrightarrow \text{Gr}_{\check{G}}$ denote the one-point embedding. The group $\check{G}(\mathbf{K}) \rtimes \mathbb{G}_m$ acts on $\text{Gr}_{\check{G}}$ on the left, where the factor \mathbb{G}_m acts by rotation of the loop.

For the rest of this section, we will use simplified notation $\text{Gr} := \text{Gr}_{\check{G}}$. The following subsets of the affine Grassmannian will play an important role. For $\lambda \in \mathbb{X}^+$, we let $\text{Gr}^\lambda := \check{G}(\mathbf{O}) \cdot \lambda$. This is a finite dimensional $(\check{G}(\mathbf{O}) \rtimes \mathbb{G}_m)$ -stable locally closed subvariety of Gr . One has a stratification $\text{Gr} = \sqcup_{\lambda \in \mathbb{X}^+} \text{Gr}^\lambda$. Further, for any $\lambda \in \mathbb{X}$, following Mirković and Vilonen one puts $\mathfrak{T}_\lambda := \check{U}^-(\mathbf{K}) \cdot \lambda$. We let $t_\lambda : \mathfrak{T}_\lambda \hookrightarrow \text{Gr}$ be the inclusion.

Let $A := \check{T} \times \mathbb{G}_m$, a toral subgroup of $\check{G}(\mathbf{K}) \rtimes \mathbb{G}_m$. The Mirković–Vilonen space \mathfrak{T}_λ is A -stable. Further, the set $\mathbb{X} \subset \text{Gr}$ is known to be equal to the set of A -fixed points in Gr . Therefore, for any object \mathcal{F} of the equivariant derived category $\mathcal{D}_A^b(\text{Gr})$, there are well-defined A -equivariant cohomology groups $H_A^*(\mathfrak{T}_\lambda, i_\lambda^! \mathcal{F})$ (resp. $H_A^*(i_\lambda^! \mathcal{F})$). These are graded modules over the graded algebra $H_A^*(\text{pt}) \cong S(\mathfrak{t})[\hbar] = S_{\hbar}$.

Let $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$ (resp. $\text{Perv}_{\check{G}(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr})$) be the category of $\check{G}(\mathbf{O})$ -equivariant (resp. $\check{G}(\mathbf{O}) \rtimes \mathbb{G}_m$ -equivariant) perverse sheaves on Gr . Let also $\text{Perv}_{\check{G}(\mathbf{O})\text{-mon}}(\text{Gr})$ be the category of perverse sheaves on Gr which are constructible with respect to the stratification by $\check{G}(\mathbf{O})$ -orbits. Recall that all three of these categories are semisimple, with simple objects parameterized by \mathbb{X}^+ . In particular, the forgetful functors

$$\text{Perv}_{\check{G}(\mathbf{O}) \rtimes \mathbb{G}_m}(\text{Gr}) \rightarrow \text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}) \rightarrow \text{Perv}_{\check{G}(\mathbf{O})\text{-mon}}(\text{Gr})$$

are equivalences of categories (see [35, Appendix A] for a similar result in a much more general situation). Let

$$\mathbb{S} : \text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}) \xrightarrow{\sim} \text{Rep}(G)$$

be the geometric Satake equivalence. By the remark above, any object of $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$ can be considered naturally as an object of $\mathcal{D}_A^b(\text{Gr})$.

The following lemma is a simple consequence of results of Kazhdan and Lusztig [30] and Mirković and Vilonen [35]; cf. also [12, 42]. It will be proved in § 6.2.

Lemma 2.2.1. *For any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ and $\lambda \in \mathbb{X}$, one has:*

- (1) *The graded S_{\hbar} -module $H_A^*(i_{\lambda}^! \mathcal{F})$ (resp. the graded $S(\mathfrak{t})$ -module $H_T^*(i_{\lambda}^! \mathcal{F})$) is free.*
- (2) *There is a canonical isomorphism of graded S_{\hbar} -modules (resp. of graded $S(\mathfrak{t})$ -modules)*

$$H_A^*(\mathfrak{T}_{\lambda}, i_{\lambda}^! \mathcal{F}) \cong (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes S_{\hbar}(\lambda(2\check{\rho})), \quad H_T^*(\mathfrak{T}_{\lambda}, i_{\lambda}^! \mathcal{F}) \cong (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes S(\mathfrak{t})(\lambda(2\check{\rho})). \tag{2.2.2}$$

One may factor the embedding $i_{\lambda} : \{\lambda\} \hookrightarrow \text{Gr}$ as a composition $\{\lambda\} \xrightarrow{i_{\lambda}} \mathfrak{T}_{\lambda} \xrightarrow{i_{\lambda}} \text{Gr}$. Hence, there is a push-forward morphism

$$(i_{\lambda})_! : H_A^*(i_{\lambda}^! \mathcal{F}) = H_A^*(i_{\lambda}^! i_{\lambda}^! \mathcal{F}) \longrightarrow H_A^*(\mathfrak{T}_{\lambda}, i_{\lambda}^! \mathcal{F}).$$

Let $\kappa_{\mathcal{F}, \lambda}^{\text{top}}$ be the following composite morphism:

$$\kappa_{\mathcal{F}, \lambda}^{\text{top}} : H_A^*(i_{\lambda}^! \mathcal{F}) \xrightarrow{(i_{\lambda})_!} H_A^*(\mathfrak{T}_{\lambda}, i_{\lambda}^! \mathcal{F}) \xrightarrow[\sim]{(2.2.2)} (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes S_{\hbar}(\lambda(2\check{\rho})). \tag{2.2.3}$$

Thus, we get a diagram of morphisms of graded S_{\hbar} -modules

$$(\mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(\lambda))^B(\lambda(2\check{\rho})) \xrightarrow[\text{(2.1.3)}]{\kappa_{\mathbb{S}(\mathcal{F}), \lambda}^{\text{alg}}(\lambda(2\check{\rho}))} (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes S_{\hbar}(\lambda(2\check{\rho})) \xleftarrow[\text{(2.2.3)}]{\kappa_{\mathcal{F}, \lambda}^{\text{top}}} H_A^*(i_{\lambda}^! \mathcal{F}).$$

One of our key results (to be proved in § 7.6) reads as follows.

Theorem 2.2.4. *For any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$ and $\lambda \in \mathbb{X}$, the morphisms $\kappa_{\mathbb{S}(\mathcal{F}), \lambda}^{\text{alg}}(\lambda(2\check{\rho}))$ and $\kappa_{\mathcal{F}, \lambda}^{\text{top}}$ are injective and have the same image. Thus, there is a natural isomorphism of graded S_{\hbar} -modules $\zeta_{\mathcal{F}, \lambda}$ that fits into the following commutative diagram:*

$$\begin{array}{ccc} (\mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(\lambda))^B(\lambda(2\check{\rho})) & \xrightarrow[\sim]{\zeta_{\mathcal{F}, \lambda}} & H_A^*(i_{\lambda}^! \mathcal{F}) \\ \downarrow \kappa_{\mathbb{S}(\mathcal{F}), \lambda}^{\text{alg}}(\lambda(2\check{\rho})) & & \downarrow \kappa_{\mathcal{F}, \lambda}^{\text{top}} \\ (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes S_{\hbar}(\lambda(2\check{\rho})) & \xlongequal{\quad} & (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes S_{\hbar}(\lambda(2\check{\rho})). \end{array} \tag{2.2.5}$$

Remark 2.2.6. We have defined in § 2.1 an action of $Z_{\hbar}(\mathfrak{g})$ on $(\mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(\lambda))^B$. On the other hand, it is explained in [9, § 2.4] that $H_A^*(i_{\lambda}^! \mathcal{F})$ also has a natural action of $Z_{\hbar}(\mathfrak{g}) = S_{\hbar}^W$ coming from the natural map $\text{Gr} = (\check{G}(\mathbf{K}) \rtimes \mathbb{G}_m) / (\check{G}(\mathbf{O}) \rtimes \mathbb{G}_m) \rightarrow \text{pt} / (\check{G}(\mathbf{O}) \rtimes \mathbb{G}_m)$. We claim that our isomorphism $\zeta_{\mathcal{F}, \lambda}$ is also $Z_{\hbar}(\mathfrak{g})$ -equivariant.

First, the action of $S_{\hbar} \otimes Z_{\hbar}(\mathfrak{g})$ on $H_A^*(i_{\lambda}^! \mathcal{F})$ factors through an action of $S_{\hbar} \otimes_{\mathbb{C}[\hbar]} Z_{\hbar}(\mathfrak{g}) = \mathbb{C}[\mathfrak{t}^* \times (\mathfrak{t}^*/W) \times \mathbb{A}^1]$. Then, the action of $S_{\hbar} \otimes_{\mathbb{C}[\hbar]} Z_{\hbar}(\mathfrak{g})$ factors through the natural action of the algebra $H_A^*(\lambda)$. Finally, it is explained in [9, § 3.2] that the $S_{\hbar} \otimes_{\mathbb{C}[\hbar]} Z_{\hbar}(\mathfrak{g})$ -algebra $H_A^*(\lambda)$ is isomorphic to the direct image under the natural quotient map of $\mathcal{O}_{\Gamma_{\lambda}}$, where

$$\Gamma_{\lambda} := \{(\eta_1, \eta_2, z) \in \mathfrak{t}^* \times \mathfrak{t}^* \times \mathbb{A}^1 \mid \eta_2 = \eta_1 + z\lambda\}.$$

The claim easily follows from these remarks and the S_{\hbar} -equivariance of $\zeta_{\mathcal{F}, \lambda}$.

Remark 2.2.7. Consider the case where $\mathcal{F} = IC_{\nu}$ is the IC-sheaf associated with the $\check{G}(\mathbf{O})$ -orbit Gr^{ν} for some $\nu \in \mathbb{X}$, and $\lambda = w_0\nu$ (where $w_0 \in W$ is the longest element). Then $V^{\nu} := \mathbb{S}(IC_{\nu})$ is a simple G -module with highest weight ν , and λ is the lowest weight of V^{ν} . In view of the right-hand isomorphism in Lemma 2.4.1 below, the image of the morphism $\kappa_{V^{\nu}, w_0\nu}^{alg}$ is computed by Kashiwara in [28]: namely, with our conventions, combining Theorem 1.7 and Proposition 1.8 in *loc. cit.*, we obtain that the image of $\kappa_{V^{\nu}, w_0\nu}^{alg}$ in $V_{w_0\nu}^{\nu} \otimes S_{\hbar} \cong S_{\hbar}$ is generated by the following element:

$$\prod_{\alpha \in R^+} \left(\prod_{j=0}^{-v(w_0\alpha)-1} (\check{\alpha} - j\hbar) \right). \tag{2.2.8}$$

The topological context is easy in this case. Namely we have

$$\mathfrak{T}_{w_0\nu} \cap \overline{Gr}^{\nu} = \mathfrak{T}_{w_0\nu} \cap Gr^{\nu} = \check{U}^-(\mathbf{O}) \cdot (w_0\nu) \cong \prod_{\alpha \in R^+} \left(\prod_{j=0}^{-v(w_0\alpha)-1} \mathbb{C}_{-\check{\alpha}+j\hbar} \right)$$

as A -varieties. One can easily deduce that the image of $\kappa_{IC_{\nu}, w_0\nu}^{top}$ is also generated by (2.2.8); see § 6.2. Hence, in this particular case, Theorem 2.2.4 can be directly deduced from these remarks.

In the case $\lambda = \nu$, one can also directly check that both $\kappa_{V^{\nu}, \nu}^{alg}$ and $\kappa_{IC_{\nu}, \nu}^{top}$ are isomorphisms.

2.3. The classical analogue

We will also prove an analogue of Theorem 2.2.4 where one replaces A by \check{T} . In this case the representation theory of the algebra $U_{\hbar}(\mathfrak{g})$ has to be replaced by the geometry of the algebraic variety \mathfrak{g}^* .

We will identify \mathfrak{t}^* with the subspace $(\mathfrak{g}/\mathfrak{u} \oplus \mathfrak{u}^-)^* \subset \mathfrak{g}^*$. In this way we obtain a canonical morphism $q : S(\mathfrak{g}/\mathfrak{u}) \rightarrow S(\mathfrak{t})$ induced by restriction of functions. For V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$, the ‘‘classical analogue’’ of the morphism $\kappa_{V, \lambda}^{alg}$, which we will denote by $\overline{\kappa}_{V, \lambda}^{alg}$, is the composition

$$(V \otimes S(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B \hookrightarrow (V \otimes S(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^T \xrightarrow{\text{id}_V \otimes q \otimes 1} (V \otimes S(\mathfrak{t}) \otimes \mathbb{C}_{-\lambda})^T = V_{\lambda} \otimes S(\mathfrak{t}).$$

This morphism is $S(\mathfrak{t})$ -equivariant, where the $S(\mathfrak{t})$ -action on the left-hand side is induced by the morphism $(\mathfrak{g}/\mathfrak{u})^* \rightarrow \mathfrak{t}^*$ given by restriction of linear maps.

Now we consider perverse sheaves on \mathbf{Gr} . For \mathcal{F} in $\mathbf{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ and $\lambda \in \mathbb{X}$, we will denote by $\bar{\kappa}_{\mathcal{F},\lambda}^{\text{top}}$ the following composite morphism:

$$\bar{\kappa}_{\mathcal{F},\lambda}^{\text{top}} : \mathbf{H}_{\check{T}}^*(i_{\lambda}^! \mathcal{F}) \xrightarrow{(i_{\lambda})!} \mathbf{H}_{\check{T}}^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) \xrightarrow[\sim]{(2.2.2)} (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes \mathbf{S}(\mathfrak{t})(\lambda(2\check{\rho})).$$

Then the classical analogue of Theorem 2.2.4 (to be proved in §7.7) reads as follows.

Theorem 2.3.1. *For any \mathcal{F} in $\mathbf{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ and $\lambda \in \mathbb{X}$, the morphisms $\bar{\kappa}_{\mathbb{S}(\mathcal{F}),\lambda}^{\text{alg}}(\lambda(2\check{\rho}))$ and $\bar{\kappa}_{\mathcal{F},\lambda}^{\text{top}}$ are injective and have the same image. Thus, there is a natural isomorphism of graded $\mathbf{S}(\mathfrak{t})$ -modules $\bar{\zeta}_{\mathcal{F},\lambda}$ that fits into the following commutative diagram:*

$$\begin{CD} (\mathbb{S}(\mathcal{F}) \otimes \mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B(\lambda(2\check{\rho})) @>[\bar{\zeta}_{\mathcal{F},\lambda}]>> \mathbf{H}_{\check{T}}^*(i_{\lambda}^! \mathcal{F}) \\ @VV\bar{\kappa}_{\mathbb{S}(\mathcal{F}),\lambda}^{\text{alg}}(\lambda(2\check{\rho}))V @VV\bar{\kappa}_{\mathcal{F},\lambda}^{\text{top}}V \\ (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes \mathbf{S}(\mathfrak{t})(\lambda(2\check{\rho})) @= (\mathbb{S}(\mathcal{F}))_{\lambda} \otimes \mathbf{S}(\mathfrak{t})(\lambda(2\check{\rho})). \end{CD} \tag{2.3.2}$$

2.4. Alternative descriptions: differential operators on G/U and intertwining operators for Verma modules

An important role in our arguments will be played by two alternative descriptions of the $\mathbb{C}[\hbar]$ -modules $(V \otimes \mathbf{M}(\lambda))^B$.

If X is a smooth algebraic variety, we write \mathcal{D}_X for the sheaf of differential operators on X . The sheaf \mathcal{D}_X comes equipped with a natural filtration by the order of differential operator. We let $\mathcal{D}_{\hbar,X}$ be the corresponding sheaf of asymptotic differential operators. As for enveloping algebras, this algebra has an alternative description as the sheaf of graded $\mathbb{C}[\hbar]$ -algebras generated (locally) by \mathcal{O}_X in degree 0 and the left \mathcal{O}_X -module \mathcal{T}_X (the tangent sheaf) in degree 2, with relations $\xi \cdot \xi' - \xi' \cdot \xi = \hbar[\xi, \xi']$ for $\xi, \xi' \in \mathcal{T}_X$ and $\xi \cdot f - f \cdot \xi = \hbar\xi(f)$ for $\xi \in \mathcal{T}_X$ and $f \in \mathcal{O}_X$. As for enveloping algebras, we will use this description of $\mathcal{D}_{\hbar,X}$ and still denote by ξ the image of an element $\xi \in \mathcal{T}_X$. (If we were using the description provided by (2.1.1), this element should rather be denoted as $\hbar\xi$.) Note that $\mathcal{D}_{\hbar,X}$ acts on $\mathcal{O}_X[\hbar]$ via $\xi \cdot f = \hbar\xi(f)$ for $\xi \in \mathcal{T}_X$ and $f \in \mathcal{O}_X$.

We put $\mathcal{D}(X) = \Gamma(X, \mathcal{D}_X)$, (resp. $\mathcal{D}_{\hbar}(X) = \Gamma(X, \mathcal{D}_{\hbar,X})$), for the corresponding algebra of global sections. The order filtration makes $\mathcal{D}(X)$ a filtered algebra; the associated Rees algebra $\mathcal{D}(X)_{\hbar}$ is canonically isomorphic to $\mathcal{D}_{\hbar}(X)$. There is also a canonical injective morphism $\mathcal{D}_{\hbar}(X)/\hbar \cdot \mathcal{D}_{\hbar}(X) \rightarrow \Gamma(X, \mathcal{D}_{\hbar,X}/\hbar \cdot \mathcal{D}_{\hbar,X})$, which is not surjective in general. Note finally that there exists a canonical algebra isomorphism $\mathcal{D}_{\hbar,X}/\hbar \cdot \mathcal{D}_{\hbar,X} \cong (p_X)_* \mathcal{O}_{T^*X}$, where T^*X is the cotangent bundle of X and $p_X : T^*X \rightarrow X$ is the projection.

Consider the quasi-affine variety $\mathcal{X} := G/U$. There is a natural $G \times T$ -action on \mathcal{X} defined as follows: $g \times t : hU \mapsto ghtU$. The T -action on \mathcal{X} also induces an action of T on $\mathcal{D}_{\hbar}(\mathcal{X})$ by algebra automorphisms. In particular, this T -action gives a weight decomposition $\mathcal{D}_{\hbar}(\mathcal{X}) = \bigoplus_{\lambda \in \mathbb{X}} \mathcal{D}_{\hbar}(\mathcal{X})_{\lambda}$. Thus, $\mathcal{D}_{\hbar}(\mathcal{X})_0 = \mathcal{D}_{\hbar}(\mathcal{X})^T$ is the algebra of right T -invariant asymptotic differential operators.

Differentiating the T -action on \mathcal{X} yields a morphism $\gamma : \mathcal{S}_\hbar \rightarrow \mathcal{D}_\hbar(\mathcal{X})$ of graded $\mathbb{C}[\hbar]$ -algebras. Using this we will consider $\mathcal{D}_\hbar(\mathcal{X})$ as an \mathcal{S}_\hbar -module where $t \in \mathfrak{t}$ acts by right multiplication by $t - \hbar\rho(t)$. Note also that differentiating the G -action on \mathcal{X} we obtain a morphism $Z_\hbar(\mathfrak{g}) \rightarrow \mathcal{D}_\hbar(\mathcal{X})$. This defines a $Z_\hbar(\mathfrak{g})$ -module structure on $\mathcal{D}_\hbar(\mathcal{X})$ (induced by multiplication on the left).

If M is an \mathcal{S}_\hbar -module and φ an algebra automorphism of \mathcal{S}_\hbar , we denote by ${}^\varphi M$ the \mathcal{S}_\hbar -module which coincides with M as a \mathbb{C} -vector space, and where $s \in \mathcal{S}_\hbar$ acts as $\varphi(m)$ acts on M . If φ, ψ are algebra automorphisms of \mathcal{S}_\hbar we have ${}^\psi({}^\varphi M) = {}^{\varphi \circ \psi} M$. This construction provides an autoequivalence of the category of \mathcal{S}_\hbar -modules, acting trivially on morphisms. We will use this notation in particular when $\varphi = w \in W$ (extended in the natural way to an automorphism of \mathcal{S}_\hbar), and for the following automorphisms: if $\mu \in \mathbb{X}$, we denote by $(\mu) : \mathcal{S}_\hbar \xrightarrow{\sim} \mathcal{S}_\hbar$ the automorphism which sends $t \in \mathfrak{t}$ to $t - \hbar\mu(t) \in \mathcal{S}_\hbar$. We will use similar notation for $\mathbf{S}(\mathfrak{t})$ -modules.

If M, N are $(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))$ -bimodules, we will write $\text{Hom}_{(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))}(M, N)$ for the space of morphisms of bimodules from M to N . It is an \mathcal{S}_\hbar -module and a $Z_\hbar(\mathfrak{g})$ -module in a natural way. If M, N are graded and M is finitely generated as a bimodule then this space is a graded \mathcal{S}_\hbar -module and a $Z_\hbar(\mathfrak{g})$ -module.

The following simple result will be proved in § 3.1 (for the first isomorphism) and § 4.3 (for the second isomorphism).

Lemma 2.4.1. *For any V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$, there are canonical isomorphisms of graded \mathcal{S}_\hbar -modules and $Z_\hbar(\mathfrak{g})$ -modules:*

$$(V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X}))_\lambda^G \cong (V \otimes \mathbf{M}(\lambda))^B \cong \text{Hom}_{(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)).$$

From Theorem 2.2.4 and Lemma 2.4.1 we deduce:

Corollary 2.4.2. *For any $\mathcal{F} \in \text{Perv}_{\check{G}(\mathbf{0})}(\text{Gr})$ and $\lambda \in \mathbb{X}$, there are natural isomorphisms of graded \mathcal{S}_\hbar - and $Z_\hbar(\mathfrak{g})$ -modules:*

$$(\mathbb{S}(\mathcal{F}) \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X}))_\lambda^G \langle \lambda(2\check{\rho}) \rangle \cong \mathbf{H}_A^*(i_\lambda^! \mathcal{F}) \cong \text{Hom}_{(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), \mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(\lambda)) \langle \lambda(2\check{\rho}) \rangle.$$

One can also give an alternative description of equivariant cohomology of cofibers in the “classical case” of § 2.3, as follows. Let $\mathcal{B} := G/B$. For $\lambda \in \mathbb{X}$, we denote by $\mathcal{O}_\mathcal{B}(\lambda)$ the line bundle on \mathcal{B} associated with the character $-\lambda$ of B (so ample line bundles correspond to dominant weights). For any variety X over \mathcal{B} , and any $\lambda \in \mathbb{X}$, we will denote by $\mathcal{O}_X(\lambda)$ the pull-back to X of $\mathcal{O}_\mathcal{B}(\lambda)$. In particular, consider the G -varieties $G/T \times \mathfrak{t}^*$ and $\tilde{\mathfrak{g}} := G \times_B (\mathfrak{g}/\mathfrak{u})^*$, which are both equipped with a natural morphism to \mathcal{B} . Both varieties are equipped with a natural action of $G \times \mathbb{C}^\times$, where the action of G is induced by left multiplication on G , and any $z \in \mathbb{C}^\times$ acts by multiplication by z^{-2} on $(\mathfrak{g}/\mathfrak{u})^*$ or \mathfrak{t}^* .

Consider the morphism

$$a : G/T \times \mathfrak{t}^* \rightarrow \tilde{\mathfrak{g}}, \quad (gT, \eta) \mapsto (g \times_B \eta)$$

where on the right-hand side η is considered as an element of \mathfrak{g}^* trivial on $\mathfrak{u} \oplus \mathfrak{u}^-$. For any $\lambda \in \mathbb{X}$ we have a canonical isomorphism $a^* \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda) \cong \mathcal{O}_{G/T \times \mathfrak{t}^*}(\lambda)$, so we obtain a pull-back morphism $a^* : \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \rightarrow \Gamma(G/T \times \mathfrak{t}^*, \mathcal{O}_{G/T \times \mathfrak{t}^*}(\lambda))$.

Note that we have a canonical isomorphism $\Gamma(G/T \times \mathfrak{t}^*, \mathcal{O}_{G/T \times \mathfrak{t}^*}(\lambda)) \cong \text{Ind}_T^G(-\lambda) \otimes \mathbf{S}(\mathfrak{t})$ where Ind is the usual induction functor for representations of algebraic groups, as defined e.g. in [26, §I.3.3]. Hence if V is in $\mathbf{Rep}(G)$, using the tensor identity and Frobenius reciprocity we obtain an isomorphism

$$(V \otimes \Gamma(G/T \times \mathfrak{t}^*, \mathcal{O}_{G/T \times \mathfrak{t}^*}(\lambda)))^G \cong (V \otimes \mathbf{S}(\mathfrak{t}) \otimes \mathbb{C}_{-\lambda})^T = V_\lambda \otimes \mathbf{S}(\mathfrak{t}). \tag{2.4.3}$$

By a similar argument, there exists a canonical isomorphism

$$(V \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda)))^G \cong (V \otimes \mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B. \tag{2.4.4}$$

Then one can easily check that, under isomorphisms (2.4.3) and (2.4.4), the morphism $\bar{\kappa}_{V,\lambda}^{\text{alg}}$ identifies with the morphism

$$(V \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda)))^G \rightarrow (V \otimes \Gamma(G/T \times \mathfrak{t}^*, \mathcal{O}_{G/T \times \mathfrak{t}^*}(\lambda)))^G$$

induced by a^* .

Using (2.4.4), from Theorem 2.3.1 we deduce the following description.

Corollary 2.4.5. *For any $\mathcal{F} \in \text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ and $\lambda \in \mathbb{X}$, there exists a natural isomorphism of graded $\mathbf{S}(\mathfrak{t})$ -modules*

$$\mathbf{H}_T^*(i_\lambda^! \mathcal{F}) \cong (\mathbb{S}(\mathcal{F}) \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda)))^G(\lambda(2\check{\rho})).$$

2.5. Weyl group symmetries

Each of the spaces in Corollaries 2.4.2 and 2.4.5 exhibits a kind of symmetry governed by the Weyl group W . These symmetries play a technical role in our proofs of Theorem 2.2.4 and 2.3.1. But we will also show that they are respected by the isomorphisms in Corollaries 2.4.2 and 2.4.5. Some of our constructions are based on isomorphisms which do not respect the gradings; hence for simplicity we just forget the gradings in this subsection.

The constructions on the side of the group G depend on the choice of root vectors for all simple roots. For these constructions to match with the constructions in perverse sheaves on \mathbf{Gr} , one has to choose the root vectors provided by the Tannakian construction of G from the tensor category $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$; see §6.5 for details.

The symmetry in the case of equivariant cohomology of cofibers of perverse sheaves is easy to construct. Namely, the normalizer $N_{\check{G}}(\check{T})$ of \check{T} in \check{G} acts naturally on \mathbf{Gr} ; we denote by $m_g : \mathbf{Gr} \xrightarrow{\sim} \mathbf{Gr}$ the action of $g \in N_{\check{G}}(\check{T})$. If we denote by $g \mapsto \bar{g}$ the projection $N_{\check{G}}(\check{T}) \rightarrow N_{\check{G}}(\check{T})/\check{T} \cong W$, for any $\lambda \in \mathbb{X}$ we have $m_g \circ i_\lambda = i_{\bar{g}\lambda}$ (where we identify the one-point varieties $\{\lambda\}$ and $\{\bar{g}\lambda\}$). If \mathcal{F} is in $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$, we deduce an isomorphism of graded \mathbf{S}_{\hbar} -modules

$$\mathbf{H}_A^*(i_\lambda^! m_g^! \mathcal{F}) \xrightarrow{\sim} \bar{\mathbf{H}}_A^*(i_{\bar{g}\lambda}^! \mathcal{F}).$$

On the other hand, since \mathcal{F} is $\check{G}(\mathbf{O})$ -equivariant (and hence in particular $N_{\check{G}}(\check{T})$ -equivariant), there exists a canonical isomorphism $m_g^! \mathcal{F} \cong \mathcal{F}$. Hence we obtain

an isomorphism $H_A^\bullet(i_\lambda^! \mathcal{F}) \xrightarrow{\sim} \bar{s}H_A^\bullet(i_{\bar{g}\lambda}^! \mathcal{F})$. Using classical arguments one can check that this isomorphism only depends on \bar{g} (and not on g); we will denote by

$$\Xi_w^{\mathcal{F}, \lambda} : H_A^\bullet(i_\lambda^! \mathcal{F}) \xrightarrow{\sim} {}^w H_A^\bullet(i_{w\lambda}^! \mathcal{F})$$

the isomorphism associated with $w \in W$. These isomorphisms generate an action of W in the sense that for any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, $\lambda \in \mathbb{X}$ and $x, y \in W$ we have

$$y(\Xi_x^{\mathcal{F}, y\lambda}) \circ \Xi_y^{\mathcal{F}, \lambda} = \Xi_{xy}^{\mathcal{F}, \lambda}. \tag{2.5.1}$$

Now, let us consider \mathcal{D} -modules on \mathcal{X} . In § 3.2 we recall a construction of Gelfand and Graev (see [8, 29]) based on “partial Fourier transforms” for \mathcal{D} -modules which provides an action of W on $\mathcal{D}_h(\mathcal{X})$ by algebra automorphisms such that for any $w \in W$ and $\lambda \in \mathbb{X}$ the action of w restricts to an isomorphism of S_h -modules:

$$({}^\lambda \mathcal{D}_h(\mathcal{X}))_\lambda \xrightarrow{\sim} w({}^{(w\lambda)} \mathcal{D}_h(\mathcal{X})_{w\lambda}). \tag{2.5.2}$$

For V in $\text{Rep}(G)$, we will denote by

$$\Phi_w^{V, \lambda} : (V \otimes ({}^\lambda \mathcal{D}_h(\mathcal{X}))_\lambda)^G \xrightarrow{\sim} w(V \otimes ({}^{(w\lambda)} \mathcal{D}_h(\mathcal{X})_{w\lambda})^G)$$

the induced isomorphism. This collection of isomorphisms satisfies the relations

$$y(\Phi_x^{V, y\lambda}) \circ \Phi_y^{V, \lambda} = \Phi_{xy}^{V, \lambda}. \tag{2.5.3}$$

Finally, in § 4.7 we will define, for any V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$, an isomorphism of S_h -modules

$$\Theta_w^{V, \lambda} : \text{Hom}_{(S_h, U_h(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \xrightarrow{\sim} {}^w \text{Hom}_{(S_h, U_h(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(w\lambda)).$$

This collection of isomorphisms naturally appears in the construction of the *dynamical Weyl group*; see § 2.6 below. As above, it satisfies the relations

$$y(\Theta_x^{V, y\lambda}) \circ \Theta_y^{V, \lambda} = \Theta_{xy}^{V, \lambda}. \tag{2.5.4}$$

Our second main result, which will be proved in § 7.6, is the following.

Theorem 2.5.5. *The isomorphisms of Corollary 2.4.2 are such that the following diagram commutes for any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, $\lambda \in \mathbb{X}$ and $w \in W$:*

$$\begin{CD} (\mathbb{S}(\mathcal{F}) \otimes ({}^\lambda \mathcal{D}_h(\mathcal{X}))_\lambda)^G @>\sim>> H_A^\bullet(i_\lambda^! \mathcal{F}) @>\sim>> \text{Hom}_{(S_h, U_h(\mathfrak{g}))}(\mathbf{M}(0), \mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(\lambda)) \\ @V \wr \Phi_w^{\mathbb{S}(\mathcal{F}), \lambda} VV @V \wr \Xi_w^{\mathcal{F}, \lambda} VV @V \wr \Theta_w^{\mathbb{S}(\mathcal{F}), \lambda} VV \\ w(\mathbb{S}(\mathcal{F}) \otimes ({}^{(w\lambda)} \mathcal{D}_h(\mathcal{X})_{w\lambda})^G @>\sim>> {}^w H_A^\bullet(i_{w\lambda}^! \mathcal{F}) @>\sim>> {}^w \text{Hom}_{(S_h, U_h(\mathfrak{g}))}(\mathbf{M}(0), \mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(w\lambda)). \end{CD}$$

One can also give a “classical” analogue of Theorem 2.5.5. First, the same construction as above provides, for any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, $\lambda \in \mathbb{X}$ and $w \in W$, an isomorphism of graded $S(\mathfrak{t})$ -modules

$$\xi_w^{\mathcal{F}, \lambda} : H_T^\bullet(i_\lambda^! \mathcal{F}) \xrightarrow{\sim} {}^w H_T^\bullet(i_{w\lambda}^! \mathcal{F}).$$

This collection of isomorphisms satisfies relations

$$y(\xi_x^{\mathcal{F}, y\lambda}) \circ \xi_y^{\mathcal{F}, \lambda} = \xi_{xy}^{\mathcal{F}, \lambda}.$$

On the other hand, let $\mathfrak{g}_r^* \subset \mathfrak{g}^*$ be the open set consisting of regular elements, and let $\tilde{\mathfrak{g}}_r$ be the inverse image of \mathfrak{g}_r^* under the morphism $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}^*$ defined by $\pi(g \times_B \eta) = g \cdot \eta$. Then there exists a canonical action of W on $\tilde{\mathfrak{g}}_r$, which induces for any $\lambda \in \mathbb{X}$ and $w \in W$ an isomorphism of $S(t)$ -modules and G -modules

$$\Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \xrightarrow{\sim} {}^w\Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda));$$

see § 5.2 for details. Hence we obtain, for any V in $\text{Rep}(G)$, an isomorphism

$$\sigma_w^{V, \lambda} : (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G \xrightarrow{\sim} {}^w(V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)))^G.$$

This collection of isomorphisms again satisfies relations

$$y(\sigma_x^{V, y\lambda}) \circ \sigma_y^{V, \lambda} = \sigma_{xy}^{V, \lambda}. \tag{2.5.6}$$

We have the following compatibility property, to be proved in § 7.7.

Theorem 2.5.7. *The isomorphism of Corollary 2.4.5 is such that the following diagram commutes for any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$, $\lambda \in \mathbb{X}$ and $w \in W$:*

$$\begin{CD} (\mathbb{S}(\mathcal{F}) \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G @>\sim>> H_{\tilde{\mathfrak{F}}}^*(i_{\lambda}^! \mathcal{F}) \\ @V \wr \sigma_w^{\mathbb{S}(\mathcal{F}), \lambda} VV @VV \wr \xi_w^{\mathcal{F}, \lambda} V \\ ({}^w\mathbb{S}(\mathcal{F}) \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)))^G @>\sim>> {}^wH_{\tilde{\mathfrak{F}}}^*(i_{w\lambda}^! \mathcal{F}). \end{CD}$$

2.6. Applications: dynamical Weyl groups and Brylinski–Kostant filtration

The first application of our results concerns a geometric realization of the *dynamical Weyl group* due to Braverman and Finkelberg ([12]). Let \mathbb{Q}_{\hbar} be the field of fractions of S_{\hbar} . If M is a \mathbb{Q}_{\hbar} -module, we define the \mathbb{Q}_{\hbar} -module wM by a formula similar to that above. In § 8.3 we will recall the definition of the dynamical Weyl group, a collection of isomorphisms of \mathbb{Q}_{\hbar} -modules

$$DW_{V, \lambda, w}^{\text{alg}} : \mathbb{Q}_{\hbar} \otimes V_{\lambda} \xrightarrow{\sim} {}^w\mathbb{Q}_{\hbar} \otimes V_{w\lambda}$$

for all V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$.

Let $\mathbf{O}^- := \mathbb{C}[z^{-1}]$, and let $\check{G}(\mathbf{O}^-)_1$ be the kernel of the morphism $\check{G}(\mathbf{O}^-) \rightarrow \check{G}$ given by evaluation at $z = \infty$. For $\lambda \in \mathbb{X}$ we set $\mathfrak{W}_{\lambda} := \check{G}(\mathbf{O}^-)_1 \cdot \lambda$; this is a locally closed (ind-)subvariety of \mathbf{Gr} which is a transverse slice to the orbit \mathbf{Gr}^{λ} at λ . We denote the inclusion by

$$s_{\lambda} : \mathfrak{W}_{\lambda} \cap \mathfrak{T}_{\lambda} \hookrightarrow \mathbf{Gr}.$$

We also set

$$n_{\lambda} := \sum_{\alpha \in R^+} |\lambda(\check{\alpha})|.$$

The following result is proved in [12]; we reproduce the proof in § 8.4 since we will need some of the details.

Lemma 2.6.1. *For any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr})$ and $\lambda \in \mathbb{X}$ there exists a canonical isomorphism of S_{\hbar} -modules*

$$H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F}) \cong (S(\mathcal{F}))_\lambda \otimes S_{\hbar}(n_\lambda).$$

The inclusion $\{\lambda\} \hookrightarrow \mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda$ induces a morphism of Q_{\hbar} -modules

$$\Delta^{\mathcal{F}, \lambda} : Q_{\hbar} \otimes_{S_{\hbar}} H_A^*(i_\lambda^! \mathcal{F}) \rightarrow Q_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F})$$

which is an isomorphism due to the localization theorem in equivariant cohomology (see e.g. [20, Theorem B.2]). For \mathcal{F} in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr})$, $\lambda \in \mathbb{X}$ and $w \in W$ we define the morphism

$$\begin{aligned} DW_{\mathcal{F}, \lambda, w}^{\text{geom}} &:= w(\Delta^{\mathcal{F}, w\lambda}) \circ \Xi_w^{\mathcal{F}, \lambda} \circ (\Delta^{\mathcal{F}, \lambda})^{-1} : \\ &Q_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F}) \xrightarrow{\sim} w(Q_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_{w\lambda} \cap \mathfrak{T}_{w\lambda}, s_{w\lambda}^! \mathcal{F})). \end{aligned}$$

The following result (which is a consequence of Theorem 2.5.5) is equivalent to the main result of [12]. Our proof, given in § 8.5, cannot really be considered as a new proof since it is based on a similar strategy (namely reduction to rank 1), but we believe that our point of view should help with understanding this question better.

Proposition 2.6.2. *For any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr})$ and $\lambda \in \mathbb{X}$ dominant, the following diagram is commutative, where the vertical isomorphisms are induced by those of Lemma 2.6.1:*

$$\begin{array}{ccc} Q_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F}) & \xrightarrow{DW_{\mathcal{F}, \lambda, w}^{\text{geom}}} & w(Q_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_{w\lambda} \cap \mathfrak{T}_{w\lambda}, s_{w\lambda}^! \mathcal{F})) \\ \downarrow \wr & & \downarrow \wr \\ Q_{\hbar} \otimes (S(\mathcal{F}))_\lambda & \xrightarrow{DW_{S(\mathcal{F}), \lambda, w}^{\text{alg}}} & w(Q_{\hbar} \otimes (S(\mathcal{F}))_{w\lambda}). \end{array}$$

The second application of our results is a new proof of a result of the first author ([22]) giving a geometric construction of the Brylinski–Kostant filtration. Namely, let $e \in \mathfrak{u}$ be a regular nilpotent element which is a sum of (non-zero) simple root vectors. If V is in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$, the Brylinski–Kostant filtration on V_λ associated with e is defined by

$$F_i^{\text{BK}}(V_\lambda) = \{v \in V_\lambda \mid e^{i+1} \cdot v = 0\} \quad \text{for } i \geq 0$$

and $F_i^{\text{BK}}(V_\lambda) = 0$ for $i < 0$. This filtration is independent of the choice of e (since all the choices for e are conjugate under the action of T).

On the other hand, for any $\varphi \in \mathfrak{t}^*$ we consider the specialized equivariant cohomology

$$H_\varphi(i_\lambda^! \mathcal{F}) := \mathbb{C}_\varphi \otimes_{S(\mathfrak{t})} H_T^*(i_\lambda^! \mathcal{F}),$$

a filtered vector space. Assume that $\varphi \in \mathfrak{t}^* \setminus \{0\}$ satisfies $(\text{ad}^* e)^2(\varphi) = 0$, where ad^* is the coadjoint representation. Then φ is regular, and hence by the localization theorem in equivariant cohomology, the morphism

$$H_\varphi(i_\lambda^! \mathcal{F}) \rightarrow \mathbb{C}_\varphi \otimes_{S(\mathfrak{t})} ((S(\mathcal{F}))_\lambda \otimes S(\mathfrak{t})) = (S(\mathcal{F}))_\lambda$$

induced by $\bar{\kappa}_{\mathcal{F}, \lambda}^{\text{top}}$ is an isomorphism. The left-hand side is equipped with a natural filtration; we denote by $F_\bullet^{\text{geom}}((S(\mathcal{F}))_\lambda)$ the resulting filtration on $(S(\mathcal{F}))_\lambda$.

The following result was first proved in [22] (see also [2] for a different proof). We observe in § 8.6 that it is an immediate consequence of Theorem 2.3.1.

Proposition 2.6.3. For any \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, $\lambda \in \mathbb{X}$ and $i \in \mathbb{Z}$ we have

$$F_i^{\text{BK}}((\mathbb{S}(\mathcal{F}))_\lambda) = F_{2i+\lambda(2\check{\rho})}^{\text{geom}}((\mathbb{S}(\mathcal{F}))_\lambda) = F_{2i+1+\lambda(2\check{\rho})}^{\text{geom}}((\mathbb{S}(\mathcal{F}))_\lambda).$$

Remark 2.6.4. Assume that G is quasi-simple. Then it follows from [31, Corollary 8.7] that the three-dimensional representation of any \mathfrak{sl}_2 -triple through e occurs only once in \mathfrak{g} , and hence in \mathfrak{g}^* . It follows that $\varphi \in \mathfrak{t}^*$ is uniquely defined, up to a scalar, by the condition $(\text{ad}^*e)^2(\varphi) = 0$. Using this remark one can easily check (for a general reductive G , and independently of Proposition 2.6.3) that the filtration F_i^{geom} is independent of the choice of φ once e is fixed. On the other hand all the possible choices for e are conjugate under the action of T , and hence our filtration is independent of any choice (other than T and B).

3. Differential operators on the basic affine space and partial Fourier transforms

In Sections 3–5 we fix a complex connected reductive group G , and we use the notation of §2.1. We also choose for any simple root α a non-zero vector $e_\alpha \in \mathfrak{g}_\alpha$. We denote by $f_\alpha \in \mathfrak{g}_{-\alpha}$ the unique vector such that $[e_\alpha, f_\alpha] = \check{\alpha}$.

3.1. The structure of $\mathcal{D}_\hbar(\mathcal{X})$

The results in this subsection are taken from [7, 37]. Below we will use the two natural actions of G on $\mathbb{C}[G]$ induced by the actions of G on itself. The action given by $(g \cdot f)(h) = f(g^{-1}h)$ for $f \in \mathbb{C}[G]$ and $g, h \in G$ will be called the *left regular representation*; it is a *left* action of G . The action given by $(f \cdot g)(h) = f(hg^{-1})$ for $f \in \mathbb{C}[G]$ and $g, h \in G$ will be called the *right regular representation*; it is a *right* action of G .

First we begin with the description of $\mathcal{D}_\hbar(G)$. Differentiating the right regular representation defines an anti-homomorphism of algebras $U_\hbar(\mathfrak{g}) \rightarrow \mathcal{D}_\hbar(G)$. Then it is well known that multiplication in $\mathcal{D}_\hbar(G)$ induces an isomorphism of $\mathbb{C}[G]$ -modules

$$\mathbb{C}[G] \otimes U_\hbar(\mathfrak{g}) \xrightarrow{\sim} \mathcal{D}_\hbar(G). \tag{3.1.1}$$

The left regular representation induces an action on $\mathcal{D}_\hbar(G)$, which will be called simply the *left action* below. Through isomorphism (3.1.1), it is given by the left regular representation of G on $\mathbb{C}[G]$ (and the trivial action on $U_\hbar(\mathfrak{g})$). Similarly, the action induced by the right regular representation (which will be called simply the *right action* below) corresponds, under isomorphism (3.1.1), to the right action on $\mathbb{C}[G] \otimes U_\hbar(\mathfrak{g})$ which is the tensor product of the right regular representation and the action on $U_\hbar(\mathfrak{g})$ which is the composition of the adjoint action with the anti-automorphism $g \mapsto g^{-1}$. There is also a natural morphism $U_\hbar(\mathfrak{g}) \rightarrow \mathcal{D}_\hbar(G)$ obtained by differentiation of the left regular representation. Under isomorphism (3.1.1), it is given by the map which sends $m \in U_\hbar(\mathfrak{g})$ to the function $G \ni g \mapsto g^{-1} \cdot m \in U_\hbar(\mathfrak{g})$, considered as an element in $\mathbb{C}[G] \otimes U_\hbar(\mathfrak{g})$. In particular, this morphism restricts to the morphism $m \mapsto 1 \otimes m$ on $Z_\hbar(\mathfrak{g}) \subset U_\hbar(\mathfrak{g})$.

Let us recall the standard description of $\mathcal{D}_\hbar(\mathcal{X})$ based on quantum Hamiltonian reduction. As \mathcal{X} is a quasi-affine variety, the natural morphism

$$\mathcal{D}_\hbar(\mathcal{X}) \rightarrow \text{End}_{\mathbb{C}[\hbar]}(\mathbb{C}[\mathcal{X}][\hbar])$$

is injective. The algebra $\mathbb{C}[\mathcal{X}]$ identifies with the subalgebra of $\mathbb{C}[G]$ given by the elements fixed by the right action of $U \subset G$. Any element D in $\mathcal{D}_\hbar(G)$ induces a morphism $\mathbb{C}[\mathcal{X}][[\hbar]] \rightarrow \mathbb{C}[G][[\hbar]]$. This morphism is trivial iff $D \in \mathbb{C}[G] \otimes (\mathfrak{u} \cdot U_\hbar(\mathfrak{g}))$, and its image is contained in $\mathbb{C}[\mathcal{X}][[\hbar]]$ iff the image of D in $\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g}))$ is U -invariant for the right action. In this way we obtain a canonical isomorphism

$$\mathcal{D}_\hbar(\mathcal{X}) \cong (\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g})))^{U_{\text{right}}}. \tag{3.1.2}$$

In this description, the action induced by the left G -action on \mathcal{X} is induced by the left regular representation on $\mathbb{C}[G]$. The morphism $Z_\hbar(\mathfrak{g}) \rightarrow \mathcal{D}_\hbar(\mathcal{X})$ obtained from the differentiation of this left action of G on \mathcal{X} corresponds to the morphism $m \mapsto 1 \otimes (m \bmod \mathfrak{u} \cdot U_\hbar(\mathfrak{g}))$.

Recall that there is also a T -action on \mathcal{X} defined by $t \cdot gU = gtU$. (Note that this action is *not* induced by the right action of G on itself considered above, but rather by its composition with $t \mapsto t^{-1}$.) This action provides a T -action on $\mathcal{D}_\hbar(\mathcal{X})$ (where the action of $t \in T$ is induced by the right action of $t^{-1} \in G$ on $\mathcal{D}_\hbar(G)$ described above) and a weight decomposition $\mathcal{D}_\hbar(\mathcal{X}) = \bigoplus_{\lambda \in \mathbb{X}} \mathcal{D}_\hbar(\mathcal{X})_\lambda$. This T -action also defines a morphism $\gamma : \mathbb{S}_\hbar \rightarrow \mathcal{D}_\hbar(\mathcal{X})$ which, under isomorphism (3.1.2), is given by $\gamma(m) = 1 \otimes (m \bmod \mathfrak{u} \cdot U_\hbar(\mathfrak{g}))$. Recall that we consider $\mathcal{D}_\hbar(\mathcal{X})$ as an \mathbb{S}_\hbar -module where $t \in \mathfrak{t}$ acts by right multiplication by $\gamma(t) - \hbar\rho(t) \cdot 1$. Under isomorphism (3.1.2), this action is given by the \mathbb{S}_\hbar -action on $U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g})$ where $t \in \mathfrak{t}$ acts by left multiplication by $t - \hbar\rho(t)$.

From this description (and isomorphism (2.1.2)) one easily obtains the following result.

Lemma 3.1.3. *For any $\lambda \in \mathbb{X}$ there exists a canonical isomorphism of graded \mathbb{S}_\hbar -modules and $Z_\hbar(\mathfrak{g})$ -modules*

$${}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda \cong \text{Ind}_B^G(\mathbf{M}(\lambda)).$$

Proof of the first isomorphism in Lemma 2.4.1. We have

$$(V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \cong (V \otimes \text{Ind}_B^G(\mathbf{M}(\lambda)))^G \cong (V \otimes \mathbf{M}(\lambda))^B$$

where the first isomorphism follows from Lemma 3.1.3 and the second one from the tensor identity and Frobenius reciprocity. □

3.2. Partial Fourier transforms for $\mathcal{D}_\hbar(\mathcal{X})$

Let us recall a construction due to Gelfand and Graev, and studied by Kazhdan and Laumon [29] in the ℓ -adic setting and by Bezrukavnikov, Braverman and Positselskii [8] in our \mathcal{D} -module setting. We choose a reductive group G^{sc} with simply connected derived subgroup and a surjective group morphism $G^{\text{sc}} \twoheadrightarrow G$ with finite central kernel denoted as Z . We denote by $T^{\text{sc}}, B^{\text{sc}}$ the inverse images of T, B in G^{sc} , and let U^{sc} be the unipotent radical of B^{sc} . We set $\mathcal{X}^{\text{sc}} := G^{\text{sc}}/U^{\text{sc}}$. Note that Z acts naturally on \mathcal{X}^{sc} , with quotient \mathcal{X} . Note also that for any simple root α there exists a unique injective morphism of algebraic groups $\varphi_\alpha : \text{SL}(2, \mathbb{C}) \rightarrow G^{\text{sc}}$ such that

$$\forall z \in \mathbb{C}^\times, \quad \varphi_\alpha \left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right) = \check{\alpha}(z) \quad \text{and} \quad d(\varphi_\alpha) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = f_\alpha, \quad d(\varphi_\alpha) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e_\alpha$$

(where we identify the Lie algebras of G^{sc} and G .)

Let α be a simple root, let P_α^{sc} be the minimal parabolic subgroup of G^{sc} containing B^{sc} associated with α , and let $Q_\alpha^{\text{sc}} := [P_\alpha^{\text{sc}}, P_\alpha^{\text{sc}}]$. Consider the projection

$$\tau_\alpha : \mathcal{X}^{\text{sc}} \rightarrow G^{\text{sc}}/Q_\alpha^{\text{sc}}.$$

It is explained in [29, §2.1] that τ_α is the complement of the zero section of a G^{sc} -equivariant vector bundle

$$\tau'_\alpha : \mathcal{V}_\alpha \rightarrow G^{\text{sc}}/Q_\alpha^{\text{sc}}$$

of rank 2. Moreover, there exists a canonical G^{sc} -equivariant symplectic form on this vector bundle (which depends on φ_α , i.e. on the choice of f_α). Hence the constructions recalled in §B.2 provide an automorphism of $\mathcal{D}_\hbar(\mathcal{V}_\alpha)$ as a $\mathbb{C}[\hbar]$ -algebra. As the complement of \mathcal{X}^{sc} in \mathcal{V}_α has codimension 2, restriction induces an isomorphism $\mathcal{D}_\hbar(\mathcal{V}_\alpha) \xrightarrow{\sim} \mathcal{D}_\hbar(\mathcal{X}^{\text{sc}})$. Hence we obtain an automorphism $\mathbf{F}'_\alpha : \mathcal{D}_\hbar(\mathcal{X}^{\text{sc}}) \xrightarrow{\sim} \mathcal{D}_\hbar(\mathcal{X}^{\text{sc}})$. This automorphism is Z -equivariant (since Z acts on \mathcal{V}_α by symplectic automorphisms), and we have $\mathcal{D}_\hbar(\mathcal{X}^{\text{sc}})^Z = \mathcal{D}_\hbar(\mathcal{X})$ in a natural way. Hence we obtain a G -equivariant $\mathbb{C}[\hbar]$ -algebra automorphism

$$\mathbf{F}_\alpha : \mathcal{D}_\hbar(\mathcal{X}) \xrightarrow{\sim} \mathcal{D}_\hbar(\mathcal{X})$$

on Z -fixed points. Using the fact that any two simply connected covers of a connected semisimple group are isomorphic (as covers of the given group), one can check that the automorphism \mathbf{F}_α does not depend on the choice of G^{sc} .

Lemma 3.2.1.

1. The automorphisms \mathbf{F}_α , α a simple root, generate an action of W on $\mathcal{D}_\hbar(\mathcal{X})$.
2. For any simple root α and any $\lambda \in \mathbb{X}$, setting $s = s_\alpha$, \mathbf{F}_α restricts to an isomorphism of G -modules and of S_{\hbar^-} - and $Z_\hbar(\mathfrak{g})$ -modules

$$\mathcal{D}_\hbar(\mathcal{X})_\lambda \xrightarrow{\sim} {}^s \mathcal{D}_\hbar(\mathcal{X})_{s\lambda}.$$

Proof. We observe that there are natural isomorphisms

$$\mathcal{D}_\hbar(\mathcal{X})[\hbar^{-1}] \cong \Gamma(\mathcal{X}, \mathcal{D}_{\hbar, \mathcal{X}}[\hbar^{-1}]) \cong \Gamma(\mathcal{X}, \mathcal{D}_{\mathcal{X}} \otimes \mathbb{C}[\hbar, \hbar^{-1}]) \cong \mathcal{D}(\mathcal{X}) \otimes \mathbb{C}[\hbar, \hbar^{-1}].$$

Moreover, under these isomorphisms, the automorphism induced by \mathbf{F}_α coincides with the tensor product of the similar automorphism of $\mathcal{D}(\mathcal{X})$ considered in [8] with $\text{id}_{\mathbb{C}[\hbar, \hbar^{-1}]}$. Hence the lemma follows from [8, Proposition 3.1 and Lemma 3.3]. \square

By (1), we can define a group morphism $w \mapsto \mathbf{F}_w$ from W to the group of $\mathbb{C}[\hbar]$ -algebra automorphisms of $\mathcal{D}_\hbar(\mathcal{X})$, such that $\mathbf{F}_{s_\alpha} = \mathbf{F}_\alpha$ for any simple root α . And by (2) these isomorphisms restrict to isomorphisms of G -modules and of S_{\hbar^-} - and $Z_\hbar(\mathfrak{g})$ -modules

$$\mathbf{F}_w^\lambda : \mathcal{D}_\hbar(\mathcal{X})_\lambda \xrightarrow{\sim} {}^w \mathcal{D}_\hbar(\mathcal{X})_{w(\lambda)}$$

which satisfy the relations

$$y(\mathbf{F}_x^{y\lambda}) \circ \mathbf{F}_y^\lambda = \mathbf{F}_{xy}^\lambda.$$

Using the relation $({}^\lambda)({}^w M) = {}^{w(\lambda)} M$ we obtain isomorphism (2.5.2), which allows us to define the collection of isomorphisms $\Phi_w^{V, \lambda}$ of §2.5.

Remark 3.2.2.

- (1) As explained in § B.1, the construction of isomorphisms $\Phi_w^{V,\lambda}$ is not compatible with the natural gradings on $(V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G$ and $(V \otimes {}^{(w\lambda)}\mathcal{D}_\hbar(\mathcal{X})_{w\lambda})^G$. However, it follows from Theorem 2.5.5 that it induces an isomorphism of graded modules

$$(V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \langle \lambda(2\check{\rho}) \rangle \xrightarrow{\sim} w(V \otimes {}^{(w\lambda)}\mathcal{D}_\hbar(\mathcal{X})_{w\lambda})^G \langle (w\lambda)(2\check{\rho}) \rangle.$$

This property is also observed in [32, Proposition 2.9].

- (2) Statement (2) of Lemma 3.2.1 (and the fact that \mathbf{F}_α is an involution) can also be proved directly as follows: by Corollary 3.3.3 below and the injectivity of the morphisms $\mathcal{R}_{G,L}^{V,\lambda}$ considered in this statement (see Lemmas 4.6.4 and 4.8.2), it is enough to prove the claim in the case where G has semisimple rank 1, which can be treated by explicit computation (see e.g. the proof of Lemma A.2.1 below).

As a consequence of these constructions we also obtain the following result, which will be needed later. This result is also proved (using different methods) in [37].

Proposition 3.2.3. *For any $\lambda \in \mathbb{X}$, the graded S_\hbar -module $\mathcal{D}_\hbar(\mathcal{X})_\lambda$ is free.*

Proof. Using isomorphisms \mathbf{F}_w^λ defined above we can assume that λ is dominant. Then using Lemma 3.1.3 it is equivalent to prove that $\text{Ind}_B^G(\mathbf{M}(\lambda))$ is free over S_\hbar .

We claim that, if λ is dominant,

$$\text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u})[\hbar] \otimes \mathbb{C}_{-\lambda}) \text{ is free over } S_\hbar \text{ and } R^{>0} \text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u})[\hbar] \otimes \mathbb{C}_{-\lambda}) = 0. \tag{3.2.4}$$

Indeed, it is sufficient to prove that the $\mathbf{S}(\mathfrak{t})$ -module $\text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})$ is free and that we have $R^{>0} \text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda}) = 0$. Consider the vector bundles

$$q_{\tilde{\mathcal{N}}} : \tilde{\mathcal{N}} \rightarrow \mathcal{B} \quad \text{and} \quad q_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \rightarrow \mathcal{B}.$$

Here $\tilde{\mathcal{N}} := G \times_B (\mathfrak{g}/\mathfrak{b})^*$ is the Springer resolution, and $\tilde{\mathfrak{g}}$ is defined in § 2.4. There is a natural inclusion of vector bundles $\tilde{\mathcal{N}} \hookrightarrow \tilde{\mathfrak{g}}$, and the quotient is the trivial vector bundle $\mathfrak{t}^* \times \mathcal{B}$. Hence there is a $\mathbb{Z}_{\geq 0}$ -filtration on $(q_{\tilde{\mathfrak{g}}})_* \mathcal{O}_{\tilde{\mathfrak{g}}}$ (as a sheaf of $\mathbf{S}(\mathfrak{t}) \otimes \mathcal{O}_{\mathcal{B}}$ -modules) with associated graded $(q_{\tilde{\mathcal{N}}})_* \mathcal{O}_{\tilde{\mathcal{N}}} \otimes \mathbf{S}(\mathfrak{t})$. By [14, Theorem 2.4], we have $H^{>0}(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = 0$. It follows that $H^{>0}(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) = 0$, and that $H^0(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda))$ has a filtration with associated graded $H^0(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \otimes \mathbf{S}(\mathfrak{t})$. Now it follows from definitions that, for any $i \geq 0$, $H^i(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \cong R^i \text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})$, which finishes the proof of (3.2.4).

For $i \geq 0$, let $M_i := S_\hbar \cdot \mathfrak{v}_\lambda \cdot U_\hbar^{\leq i}(\mathfrak{g}) \subset \mathbf{M}(\lambda)$, where $U_\hbar^{\leq i}(\mathfrak{g})$ is the PBW filtration of $U_\hbar(\mathfrak{g})$. Then M_\bullet is a B -stable and S_\hbar -stable exhaustive filtration of $\mathbf{M}(\lambda)$, and the associated graded is isomorphic to $\mathbf{S}(\mathfrak{g}/\mathfrak{u})[\hbar] \otimes \mathbb{C}_{-\lambda}$. From the second claim in (3.2.4) it follows that $\text{Ind}_B^G(\mathbf{M}(\lambda))$ has a filtration with associated graded $\text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u})[\hbar] \otimes \mathbb{C}_{-\lambda})$, and then the corollary follows from the first claim in (3.2.4). □

Remark 3.2.5. The arguments in the proof of Proposition 3.2.3 also prove that, when λ is dominant, we have $R^{>0} \text{Ind}_B^G(\mathbf{M}(\lambda)) = 0$.

3.3. Restriction to a Levi subgroup

Fix a subset I of the set of simple roots and let \mathfrak{l} be the Levi subalgebra containing \mathfrak{t} that has the set I as simple roots. Note that our choice of a Borel subgroup, a maximal torus and simple root vectors for \mathfrak{g} determines a similar choice for \mathfrak{l} ; hence the constructions of the present section make sense both for \mathfrak{g} and for \mathfrak{l} .

We put

$$\mathfrak{u}_L := \bigoplus_{\substack{\alpha \in R^+ \\ \alpha \in \mathbb{Z}I}} \mathfrak{g}_\alpha, \quad \mathfrak{b}_L := \mathfrak{t} \oplus \mathfrak{u}_L, \quad \mathfrak{n}_L^+ := \bigoplus_{\substack{\alpha \in R^+ \\ \alpha \notin \mathbb{Z}I}} \mathfrak{g}_\alpha, \quad \mathfrak{n}_L^- := \bigoplus_{\substack{\alpha \in R^- \\ \alpha \notin \mathbb{Z}I}} \mathfrak{g}_\alpha.$$

Thus, one has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_L^+ \oplus \mathfrak{l} \oplus \mathfrak{n}_L^-$, and $\mathfrak{u}_L = \mathfrak{u} \cap \mathfrak{l}$ is the nilradical of the Borel subalgebra $\mathfrak{b}_L = \mathfrak{b} \cap \mathfrak{l}$ of \mathfrak{l} . Further, let $\mathfrak{p}_\pm := \mathfrak{l} \oplus \mathfrak{n}_L^\pm$ and $\mathfrak{b}_\pm := \mathfrak{b}_L \oplus \mathfrak{n}_L^\pm$ (resp. $\mathfrak{u}_\pm := \mathfrak{u}_L \oplus \mathfrak{n}_L^\pm$). Thus, \mathfrak{p}_\pm is a pair of opposite parabolic subalgebras of \mathfrak{g} such that $\mathfrak{p}_+ \cap \mathfrak{p}_- = \mathfrak{l}$, and \mathfrak{b}_\pm is a pair of Borel subalgebras of \mathfrak{g} such that $\mathfrak{b}_+ \cap \mathfrak{b}_- = \mathfrak{b}_L$, with respective nilpotent radicals \mathfrak{u}_\pm . Let $L, P_\pm, B_\pm, U_\pm, N_L^\pm, B_L, U_L$ be the subgroups of G corresponding to the Lie algebras $\mathfrak{l}, \mathfrak{p}_\pm, \mathfrak{b}_\pm, \mathfrak{u}_\pm, \mathfrak{n}_L^\pm, \mathfrak{b}_L, \mathfrak{u}_L$, respectively. By definition, we have $\mathcal{X}_G = G/U_+$ and $\mathcal{X}_L = L/U_L$. (Observe that $B_+, U_+, \mathfrak{b}_+, \mathfrak{u}_+$ coincide with the objects denoted by $B, U, \mathfrak{b}, \mathfrak{u}$ in the preceding sections).

Now we construct a morphism of L -modules

$$r_L^G : \mathcal{D}_\hbar(\mathcal{X}_G) \rightarrow \mathcal{D}_\hbar(\mathcal{X}_L)$$

as follows. Note that the right $U_\hbar(\mathfrak{l})$ -action on $U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g})$ descends to a well-defined action on $U_\hbar(\mathfrak{g})/(\mathfrak{u} \cdot U_\hbar(\mathfrak{g}) + U_\hbar(\mathfrak{g}) \cdot \mathfrak{n}_L^-)$. Using this, from the diagram $\mathfrak{g} \leftarrow \mathfrak{p}_- \rightarrow \mathfrak{l}$ of natural Lie algebra morphisms, one obtains the following morphisms of right $U_\hbar(\mathfrak{l})$ -modules:

$$U_\hbar(\mathfrak{g})/(\mathfrak{u} \cdot U_\hbar(\mathfrak{g}) + U_\hbar(\mathfrak{g}) \cdot \mathfrak{n}_L^-) \cong U_\hbar(\mathfrak{p}_-)/(\mathfrak{u}_L \cdot U_\hbar(\mathfrak{p}_-) + U_\hbar(\mathfrak{p}_-) \cdot \mathfrak{n}_L^-) \cong U_\hbar(\mathfrak{l})/\mathfrak{u}_L \cdot U_\hbar(\mathfrak{l}). \tag{3.3.1}$$

All the above maps are bijections since the linear maps

$$\mathfrak{g}/(\mathfrak{u} \oplus \mathfrak{n}_L^-) \xleftarrow{\sim} \mathfrak{p}_-/\mathfrak{u}_- \xrightarrow{\sim} \mathfrak{l}/\mathfrak{u}_L$$

are clearly vector space isomorphisms. We deduce the following chain of maps:

$$\begin{aligned} (\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g})))^U &\hookrightarrow (\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g})))^{U_L} \\ &\rightarrow (\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{g})/(\mathfrak{u} \cdot U_\hbar(\mathfrak{g}) + U_\hbar(\mathfrak{g}) \cdot \mathfrak{n}_L^-)))^{U_L} \\ &\rightarrow (\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{l})/\mathfrak{u}_L \cdot U_\hbar(\mathfrak{l})))^{U_L} \\ &\rightarrow (\mathbb{C}[L] \otimes (U_\hbar(\mathfrak{l})/\mathfrak{u}_L \cdot U_\hbar(\mathfrak{l})))^{U_L}, \end{aligned}$$

where the third morphism is induced by (3.3.1), and the last one is induced by the restriction of functions $\mathbb{C}[G] \rightarrow \mathbb{C}[L]$. Using isomorphism (3.1.2), this allows us to define the desired morphism

$$r_L^G : \mathcal{D}_\hbar(\mathcal{X}_G) \cong (\mathbb{C}[G] \otimes (U_\hbar(\mathfrak{g})/\mathfrak{u} \cdot U_\hbar(\mathfrak{g})))^U \rightarrow (\mathbb{C}[L] \otimes (U_\hbar(\mathfrak{l})/\mathfrak{u}_L \cdot U_\hbar(\mathfrak{l})))^{U_L} \cong \mathcal{D}_\hbar(\mathcal{X}_L).$$

We also define an automorphism of L -modules

$$t_L^G : \mathcal{D}_\hbar(\mathcal{X}_L) \xrightarrow{\sim} \mathcal{D}_\hbar(\mathcal{X}_L)$$

as follows. The linear map $\rho_G - \rho_L$ on \mathfrak{t} extends naturally to a Lie algebra morphism $\mathfrak{l} \rightarrow \mathbb{C}$, which we denote similarly. Then the assignment $\mathfrak{l} \ni x \mapsto x + \hbar(\rho_G - \rho_L)(x)$ defines a graded $\mathbb{C}[\hbar]$ -algebra automorphism t_L^G of $U_\hbar(\mathfrak{l})$, which descends to a U_L -equivariant automorphism of the quotient $U_\hbar(\mathfrak{l})/(\mathfrak{u}_L \cdot U_\hbar(\mathfrak{l}))$. Using the isomorphism $\mathcal{D}_\hbar(\mathcal{X}_L) \cong (\mathbb{C}[L] \otimes (U_\hbar(\mathfrak{l})/\mathfrak{u}_L \cdot U_\hbar(\mathfrak{l})))^{U_L}$ as above, we obtain the wished for automorphism t_L^G . This morphism can also be described in more geometric terms as follows: the linear form $\rho_G - \rho_L$ on \mathfrak{t} can be considered as a character of T , which extends in a natural way to a character of L , and then descends to an invertible function f_L^G on L/U_L . Then one can easily check that t_L^G is the automorphism of $\mathcal{D}_\hbar(\mathcal{X}_L)$ sending D to $(f_L^G)^{-1} \cdot D \cdot f_L^G$.

Finally we define the morphism of L -modules

$$\text{res}_L^G := t_L^G \circ r_L^G : \mathcal{D}_\hbar(\mathcal{X}_G) \rightarrow \mathcal{D}_\hbar(\mathcal{X}_L).$$

One can easily check that res_L^G is S_\hbar -equivariant, and T -equivariant for the $T = \{1\} \times T$ -actions. The following result (in which we use superscripts on the left to indicate which reductive group we consider) will be proved in §3.4 below.

Proposition 3.3.2. *Let $\alpha \in I$. Then the following diagram commutes:*

$$\begin{CD} \mathcal{D}_\hbar(\mathcal{X}_G) @>{G\mathbf{F}_\alpha}>> \mathcal{D}_\hbar(\mathcal{X}_G) \\ @V{\text{res}_L^G}VV @VV{\text{res}_L^G}V \\ \mathcal{D}_\hbar(\mathcal{X}_L) @>{L\mathbf{F}_\alpha}>> \mathcal{D}_\hbar(\mathcal{X}_L). \end{CD}$$

As a corollary we obtain the following result. For $\lambda \in \mathbb{X}$ and V in $\text{Rep}(G)$, we denote by

$$\mathcal{R}_{G,L}^{V,\lambda} : (V \otimes^{(\lambda)} \mathcal{D}_\hbar(\mathcal{X}_G)_\lambda)^G \rightarrow (V|_L \otimes^{(\lambda)} \mathcal{D}_\hbar(\mathcal{X}_L)_\lambda)^L$$

the morphism induced by res_L^G .

Corollary 3.3.3. *For any $\lambda \in \mathbb{X}$, V in $\text{Rep}(G)$ and $w \in W_L \subset W_G$, the following diagram commutes:*

$$\begin{CD} (V \otimes^{(\lambda)} \mathcal{D}_\hbar(\mathcal{X}_G)_\lambda)^G @>{G\Phi_w^{V,\lambda}}>> w(V \otimes^{(w\lambda)} \mathcal{D}_\hbar(\mathcal{X}_G)_{w\lambda})^G \\ @V{\mathcal{R}_{G,L}^{V,\lambda}}VV @VV{w\mathcal{R}_{G,L}^{V,w\lambda}}V \\ (V|_L \otimes^{(\lambda)} \mathcal{D}_\hbar(\mathcal{X}_L)_\lambda)^L @>{L\Phi_w^{V|_L,\lambda}}>> w(V|_L \otimes^{(w\lambda)} \mathcal{D}_\hbar(\mathcal{X}_L)_{w\lambda})^L. \end{CD}$$

3.4. Proof of Proposition 3.3.2

If G has simply connected derived subgroup, then so does L . Hence we can assume that $G = G^{\text{sc}}$.

In the next lemma, if X is any variety we consider $\mathbb{C}[X][\hbar]$ as the algebra of functions on X with values in $\mathbb{C}[\hbar]$. The subset $P_- \cdot P_+ \subset G$ is an open subvariety, so if x is in $U_{\hbar}(\mathfrak{g})$ and f is in $\mathbb{C}[P_- \cdot P_+][\hbar]$, it makes sense to consider $x \cdot f \in \mathbb{C}[P_- \cdot P_+][\hbar]$, and also the restriction $(x \cdot f)|_L \in \mathbb{C}[L][\hbar]$. (Here we consider the *right* action of $U_{\hbar}(\mathfrak{g})$ on $\mathbb{C}[G][\hbar]$ obtained by differentiating the *right regular representation* of § 3.1.)

Lemma 3.4.1. *Let $f \in \mathbb{C}[P_- \cdot P_+][\hbar]$ be a left N_L^- -invariant function. Then for any $x \in U_{\hbar}(\mathfrak{g}) \cdot n_L^-$ we have*

$$(x \cdot f)|_L = 0.$$

Proof. For any $y \in U_{\hbar}(\mathfrak{g})$ the function $y \cdot f$ is again left N_L^- -invariant, so we can assume that $x \in n_L^-$. Then the result follows from the observation that for $g \in L$ we have $g \cdot N_L^- = N_L^- \cdot g$. □

In the next lemma we use the embedding $\mathcal{X}_L = L/U_L = P_+/U_+ \hookrightarrow P_- \cdot P_+/U_+ \subset G/U_+ = \mathcal{X}_G$.

Lemma 3.4.2. *For any left N_L^- -invariant function $f \in \mathbb{C}[P_- \cdot P_+/U_+][\hbar]$ and any $D \in \mathcal{D}_{\hbar}(\mathcal{X}_G)$ we have*

$$D(f)|_{\mathcal{X}_L} = (r_L^G D)(f|_{\mathcal{X}_L}).$$

Proof. The element $D \in \mathcal{D}_{\hbar}(\mathcal{X}_G)$ induces a morphism $\mathbb{C}[P_- \cdot P_+/U_+][\hbar] \rightarrow \mathbb{C}[P_- \cdot P_+][\hbar]$. Similarly, the restriction morphism $\mathbb{C}[P_- \cdot P_+/U_+][\hbar] \rightarrow \mathbb{C}[\mathcal{X}_L][\hbar]$ is the restriction to right U_+ -invariants of the restriction morphism $\mathbb{C}[P_- \cdot P_+][\hbar] \rightarrow \mathbb{C}[P_+][\hbar]$. Hence it is enough to show that if $f \in \mathbb{C}[P_- \cdot P_+][\hbar]$ is left N_L^- -invariant then the morphism $\mathcal{D}_{\hbar}(G) \rightarrow \mathbb{C}[L][\hbar]$ sending D' to $D'(f)|_L$ factors (via isomorphism (3.1.1)) through the quotient

$$\mathbb{C}[G] \otimes U_{\hbar}(\mathfrak{g}) \rightarrow \mathbb{C}[G] \otimes (U_{\hbar}(\mathfrak{g})/(U_{\hbar}(\mathfrak{g}) \cdot n_L^-)).$$

This fact follows from Lemma 3.4.1. □

We obtain as a corollary of Lemma 3.4.2 the following description of the morphism r_L^G . We denote by $\varepsilon_L : \mathcal{D}_{\hbar}(N_L^-) \rightarrow \mathbb{C}[\hbar]$ the morphism sending a differential operator D to the value at $1 \in N_L^-$ of the function $D(1_{N_L^-})$, where $1_{N_L^-}$ is the constant function with value 1.

Corollary 3.4.3. *The morphism r_L^G coincides with the composition*

$$\mathcal{D}_{\hbar}(\mathcal{X}_G) \hookrightarrow \mathcal{D}_{\hbar}(P_- \cdot P_+/U_+) \xrightarrow{\sim} \mathcal{D}_{\hbar}(N_L^-) \otimes_{\mathbb{C}[\hbar]} \mathcal{D}_{\hbar}(\mathcal{X}_L) \xrightarrow{D^- \otimes D \mapsto \varepsilon_L(D^-) \cdot D} \mathcal{D}_{\hbar}(\mathcal{X}_L)$$

where the first morphism is restriction to the open subset $P_- \cdot P_+/U_+ \subset \mathcal{X}_G$, and the second morphism uses the isomorphism $P_- \cdot P_+/U_+ \cong N_L^- \times \mathcal{X}_L$ induced by the action of N_L^- on \mathcal{X}_G .

Proof. As the action of $\mathcal{D}_{\hbar}(\mathcal{X}_L)$ on $\mathbb{C}[\mathcal{X}_L][\hbar]$ is faithful, it is enough to check the claim after acting on any $f \in \mathbb{C}[\mathcal{X}_L]$. However one can write $f = \tilde{f}|_{\mathcal{X}_L}$ where $\tilde{f} = 1_{N_L^-} \otimes f \in \mathbb{C}[P_- \cdot P_+/U_+]$, and then the claim follows from Lemma 3.4.2. □

Finally we can give the proof of Proposition 3.3.2.

Proof of Proposition 3.3.2. It is enough to prove the commutativity of the following two diagrams:

$$\begin{array}{ccc}
 \mathcal{D}_\hbar(\mathcal{X}_G) & \xrightarrow{G\mathbf{F}_\alpha} & \mathcal{D}_\hbar(\mathcal{X}_G) \\
 r_L^G \downarrow & & \downarrow r_L^G \\
 \mathcal{D}_\hbar(\mathcal{X}_L) & \xrightarrow{L\mathbf{F}_\alpha} & \mathcal{D}_\hbar(\mathcal{X}_L),
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{D}_\hbar(\mathcal{X}_L) & \xrightarrow{L\mathbf{F}_\alpha} & \mathcal{D}_\hbar(\mathcal{X}_L) \\
 i_L^G \downarrow & & \downarrow i_L^G \\
 \mathcal{D}_\hbar(\mathcal{X}_L) & \xrightarrow{L\mathbf{F}_\alpha} & \mathcal{D}_\hbar(\mathcal{X}_L).
 \end{array}$$

The commutativity of the right-hand diagram follows from Lemma B.2.2 below; hence we only have to consider the left-hand diagram. Now we observe that (since the construction of the partial Fourier transform is local on the base of the vector bundle) the automorphism $G\mathbf{F}_\alpha$ extends to an automorphism of $\mathcal{D}_\hbar(P_- \cdot P_+ / U_+)$ denoted similarly, which makes the following diagram commutative, where vertical morphisms are induced by restriction:

$$\begin{array}{ccc}
 \mathcal{D}_\hbar(\mathcal{X}_G) & \xrightarrow{G\mathbf{F}_\alpha} & \mathcal{D}_\hbar(\mathcal{X}_G) \\
 \downarrow & & \downarrow \\
 \mathcal{D}_\hbar(P_- \cdot P_+ / U_+) & \xrightarrow{G\mathbf{F}_\alpha} & \mathcal{D}_\hbar(P_- \cdot P_+ / U_+).
 \end{array}$$

Next, by construction the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{D}_\hbar(P_- \cdot P_+ / U_+) & \xrightarrow{G\mathbf{F}_\alpha} & \mathcal{D}_\hbar(P_- \cdot P_+ / U_+) \\
 \wr \downarrow & & \downarrow \wr \\
 \mathcal{D}_\hbar(N_L^-) \otimes_{\mathbb{C}[\hbar]} \mathcal{D}_\hbar(\mathcal{X}_L) & \xrightarrow{\text{id} \otimes L\mathbf{F}_\alpha} & \mathcal{D}_\hbar(N_L^-) \otimes_{\mathbb{C}[\hbar]} \mathcal{D}_\hbar(\mathcal{X}_L).
 \end{array}$$

Then the commutativity follows from Corollary 3.4.3. □

3.5. The classical analogue

Consider the sheaf of algebras

$$\mathcal{A}_{\mathcal{X}} := \mathcal{D}_{\hbar, \mathcal{X}} / \hbar \cdot \mathcal{D}_{\hbar, \mathcal{X}}$$

on \mathcal{X} . This sheaf is canonically isomorphic to $(p_{\mathcal{X}})_* \mathcal{O}_{T^* \mathcal{X}}$, where $p_{\mathcal{X}} = T^* \mathcal{X} \rightarrow \mathcal{X}$ is the natural projection. We also set $\mathcal{A}(\mathcal{X}) := \Gamma(\mathcal{X}, \mathcal{A}_{\mathcal{X}})$. If α is a simple root, we define similarly the sheaf of rings $\mathcal{A}_{\mathcal{V}_\alpha}$ on \mathcal{V}_α , and the ring $\mathcal{A}(\mathcal{V}_\alpha)$. In §3.2 we have defined an automorphism of $\mathcal{D}_{\hbar, \mathcal{V}_\alpha}$ as a sheaf of $\mathbb{C}[\hbar]$ -algebras. Specializing to $\hbar = 0$ we deduce an automorphism of $\mathcal{A}_{\mathcal{V}_\alpha}$. By the same arguments as in §3.2, restriction induces an isomorphism $\mathcal{A}(\mathcal{V}_\alpha) \xrightarrow{\sim} \mathcal{A}(\mathcal{X})$, and hence we obtain an algebra automorphism

$$\mathcal{F}_\alpha : \mathcal{A}(\mathcal{X}) \xrightarrow{\sim} \mathcal{A}(\mathcal{X})$$

such that the following diagram commutes, where vertical arrows are induced by the natural quotient morphism $\mathcal{D}_{\hbar, \mathcal{X}} \rightarrow \mathcal{A}(\mathcal{X})$:

$$\begin{array}{ccc}
 \mathcal{D}_{\hbar}(\mathcal{X}) & \xrightarrow[\sim]{\mathbf{F}_{\alpha}} & \mathcal{D}_{\hbar}(\mathcal{X}) \\
 \downarrow & & \downarrow \\
 \mathcal{A}(\mathcal{X}) & \xrightarrow[\sim]{\mathcal{F}_{\alpha}} & \mathcal{A}(\mathcal{X}).
 \end{array} \tag{3.5.1}$$

The following lemma will be used to deduce Theorem 2.3.1 from Theorem 2.2.4.

Lemma 3.5.2. *The natural morphism $\mathcal{D}_{\hbar}(\mathcal{X}) \rightarrow \mathcal{A}(\mathcal{X})$ induces an isomorphism*

$$\mathcal{D}_{\hbar}(\mathcal{X})/\hbar \cdot \mathcal{D}_{\hbar}(\mathcal{X}) \xrightarrow{\sim} \mathcal{A}(\mathcal{X}).$$

In other words, for any V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$, the morphism

$$(V \otimes \mathbf{M}(\lambda))^B \rightarrow (V \otimes \mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B$$

induced by the quotient morphism $\mathbf{M}(\lambda) \rightarrow \mathbf{M}(\lambda)/(\hbar \cdot \mathbf{M}(\lambda)) \cong \mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda}$ induces an isomorphism

$$(V \otimes \mathbf{M}(\lambda))^B/\hbar \xrightarrow{\sim} (V \otimes \mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B.$$

Proof. By Lemma 3.1.3, both statements are equivalent to the fact that for any $\lambda \in \mathbb{X}$ the morphism

$$\text{Ind}_B^G(\mathbf{M}(\lambda))/\hbar \rightarrow \text{Ind}_B^G(\mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})$$

is an isomorphism. In the case λ is dominant, this property follows from the exact sequence of B -modules

$$\mathbf{M}(\lambda) \xrightarrow{\hbar} \mathbf{M}(\lambda) \twoheadrightarrow \mathbf{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda}$$

and the cohomology vanishing, $R^1 \text{Ind}_B^G(\mathbf{M}(\lambda)) = 0$; see Remark 3.2.5. Since \mathbf{F}_{α} induces an isomorphism $\mathcal{D}_{\hbar}(\mathcal{X})_{\lambda} \xrightarrow{\sim} \mathcal{D}_{\hbar}(\mathcal{X})_{s_{\alpha}\lambda}$ (see Lemma 3.2.1), using diagram (3.5.1) we deduce the general case from the case where λ is dominant. □

It follows in particular from (3.5.1) and Lemma 3.5.2 (using Lemma 3.2.1) that the assignment $s_{\alpha} \mapsto \mathcal{F}_{\alpha}$ defines an action of W on $\mathcal{A}(\mathcal{X})$ by algebra automorphisms, which we denote by $w \mapsto \mathcal{F}_w$. Moreover, for any $w \in W$ and $\lambda \in \mathbb{X}$, \mathcal{F}_w defines an isomorphism of $\mathbf{S}(\mathfrak{t})$ -modules $\mathcal{A}(\mathcal{X})_{\lambda} \xrightarrow{\sim} {}^w\mathcal{A}(\mathcal{X})_{w\lambda}$.

3.6. Complementary results on the structure of $\mathcal{D}(\mathcal{X})$

In this subsection we observe that Lemma 3.5.2 has some interesting consequences for the structure of $\mathcal{D}(\mathcal{X})$. These results will not be used in the rest of the paper.

We begin with the following direct consequence of Lemma 3.5.2 (using the natural isomorphism $\mathcal{A}(\mathcal{X}) \cong \mathbb{C}[T^*\mathcal{X}]$), which appears to be new.

Corollary 3.6.1. *The canonical graded algebra morphism $\text{gr } \mathcal{D}(\mathcal{X}) \rightarrow \mathbb{C}[T^*\mathcal{X}]$ is an isomorphism.*

This corollary allows us to give new proofs of some results of [32] and [8]. These proofs use the following simple lemma.

Lemma 3.6.2. *The \mathbb{C} -algebra $\mathbb{C}[T^*\mathcal{X}]$ is finitely generated, and hence Noetherian.*

Proof. First we observe that there exists a canonical isomorphism $T^*\mathcal{X} \cong G \times_U (\mathfrak{g}/\mathfrak{u})^*$. Hence $T^*\mathcal{X}$ is a T -torsor over $\tilde{\mathfrak{g}}$, which implies that we have a natural algebra isomorphism

$$\mathbb{C}[T^*\mathcal{X}] \cong \bigoplus_{\lambda \in \mathbb{X}} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)).$$

By the same observation, there exists a natural morphism $T^*\mathcal{X} \rightarrow \mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$, $g \times_U \eta \mapsto (g \cdot \eta, \eta|_{\mathfrak{t}})$, which induces an algebra morphism $\mathbf{S}(\mathfrak{g}) \otimes_{\mathbf{S}(\mathfrak{t})^W} \mathbf{S}(\mathfrak{t}) \rightarrow \mathbb{C}[T^*\mathcal{X}]$. Note also that if $\lambda \in \mathbb{X}^+$, then $\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\lambda))$ identifies naturally with a subspace of $\mathbb{C}[\mathcal{X}]$, and hence also defines a subspace X_λ of $\mathbb{C}[T^*\mathcal{X}]$ using the projection $T^*\mathcal{X} \rightarrow \mathcal{X}$.

Let $\lambda_1, \dots, \lambda_r$ be a finite collection of dominant weights such that $\mathbb{X}^+ = \sum_{i=1}^r \mathbb{Z}_{\geq 0} \lambda_i$. We claim that $\mathbb{C}[T^*\mathcal{X}]$ is generated (as an algebra) by $\mathbf{S}(\mathfrak{g}) \otimes_{\mathbf{S}(\mathfrak{t})^W} \mathbf{S}(\mathfrak{t})$ and the G -modules X_{λ_i} for $i \in \{1, \dots, r\}$, together with the images of these subspaces under the automorphisms \mathcal{F}_w for all $w \in W$. This claim clearly implies the statement of the lemma.

To prove the claim we first observe that if $\lambda \in \mathbb{X}^+$ the morphism

$$(\mathbf{S}(\mathfrak{g}) \otimes_{\mathbf{S}(\mathfrak{t})^W} \mathbf{S}(\mathfrak{t})) \otimes X_\lambda \rightarrow \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda))$$

induced by the product in $\mathbb{C}[T^*\mathcal{X}]$ is surjective. Indeed by the graded Nakayama lemma it is enough to prove surjectivity after tensoring with the trivial $\mathbf{S}(\mathfrak{g}) \otimes_{\mathbf{S}(\mathfrak{t})^W} \mathbf{S}(\mathfrak{t})$ -module; hence it is also enough to prove surjectivity after tensoring with the trivial $\mathbf{S}(\mathfrak{t})$ -module. However the arguments in the proof of Proposition 3.2.3 imply that the natural morphism

$$\mathbb{C} \otimes_{\mathbf{S}(\mathfrak{t})} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \rightarrow \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))$$

induced by restriction is an isomorphism, and hence the latter surjectivity statement follows from [14, Proposition 2.6]. From this observation, together with the fact that if $\lambda, \mu \in \mathbb{X}^+$ the natural morphism

$$\Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\lambda)) \otimes \Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\mu)) \rightarrow \Gamma(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\lambda + \mu))$$

is surjective (see e.g. [13, Theorem 3.1.2]), it follows that the subalgebra of $\mathbb{C}[T^*\mathcal{X}]$ generated by $\mathbf{S}(\mathfrak{g}) \otimes_{\mathbf{S}(\mathfrak{t})^W} \mathbf{S}(\mathfrak{t})$ and the modules X_{λ_i} contains

$$\bigoplus_{\lambda \in \mathbb{X}^+} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)).$$

The claim follows, using the W -action and the fact that every weight is W -conjugate to a dominant weight. □

In the following corollary, statement (1) is due to Levasseur and Stafford (see [32, Theorem 3.3]). The present proof is suggested in [32, Remark 3.4], but the authors did not have Corollary 3.6.1 to complete the argument. Statement (2) is due to Bezrukavnikov, Braverman and Positselskii (see [8, Theorem 1.1]).

Corollary 3.6.3. *The algebra $\mathcal{D}(\mathcal{X})$ is*

- (1) *Auslander–Gorenstein and Cohen–Macaulay;*
- (2) *left and right Noetherian.*

Proof. (1) Choose some elements $\check{\mu}_i \in X_*(T)$, $i = 1, \dots, s$, such that if $\lambda(\check{\mu}_i) = 0$ for all i then $\lambda = 0$, and set $\|\lambda\| := \sum_i |\lambda(\check{\mu}_i)|$. Consider the filtration on $\mathcal{D}(\mathcal{X})$ defined for $n \geq 0$ by

$$F_n \mathcal{D}(\mathcal{X}) := \mathcal{D}(\mathcal{X})^{\leq n} \cap \left(\bigoplus_{\|\lambda\| \leq n} \mathcal{D}(\mathcal{X})_\lambda \right),$$

where $\mathcal{D}(\mathcal{X})^{\leq \cdot}$ is the filtration by order of differential operators. Then this filtration is connected (in the sense that $F_0 \mathcal{D}(\mathcal{X}) = \mathbb{C}$) and the associated graded is canonically isomorphic to $\mathbb{C}[T^* \mathcal{X}]$ by Corollary 3.6.1. Hence the claim follows from [41, Corollary 0.3], using the simplicity of $\mathcal{D}(\mathcal{X})$ proved in [32, Proposition 3.1] and Lemma 3.6.2.

(2) It is enough to prove that $\mathcal{D}_\hbar(\mathcal{X})$ is (left and right) Noetherian. But this follows from [4, Lemma 8.2] (for $g = \hbar$) and Lemma 3.6.2. □

4. Morphisms between asymptotic universal Verma modules

In this section we will use the following convention. If $\mathcal{P}(\mu)$ is a property depending on $\mu \in \mathfrak{t}^*$, we will say that $\mathcal{P}(\mu)$ holds “for $\mu \in \mathfrak{t}^*$ sufficiently large” if there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{P}(\mu)$ holds for any $\mu \in \mathfrak{t}^*$ satisfying $|\mu(\check{\alpha})| \geq n$ for all simple roots α . We will use similar conventions for subsets of \mathfrak{t}^* (e.g. \mathbb{X}).

4.1. A reminder on Verma modules

The results in this subsection are well known; see e.g. [19, 39]. We include (short) proofs for the reader’s convenience. We will use the “dot-action” defined by $w \bullet \mu = w(\mu + \rho) - \rho$.

For any $\mu \in \mathfrak{t}^*$ we consider the Verma module

$$\mathbf{V}(\mu) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\mu$$

(a left $U(\mathfrak{g})$ -module). We also set $1_\mu := 1 \otimes 1 \in \mathbf{V}(\mu)$. We set $\mathbf{V}(\mu)_- := U(\mathfrak{u}^-) \mathfrak{u}^- \cdot 1_\mu$, so we have $\mathbf{V}(\mu) = \mathbb{C} \cdot 1_\mu \oplus \mathbf{V}(\mu)_-$.

If V is in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$, $\mu \in \mathfrak{t}^*$, and $\phi \in \text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\mu), V \otimes \mathbf{V}(\mu - \lambda))$, then one can write

$$\phi(1_\mu) = u \otimes 1_{\mu - \lambda} + x$$

for unique $u \in V_\lambda$ and $x \in V \otimes \mathbf{V}(\mu - \lambda)_-$. We set $u := E_\mu^{V, \lambda}(\phi)$; it is called the *expectation value* of ϕ .

Let $\mu \in \mathfrak{t}^*$ and let α be a simple root. If $n := \mu(\check{\alpha}) + 1 \in \mathbb{Z}_{\geq 0}$, then as in [25, § 1.4] there exists a unique embedding of $U(\mathfrak{g})$ -modules

$$\mathbf{V}(s_\alpha \bullet \mu) \hookrightarrow \mathbf{V}(\mu)$$

which sends $1_{s_\alpha \bullet \mu}$ to $\frac{(f_\alpha)^n}{n!} \cdot 1_\mu$. Iterating, we perform the following construction. Let $\mu \in \mathbb{X}^+ - \rho$, and let $w \in W$. Choose a reduced decomposition $w = s_k \cdots s_1$, where $\alpha_1, \dots, \alpha_k$

is a sequence of simple roots (possibly with repetitions) and s_i is the reflection associated with α_i . Then for any $i = 1, \dots, k$, $n_i := (s_{i-1} \cdots s_1 \bullet \mu)(\check{\alpha}_i) + 1 \in \mathbb{Z}_{\geq 0}$. Moreover the collection (n_1, \dots, n_k) and the product $(f_{\alpha_k})^{n_k} \cdots (f_{\alpha_1})^{n_1}$ do not depend on the reduced decomposition of w (see [39, Lemma 4]). Hence there exists a unique embedding of graded right $U(\mathfrak{g})$ -modules

$$\iota_\mu^w : \mathbf{V}(w \bullet \mu) \hookrightarrow \mathbf{V}(\mu) \quad \text{such that} \quad \iota_\mu^w(1_{w \bullet \mu}) = \frac{(f_{\alpha_k})^{n_k}}{n_k!} \cdots \frac{(f_{\alpha_1})^{n_1}}{n_1!} \cdot 1_\mu.$$

Lemma 4.1.1. *Let V in $\text{Rep}(G)$, and $\lambda \in \mathbb{X}$.*

(1) *For $\mu \in \mathfrak{t}^*$ sufficiently large, the morphism*

$$\mathbf{E}_\mu^{V, \lambda} : \text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\mu), V \otimes \mathbf{V}(\mu - \lambda)) \rightarrow V_\lambda$$

is an isomorphism.

(2) *For $w \in W$ and $\mu \in \mathbb{X}^+$ sufficiently large, the morphism*

$$\text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(w \bullet \mu), V \otimes \mathbf{V}(w \bullet (\mu - \lambda))) \rightarrow \text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(w \bullet \mu), V \otimes \mathbf{V}(\mu - \lambda))$$

defined by $\phi \mapsto (\text{id}_V \otimes \iota_{\mu - \lambda}^w) \circ \phi$ is an isomorphism.

Proof. (1) It is well known (see e.g. [25, Theorem 3.6]) that there exists an enumeration ν_1, \dots, ν_k of the T -weights of V and a filtration (as a $U(\mathfrak{g})$ -module)

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_k = V \otimes \mathbf{V}(\mu - \lambda)$$

where for all $i = 1, \dots, k$, $M_i/M_{i-1} \cong V_{\nu_i} \otimes \mathbf{V}(\mu - \lambda + \nu_i)$. Consider, for any $i = 1, \dots, n$, the associated exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\mu), M_{i-1}) &\rightarrow \text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\mu), M_i) \\ &\rightarrow \text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\mu), V(\mu - \lambda + \nu_i) \otimes V_{\nu_i}) \rightarrow \text{Ext}_{U(\mathfrak{g})}^1(\mathbf{V}(\mu), M_{i-1}). \end{aligned}$$

It is also well known that

$$\text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\eta_1), \mathbf{V}(\eta_2)) = \text{Ext}_{U(\mathfrak{g})}^1(\mathbf{V}(\eta_1), \mathbf{V}(\eta_2)) = 0$$

unless $\eta_1 \in W \bullet \eta_2$, and $\text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\eta), \mathbf{V}(\eta)) = \mathbb{C}$. Now if $\mu \in \mathfrak{t}^*$ is sufficiently large, the property $\mu \in W \bullet (\mu - \lambda + \nu_i)$ implies that $w = 1$ and $\lambda = \nu_i$, and the result follows.

(2) First we remark that the morphism under consideration is indeed well defined if $\mu \in \mathbb{X}^+$ is sufficiently large. As $\iota_{\mu - \lambda}^w$ is injective, our morphism is injective. Hence it is enough to prove that the two sides have the same dimension for $\mu \in \mathbb{X}^+$ sufficiently large.

By (1), if μ is sufficiently large the left-hand side is isomorphic to $V_{w\lambda}$. Now similar arguments, using the property that if $\eta \in \mathbb{X}^+ - \rho$ then

$$\text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(w \bullet \eta), \mathbf{V}(\eta)) = \mathbb{C},$$

show that, again if μ is sufficiently large, the right-hand side is isomorphic to V_λ . As $\dim_{\mathbb{C}}(V_\lambda) = \dim_{\mathbb{C}}(V_{w\lambda})$, this finishes the proof. □

4.2. Asymptotic Verma modules

If $\mu \in \mathfrak{t}^*$, we denote by $\mathbb{C}[\hbar]\langle\langle\mu\rangle\rangle$ the graded right S_{\hbar} -module where any $t \in \mathfrak{t}$ acts by multiplication by $\hbar\mu(t)$.

For $\mu \in \mathfrak{t}^*$ we define the graded right $U_{\hbar}(\mathfrak{g})$ -module

$$\overline{\mathbf{M}}(\mu) := \mathbb{C}[\hbar]\langle\langle 0 \rangle\rangle \otimes_{S_{\hbar}} \mathbf{M}(\mu).$$

We also set $\overline{\mathbf{v}}_{\mu} := 1 \otimes \mathbf{v}_{\mu} \in \overline{\mathbf{M}}(\mu)$. Any $t \in \mathfrak{t}$ acts on $\overline{\mathbf{v}}_{\mu}$ by multiplication by $\hbar(\mu + \rho)(t)$, and $\overline{\mathbf{M}}(\mu)$ admits a basis (as a $\mathbb{C}[\hbar]$ -module) such that, for any vector v in this basis, there exists $\gamma \in \mathbb{Z}_{\geq 0}R^+$ such that $v \cdot t = (\mu + \rho + \gamma)(t)$ for any $t \in \mathfrak{t}$. We set $\overline{\mathbf{M}}(\mu)_{-} := \overline{\mathbf{v}}_{\mu} \cdot \mathbf{u}^{-} U_{\hbar}(\mathbf{u}^{-})$. Then we have

$$\overline{\mathbf{M}}(\mu) = \mathbb{C}[\hbar] \cdot \overline{\mathbf{v}}_{\mu} \oplus \overline{\mathbf{M}}(\mu)_{-}.$$

Note also that for any $\lambda, \mu \in \mathfrak{t}^*$ there exists a natural isomorphism

$$\mathbb{C}[\hbar]\langle\langle\mu\rangle\rangle \otimes_{S_{\hbar}} \mathbf{M}(\lambda) \cong \overline{\mathbf{M}}(\lambda + \mu) \tag{4.2.1}$$

sending $1 \otimes \mathbf{v}_{\lambda}$ to $\overline{\mathbf{v}}_{\lambda+\mu}$.

It will be convenient to invert \hbar . To simplify notation we set

$$\overline{\mathbf{M}}_{\text{loc}}(\mu) := \mathbb{C}[\hbar, \hbar^{-1}] \otimes_{\mathbb{C}[\hbar]} \overline{\mathbf{M}}(\mu), \quad U_{\text{loc}}(\mathfrak{g}) := \mathbb{C}[\hbar, \hbar^{-1}] \otimes_{\mathbb{C}[\hbar]} U_{\hbar}(\mathfrak{g}).$$

We define $\overline{\mathbf{M}}_{\text{loc}}(\mu)_{-}$ in the obvious way. There exists an isomorphism of graded $\mathbb{C}[\hbar, \hbar^{-1}]$ -algebras

$$U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}[\hbar, \hbar^{-1}] \xrightarrow{\sim} U_{\text{loc}}(\mathfrak{g})^{\text{op}}, \quad \mathfrak{g} \ni x \mapsto -\frac{1}{\hbar}x.$$

(In the left-hand side, $U(\mathfrak{g})$ is in degree 0.) Using this isomorphism one can regard $\overline{\mathbf{M}}_{\text{loc}}(\mu)$ as a (left) module over the algebra $U(\mathfrak{g}) \otimes_{\mathbb{C}} \mathbb{C}[\hbar, \hbar^{-1}]$. With this structure it is isomorphic to $\mathbf{V}(-\mu - \rho) \otimes_{\mathbb{C}} \mathbb{C}[\hbar, \hbar^{-1}]$.

Let $\mu \in \mathfrak{t}^*$ and let α be a simple root. If $n := -\mu(\check{\alpha}) \in \mathbb{Z}_{\geq 0}$, then as in § 4.1 there exists a unique embedding of graded $U_{\text{loc}}(\mathfrak{g})$ -modules

$$\overline{\mathbf{M}}_{\text{loc}}(s_{\alpha}\mu) \hookrightarrow \overline{\mathbf{M}}_{\text{loc}}(\mu)$$

which sends $\overline{\mathbf{v}}_{s_{\alpha}\mu}$ to $\overline{\mathbf{v}}_{\mu} \cdot \frac{(-f_{\alpha})^n}{\hbar^n n!}$. Iterating, we perform the following construction. Let $\mu \in \mathbb{X}^{-}$, and let $w \in W$. Choose a reduced decomposition $w = s_k \cdots s_1$, where s_i is the reflection associated with the simple root α_i . Then for any $i = 1, \dots, k$, $n_i := -(s_{i-1} \cdots s_1 \mu, \check{\alpha}_i) \in \mathbb{Z}_{\geq 0}$, and hence we can define

$$\overline{\mathbf{v}}_{\mu}^w := \overline{\mathbf{v}}_{\mu} \cdot \frac{(-f_{\alpha_{i_1}})^{n_1}}{\hbar^{n_1} \cdot n_1!} \cdots \frac{(-f_{\alpha_{i_k}})^{n_k}}{\hbar^{n_k} \cdot n_k!} \in \overline{\mathbf{M}}_{\text{loc}}(\lambda),$$

and there exists a unique embedding of graded right $U_{\text{loc}}(\mathfrak{g})$ -modules

$$\mathbf{i}_{\mu}^w : \overline{\mathbf{M}}_{\text{loc}}(w\mu) \hookrightarrow \overline{\mathbf{M}}_{\text{loc}}(\mu) \quad \text{such that} \quad \mathbf{i}_{\mu}^w(\overline{\mathbf{v}}_{w\mu}) = \overline{\mathbf{v}}_{\mu}^w.$$

4.3. Specialization

Let $\lambda \in \mathbb{X}$. Recall that in §2.1 we have defined a structure of a B -module and of a $(U_{\hbar}(\mathfrak{b}), U_{\hbar}(\mathfrak{g}))$ -bimodule on $\mathbf{M}(\lambda)$. In particular, if V is in $\mathbf{Rep}(G)$, we have a (diagonal) B -module structure on $V \otimes \mathbf{M}(\lambda)$.

Lemma 4.3.1. *For V in $\mathbf{Rep}(G)$ and $\lambda \in \mathbb{X}$, the assignment $\phi \mapsto \phi(\mathbf{v}_0)$, induces an isomorphism*

$$\mathrm{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \xrightarrow{\sim} (V \otimes \mathbf{M}(\lambda))^B.$$

Proof. By definition, $\mathbf{M}(0)$ is an induced right $U_{\hbar}(\mathfrak{g})$ -module. Hence we have

$$\mathrm{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \cong \mathrm{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{b}))}(S_{\hbar}\langle\langle 0 \rangle\rangle, V \otimes \mathbf{M}(\lambda)).$$

It follows that the left-hand side is isomorphic to the space of vectors x in $V \otimes \mathbf{M}(\lambda)$ which satisfy

$$x \cdot b = (b + \hbar\rho(b)) \cdot x$$

for all $b \in \mathfrak{b} \subset U_{\hbar}(\mathfrak{b})$. This is exactly the condition for being $U(\mathfrak{b})$ -invariant (i.e. B -invariant); see §2.1. □

If $\mu \in \mathfrak{t}^*$, we denote by $P \mapsto P(\mu)$ the unique algebra morphism $S_{\hbar} \rightarrow \mathbb{C}[\hbar]$ which sends $x \in \mathfrak{t}$ to $\hbar\mu(x)$. Note that if $P \in S_{\hbar}$ and if for some $w \in W$ we have $P(\mu) = 0$ for all $\mu \in w(\mathbb{X}^-)$ sufficiently large, then $P = 0$.

If $\lambda, \mu \in \mathfrak{t}^*$, we denote by

$$\underline{\mathrm{Sp}}_{\mu} : \mathbf{M}(\lambda) \rightarrow \overline{\mathbf{M}}_{\mathrm{loc}}(\lambda + \mu)$$

the natural morphism induced by isomorphism (4.2.1). The following lemma follows from the fact that $\mathbf{M}(\lambda)$ is a free S_{\hbar} -module and the remarks above on the map $P \mapsto P(\mu)$.

Lemma 4.3.2. *Let $\lambda \in \mathbb{X}$ and $w \in W$. Let $m \in \mathbf{M}(\lambda)$, and assume that $\underline{\mathrm{Sp}}_{\mu}(m) = 0$ for all $\mu \in w(\mathbb{X}^-)$ sufficiently large. Then $m = 0$.*

We will derive several useful corollaries.

Corollary 4.3.3. *Let V in $\mathbf{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$. Let $m \in V \otimes \mathbf{M}(\lambda)$, and assume that for all $\mu \in w(\mathbb{X}^-)$ sufficiently large we have*

$$(\mathrm{id}_V \otimes \underline{\mathrm{Sp}}_{\mu})(m) \cdot b = \hbar(\mu + \rho)(b) \cdot (\mathrm{id}_V \otimes \underline{\mathrm{Sp}}_{\mu})(m)$$

for all $b \in \mathfrak{b}$. Then there exists a unique $\phi \in \mathrm{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))$ such that $m = \phi(\mathbf{v}_0)$.

Proof. By Lemma 4.3.1 we only have to prove that m is fixed by B , i.e. that for any $b \in \mathfrak{b}$ we have

$$m \cdot b - (b + \hbar\rho(b)) \cdot m = 0$$

(see the proof of Lemma 4.3.1). However, our assumption implies that this vector is annihilated by $(\mathrm{id}_V \otimes \underline{\mathrm{Sp}}_{\mu})$ for all $\mu \in w(\mathbb{X}^-)$ sufficiently large. Hence we conclude using Lemma 4.3.2. □

If $\phi \in \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))$ we denote by $\text{Sp}_{\mu}(\phi) : \overline{\mathbf{M}}_{\text{loc}}(\mu) \rightarrow V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu)$ the morphism obtained by tensoring with $\mathbb{C}[\hbar]\langle\langle\mu\rangle\rangle$, using isomorphism (4.2.1), and inverting \hbar . This construction induces a morphism of $\mathbb{C}[\hbar, \hbar^{-1}]$ -modules

$$\text{Sp}_{\mu} : \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \rightarrow \text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu)).$$

Corollary 4.3.4. *Let V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$, $w \in W$, and $\phi \in \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))$. If $\text{Sp}_{\mu}(\phi) = 0$ for all $\mu \in w(\mathbb{X}^-)$ sufficiently large, then $\phi = 0$.*

Proof. This follows from the commutativity of the following diagram:

$$\begin{array}{ccc} \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) & \xrightarrow{\phi \mapsto \phi(v_0)} & V \otimes \mathbf{M}(\lambda) \\ \text{Sp}_{\mu} \downarrow & & \downarrow \text{id}_V \otimes \text{Sp}_{\mu} \\ \text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu)) & \xrightarrow{\phi \mapsto \phi(\bar{v}_{\mu})} & V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu) \end{array}$$

and Lemma 4.3.2. □

4.4. Intertwining operators

Let V in $\text{Rep}(G)$, and let $\phi \in \text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu))$ be an intertwining operator. Then we can write

$$\phi(\bar{v}_{\mu}) = u \otimes f(\hbar) \cdot \bar{v}_{\lambda + \mu} + x$$

for unique $u \in V_{\lambda}$, $f(\hbar) \in \mathbb{C}[\hbar, \hbar^{-1}]$ and $x \in V \otimes \overline{\mathbf{M}}(\lambda + \mu)_{-}$. The vector $u \otimes f(\hbar) \in V_{\lambda} \otimes \mathbb{C}[\hbar, \hbar^{-1}]$ is called the *expectation value* of ϕ . In this way we have defined a morphism of graded $\mathbb{C}[\hbar, \hbar^{-1}]$ -modules

$$\text{E}_{\mu}^{V, \lambda} : \text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu)) \rightarrow V_{\lambda} \otimes \mathbb{C}[\hbar, \hbar^{-1}].$$

Lemma 4.4.1. *Let V in $\text{Rep}(G)$, and $\lambda \in \mathbb{X}$.*

- (1) *For $\mu \in \mathfrak{t}^*$ sufficiently large, $\text{E}_{\mu}^{V, \lambda}$ is an isomorphism.*
- (2) *For $w \in W$ and $\mu \in \mathbb{X}^-$ sufficiently large, the morphism*

$$\begin{aligned} &\text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(w\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(w(\lambda + \mu))) \\ &\rightarrow \text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(w\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\lambda + \mu)) \end{aligned}$$

defined by $\phi \mapsto (\text{id}_V \otimes \mathbf{i}_{\lambda + \mu}^w) \circ \phi$ is an isomorphism.

Proof. By the remarks above we have an isomorphism $U_{\text{loc}}(\mathfrak{g})^{\text{op}} \cong U(\mathfrak{g}) \otimes \mathbb{C}[\hbar, \hbar^{-1}]$, which induces an isomorphism

$$\text{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V' \otimes \overline{\mathbf{M}}_{\text{loc}}(v)) \cong \text{Hom}_{U(\mathfrak{g})}(V(-\mu - \rho), V' \otimes V(-v - \rho)) \otimes \mathbb{C}[\hbar, \hbar^{-1}]$$

for any V' in $\text{Rep}(G)$ and $\mu, v \in \mathfrak{t}^*$. Moreover, under these isomorphisms the morphisms considered in the lemma are induced by those of Lemma 4.1.1. Hence the claims follow from Lemma 4.1.1. □

Fix V in $\mathbf{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$. For any $\mu \in \mathbb{X}^-$ sufficiently large, we define the morphism of graded $\mathbb{C}[\hbar, \hbar^{-1}]$ -modules

$$\begin{aligned} \Psi_{w,\mu}^{V,\lambda} &: \mathbf{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\mu + \lambda)) \\ &\rightarrow \mathbf{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(w\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(w(\mu + \lambda))), \end{aligned}$$

in such a way that for any $\phi \in \mathbf{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\mu + \lambda))$ we have

$$\phi \circ \mathbf{i}_\mu^w = (\mathbf{i}_{\lambda+\mu}^w \otimes \text{id}_V) \circ \Psi_{w,\mu}^{V,\lambda}(\phi).$$

This morphism is well defined by Lemma 4.4.1.

Let now α be a simple root, and consider the case $w = s_\alpha$. Then, even if μ is not in \mathbb{X}^- , one can define a morphism

$$\begin{aligned} \Psi_{s_\alpha,\mu}^{V,\lambda} &: \mathbf{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(\mu + \lambda)) \\ &\rightarrow \mathbf{Hom}_{-U_{\text{loc}}(\mathfrak{g})}(\overline{\mathbf{M}}_{\text{loc}}(s_\alpha\mu), V \otimes \overline{\mathbf{M}}_{\text{loc}}(s_\alpha(\mu + \lambda))) \end{aligned}$$

with the same properties as above as soon as $\mu \in \mathbb{X}$ is sufficiently large and $\mu(\check{\alpha}) < 0$. With this extension of the definition, consider again some $w \in W$, and let $w = s_k \cdots s_1$ be a reduced decomposition. Note that if $\mu \in \mathbb{X}^-$, for any $i = 1, \dots, k$ we have $(s_{i-1} \cdots s_1\mu)(\check{\alpha}_i) \leq 0$. Then, by definition, for $\mu \in \mathbb{X}^-$ sufficiently large we have

$$\Psi_{w,\mu}^{V,\lambda} = \Psi_{s_k, s_{k-1} \cdots s_1(\mu)}^{V, s_{k-1} \cdots s_1(\lambda)} \circ \dots \circ \Psi_{s_2, s_1(\mu)}^{V, s_1(\lambda)} \circ \Psi_{s_1, \mu}^{V, \lambda}. \tag{4.4.2}$$

4.5. Simple reflections

In this subsection we fix a simple root α , and set $s = s_\alpha$.

Proposition 4.5.1. *Let V in $\mathbf{Rep}(G)$ and $\lambda \in \mathbb{X}$. There exists a unique morphism of graded S_\hbar -modules*

$$\begin{aligned} \Theta_s^{V,\lambda} &: \mathbf{Hom}_{(S_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \langle \lambda(2\check{\rho}) \rangle \\ &\rightarrow {}^s\mathbf{Hom}_{(S_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(s\lambda)) \langle (s\lambda)(2\check{\rho}) \rangle \end{aligned}$$

such that for any $\mu \in \mathbb{X}$ sufficiently large such that $\mu(\check{\alpha}) < 0$ we have

$$\begin{aligned} \text{Sp}_{s\mu} \circ \Theta_s^{V,\lambda} &= \\ &\begin{cases} (-\hbar)^{\lambda(\check{\alpha})} (-\mu(\check{\alpha})) (-\mu(\check{\alpha}) - 1) \cdots (-\mu(\check{\alpha}) - \lambda(\check{\alpha}) + 1) \cdot \Psi_{s,\mu}^{V,\lambda} \circ \text{Sp}_\mu & \text{if } \lambda(\check{\alpha}) \geq 0, \\ \frac{1}{(-\hbar)^{-\lambda(\check{\alpha})} (-\mu(\check{\alpha}) - \lambda(\check{\alpha})) \cdots (-\mu(\check{\alpha}) + 1)} \cdot \Psi_{s,\mu}^{V,\lambda} \circ \text{Sp}_\mu & \text{if } \lambda(\check{\alpha}) \leq 0. \end{cases} \end{aligned} \tag{4.5.2}$$

Proof. Unicity follows from Corollary 4.3.4. Let us prove existence. Choose an enumeration of positive roots $\alpha_1, \dots, \alpha_n$ such that $\alpha_n = \alpha$ and, for any $i = 1, \dots, n - 1$, a non-zero vector $f_i \in \mathfrak{g}_{-\alpha_i}$. (These choices can be arbitrary.) For any multi-index

$\underline{k} = (k_1, \dots, k_{n-1})$ we set $f^{\underline{k}} := f_1^{k_1} \cdots f_{n-1}^{k_{n-1}} \in U_{\text{loc}}(\mathfrak{g})$. Then any vector of $\mathbf{M}(\lambda)$ can be written in a unique way in the form

$$\sum_{\underline{k}, i} P_{\underline{k}, i} \otimes f^{\underline{k}} f_{\alpha}^i$$

where $P_{\underline{k}, i} \in S_{\hbar}$ (and where only finitely many terms are non-zero). Choose also a basis $\{u_j, j \in J\}$ of V . To simplify the notation, for $\mu \in \mathfrak{t}^*$ we set $n(\mu) := -\mu(\check{\alpha})$.

For P in S_{\hbar} we set

$$\binom{P}{m}_{\hbar} := \frac{P(P - \hbar) \cdots (P - (m - 1)\hbar)}{m(m - 1) \cdots 1}.$$

With this notation, for $\mu \in \mathbb{X}$ such that $n(\mu) \geq m$ we have

$$\binom{\check{\alpha}}{m}_{\hbar} (s\mu) = \hbar^m \binom{n(\mu)}{m}. \tag{4.5.3}$$

Let $\phi : \mathbf{M}(0) \rightarrow V \otimes \mathbf{M}(\lambda)$ be a morphism of bimodules. Write

$$\phi(\mathbf{v}_0) = \sum_{j, \underline{k}, i} u_j \otimes P_{j, \underline{k}, i} \otimes f^{\underline{k}} f_{\alpha}^i$$

where $P_{j, \underline{k}, i} \in S_{\hbar}$. Then for $\mu \in \mathbb{X}$ such that $n(\mu) \geq 0$ we have

$$\text{Sp}_{\mu}(\phi)(\bar{\mathbf{v}}_{\mu}) = \sum_{j, \underline{k}, i} u_j \otimes P_{j, \underline{k}, i}(\mu) \otimes f^{\underline{k}} f_{\alpha}^i \in V \otimes \bar{\mathbf{M}}_{\text{loc}}(\lambda + \mu),$$

and hence

$$\begin{aligned} \text{Sp}_{\mu}(\phi)(\bar{\mathbf{v}}_{\mu}^s) &= \left(\sum_{j, \underline{k}, i} u_j \otimes P_{j, \underline{k}, i}(\mu) \otimes f^{\underline{k}} f_{\alpha}^i \right) \cdot \frac{(-f_{\alpha})^{n(\mu)}}{\hbar^{n(\mu)} n(\mu)!} \\ &= \frac{(-1)^{n(\mu)}}{\hbar^{n(\mu)} n(\mu)!} \left(\sum_{m=0}^{n(\mu)} (-\hbar)^m \binom{n(\mu)}{m} \sum_{j, \underline{k}, i} f_{\alpha}^m \cdot u_j \otimes P_{j, \underline{k}, i}(\mu) \otimes f^{\underline{k}} f_{\alpha}^{i+n(\mu)-m} \right). \end{aligned}$$

By Lemma 4.4.1(2) (or more precisely an obvious generalization, when $w = s$, to the case $\mu(\check{\alpha}) \leq 0$ instead of $\mu \in \mathbb{X}^-$), if μ is sufficiently large then the terms for which $i + n(\mu) - m < n(\lambda + \mu)$ vanish. Hence we obtain that in this case $\text{Sp}_{\mu}(\phi)(\bar{\mathbf{v}}_{\mu}^s)$ equals

$$\frac{(-1)^{n(\mu)}}{\hbar^{n(\mu)} n(\mu)!} \left(\sum_{\substack{j, \underline{k}, i, m \\ 0 \leq m \leq i - n(\lambda)}} (-\hbar)^m \binom{n(\mu)}{m} f_{\alpha}^m \cdot u_j \otimes P_{j, \underline{k}, i}(\mu) \otimes f^{\underline{k}} f_{\alpha}^{i+n(\mu)-m} \right),$$

i.e. equals

$$\frac{(-\hbar)^{n(\lambda)} n(\lambda + \mu)!}{n(\mu)!} (\text{id}_V \otimes \mathbf{i}_{\lambda + \mu}^s) \left(\sum_{\substack{j, \underline{k}, i, m \\ 0 \leq m \leq i - n(\lambda)}} (-\hbar)^m \binom{n(\mu)}{m} f_{\alpha}^m \cdot u_j \otimes P_{j, \underline{k}, i}(\mu) \otimes f^{\underline{k}} f_{\alpha}^{i-n(\lambda)-m} \right).$$

Note that in this sum the indices do not depend on μ . Hence we can consider the element

$$x := \sum_{\substack{j,k,i,m \\ 0 \leq m \leq i-n(\lambda)}} (-1)^m f_\alpha^m \cdot u_j \otimes \binom{\check{\alpha}}{m}_\hbar s(P_{j,k,i}) \otimes f^k f_\alpha^{i-n(\lambda)-m} \in V \otimes \mathbf{M}(s\lambda). \tag{4.5.4}$$

By construction (and using (4.5.3)), for any $\mu \in \mathbb{X}$ sufficiently large with $n(\mu) \geq 0$, $(\text{id}_V \otimes \underline{\mathbf{Sp}}_{s\mu})(x)$ is a multiple of $\mathbf{Sp}_\mu(\phi)(\bar{\mathbf{v}}_\mu^s)$; hence it satisfies

$$(\text{id}_V \otimes \underline{\mathbf{Sp}}_{s\mu})(x) \cdot b = \hbar(s\mu + \rho)(b) \cdot (\text{id}_V \otimes \underline{\mathbf{Sp}}_{s\mu})(x)$$

for all $b \in \mathfrak{b}$. Hence by Corollary 4.3.3 there exists a unique $\psi \in \text{Hom}_{(\mathbf{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(s\lambda))$ such that $x = \psi(\mathbf{v}_0)$. We set $\Theta_s^{V,\lambda}(\phi) := \psi$. With this definition, for any ϕ and any sufficiently large μ with $n(\mu) \geq 0$ we have

$$\mathbf{Sp}_\mu(\phi) \circ \mathbf{i}_\mu^s = \frac{(-\hbar)^{n(\lambda)} n(\lambda + \mu)!}{n(\mu)!} (\text{id}_V \otimes \mathbf{i}_{\lambda+\mu}^s) \circ \mathbf{Sp}_{s\mu}(\Theta_s^{V,\lambda}(\phi)),$$

which implies (4.5.2). □

4.6. Restriction to a Levi subgroup

Fix a subset I of the set of simple roots, and recall the notation of § 3.3. As in § 3.3 we can consider our constructions both for G and for L ; we add superscripts or subscripts to indicate which reductive group we consider.

The projection $\mathfrak{g} \rightarrow \mathfrak{l}$, along $\mathfrak{n}_L^+ \oplus \mathfrak{n}_L^-$, is L -equivariant and it induces a morphism of graded $\mathbb{C}[\hbar]$ -modules $\pi_L^G : U_\hbar(\mathfrak{g}) \rightarrow U_\hbar(\mathfrak{g})/(\mathfrak{n}_L^+ \cdot U_\hbar(\mathfrak{g}) + U_\hbar(\mathfrak{g}) \cdot \mathfrak{n}_L^-) \cong U_\hbar(\mathfrak{l})$. We consider the morphism of graded \mathbf{S}_\hbar -modules

$$\mathbf{M}^G(\lambda) \rightarrow \mathbf{M}^L(\lambda) \tag{4.6.1}$$

sending $p \otimes u \in \mathbf{M}^G(\lambda)$ to $p \otimes (i_L^G \circ \pi_L^G(u))$. One can easily check that (4.6.1) is also a morphism of B_L -modules, so it induces a morphism of graded $\mathbb{C}[\hbar]$ -modules

$$\mathbf{R}_{G,L}^{V,\lambda} : (V \otimes \mathbf{M}^G(\lambda))^B \rightarrow (V|_L \otimes \mathbf{M}^L(\lambda))^{B_L} \tag{4.6.2}$$

for any V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$.

Note that if $K \subset L$ is a smaller Levi subgroup containing T , then for any V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$ we have

$$\mathbf{R}_{L,K}^{V|_L,\lambda} \circ \mathbf{R}_{G,L}^{V,\lambda} = \mathbf{R}_{G,K}^{V,\lambda}. \tag{4.6.3}$$

If $L = T$, then we have $\mathbf{M}^T(\lambda) = \mathbf{S}_\hbar \langle \langle \lambda \rangle \rangle$, where $B_T = T$ acts via $-\lambda$. Hence we have an isomorphism of graded \mathbf{S}_\hbar -modules

$$(V|_T \otimes \mathbf{M}^T(\lambda))^{B_T} = V_\lambda \otimes \mathbf{S}_\hbar.$$

Under this isomorphism, one can easily check that $\mathbf{R}_{G,T}^{V,\lambda} = \kappa_{V,\lambda}^{\text{alg}}$.

Using Lemma 4.3.1, $R_{G,L}^{V,\lambda}$ induces a morphism of graded S_{\hbar} -modules

$$R_{G,L}^{V,\lambda} : \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}^G(0), V \otimes \mathbf{M}^G(\lambda)) \rightarrow \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{l}))}(\mathbf{M}^L(0), V_L \otimes \mathbf{M}^L(\lambda)).$$

Lemma 4.6.4. *For all V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$, the morphism $R_{G,L}^{V,\lambda}$ is injective.*

Proof. By (4.6.3), it is enough to prove the claim when $L = T$. Then by construction, if we identify $\text{Hom}_{-U_{\text{loc}}(\mathfrak{t})}(\overline{\mathbf{M}}_{\text{loc}}^T(0), V|_T \otimes \overline{\mathbf{M}}_{\text{loc}}^T(\lambda))$ with $V_{\lambda} \otimes \mathbb{C}[\hbar, \hbar^{-1}]$ in the natural way, for any $\mu \in \mathfrak{t}^*$ we have $\text{Sp}_{\mu}^T \circ R_{G,T}^{V,\lambda} = E_{\mu}^{V,\lambda} \circ \text{Sp}_{\mu}^G$.

Let $\phi \in \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}^G(0), V \otimes \mathbf{M}^G(\lambda))$ be such that $R_{G,T}^{V,\lambda}(\phi) = 0$. Then for any $\mu \in \mathfrak{t}^*$ we have $E_{\mu}^{V,\lambda}(\text{Sp}_{\mu}^G(\phi)) = 0$. Using Lemma 4.4.1(1) we deduce that for μ sufficiently large we have $\text{Sp}_{\mu}^G(\phi) = 0$. By Corollary 4.3.4 we deduce that $\phi = 0$, which finishes the proof. \square

The following result follows from the construction (see the proof of Proposition 4.5.1), using the fact that if α is a simple root of L , then $[f_{\alpha}, \mathfrak{n}_L^-] \subset \mathfrak{n}_L^-$. For simplicity, in this statement we neglect the gradings.

Lemma 4.6.5. *Let V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$. If $\alpha \in I$ the following diagram commutes:*

$$\begin{CD} \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}^G(0), V \otimes \mathbf{M}^G(\lambda)) @>G \Theta_{s_{\alpha}}^{V,\lambda}>> {}^{s_{\alpha}}\text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}^G(0), V \otimes \mathbf{M}^G(s_{\alpha}\lambda)) \\ @V R_{G,L}^{V,\lambda} VV @VV s_{\alpha} R_{G,L}^{V,s_{\alpha}\lambda} V \\ \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{l}))}(\mathbf{M}^L(0), V_L \otimes \mathbf{M}^L(\lambda)) @>L \Theta_{s_{\alpha}}^{V_L,\lambda}>> {}^{s_{\alpha}}\text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{l}))}(\mathbf{M}^L(0), V_L \otimes \mathbf{M}^L(s_{\alpha}\lambda)). \end{CD}$$

4.7. Definition of operators Θ_w

Lemma 4.7.1. *Let V in $\text{Rep}(G)$, and $\lambda \in \mathbb{X}$. For any simple root α , we have*

$${}^{s_{\alpha}}(\Theta_{s_{\alpha}}^{V,s_{\alpha}\lambda}) \circ \Theta_{s_{\alpha}}^{V,\lambda} = \text{id}$$

as endomorphisms of $\text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))\langle\lambda(2\check{\rho})\rangle$. In particular, $\Theta_s^{V,\lambda}$ is an isomorphism.

Proof. Let L_{α} be the Levi subgroup containing T with roots $\{\alpha, -\alpha\}$. Using Lemma 4.6.5 and Lemma 4.6.4 it is enough to prove the equality when $G = L_{\alpha}$. In this case it is checked in Corollary A.3.2 below. \square

Let again V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$. Let $w \in W$, and choose a reduced expression $w = s_k \cdots s_1$. We define the isomorphism of graded S_{\hbar} -modules

$$\begin{aligned} \Theta_w^{V,\lambda} : \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))\langle\lambda(2\check{\rho})\rangle \\ \xrightarrow{\sim} {}^w\text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(w\lambda))\langle(w\lambda)(2\check{\rho})\rangle \end{aligned}$$

by the formula

$$\Theta_w^{V,\lambda} := (s_{k-1} \cdots s_1) \Theta_{s_k}^{V, s_{k-1} \cdots s_1(\lambda)} \circ \cdots \circ s_1 \Theta_{s_2}^{V, s_1 \lambda} \circ \Theta_{s_1}^{V,\lambda}.$$

Lemma 4.7.2. *The operator $\Theta_w^{V,\lambda}$ does not depend on the choice of the reduced decomposition.*

Proof. Using (4.5.2) and (4.4.2) we obtain that for $\mu \in \mathbb{X}^-$ sufficiently large we have

$$\mathrm{Sp}_{w\mu} \circ \Theta_w^{V,\lambda} = n(w, \lambda, \mu) \cdot \Psi_{w,\mu}^{V,\lambda} \circ \mathrm{Sp}_\mu$$

where

$$n(w, \lambda, \mu) = \frac{\prod_{\check{\alpha} > 0, w(\check{\alpha}) < 0, \lambda(\check{\alpha}) > 0} (-\hbar)^{\lambda(\check{\alpha})} (-\mu(\check{\alpha})) \cdots (-\mu(\check{\alpha}) - \lambda(\check{\alpha}) + 1)}{\prod_{\check{\alpha} > 0, w(\check{\alpha}) < 0, \lambda(\check{\alpha}) < 0} (-\hbar)^{-\lambda(\check{\alpha})} (-\mu(\check{\alpha}) - \lambda(\check{\alpha})) \cdots (-\mu(\check{\alpha}) + 1)}.$$

The right-hand side is independent of the reduced decomposition; hence we conclude by Corollary 4.3.4. □

By construction and Lemma 4.7.1, our collection of isomorphisms satisfies condition (2.5.4) for all V in $\mathrm{Rep}(G)$, $\lambda \in \mathbb{X}$ and $x, y \in W$.

4.8. The relation to the operators Φ

As explained in § 3.1, for any V in $\mathrm{Rep}(G)$ and $\lambda \in \mathbb{X}$ there exists a canonical isomorphism of graded S_\hbar -modules

$$(V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \cong (V \otimes \mathbf{M}(\lambda))^B.$$

Using Lemma 4.3.1 we deduce a canonical isomorphism

$$(V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \cong \mathrm{Hom}_{(S_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)). \tag{4.8.1}$$

The following result is clear from the definitions.

Lemma 4.8.2. *Let us have V in $\mathrm{Rep}(G)$, $\lambda \in \mathbb{X}$, and $L \subset G$ a Levi subgroup containing T . The following diagram commutes:*

$$\begin{CD} \mathrm{Hom}_{(S_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}^G(0), V \otimes \mathbf{M}^G(\lambda)) @>{(4.8.1)}>> (V \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X}_G)_\lambda)^G \\ @V{\mathbb{R}_{G,L}^{V,\lambda}}VV @VV{\mathcal{R}_{G,L}^{V,\lambda}}V \\ \mathrm{Hom}_{(S_\hbar, U_\hbar(\mathfrak{l}))}(\mathbf{M}^L(0), V|_L \otimes \mathbf{M}^L(\lambda)) @>{(4.8.1)}>> (V|_L \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X}_L)_\lambda)^L \end{CD}$$

In the following proposition we assume that the vectors f_α of § 3 are the same as the vectors f_α of the present § 4.

Proposition 4.8.3. *For any V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$ the following diagram commutes:*

$$\begin{array}{ccc}
 (V \otimes {}^{(\lambda)}\mathcal{D}_{\hbar}(\mathcal{X})_{\lambda})^G & \xrightarrow[\sim]{(4.8.1)} & \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \\
 \Phi_w^{V,\lambda} \downarrow & & \downarrow \Theta_w^{V,\lambda} \\
 w(V \otimes {}^{(w\lambda)}\mathcal{D}_{\hbar}(\mathcal{X})_{w\lambda})^G & \xrightarrow[\sim]{(4.8.1)} & w \text{Hom}_{(S_{\hbar}, U_{\hbar}(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(w\lambda)).
 \end{array}$$

Proof. By (2.5.3) and (2.5.4) it is enough to prove commutativity when w is a simple reflection.

By Corollary 3.3.3, Lemma 4.6.5 and Lemma 4.8.2, our constructions are compatible with restriction to a Levi subgroup. Using Lemma 4.6.4, we deduce that it is enough to prove the claim when G has semisimple rank 1. In this case it is proved in Lemma A.4.1 below. \square

It follows from this proposition that for any V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$ there exists a unique isomorphism

$$\Theta_w^{V,\lambda} : (V \otimes \mathbf{M}(\lambda))^B \xrightarrow{\sim} w(V \otimes \mathbf{M}(w\lambda))^B$$

which corresponds to $\Phi_w^{V,\lambda}$ under the left-hand isomorphism of Lemma 2.4.1, and to $\Theta_w^{V,\lambda}$ under the right-hand isomorphism of Lemma 2.4.1.

5. The geometry of the Grothendieck–Springer resolution

5.1. The W -action on the regular part of $\tilde{\mathfrak{g}}$

In this section we are interested in the geometry of the Grothendieck–Springer resolution $\tilde{\mathfrak{g}}$ (see § 2.4). We have a standard commutative diagram

$$\begin{array}{ccc}
 \tilde{\mathfrak{g}} & \xrightarrow{\pi} & \mathfrak{g}^* \\
 \delta \downarrow & & \downarrow \\
 \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^* // W \cong \mathfrak{g}^* // G
 \end{array}$$

where the right vertical and the lower horizontal maps are the natural quotient maps, π is defined in § 2.5, and δ is defined by $\delta(g \times_B \eta) = \eta|_{\mathfrak{b}} \in (\mathfrak{b}/\mathfrak{u})^* \cong \mathfrak{t}^*$.

It is well known that there exists an action of W on $\tilde{\mathfrak{g}}_{\mathfrak{r}}$ which commutes with the natural G -action, and such that the restrictions of π and δ are W -equivariant, where W acts naturally on \mathfrak{t}^* and trivially on \mathfrak{g}^* (see e.g. [11, Proposition 1.9.2]). We denote the action of w by $\theta_w : \tilde{\mathfrak{g}}_{\mathfrak{r}} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{\mathfrak{r}}$. In the whole section we will use [11] as a convenient reference for the properties of this action, though much of this material was known before.

Recall that for any $\lambda \in \mathbb{X}$ we have a line bundle $\mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)$ on $\tilde{\mathfrak{g}}$ and its restriction $\mathcal{O}_{\tilde{\mathfrak{g}}_{\mathfrak{r}}}(\lambda)$ to $\tilde{\mathfrak{g}}_{\mathfrak{r}}$.

Lemma 5.1.1. *For any $w \in W$ and $\lambda \in \mathbb{X}$, there exists an isomorphism of $G \times \mathbb{G}_m$ -equivariant line bundles*

$$(\theta_{w^{-1}})^* \mathcal{O}_{\tilde{\mathfrak{g}}_{\mathfrak{r}}}(\lambda) \langle \lambda(2\check{\rho}) \rangle \cong \mathcal{O}_{\tilde{\mathfrak{g}}_{\mathfrak{r}}}(w\lambda) \langle (w\lambda)(2\check{\rho}) \rangle.$$

Proof. It is enough to prove the isomorphism when $w = s_\alpha$ is a simple reflection. Next, by the compatibility of $\theta_{w^{-1}}^*$ with tensor products (and passing to a simply connected cover if necessary), it is enough to prove the isomorphism when $\lambda(\check{\alpha}) \in \{-1, 0, 1\}$. Finally using duality one can assume that $\lambda(\check{\alpha}) \in \{-1, 0\}$. In these cases the isomorphism follows from [11, Lemma 1.11.3(2) and Remark 1.11.7(2)]. \square

5.2. Construction of the operators σ

The isomorphism of Lemma 5.1.1 is unique up to a scalar since we have

$$\Gamma(\tilde{\mathfrak{g}}_r, \mathcal{O}_{\tilde{\mathfrak{g}}_r})^{G \times \mathbb{G}_m} \cong \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}})^{G \times \mathbb{G}_m} = \mathbb{C}$$

(see e.g. equation (5.2.2) below). We will need to fix a normalization of this isomorphism, using our choice of vectors f_α . Let us denote by $\eta_0 \in \mathfrak{g}^*$ the element which is zero on \mathfrak{t} and on any \mathfrak{g}_β where β is not opposite to a simple root, and such that $\eta_0(f_\alpha) = 1$ for any simple root α . Then $(1 \times_B \eta_0) \in \tilde{\mathfrak{g}}_r$, and this point is W -invariant. For any $\lambda \in \mathbb{X}$, $\mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)$ is the sheaf of sections of the line bundle

$$\mathcal{L}(\lambda) := G \times_B ((\mathfrak{g}/\mathfrak{u})^* \times \mathbb{C}_{-\lambda})$$

over $\tilde{\mathfrak{g}}$. The fiber of this line bundle over $(B/B, \eta_0)$ can be canonically identified with \mathbb{C} through the morphism $x \mapsto (1 \times_B (\eta_0, x))$. Then there exists a unique isomorphism of $G \times \mathbb{G}_m$ -equivariant line bundles

$$(\theta_{w^{-1}})^* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda) \langle \lambda(2\check{\rho}) \rangle \xrightarrow{\sim} \mathcal{O}_{\tilde{\mathfrak{g}}_r}(w\lambda) \langle (w\lambda)(2\check{\rho}) \rangle \tag{5.2.1}$$

whose restriction to $(B/B, \eta_0) \in \tilde{\mathfrak{g}}_r$ is $\text{id}_{\mathbb{C}}$ via the identifications above.

Let $j_r : \tilde{\mathfrak{g}}_r \hookrightarrow \tilde{\mathfrak{g}}$ be the inclusion. As the codimension of $\tilde{\mathfrak{g}} \setminus \tilde{\mathfrak{g}}_r$ in $\tilde{\mathfrak{g}}$ is at least 2 (see e.g. [11, Proposition 1.9.3]), for any $\lambda \in \mathbb{X}$ the morphism $\mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda) \rightarrow (j_r)_* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)$ induced by adjunction is an isomorphism (see [34, Theorem 11.5.(ii)]). We deduce that the restriction induces an isomorphism

$$\Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \xrightarrow{\sim} \Gamma(\tilde{\mathfrak{g}}_r, \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)). \tag{5.2.2}$$

As $\theta_{w^{-1}}$ is an isomorphism, the adjunction morphism $\mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda) \rightarrow (\theta_{w^{-1}})_* (\theta_{w^{-1}})^* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)$ is also an isomorphism; in particular there is a canonical isomorphism

$$\Gamma(\tilde{\mathfrak{g}}_r, \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)) \xrightarrow{\sim} \Gamma(\tilde{\mathfrak{g}}_r, \theta_{w^{-1}}^* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)).$$

Putting these remarks together with isomorphism (5.2.1), we obtain for any $w \in W$ and $\lambda \in \mathbb{X}$ a canonical isomorphism of graded $\mathbf{S}(\mathfrak{t})$ -modules and G -modules

$$\Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \langle \lambda(2\check{\rho}) \rangle \xrightarrow{\sim} {}^w \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)) \langle (w\lambda)(2\check{\rho}) \rangle. \tag{5.2.3}$$

Hence for any V in $\text{Rep}(G)$ we obtain an isomorphism of graded $\mathbf{S}(\mathfrak{t})$ -modules

$$\sigma_w^{V, \lambda} : (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G \langle \lambda(2\check{\rho}) \rangle \xrightarrow{\sim} {}^w (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)))^G \langle (w\lambda)(2\check{\rho}) \rangle.$$

By construction, this collection of isomorphisms satisfies relations (2.5.6).

As explained above, for any $\mu \in \mathbb{X}$ we can describe $\Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\mu))$ as the space of sections of the line bundle $\mathcal{L}(\mu)$ over $\tilde{\mathfrak{g}}$. In particular, for any $\eta \in (\mathfrak{g}/\mathfrak{u})^*$ there exists a unique morphism

$$\text{ev}_\eta^\mu : \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\mu)) \rightarrow \mathbb{C}$$

(“evaluation at $(1, \eta)$ ”) such that if $f \in \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\mu))$ is considered as a section $\tilde{\mathfrak{g}} \rightarrow \mathcal{L}(\mu)$ we have

$$f(1 \times_B \eta) = 1 \times_B (\eta, \text{ev}_\eta^\mu(f)).$$

With this definition, (5.2.1) can be characterized as the unique isomorphism of $G \times \mathbb{C}^\times$ -equivariant line bundles $(\theta_{w^{-1}})^* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)(\lambda(2\check{\rho})) \xrightarrow{\sim} \mathcal{O}_{\tilde{\mathfrak{g}}_r}(w\lambda)((w\lambda)(2\check{\rho}))$ such that the following diagram commutes:

$$\begin{array}{ccc} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) & \xrightarrow[\sim]{(5.2.3)} & \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)) \\ & \searrow \text{ev}_{\eta_0}^\lambda & \swarrow \text{ev}_{\eta_0}^{w\lambda} \\ & \mathbb{C} & \end{array} \tag{5.2.4}$$

In fact the diagram commutes by construction. To prove unicity it suffices to prove that the morphism $\text{ev}_{\eta_0}^\lambda$ is non-zero. However, if λ is dominant, this property follows from the fact that $\mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)$ is globally generated (which itself follows from the similar claim for \mathcal{B}), and the general case follows from commutativity of (5.2.4) (and the fact that every weight is W -conjugate to a dominant weight).

Below we will need a refinement of this characterization in the case $w = s_\alpha$ for a simple root α . We denote by $\eta_\alpha \in \mathfrak{g}^*$ the element which is zero on \mathfrak{t} and on any \mathfrak{g}_β with $\beta \neq -\alpha$, and such that $\eta_\alpha(f_\alpha) = 1$.

Lemma 5.2.5. *When $w = s_\alpha$, (5.2.1) is the unique isomorphism of $G \times \mathbb{C}^\times$ -equivariant line bundles $(\theta_{s_\alpha})^* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda)(\lambda(2\check{\rho})) \rightarrow \mathcal{O}_{\tilde{\mathfrak{g}}_r}(s_\alpha\lambda)((s_\alpha\lambda)(2\check{\rho}))$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) & \xrightarrow[\sim]{(5.2.3)} & \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(s_\alpha\lambda)) \\ & \searrow \text{ev}_{\eta_\alpha}^\lambda & \swarrow \text{ev}_{\eta_\alpha}^{s_\alpha\lambda} \\ & \mathbb{C} & \end{array}$$

Proof. As for the similar claim concerning diagram (5.2.4), we only have to check that the diagram commutes. In this proof we denote the isomorphism (5.2.3) by $\vartheta_\alpha : \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \xrightarrow{\sim} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(s_\alpha\lambda))$. Choose a coweight $\check{\mu} \in X_*(T)$ such that $\alpha(\check{\mu}) = 0$ and $\beta(\check{\mu}) > 0$ for all simple roots $\beta \neq \alpha$. Then $\lim_{z \rightarrow 0} \check{\mu}(z) \cdot \eta_0 = \eta_\alpha$, so it is enough to prove that the following diagram commutes for all $z \in \mathbb{C}^\times$:

$$\begin{array}{ccc} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) & \xrightarrow[\sim]{\vartheta_\alpha} & \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(s_\alpha\lambda)) \\ & \searrow \text{ev}_{\check{\mu}(z)\cdot\eta_0}^\lambda & \swarrow \text{ev}_{\check{\mu}(z)\cdot\eta_0}^{s_\alpha\lambda} \\ & \mathbb{C} & \end{array}$$

However for $v \in \mathbb{X}$ and $f \in \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(v))$ we have

$$\begin{aligned} f(1 \times_B (\check{\mu}(z) \cdot \eta_0)) &= f(\check{\mu}(z) \times_B \eta_0) = \check{\mu}(z) \cdot \left((\check{\mu}(z^{-1}) \cdot f)(1 \times_B \eta_0) \right) \\ &= \check{\mu}(z) \cdot \left(1 \times_B (\eta_0, \text{ev}_{\eta_0}^\lambda(\check{\mu}(z^{-1}) \cdot f)) \right) = \left(\check{\mu}(z) \times_B (\eta_0, \text{ev}_{\eta_0}^\lambda(\check{\mu}(z^{-1}) \cdot f)) \right) \\ &= \left(1 \times_B (\check{\mu}(z) \cdot \eta_0, z^{\lambda(\check{\mu})} \cdot \text{ev}_{\eta_0}^\lambda(\check{\mu}(z^{-1}) \cdot f)) \right). \end{aligned}$$

On the other hand we have

$$f(1 \times_B (\check{\mu}(z) \cdot \eta_0)) = \left(1 \times_B (\check{\mu}(z) \cdot \eta_0, \text{ev}_{\check{\mu}(z) \cdot \eta_0}^\lambda(f)) \right),$$

which implies that

$$\text{ev}_{\check{\mu}(z) \cdot \eta_0}^\lambda(f) = z^{\lambda(\check{\mu})} \cdot \text{ev}_{\eta_0}^\lambda(\check{\mu}(z^{-1}) \cdot f).$$

Hence we obtain

$$\text{ev}_{\check{\mu}(z) \cdot \eta_0}^{s_\alpha \lambda}(\vartheta_\alpha f) = z^{(s_\alpha \lambda)(\check{\mu})} \cdot \text{ev}_{\eta_0}^{s_\alpha \lambda}(\check{\mu}(z^{-1}) \cdot (\vartheta_\alpha f)) = z^{\lambda(\check{\mu})} \cdot \text{ev}_{\eta_0}^\lambda(\check{\mu}(z^{-1}) \cdot f) = \text{ev}_{\check{\mu}(z) \cdot \eta_0}^\lambda(f)$$

since $(s_\alpha \lambda)(\check{\mu}) = \lambda(\check{\mu})$ and ϑ_α is G -equivariant. This finishes the proof. □

5.3. Restriction to a Levi subgroup

Fix a subset I of the set of the set of simple roots, and recall the notation of § 3.3. We can consider the Grothendieck–Springer resolution $\tilde{\mathfrak{l}}$ associated with L , and there exists a natural morphism $\varpi_L^G : G \times_L \tilde{\mathfrak{l}} = G \times_{B_L} (\mathfrak{l}/\mathfrak{u}_L)^* \rightarrow \tilde{\mathfrak{g}}$ induced by the identification $\mathfrak{l}^* \cong (\mathfrak{g}/(\mathfrak{n}_L^+ \oplus \mathfrak{n}_L^-))^*$. In particular for $L = T$ we have $\tilde{\mathfrak{t}} = \mathfrak{t}^*$, and the morphism $\varpi_T^G : G \times_T \tilde{\mathfrak{t}} = G/T \times \mathfrak{t}^* \rightarrow \tilde{\mathfrak{g}}$ identifies with the morphism denoted as “ a ” in § 2.4. The following diagram commutes by construction:

$$\begin{array}{ccc} & \xrightarrow{\varpi_T^G} & \\ G \times_T \tilde{\mathfrak{t}} & \xrightarrow{G \times_L \varpi_T^G} & G \times_L \tilde{\mathfrak{l}} \xrightarrow{\varpi_L^G} \tilde{\mathfrak{g}}. \end{array} \tag{5.3.1}$$

We have $(\varpi_L^G)^{-1}(\tilde{\mathfrak{g}}_r) \subset G \times_L \tilde{\mathfrak{l}}_r$. (Note that this inclusion is strict in general.) This open subset is W_L -invariant (for the W_L -action on $G \times_L \tilde{\mathfrak{l}}_r$ induced by the action on $\tilde{\mathfrak{l}}_r$), and the morphism $(\varpi_L^G)^{-1}(\tilde{\mathfrak{g}}_r) \rightarrow \tilde{\mathfrak{g}}_r$ induced by ϖ_L^G is W_L -equivariant.

Adjunction for the morphism ϖ_L^G induces an injective morphism

$$\Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \hookrightarrow \Gamma(G \times_L \tilde{\mathfrak{l}}, \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}}(\lambda)) \cong \text{Ind}_L^G(\Gamma(\tilde{\mathfrak{l}}, \mathcal{O}_{\tilde{\mathfrak{l}}}(\lambda))). \tag{5.3.2}$$

For simplicity, in the next statement we forget about the gradings (i.e. the \mathbb{C}^\times -actions).

Lemma 5.3.3. *The following diagram commutes for any V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W_L$, where vertical maps are induced by (5.3.2):*

$$\begin{array}{ccc} (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G & \xrightarrow{\sigma_w^{V, \lambda}} & w(V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)))^G \\ \downarrow & & \downarrow \\ (V|_L \otimes \Gamma(\tilde{\mathfrak{l}}, \mathcal{O}_{\tilde{\mathfrak{l}}}(\lambda)))^L & \xrightarrow{\sigma_w^{V|_L, \lambda}} & w(V|_L \otimes \Gamma(\tilde{\mathfrak{l}}, \mathcal{O}_{\tilde{\mathfrak{l}}}(w\lambda)))^L. \end{array}$$

Proof. It is enough to prove the result when $w = s_\alpha$ is a simple reflection associated with a simple root $\alpha \in I$. Let $\mathfrak{sl}_{2,\alpha} \subset \mathfrak{g}$ be the Lie subalgebra generated by \mathfrak{g}_α and $\mathfrak{g}_{-\alpha}$, and consider the open subset

$$\tilde{\mathfrak{g}}_{\alpha-r} := \{(g \times_B \eta) \in \tilde{\mathfrak{g}} \mid \eta|_{\mathfrak{sl}_{2,\alpha}} \neq 0\} \subset \tilde{\mathfrak{g}}.$$

By [11, Lemma 1.9.1] we have $\tilde{\mathfrak{g}}_r \subset \tilde{\mathfrak{g}}_{\alpha-r}$, and it follows from [11, Lemma 2.9.1] that the action map θ_{s_α} is the restriction of an isomorphism (denoted similarly) $\theta_{s_\alpha} : \tilde{\mathfrak{g}}_{\alpha-r} \xrightarrow{\sim} \tilde{\mathfrak{g}}_{\alpha-r}$. Moreover, isomorphism (5.2.1) is (a shift of) the restriction of an isomorphism of $G \times \mathbb{C}^\times$ -equivariant line bundles

$$\zeta_{s_\alpha}^{G,\lambda} : (\theta_{s_\alpha})^* \mathcal{O}_{\tilde{\mathfrak{g}}_{\alpha-r}}(\lambda) \xrightarrow{\sim} \mathcal{O}_{\tilde{\mathfrak{g}}_{\alpha-r}(s_\alpha\lambda)}(-2\lambda(\check{\alpha})).$$

The same assertions are of course true for the Levi subgroup L , and we obtain a similar isomorphism $\zeta_{s_\alpha}^{L,\lambda}$.

The morphism ϖ_L^G restricts to a morphism $G \times_L \tilde{\mathfrak{l}}_{\alpha-r} \rightarrow \tilde{\mathfrak{g}}_{\alpha-r}$ (denoted similarly) which satisfies $\varpi_L^G \circ (G \times_L \theta_{s_\alpha}^L) = \theta_{s_\alpha}^G \circ \varpi_L^G$. What we have to prove is that the two isomorphisms

$$(\varpi_L^G)^* \zeta_{s_\alpha}^{G,\lambda}, \text{Ind}_L^G(\zeta_{s_\alpha}^{L,\lambda}) : (G \times_L \theta_{s_\alpha}^L)^* \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}_{\alpha-r}}(\lambda) \xrightarrow{\sim} \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}_{\alpha-r}(s_\alpha\lambda)}$$

coincide. As both isomorphisms are $G \times \mathbb{C}^\times$ -equivariant, and as

$$\Gamma(G \times_L \tilde{\mathfrak{l}}_{\alpha-r}, \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}_{\alpha-r}})^{G \times \mathbb{C}^\times} \cong \Gamma(G \times_L \tilde{\mathfrak{l}}, \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}})^{G \times \mathbb{C}^\times} \cong \mathbb{C},$$

we know that these isomorphisms coincide up to multiplication by a scalar $c \in \mathbb{C}^\times$. Consider the following diagram:

$$\begin{array}{ccc}
 \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) & \xrightarrow{\sim} & \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(s_\alpha\lambda)) \\
 \downarrow (5.3.2) & & \downarrow (5.3.2) \\
 \Gamma(G \times_L \tilde{\mathfrak{l}}, \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}}(\lambda)) & \xrightarrow{\sim} & \Gamma(G \times_L \tilde{\mathfrak{l}}, \mathcal{O}_{G \times_L \tilde{\mathfrak{l}}}(s_\alpha\lambda)) \\
 \downarrow \text{ev}_{\eta_\alpha}^{\lambda,L} & & \downarrow \text{ev}_{\eta_\alpha}^{s_\alpha\lambda,L} \\
 \mathbb{C} & \xlongequal{c} & \mathbb{C}
 \end{array}
 \tag{5.3.4}$$

where the upper horizontal morphism is induced by $\zeta_{s_\alpha}^{G,\lambda}$, the middle horizontal morphism is induced by $\text{Ind}_L^G(\zeta_{s_\alpha}^{L,\lambda})$, and the lower vertical maps are induced by evaluation at $(1 \times_{B_L} \eta_\alpha)$. We know that the upper square commutes up to multiplication by c , that the lower square and the exterior square both commute and finally that the morphism $\text{ev}_{\eta_\alpha}^{\lambda,G}$ is non-zero (see Lemma 5.2.5 and its proof). We deduce that $c = 1$, which finishes the proof. \square

5.4. The relation to the operators Φ

Recall that if $\lambda \in \mathbb{X}$, there is a natural morphism $(\lambda)\mathcal{D}_h(\mathcal{X})_\lambda \rightarrow \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda))$ sending h to 0 (see e.g. §3.5).

Proposition 5.4.1. *Let V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$. The following diagram commutes, where vertical maps are the natural morphisms sending \mathfrak{h} to 0:*

$$\begin{CD} (V \otimes {}^{(\lambda)}\mathcal{D}_{\mathfrak{h}}(\mathcal{X})_{\lambda})^G @>\Phi_w^{V,\lambda}>> w(V \otimes {}^{(w\lambda)}\mathcal{D}_{\mathfrak{h}}(\mathcal{X})_{w\lambda})^G \\ @VVV @VVV \\ (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G @>\sigma_w^{V,\lambda}>> w(V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(w\lambda)))^G \end{CD}$$

Proof. First, using relations (2.5.3) and (2.5.6) it is enough to prove the lemma when $w = s_{\alpha}$ is a simple reflection. Then using the compatibility of our constructions with restriction to a Levi subgroup (see Corollary 3.3.3 and Lemma 5.3.3) and the injectivity of morphism (5.3.2), it is enough to prove the lemma when G has semisimple rank 1 (with unique simple root α). In this case it is proved in Corollary A.5.2 below. \square

Remark 5.4.2. By Lemma 3.5.2, the vertical arrows in the diagram of Proposition 5.4.1 induce isomorphisms

$$\begin{aligned} (V \otimes {}^{(\lambda)}\mathcal{D}_{\mathfrak{h}}(\mathcal{X})_{\lambda})^G / \langle \mathfrak{h} \rangle &\xrightarrow{\sim} (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G, \\ (V \otimes {}^{(s\lambda)}\mathcal{D}_{\mathfrak{h}}(\mathcal{X})_{s\lambda})^G / \langle \mathfrak{h} \rangle &\xrightarrow{\sim} (V \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(s\lambda)))^G. \end{aligned}$$

Hence the proposition implies that the operators $\sigma_w^{V,\lambda}$ can be completely recovered from the operators $\Phi_w^{V,\lambda}$ (or equivalently the operators $\Theta_w^{V,\lambda}$; see Proposition 4.8.3).

5.5. Geometric interpretation: W -action on the regular part of $T^*\mathcal{X}$

The results in this subsection will not be used in the rest of the paper.

Consider the natural morphism $T^*\mathcal{X} = G \times_U (\mathfrak{g}/\mathfrak{u})^* \rightarrow \mathfrak{g}^*$, and denote by $(T^*\mathcal{X})_{\mathfrak{r}}$ the inverse image of the open subset of regular elements in \mathfrak{g}^* . Note that the $G \times T$ -action on \mathcal{X} defined in § 2.4 induces an action on $T^*\mathcal{X}$ which stabilizes $(T^*\mathcal{X})_{\mathfrak{r}}$, and also a moment map $T^*\mathcal{X} \rightarrow \mathfrak{t}^*$.

The existence of the collection of isomorphisms (5.2.1) has the following quite surprising consequence. This construction will be reinterpreted and studied further in [24].

Proposition 5.5.1. *There exists an action of W on $(T^*\mathcal{X})_{\mathfrak{r}}$ (which depends on the choice of the e_{α}), denoted as \odot , which satisfies the following properties:*

- (1) for any $w \in W$ the morphism $w \odot (-)$ is G -equivariant;
- (2) for $x \in (T^*\mathcal{X})_{\mathfrak{r}}$ and $t \in T$ we have $w \odot (t \cdot x) = w(t) \cdot (w \odot x)$;
- (3) the natural morphism $(T^*\mathcal{X})_{\mathfrak{r}} \rightarrow \tilde{\mathfrak{g}}_{\mathfrak{r}}$ is W -equivariant;
- (4) the restriction $(T^*\mathcal{X})_{\mathfrak{r}} \rightarrow \mathfrak{t}^*$ of the moment map is W -equivariant.

Proof. The morphism $p_{\mathfrak{r}} : (T^*\mathcal{X})_{\mathfrak{r}} \rightarrow \tilde{\mathfrak{g}}_{\mathfrak{r}}$ is a T -torsor; in particular it is affine. Hence to prove the proposition it is enough to construct a collection of isomorphisms of G -equivariant sheaves of algebras

$$(\theta_{w^{-1}})^* \left((p_{\mathfrak{r}})_* \mathcal{O}_{(T^*\mathcal{X})_{\mathfrak{r}}} \right) \cong (p_{\mathfrak{r}})_* \mathcal{O}_{(T^*\mathcal{X})_{\mathfrak{r}}}$$

for all $w \in W$, which are compatible with composition in W . Now we have a natural isomorphism of G -equivariant sheaves of algebras

$$(p_r)_* \mathcal{O}_{(T^* \mathcal{X})_r} \cong \bigoplus_{\lambda \in \mathbb{X}} \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda),$$

and hence it is enough to construct a collection of isomorphisms of G -equivariant line bundles

$$(\theta_{w^{-1}})^* \mathcal{O}_{\tilde{\mathfrak{g}}_r}(\lambda) \cong \mathcal{O}_{\tilde{\mathfrak{g}}_r}(w\lambda)$$

compatible with the tensor product and composition in W . However one can easily check that the collection constructed in (5.2.1) satisfies these requirements. \square

6. A reminder on the Satake equivalence

In §§ 6–8 we let \check{G} be a complex connected reductive group. We choose a maximal torus $\check{T} \subset \check{G}$ (with Lie algebra $\check{\mathfrak{t}}$), and a Borel subgroup $\check{B} \subset \check{G}$. We let \check{U} (resp. \check{U}^-) be the unipotent radical of \check{B} (resp. of the opposite Borel subgroup), with respect to \check{T} . We set $\mathbb{X} := X_*(\check{T})$, and let $\mathbb{X}^+ \subset \mathbb{X}$ be the subsemigroup of dominant coweights of \check{T} (where positive roots of \check{G} are those appearing in $\text{Lie}(\check{U})$). We also set $S_{\hbar} := S(\check{\mathfrak{t}}^*[\hbar])$, considered as a graded algebra where elements of $\check{\mathfrak{t}}^*$ and \hbar are in degree 2. Finally we denote by $2\check{\rho} \in X^*(\check{T})$ the sum of the positive roots.

6.1. Satake equivalence

Recall (see § 2.2) that the affine Grassmannian attached to \check{G} is the ind-variety

$$\text{Gr}_{\check{G}} := \check{G}(\mathbf{K})/\check{G}(\mathbf{O})$$

(equipped with the *reduced* scheme structure). This ind-variety is equipped with an action of the group scheme $\check{G}(\mathbf{O})$. Recall (see [22, 35]) that the category

$$\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$$

of $\check{G}(\mathbf{O})$ -equivariant perverse sheaves on $\text{Gr}_{\check{G}}$ (with coefficients in \mathbb{C}) can be endowed with a natural convolution product \star which makes it a tensor category, and that the functor

$$F_{\check{G}} := H^*(\text{Gr}_{\check{G}}, -) : \text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}}) \rightarrow \text{Vect}(\mathbb{C})$$

(where $\text{Vect}(\mathbb{C})$ is the category of finite dimensional \mathbb{C} -vector spaces) is a tensor functor. (As usual, perverse sheaves on $\text{Gr}_{\check{G}}$ are assumed to be supported on a finite union of $\check{G}(\mathbf{O})$ -orbits.) We let

$$G := \text{Aut}^*(F_{\check{G}})$$

be the \mathbb{C} -group scheme of automorphisms of this tensor functor. It is well known (see [22, 35]) that G is a (complex) connected reductive group, with root datum dual to that of \check{G} . Moreover, the functor $F_{\check{G}}$ lifts to an equivalence of tensor categories

$$\mathbb{S}_{\check{G}} : \text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}}) \xrightarrow{\sim} \text{Rep}(G)$$

known as the geometric Satake equivalence.

In §2.2 we have defined the embedding $\mathbb{X} = \text{Gr}_{\check{T}} \hookrightarrow \text{Gr}_{\check{G}}$, the points λ (for $\lambda \in \mathbb{X}$), the orbits $\text{Gr}_{\check{G}}^\lambda$ (for $\lambda \in \mathbb{X}^+$), the semi-infinite orbits \mathfrak{T}_λ (for $\lambda \in \mathbb{X}$), and the morphisms i_λ and t_λ .

Using the identification of $\text{Gr}_{\check{T}}$ with \mathbb{X} , the group T (of automorphisms of the tensor functor $F_{\check{T}}$) is identified with the torus $\text{Hom}_{\mathbb{Z}}(\mathbb{X}, \mathbb{C}^\times)$. In particular, the character lattice $X^*(T)$ is canonically identified with \mathbb{X} ; hence the category $\text{Rep}(T)$ identifies with the category of finite dimensional \mathbb{X} -graded vector spaces. Define the functor

$$F^{\mathbb{X}} := \bigoplus_{\lambda \in \mathbb{X}} H^{\lambda(2\check{\rho})}(\mathfrak{T}_\lambda, t_\lambda^!(-)) : \text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}}) \rightarrow \text{Rep}(T).$$

By [35, Theorems 3.5 and 3.6], we have a canonical isomorphism

$$\text{For}^T \circ F^{\mathbb{X}} \xrightarrow{\sim} F_{\check{G}}, \tag{6.1.1}$$

where $\text{For}^T : \text{Rep}(T) \rightarrow \text{Vect}(\mathbb{C})$ is the forgetful functor. Moreover, $F^{\mathbb{X}}$ is a tensor functor, and (6.1.1) is an isomorphism of tensor functors. So $F^{\mathbb{X}}$ is the composition of $\mathbb{S}_{\check{G}}$ with a tensor functor $\text{Rep}(G) \rightarrow \text{Rep}(T)$ compatible with forgetful functors. By [17, Corollary 2.9], the latter functor is induced by a group morphism $T \rightarrow G$. It is proved in [35] that this morphism is injective, and identifies T with a maximal torus of G . Hence from now on we will consider T as a subgroup of G . By construction, if $\lambda \in \mathbb{X}$ and if \mathcal{F} is in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$, we have a canonical isomorphism of \mathbb{C} -vector spaces

$$H^{\lambda(2\check{\rho})}(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \cong (\mathbb{S}_{\check{G}}(\mathcal{F}))_\lambda. \tag{6.1.2}$$

The choice of $\check{B} \subset \check{G}$ determines a set of positive roots for \check{G} , and hence also a set of positive roots for G . We denote by B the Borel subgroup of G containing T associated with this set of positive roots.

6.2. Equivariant cohomology

For the results stated below, see e.g. [33, § 1].

For any complex algebraic variety X endowed with an algebraic action of an algebraic group H , recall that the equivariant cohomology (resp. Borel–Moore homology) is defined by

$$H_H^\bullet(X) := \text{Ext}_{\mathcal{D}_H(X)}^\bullet(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X), \quad H_H^H(X) := \text{Ext}_{\mathcal{D}_H(X)}^{\bullet-2 \dim(X)}(\underline{\mathbb{C}}_X, \underline{\mathbb{D}}_X),$$

where $\mathcal{D}_H(X)$ is the H -equivariant derived category of X , $\underline{\mathbb{C}}_X$ is the (equivariant) constant sheaf on X and $\underline{\mathbb{D}}_X$ is the (equivariant) dualizing sheaf. Then $H_H^\bullet(X)$ is a (graded commutative) algebra for the Yoneda product (or cup product) and $H_H^H(X)$ is a right module over this algebra, again for the Yoneda product. If X is smooth, then this module is free of rank 1.

Let now K be a torus, with Lie algebra \mathfrak{k} . Let $\lambda \in X^*(K)$, and consider the one-dimensional K -module \mathbb{C}_λ , considered as a K -variety. Consider the K -equivariant morphisms

$$\{\text{pt}\} \xrightarrow{\iota} \mathbb{C}_\lambda \xrightarrow{\pi} \{\text{pt}\}$$

(where $\iota(\text{pt}) = 0$), and the induced morphisms in equivariant homology

$$H_K^K(\text{pt}) \xrightarrow{\iota} H_{\bullet+2}^K(\mathbb{C}_\lambda) \xrightarrow[\sim]{(\pi^*)^{-1}} H_{\bullet+2}^K(\text{pt}),$$

which are morphisms of right $H_K^\bullet(\text{pt})$ -modules. Since the right $H_K^\bullet(\text{pt})$ -module $H_K^K(\text{pt})$ is free of rank 1, there exists a unique element $c(\lambda) \in H_K^2(\text{pt})$ such that the composition above is the action of $c(\lambda)$, and $c : X^*(K) \rightarrow H_K^2(\text{pt})$ is a morphism of Abelian groups. There exists a unique isomorphism $a : \mathfrak{k}^* \xrightarrow{\sim} H_K^2(\text{pt})$ such that the following diagram commutes:

$$\begin{array}{ccc} & X^*(K) & \\ d \swarrow & & \searrow c \\ \mathfrak{k}^* & \xrightarrow[\sim]{a} & H_K^2(\text{pt}), \end{array}$$

where d is the differential. Moreover, this isomorphism extends to an isomorphism of graded \mathbb{C} -algebras

$$S(\mathfrak{k}^*) \xrightarrow{\sim} H_K^\bullet(\text{pt}), \tag{6.2.1}$$

where in the left-hand side \mathfrak{k}^* is in degree 2. We will use this isomorphism throughout the paper without further details. In particular we can make the identification

$$H_A^\bullet(\text{pt}) \cong S_{\hbar}, \quad H_{\check{T}}^\bullet(\text{pt}) \cong S(\mathfrak{t}),$$

where $\hbar \in S_{\hbar}$ corresponds to the natural generator of $H_{\mathbb{C}^\times}^2(\text{pt}) \cong \mathbb{C}$.

If V is any K -module, with K -weights $\lambda_1, \dots, \lambda_n$ (counted with multiplicities), then if as above ι denotes inclusion of 0 and π projection to 0, via isomorphism (6.2.1), the composition

$$H_K^K(\text{pt}) \xrightarrow{\iota} H_{\bullet+2n}^K(V) \xrightarrow[\sim]{(\pi^*)^{-1}} H_{\bullet+2n}^K(\text{pt})$$

identifies with the action of $d(\lambda_1) \cdots d(\lambda_n)$. Note that the action map induces an isomorphism of $H_K^\bullet(\text{pt})$ -modules $H_0^K(V) \otimes H_K^\bullet(\text{pt}) \xrightarrow{\sim} H_K^K(V)$, and that the forgetful map $H_0^K(V) \rightarrow H_0(V)$ is an isomorphism. Hence we obtain a canonical isomorphism $H_K^K(V) \cong H_0(V) \otimes H_K^\bullet(\text{pt})$. The Borel–Moore homology $H_0(V)$ contains the canonical class $[V]$, which can therefore be viewed as a generator of $H_K^K(V)$. Similarly we have the canonical class $[\text{pt}] \in H_K^K(\text{pt})$. Then we have $\pi^*[\text{pt}] = [V]$, and hence the morphism $\iota_!$ has the property that

$$\iota_!([\text{pt}]) = [V] \cdot d(\lambda_1) \cdots d(\lambda_n).$$

We can now give a proof of Lemma 2.2.1.

Proof of Lemma 2.2.1. To fix notation we treat the case of A ; the case of \check{T} is similar. Let \mathcal{F} in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr}_{\check{G}})$ and $\lambda \in \mathbb{X}$.

(1) Using the Leray–Serre spectral sequence for an appropriate fibration, there exists a spectral sequence which computes $H_A^\bullet(i_\lambda^! \mathcal{F})$ and with E_2 -term

$$E_2^{p,q} = H_A^p(\text{pt}) \otimes H^q(i_\lambda^! \mathcal{F}) \tag{6.2.2}$$

(see e.g. [33, Proof of Proposition 7.2]). It is well known also (see [30, Theorem 5.5] or [38, Corollaire 2.10]) that the ordinary cohomology $H^\bullet(i_\lambda^! \mathcal{F})$ is concentrated in degrees of constant parity. Hence the spectral sequence (6.2.2) degenerates at E_2 . It follows that there exists a non-canonical isomorphism of graded S_{\hbar} -modules

$$H_A^\bullet(i_\lambda^! \mathcal{F}) \cong H_A^\bullet(\text{pt}) \otimes H^\bullet(i_\lambda^! \mathcal{F}).$$

In particular, the left-hand side is free over $S_{\hbar} = H_A^\bullet(\text{pt})$.

(2) By [35, Theorem 3.4], $H^\bullet(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F})$ is concentrated in degree $\lambda(2\check{\rho})$. Hence the same spectral sequence argument as in (1) shows that $H_A^\bullet(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F})$ is free over $H_A^\bullet(\text{pt})$. Moreover, the morphism

$$H_A^\bullet(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \rightarrow H^\bullet(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F})$$

induced by the forgetful functor is an isomorphism in degree $\lambda(2\check{\rho})$. Hence we obtain an inclusion $(S_{\check{G}}(\mathcal{F}))_\lambda \stackrel{(6.1.2)}{\cong} H^{(\lambda, 2\check{\rho})}(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \hookrightarrow H_A^\bullet(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F})$. Using again the spectral sequence argument, the morphism

$$H_A^\bullet(\text{pt}) \otimes (S_{\check{G}}(\mathcal{F}))_\lambda \rightarrow H_A^\bullet(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F})$$

induced by the cup product is an isomorphism of $H_A^\bullet(\text{pt})$ -modules, which finishes the proof. □

We deduce the following result from Lemma 2.2.1.

Corollary 6.2.3. *For $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$, the morphism $\kappa_{\mathcal{F}, \lambda}^{\text{top}}$ is injective.*

Proof. By the localization theorem in equivariant cohomology, the morphism

$$Q_{\hbar} \otimes_{S_{\hbar}} H_A^\bullet(i_\lambda^! \mathcal{F}) \rightarrow Q_{\hbar} \otimes_{S_{\hbar}} H_A^\bullet(\mathfrak{T}_\lambda, i_\lambda^! \mathcal{F})$$

induced by $(t_\lambda)_!$ is an isomorphism (since λ is the only A -fixed point in \mathfrak{T}_λ). As $H_A^\bullet(i_\lambda^! \mathcal{F})$ is free over S_{\hbar} (see Lemma 2.2.1(1)), we deduce that $(t_\lambda)_!$ is injective, which implies that $\kappa_{\mathcal{F}, \lambda}^{\text{top}}$ is also injective. □

We will also need the following result, which again follows from the fact that the spectral sequence (6.2.2) degenerates.

Lemma 6.2.4. *Let \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$ and $\lambda \in \mathbb{X}$.*

The forgetful morphism $H_A^\bullet(i_\lambda^! \mathcal{F}) \rightarrow H_T^\bullet(i_\lambda^! \mathcal{F})$ induces an isomorphism

$$H_A^\bullet(i_\lambda^! \mathcal{F}) / (\hbar \cdot H_A^\bullet(i_\lambda^! \mathcal{F})) \xrightarrow{\sim} H_T^\bullet(i_\lambda^! \mathcal{F}).$$

6.3. Restriction to a Levi subgroup

Below we will make extensive use of the geometric description of the functor of restriction to a Levi subgroup, due to Mirković and Vilonen [35] in the (crucial) case of the maximal torus, and to Beilinson and Drinfeld [6] in the general case.

Let $\check{P} \subset \check{G}$ be a parabolic subgroup containing \check{B} , and let $\check{L} \subset \check{P}$ be the (unique) Levi factor containing \check{T} . Let also \check{P}^- be the opposite parabolic subgroup. Note that \check{T} is also a maximal torus of \check{L} , and $\check{B}_{\check{L}} := \check{B} \cap \check{L}$ is a Borel subgroup containing \check{T} . We have dual groups G and L defined as in § 6.1. Consider the diagram

$$\mathrm{Gr}_{\check{L}} \xleftarrow{q_{\check{P}^-}} \mathrm{Gr}_{\check{P}^-} \xrightarrow{i_{\check{P}^-}} \mathrm{Gr}_{\check{G}} \tag{6.3.1}$$

where $q_{\check{P}^-}$ is induced by the projection $\check{P}^- \rightarrow \check{L}$ whose kernel is the unipotent radical of \check{P}^- , and $i_{\check{P}^-}$ is induced by the embedding $\check{P}^- \hookrightarrow \check{G}$. Define the functor

$$\mathfrak{R}_{\check{L}}^{\check{G}} := (q_{\check{P}^-})_* \circ (i_{\check{P}^-})^! : \mathcal{D}_c^b(\mathrm{Gr}_{\check{G}}) \rightarrow \mathcal{D}_c^b(\mathrm{Gr}_{\check{L}}),$$

where $\mathcal{D}_c^b(\mathrm{Gr}_{\check{G}})$ is the derived category of constructible complexes of \mathbb{C} -vector spaces on the ind-variety $\mathrm{Gr}_{\check{G}}$ which are supported on a finite union of $\check{G}(\mathbf{O})$ -orbits, and similarly for $\mathcal{D}_c^b(\mathrm{Gr}_{\check{L}})$. The functor $\mathfrak{R}_{\check{L}}^{\check{G}}$ does not map the subcategory $\mathrm{Perv}_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}})$ of $\mathcal{D}_c^b(\mathrm{Gr}_{\check{G}})$ into the subcategory $\mathrm{Perv}_{\check{L}(\mathbf{O})}(\mathrm{Gr}_{\check{L}})$ of $\mathcal{D}_c^b(\mathrm{Gr}_{\check{L}})$; however, the following modification of this functor has this property.

Recall that the connected components of $\mathrm{Gr}_{\check{L}}$ are parameterized by the quotient $\mathbb{X}/(\mathbb{Z}\check{R}_{\check{L}})$; see [6, Proposition 4.5.4]. (Here $\check{R}_{\check{L}}$ denotes the coroots of \check{L} , and $\mathbb{Z}\check{R}_{\check{L}}$ is the lattice that they generate). If M is in $\mathcal{D}_c^b(\mathrm{Gr}_{\check{L}})$ and $\chi \in \mathbb{X}/(\mathbb{Z}\check{R}_{\check{L}})$, we denote by M_{χ} the restriction of M to the corresponding connected component. Define the functor $\mathfrak{R}_{\check{L}}^{\check{G}} : \mathcal{D}_c^b(\mathrm{Gr}_{\check{G}}) \rightarrow \mathcal{D}_c^b(\mathrm{Gr}_{\check{L}})$ by the formula

$$\mathfrak{R}_{\check{L}}^{\check{G}}(M) = \bigoplus_{\chi \in \mathbb{X}/(\mathbb{Z}\check{R}_{\check{L}})} (\mathfrak{R}_{\check{L}}^{\check{G}}(M))_{\chi}[\chi(2\rho_{\check{G}} - 2\rho_{\check{L}})],$$

where $\rho_{\check{G}}$ and $\rho_{\check{L}}$ are the half-sums of positive roots of \check{G} and \check{L} . It is proved in [6, Proposition 5.3.29] that $\mathfrak{R}_{\check{L}}^{\check{G}}$ restricts to a functor

$$\mathfrak{R}_{\check{L}}^{\check{G}} : \mathrm{Perv}_{\check{G}(\mathbf{O})}(\mathrm{Gr}_{\check{G}}) \rightarrow \mathrm{Perv}_{\check{L}(\mathbf{O})}(\mathrm{Gr}_{\check{L}}).$$

Moreover, it is explained in [6, § 5.3.30] that this functor is a tensor functor.

Using the base change theorem one can easily construct an isomorphism of tensor functors

$$F_{\check{G}} \xrightarrow{\sim} F_{\check{L}} \circ \mathfrak{R}_{\check{L}}^{\check{G}};$$

see e.g. [1, § 4.1] for details. We deduce a morphism of algebraic groups

$$L = \mathrm{Aut}^*(F_{\check{L}}) \rightarrow \mathrm{Aut}^*(F_{\check{L}} \circ \mathfrak{R}_{\check{L}}^{\check{G}}) \cong \mathrm{Aut}^*(F_{\check{G}}) = G.$$

It is known that this morphism is injective, and identifies L with the Levi subgroup of G whose root system is the system of coroots of \check{L} . Hence we will consider L as a subgroup

of G . By construction the following diagram is commutative:

$$\begin{CD}
 \text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}}) @>\mathfrak{R}_{\check{L}}^{\check{G}}>> \text{Perv}_{\check{L}(\mathbf{O})}(\text{Gr}_{\check{L}}) \\
 @V\mathbb{S}_{\check{G}}VV @VV\mathbb{S}_{\check{L}}V \\
 \text{Rep}(G) @>V \mapsto V_{|L}>> \text{Rep}(L).
 \end{CD}$$

Note that by the constructions of § 6.1 we have identified T with a subgroup of L , but also with a subgroup of G . These identifications are compatible with the identification of L as a subgroup of G ; see e.g. [1, § 4.1]. Moreover, by the base change theorem there exists a canonical isomorphism of functors

$$\mathfrak{R}_T^{\check{L}} \circ \mathfrak{R}_L^{\check{G}} \cong \mathfrak{R}_T^{\check{G}}. \tag{6.3.2}$$

6.4. Restriction to a Levi subgroup: cofibers

Let $\check{P}, \check{P}^-, \check{L}$ be as in § 6.3. Let also $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$. We want to compare the $\mathbb{S}_{\check{h}}$ -modules

$$\mathbf{H}_A^*((i_{\lambda}^{\check{G}})^!\mathcal{F}) \quad \text{and} \quad \mathbf{H}_A^*((i_{\lambda}^{\check{L}})^!\mathfrak{R}_L^{\check{G}}(\mathcal{F})),$$

where $i_{\lambda}^{\check{G}}$ and $i_{\lambda}^{\check{L}}$ are the inclusions of λ in $\text{Gr}_{\check{G}}$ and $\text{Gr}_{\check{L}}$ respectively. Let $t_{\lambda}^{\check{P}}$ be the inclusion of $(q_{\check{P}^-})^{-1}(\lambda)$ in $\text{Gr}_{\check{G}}$. Then by the base change theorem there is a canonical isomorphism of graded $\mathbb{S}_{\check{h}}$ -modules

$$\mathbf{H}_A^*((i_{\lambda}^{\check{L}})^!\mathfrak{R}_L^{\check{G}}(\mathcal{F})) \cong \mathbf{H}_A^*((q_{\check{P}^-})^{-1}(\lambda); (t_{\lambda}^{\check{P}})^!\mathcal{F})(\lambda(2\check{\rho}_L - 2\check{\rho}_G)).$$

As $\{\lambda\}$ is closed in $(q_{\check{P}^-})^{-1}(\lambda)$, the $(, ^!)$ -adjunction for the inclusion $\{\lambda\} \hookrightarrow (q_{\check{P}^-})^{-1}(\lambda)$ induces a canonical morphism of $\mathbb{S}_{\check{h}}$ -modules

$$\mathbf{H}_A^*((i_{\lambda}^{\check{L}})^!\mathcal{F}) \rightarrow \mathbf{H}_A^*((i_{\lambda}^{\check{L}})^!\mathfrak{R}_L^{\check{G}}(\mathcal{F}))(\lambda(2\check{\rho}_G - 2\check{\rho}_L)). \tag{6.4.1}$$

If $\check{P} = \check{B}$ (and so $\check{L} = \check{T}$) we have canonical isomorphisms

$$\mathbf{H}_A^*((i_{\lambda}^{\check{T}})^!\mathfrak{R}_T^{\check{G}}(\mathcal{F})) \cong \mathbf{H}_A^*(t_{\lambda}^!\mathcal{F})(-\lambda(2\check{\rho}_G)) \cong (\mathbb{S}_{\check{G}}(\mathcal{F}))_{\lambda} \otimes \mathbb{S}_{\check{h}}.$$

Via this isomorphism, (6.4.1) identifies with the morphism $\kappa_{\mathcal{F}, \lambda}^{\text{top}}$ defined in (2.2.3). Moreover, one can easily check that the following diagram is commutative:

$$\begin{CD}
 \mathbf{H}_A^*((i_{\lambda}^{\check{G}})^!\mathcal{F}) @>\mathfrak{R}_{\check{L}}^{\check{G}}>> \mathbf{H}_A^*((i_{\lambda}^{\check{L}})^!\mathfrak{R}_L^{\check{G}}(\mathcal{F}))(\lambda(2\check{\rho}_G - 2\check{\rho}_L)) @>\kappa_{\mathcal{F}, \lambda}^{\text{top}}>> (\mathbb{S}_{\check{G}}(\mathcal{F}))_{\lambda} \otimes \mathbb{S}_{\check{h}}(\lambda(2\check{\rho}_G)) \\
 @. @. @. \\
 @. @. @.
 \end{CD}$$

(where for simplicity we neglect shifts of morphisms). As the morphism $\check{\kappa}_{\mathcal{F}, \lambda}^{\text{top}}$ is injective (see Corollary 6.2.3), we deduce that (6.4.1) is also injective.

We will also need the following result, which is clear by construction.

Lemma 6.4.2. *Let $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr}_{\check{G}})$. For any $w \in W_L \subset W_G$ the following diagram commutes:*

$$\begin{CD}
 \mathrm{H}_A^{\bullet}((i_{\lambda}^{\check{G}})^!\mathcal{F}) @>(6.4.1)>> \mathrm{H}_A^{\bullet}((i_{\lambda}^{\check{L}})^!\mathfrak{R}_{\check{L}}^{\check{G}}(\mathcal{F})) \\
 @V\check{\Xi}_w^{\mathcal{F},\lambda}VV @VV\check{L}_{\Xi_w}^{\mathfrak{R}_{\check{L}}^{\check{G}}(\mathcal{F}),\lambda}V \\
 w\mathrm{H}_A^{\bullet}((i_{w\lambda}^{\check{G}})^!\mathcal{F}) @>(6.4.1)>> \mathrm{H}_A^{\bullet}((i_{\lambda}^{\check{L}})^!\mathfrak{R}_{\check{L}}^{\check{G}}(\mathcal{F})).
 \end{CD}$$

6.5. Construction of root vectors

There are several ways to construct simple root vectors in $\mathfrak{g} := \text{Lie}(G)$ out of the equivalence \mathbb{S} using [22] and [35]; see e.g. [5, 42]. Here we recall a simple version essentially explained in [40], which will be sufficient for our purposes.

Let α be a simple root. Let \check{P}^{α} be the minimal parabolic subgroup of \check{G} containing \check{B} associated with α , and let \check{L}^{α} be the Levi factor of \check{P}^{α} containing \check{T} . Then as explained in § 6.3, the “dual” group L^{α} associated with \check{L}^{α} can be canonically identified with the Levi subgroup of G whose Lie algebra is $\mathfrak{g}_{-\alpha} \oplus \mathfrak{t} \oplus \mathfrak{g}_{\alpha}$, so to construct a root vector in \mathfrak{g}_{α} it is enough to construct a root vector in the α -weight space of the Lie algebra \mathfrak{l}^{α} of L^{α} .

Let $\text{Gr}_{\check{L}^{\alpha}}^{\circ}$ be the connected component of $\mathbf{0}$ in $\text{Gr}_{\check{L}^{\alpha}}$. Then the subcategory $\text{Perv}_{\check{L}^{\alpha}(\mathbf{O})}(\text{Gr}_{\check{L}^{\alpha}}^{\circ})$ of $\text{Perv}_{\check{L}^{\alpha}(\mathbf{O})}(\text{Gr}_{\check{L}^{\alpha}})$ is closed under convolution. If we denote by M^{α} the group of automorphisms of the restriction $\mathbf{F}_{\check{L}^{\alpha}}^{\circ}$ of the fiber functor $\mathbf{F}_{\check{L}^{\alpha}}$ to this subcategory, then by definition we have a natural morphism $L^{\alpha} \rightarrow M^{\alpha}$, which induces a morphism $\mathfrak{l}^{\alpha} \rightarrow \mathfrak{m}^{\alpha}$ whose restriction to α -weight spaces is an isomorphism. Hence to construct a root vector in \mathfrak{g}_{α} it is enough to construct a root vector in the α -weight space of \mathfrak{m}^{α} .

Now let \mathcal{L} be the positive generator of the Picard group of $\text{Gr}_{\check{L}^{\alpha}}^{\circ}$. (See [40, § 1.4] or [5, § 3.3] for the explicit construction of this line bundle.) The cup product with the first Chern class of \mathcal{L} defines an endomorphism of the functor $\mathbf{F}_{\check{L}^{\alpha}}^{\circ}$. By the arguments of [42, § 3.4], this endomorphism defines an element in the α -weight space of \mathfrak{m}^{α} , which finishes the construction.

This construction of the root vectors is clearly compatible with restriction to a Levi subgroup in the sense of § 6.3.

7. Proofs of the main results

In §§ 7–8 we will use the results of §§ 3–5 for the datum $T \subset B \subset G$ constructed in § 6.1, and for the root vectors constructed in § 6.5. Note that all the objects which are denoted by the same symbol in §§ 3–5 and in § 6 (e.g. \mathbb{X} , \mathbb{X}^+ , \mathbf{S}_h) get identified canonically.

7.1. Preliminaries on Verma modules

Recall the Verma modules $\mathbf{V}(\mu)$ defined in § 4.1.

Let $\mu \in \mathfrak{t}^*$, and assume that for all $\alpha \in R$, $\mu(\check{\alpha}) \notin \mathbb{Z}$. We claim that for $\nu \in \mathfrak{t}^*$ we have

$$\text{Hom}_{U(\mathfrak{g})}(\mathbf{V}(\nu), \mathbf{V}(\mu)) = \mathbb{C}^{\delta_{\mu,\nu}}. \tag{7.1.1}$$

Indeed, if there exists a non-zero morphism $V(\nu) \rightarrow V(\mu)$, then by [25, § 3.4], ν must be in $W_{[\mu]} \bullet \mu$, where $W_{[\mu]} = \{w \in W \mid w(\mu) - \mu \in \mathbb{Z}R\}$. However, by [25, Theorem 3.4] and our assumption on μ , $W_{[\mu]} = \{1\}$, which implies (7.1.1).

If $\lambda \in \mathbb{X}$, then the \mathfrak{t} -weights of the $U(\mathfrak{b})$ -module $V(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$ (where the $U(\mathfrak{b})$ -action is diagonal, the action on $V(\mu + \lambda)$ being the restriction of the $U(\mathfrak{g})$ -action) are in \mathbb{X} , and the action of $U(\mathfrak{u})$ is locally finite. We deduce that the $U(\mathfrak{b})$ -action can be integrated to a B -action.

Lemma 7.1.2. *Let $\mu \in \mathfrak{t}^*$ be such that $\mu(\check{\alpha}) \notin \mathbb{Z}$ for all $\alpha \in R$, and let $\lambda \in \mathbb{X}$. Then there exists a natural isomorphism of B -modules*

$$V(\mu + \lambda) \otimes \mathbb{C}_{-\mu} \xrightarrow{\sim} \text{Ind}_T^B(\lambda).$$

Proof. First, let us explain how this morphism is constructed. The projection of \mathfrak{t} -modules $V(\mu + \lambda) \rightarrow \mathbb{C}_{\mu+\lambda}$ (with kernel $V(\mu + \lambda)_-$) induces a morphism of T -modules $V(\mu + \lambda) \otimes \mathbb{C}_{-\mu} \rightarrow \mathbb{C}_\lambda$. By Frobenius reciprocity we deduce a morphism of B -modules as in the statement of the lemma.

Now we prove that this morphism is injective. For this, it suffices to prove that its restriction to the socle of the left-hand side (as a B -module, or equivalently as a $U(\mathfrak{b})$ -module) is injective. We claim that this socle is isomorphic to \mathbb{C}_λ , and has a basis consisting of the vector $1_{\lambda+\mu} \otimes 1 \in V(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$. Indeed, this is equivalent to saying that the socle of $V(\mu + \lambda)$ is isomorphic to $\mathbb{C}_{\mu+\lambda}$. However, if there exists a non-zero morphism $\mathbb{C}_\nu \rightarrow V(\mu + \lambda)$ for some $\nu \in \mathfrak{t}^*$, then we obtain a non-zero morphism of $U(\mathfrak{g})$ -modules $V(\nu) \rightarrow V(\mu + \lambda)$. By (7.1.1), this implies that $\nu = \mu + \lambda$, proving the claim and the injectivity of the morphism.

Now, it is easy to see that the T -modules $V(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$ and $\text{Ind}_T^B(\lambda)$ have the same weights, with the same (finite) multiplicities. Hence our morphism must be an isomorphism. □

Below we will use the standard order on \mathfrak{t}^* , defined by $\nu \leq \mu$ iff $\mu - \nu \in \mathbb{Z}_{\geq 0}R^+$.

Let α be a simple root, $P^\alpha \subset G$ the corresponding minimal parabolic subgroup containing B , and L^α the Levi factor of P^α containing T . Let also $B^\alpha := L^\alpha \cap B$, and $\mathfrak{b}^\alpha := \text{Lie}(B^\alpha)$. Let $\mu \in \mathfrak{t}^*$ be such that $\mu(\check{\beta}) \notin \mathbb{Z}$ for any $\beta \in R \setminus \{\pm\alpha\}$. We claim that for $\nu \in \mathfrak{t}^*$ we have

$$\text{Hom}_{U(\mathfrak{g})}(V(\nu), V(\mu)) \cong \begin{cases} \mathbb{C} & \text{if } \nu = \mu; \\ \mathbb{C} & \text{if } \nu = s_\alpha \bullet \mu \text{ and } \nu < \mu; \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.3}$$

Indeed, if there exists a non-zero morphism $V(\nu) \rightarrow V(\mu)$ with $\nu \neq \mu$, then, with the same notation as above, we must have $\nu \in W_{[\mu]} \bullet \mu$ and $\nu < \mu$. Again by [25, Theorem 3.4], this implies that $\nu = s_\alpha \bullet \mu$. On the other hand, if $\nu = s_\alpha \bullet \mu$ and $\nu < \mu$, then the ν -weight space of $V(\mu)$ is one-dimensional, and consists of singular vectors by [25, Proposition 1.4].

In the following lemma, for $\mu \in \mathfrak{t}^*$ we denote by $V^\alpha(\mu)$ the Verma module associated with μ for the reductive group L^α with Borel subgroup B^α .

Lemma 7.1.4. *Let $\mu \in \mathfrak{t}^*$ be such that $\mu(\check{\beta}) \notin \mathbb{Z}$ for all $\beta \in R \setminus \{\pm\alpha\}$, and let $\lambda \in \mathbb{X}$. Then there exists a natural isomorphism of B -modules*

$$\mathbf{V}(\mu + \lambda) \otimes \mathbb{C}_{-\mu} \xrightarrow{\sim} \text{Ind}_{B^\alpha}^B(\mathbf{V}^\alpha(\mu + \lambda) \otimes \mathbb{C}_{-\mu}).$$

Proof. First, let us explain the construction of this morphism. As in the proof of Lemma 7.1.2, $\mathbf{V}(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$ has a natural structure of a B -module, and $\mathbf{V}^\alpha(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$ has a natural structure of a B^α -module. The subspace of $\mathbf{V}(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$ spanned by weight spaces whose weight is not in $\lambda + \mathbb{Z}\alpha$ is stable under the action of B^α , and the quotient by this subspace is clearly isomorphic to $\mathbf{V}^\alpha(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$. Hence we have constructed a morphism of B^α -modules $\mathbf{V}(\mu + \lambda) \otimes \mathbb{C}_{-\mu} \rightarrow \mathbf{V}^\alpha(\mu + \lambda) \otimes \mathbb{C}_{-\mu}$. Using Frobenius reciprocity we obtain the desired morphism of B -modules.

Now we prove that this morphism is injective. As in Lemma 7.1.2, it is enough to prove that its restriction to the socle of the left-hand side is injective. But it follows from (7.1.3) that this socle has dimension 1 or 2, and injectivity is clear by construction.

Finally, one can deduce surjectivity as in the proof of Lemma 7.1.2 by comparing characters. □

7.2. Generic and subgeneric situations: the classical case

Let $\mathfrak{t}_{\text{rs}}^*$ and $(\mathfrak{g}/\mathfrak{u})_{\text{rs}}^*$ be the sets of elements in \mathfrak{t}^* and $(\mathfrak{g}/\mathfrak{u})^*$ which are regular semisimple (as elements of \mathfrak{g}^*). Then $(\mathfrak{g}/\mathfrak{u})_{\text{rs}}^* = (\mathfrak{g}/\mathfrak{u})^* \times_{\mathfrak{t}^*} \mathfrak{t}_{\text{rs}}^*$, and the action of B on $(\mathfrak{g}/\mathfrak{u})^*$ induces an isomorphism of B -varieties

$$B/T \times \mathfrak{t}_{\text{rs}}^* \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{u})_{\text{rs}}^* \tag{7.2.1}$$

where the B -action on the left-hand side is trivial on $\mathfrak{t}_{\text{rs}}^*$, and given by left multiplication on B/T (see e.g. [27, p. 188]). In particular, we deduce that for any $\lambda \in \mathbb{X}$ there is a natural isomorphism of B -modules

$$\mathbb{C}[\mathfrak{t}_{\text{rs}}^*] \otimes_{S(\mathfrak{t})} S(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda} \cong \mathbb{C}[\mathfrak{t}_{\text{rs}}^*] \otimes \text{Ind}_T^B(-\lambda). \tag{7.2.2}$$

Now let α be a simple root, and let $P^\alpha, L^\alpha, B^\alpha$ be defined as in § 7.1. Let also \mathfrak{l}^α be the Lie algebra of L^α , and $\mathfrak{u}^\alpha := \mathfrak{l}^\alpha \cap \mathfrak{u}$. Let $\mathfrak{t}_{\alpha-\text{rs}}^*$ be the complement in \mathfrak{t}^* of the collection of hyperplanes defined by the equations $\check{\beta}$ for $\beta \in R \setminus \{\pm\alpha\}$. Let also $(\mathfrak{g}/\mathfrak{u})_{\alpha-\text{rs}}^* = (\mathfrak{g}/\mathfrak{u})^* \times_{\mathfrak{t}^*} \mathfrak{t}_{\alpha-\text{rs}}^*$, and $(\mathfrak{l}^\alpha/\mathfrak{u}^\alpha)_{\alpha-\text{rs}}^* = (\mathfrak{l}^\alpha/\mathfrak{u}^\alpha)^* \times_{\mathfrak{t}^*} \mathfrak{t}_{\alpha-\text{rs}}^*$.

Lemma 7.2.3. *The (coadjoint) action of B on $(\mathfrak{g}/\mathfrak{u})^*$ induces an isomorphism of B -varieties*

$$B \times_{B^\alpha} (\mathfrak{l}^\alpha/\mathfrak{u}^\alpha)_{\alpha-\text{rs}}^* \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{u})_{\alpha-\text{rs}}^*.$$

Proof. Both varieties under consideration are smooth complex varieties; hence it is enough to prove that the map is bijective. We use the Killing form to get identifications $(\mathfrak{g}/\mathfrak{u})^* \cong \mathfrak{b}$ and $(\mathfrak{l}^\alpha/\mathfrak{u}^\alpha)^* \cong \mathfrak{b}^\alpha$ (where $\mathfrak{b}^\alpha := \text{Lie}(B^\alpha)$), and define $\mathfrak{b}_{\alpha-\text{rs}}, \mathfrak{b}_{\alpha-\text{rs}}^\alpha$ in an obvious way.

First, let $x \in \mathfrak{b}_{\alpha-\text{rs}}$, and consider the Jordan decomposition $x = s + n$. There exists $u \in B$ such that $u \cdot s \in \mathfrak{t}$, and we must have $u \cdot s \in \mathfrak{t}_{\alpha-\text{rs}}$. Then $u \cdot n \in Z_{\mathfrak{b}}(u \cdot s) \subset \mathfrak{b}^\alpha$, which implies that $u \cdot x \in \mathfrak{b}_{\alpha-\text{rs}}^\alpha$. This proves surjectivity of our map.

Next, we prove injectivity. Let $u_1, u_2 \in B$ and $x_1, x_2 \in \mathfrak{b}_{\alpha-rs}^\alpha$, and assume that $u_1 \cdot x_1 = u_2 \cdot x_2$. Consider the Jordan decompositions $x_1 = s_1 + n_1, x_2 = s_2 + n_2$, such that $u_1 \cdot s_1 = u_2 \cdot s_2$. Conjugating if necessary x_1 by an element of B^α (and modifying u_1 accordingly), one can assume that $s_1 \in \mathfrak{t}_{\alpha-rs}$. Then the fact that $(u_2^{-1}u_1) \cdot s_1 = s_2 \in \mathfrak{b}^\alpha$ implies that $u_2^{-1}u_1 \in B^\alpha$. We deduce that

$$(u_1 \times_{B^\alpha} x_1) = (u_2(u_2^{-1}u_1) \times_{B^\alpha} (u_1^{-1}u_2) \cdot x_2) = (u_2 \times_{B^\alpha} x_2)$$

in $B \times_{B^\alpha} \mathfrak{b}_{\alpha-rs}^\alpha$, which finishes the proof. □

In particular it follows from Lemma 7.2.3 that for any $\lambda \in \mathbb{X}$ there is a natural isomorphism of B -modules

$$\mathbb{C}[\mathfrak{t}_{\alpha-rs}^*] \otimes_{S(\mathfrak{t})} S(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda} \cong \text{Ind}_{B^\alpha}^B (\mathbb{C}[\mathfrak{t}_{\alpha-rs}^*] \otimes_{S(\mathfrak{t})} S(\mathfrak{l}^\alpha/\mathfrak{u}^\alpha) \otimes \mathbb{C}_{-\lambda}). \tag{7.2.4}$$

Remark 7.2.5. The same arguments as in Lemma 7.2.3 can be used to prove the following more general claim, which will not be used in this paper. Let I be a set of simple roots, and consider the associated Levi subgroup L^I containing T . Let $B^I = B \cap L^I, \mathfrak{u}^I = \text{Lie}(U \cap L^I)$. Let \mathfrak{t}_{I-rs}^* be the complement in \mathfrak{t}^* of the collection of hyperplanes defined by the coroots associated with roots in $R \setminus \mathbb{Z}I$. Let $(\mathfrak{g}/\mathfrak{u})_{I-rs}^* = (\mathfrak{g}/\mathfrak{u})^* \times_{\mathfrak{t}^*} \mathfrak{t}_{rs}^*$, and similarly for $(\mathfrak{l}^I/\mathfrak{u}^I)_{I-rs}^*$. Then the coadjoint action of B induces an isomorphism of varieties $B \times_{B^I} (\mathfrak{l}^I/\mathfrak{u}^I)_{I-rs}^* \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{u})_{I-rs}^*$.

7.3. Generic and subgeneric situations: the quantum case

Set $\mathfrak{a} := \text{Lie}(A) = \mathfrak{t} \times \mathbb{A}^1$. We will identify \mathfrak{a}^* with $\mathfrak{t}^* \times \mathbb{A}^1$ in the natural way. We denote by $\mathbb{C}[\mathfrak{a}_{rs}^*]$ the localization of S_\hbar with respect to the collection $\{\check{\alpha} + n\hbar \mid \alpha \in R, n \in \mathbb{Z}\}$. If $(\nu, a) \in \mathfrak{a}^*$, we denote by $\mathbb{C}_{S_\hbar}(\nu, a)$ the one-dimensional S_\hbar -module associated with (ν, a) .

Lemma 7.3.1. *For any $\lambda \in \mathbb{X}$, there exists a natural isomorphism of B -modules and $\mathbb{C}[\mathfrak{a}_{rs}^*]$ -modules,*

$$\mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_\hbar} \mathbf{M}(\lambda) \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes \text{Ind}_T^B(-\lambda).$$

Proof. Recall that there exists a morphism of T -modules and S_\hbar -modules $\mathbf{M}^G(\lambda) \rightarrow \mathbf{M}^T(\lambda) = S_\hbar \otimes \mathbb{C}_{-\lambda}$; see (4.6.1). Using Frobenius reciprocity we deduce a morphism as in the statement of the lemma.

First, we claim that for any $(\nu, a) \in \mathfrak{a}^*$ such that $\nu(\check{\alpha}) \notin a\mathbb{Z}$ for all $\alpha \in R$, the induced morphism of B -modules

$$\mathbb{C}_{S_\hbar}(\nu, a) \otimes_{S_\hbar} \mathbf{M}(\lambda) \rightarrow \text{Ind}_T^B(-\lambda) \tag{7.3.2}$$

is an isomorphism. Indeed, if $a = 0$, then $\nu \in \mathfrak{t}_{rs}^*$, and morphism (7.3.2) can be identified with the specialization of isomorphism (7.2.2) at ν ; hence we are done. Now, assume that $a \neq 0$. Then there is an algebra isomorphism $U_\hbar(\mathfrak{g})^{\text{op}} \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_a \cong U(\mathfrak{g})$ which maps $x \in \mathfrak{g} \subset U_\hbar(\mathfrak{g})$ to $-ax$ (see §4.2). Using this isomorphism, the left-hand side of (7.3.2) identifies with the $U(\mathfrak{b})$ -module $\mathbf{V}(-\frac{1}{a}\nu - \lambda - \rho) \otimes \mathbb{C}_{\frac{1}{a}\nu + \rho}$. By our assumption, $-\frac{1}{a}\nu - \rho$ satisfies the assumptions of Lemma 7.1.2, and hence (7.3.2) is an isomorphism in this case also.

Now we deduce that our morphism of $\mathbb{C}[\mathfrak{a}_{rs}^*]$ -modules is an isomorphism. As this morphism is B -equivariant, it is sufficient to prove that its restriction to each T -weight space is an isomorphism. It is easily checked that for any $\lambda \in \mathbb{X}$ the λ -weight spaces of both modules are free $\mathbb{C}[\mathfrak{a}_{rs}^*]$ -modules, of the same rank. Choose a basis of each of these spaces, and consider the determinant $d_\lambda \in \mathbb{C}[\mathfrak{a}_{rs}^*]$ of the restriction of our morphism in these bases. Then for each $(\nu, a) \in \mathfrak{a}^*$ such that $\nu(\check{\alpha}) \notin a\mathbb{Z}$ for all $\alpha \in R$, we have $d_\lambda(\nu, a) \neq 0$. However a polynomial $P \in S_{\mathfrak{h}}$ which does not vanish on any hyperplane $\check{\alpha} + n\mathfrak{h} = 0$ (with $\alpha \in R$, $n \in \mathbb{Z}$) is necessarily a scalar multiple of a product of polynomials of the form $\check{\alpha} + n\mathfrak{h}$. Hence d_λ is invertible in the algebra $\mathbb{C}[\mathfrak{a}_{rs}^*]$, which proves that our morphism is indeed an isomorphism. \square

Let now α be a simple root, and let L^α be the Levi subgroup defined in § 7.1, with its Borel subgroup B^α . For $\lambda \in \mathbb{X}$, we denote by $\mathbf{M}^\alpha(\lambda)$ the asymptotic universal Verma module associated with λ for the group L^α and its Borel subgroup B^α . Let $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]$ be the localization of $S_{\mathfrak{h}}$ with respect to the collection $\{\check{\beta} + n\mathfrak{h} \mid \beta \in R \setminus \{\pm\alpha\}, n \in \mathbb{Z}\}$.

Lemma 7.3.3. *For any $\lambda \in \mathbb{X}$, there exists a natural isomorphism of B -modules and $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]$ -modules*

$$\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\mathfrak{h}}} \mathbf{M}(\lambda) \rightarrow \text{Ind}_{B^\alpha}^B(\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\mathfrak{h}}} \mathbf{M}^\alpha(\lambda))$$

where in the right-hand side B^α acts trivially on $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]$.

Proof. As in Lemma 7.3.1, the morphism in question is constructed using the morphism $\mathbf{M}(\lambda) \rightarrow \mathbf{M}^\alpha(\lambda)$ of (4.6.1) and Frobenius reciprocity.

To prove that this morphism is an isomorphism it is enough to prove that for any $(\nu, a) \in \mathfrak{a}^*$ such that $\nu(\check{\beta}) \notin a\mathbb{Z}$ for all $\beta \in R \setminus \{\pm\alpha\}$, the natural morphism of B -modules

$$\mathbb{C}_{S_{\mathfrak{h}}}(\nu, a) \otimes_{S_{\mathfrak{h}}} \mathbf{M}(\lambda) \rightarrow \text{Ind}_{B^\alpha}^B(\mathbb{C}_{S_{\mathfrak{h}}}(\nu, a) \otimes_{S_{\mathfrak{h}}} \mathbf{M}^\alpha(\lambda)) \tag{7.3.4}$$

is an isomorphism. (Note that the functor $\text{Ind}_{B^\alpha}^B$ is exact by [26, Corollary I.5.13]; hence specialization commutes with induction here.)

If $a = 0$, then morphism (7.3.4) can be identified with the specialization of isomorphism (7.2.4) at ν ; hence we are done. If $a \neq 0$, by the same arguments as in the proof of Lemma 7.3.1, morphism (7.3.4) can be identified with the morphism

$$V\left(-\frac{1}{a}\nu - \lambda - \rho\right) \otimes \mathbb{C}_{\frac{1}{a}\nu + \rho} \rightarrow \text{Ind}_{B^\alpha}^B\left(V^\alpha\left(-\frac{1}{a}\nu - \lambda - \rho\right) \otimes \mathbb{C}_{\frac{1}{a}\nu + \rho}\right)$$

considered in Lemma 7.1.4. Hence it is an isomorphism in this case also. \square

7.4. Generic and subgeneric situations: isomorphism

Fix $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr}_{\check{G}})$. To simplify the notation, we set $V := \mathbb{S}_{\check{G}}(\mathcal{F})$.

As a first step towards proving Theorem 2.2.4, we construct a canonical isomorphism of $\mathbb{C}[\mathfrak{a}_{rs}^*]$ -modules

$$\mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\mathfrak{h}}} (V \otimes \mathbf{M}(\lambda))^B \cong \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\mathfrak{h}}} H_A^*(i_\lambda^! \mathcal{F}). \tag{7.4.1}$$

In fact, we deduce this isomorphism from the fact that the morphism

$$\mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} H_A^*(i_{\lambda}^! \mathcal{F}) \rightarrow \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes V_{\lambda} \quad (\text{resp. } \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} (V \otimes \mathbf{M}(\lambda))^B \rightarrow \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes V_{\lambda}) \tag{7.4.2}$$

induced by $\kappa_{\mathcal{F},\lambda}^{\text{top}}$ (resp. $\kappa_{V,\lambda}^{\text{alg}}$) is an isomorphism. For the first morphism this follows from the localization theorem in equivariant cohomology. Let us now prove that the second morphism is also an isomorphism. In fact we have a series of isomorphisms

$$\begin{aligned} \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} (V \otimes \mathbf{M}(\lambda))^B &\cong (V \otimes (\mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} \mathbf{M}(\lambda)))^B \cong (V \otimes \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes \text{Ind}_T^B(-\lambda))^B \\ &\cong \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes (\text{Ind}_T^B(V \otimes \mathbb{C}_{-\lambda}))^B \cong \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes V_{\lambda}. \end{aligned}$$

Here the second isomorphism follows from Lemma 7.3.1, the third one from the tensor identity, and the last one from the isomorphism $(\text{Ind}_T^B(V \otimes \mathbb{C}_{-\lambda}))^B = (V \otimes \mathbb{C}_{-\lambda})^T$ given by Frobenius reciprocity. By construction, the composition of these isomorphisms is precisely the right-hand morphism in (7.4.2).

Now, let α be a simple root. As a second step towards proving Theorem 2.2.4, we want to show that (7.4.1) restricts to a canonical isomorphism of $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]$ -modules

$$\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} (V \otimes \mathbf{M}(\lambda))^B \cong \mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} H_A^*(i_{\lambda}^! \mathcal{F}). \tag{7.4.3}$$

Let \check{P}^{α} be the minimal parabolic subgroup of \check{G} containing \check{B} associated with α , and let \check{L}^{α} be the unique Levi factor of \check{P}^{α} containing \check{T} . By the constructions of § 6.3, these data determine a Levi factor L^{α} in G , and hence the corresponding minimal parabolic subgroup P^{α} containing B .

Consider first the right-hand side of (7.4.3). We will use the constructions of § 6.4 for the Levi subgroup \check{L}^{α} . By the localization theorem in equivariant cohomology, the morphism

$$\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} H_A^*(i_{\lambda}^! \mathcal{F}) \rightarrow \mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} H_A^*((i_{\lambda}^{\check{L}^{\alpha}})^! \mathfrak{R}_{\check{L}^{\alpha}}^{\check{G}}(\mathcal{F}))$$

induced by (6.4.1) is an isomorphism.

Consider now the left-hand side of (7.4.3). Here we will use the constructions of § 7.3. We have a series of isomorphisms

$$\begin{aligned} \mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} (V \otimes \mathbf{M}(\lambda))^B &\cong (V \otimes (\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} \mathbf{M}(\lambda)))^B \cong (V \otimes \text{Ind}_{B^{\alpha}}^B(\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} \mathbf{M}^{\alpha}(\lambda)))^B \\ &\cong \mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} (\text{Ind}_{B^{\alpha}}^B(V_{|L^{\alpha}} \otimes \mathbf{M}^{\alpha}(\lambda)))^B \cong \mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} (V_{|L^{\alpha}} \otimes \mathbf{M}^{\alpha}(\lambda))^{B^{\alpha}}. \end{aligned}$$

Here the second isomorphism follows from Lemma 7.3.3, the third one from the tensor identity, and the last one from the isomorphism $(\text{Ind}_{B^{\alpha}}^B(V_{|L^{\alpha}} \otimes \mathbf{M}^{\alpha}(\lambda)))^B \cong (V_{|L^{\alpha}} \otimes \mathbf{M}^{\alpha}(\lambda))^{B^{\alpha}}$ given by Frobenius reciprocity.

Using these isomorphisms, to construct (7.4.3) we only have to construct a canonical isomorphism of S_{\hbar} -modules

$$(V_{|L^{\alpha}} \otimes \mathbf{M}^{\alpha}(\lambda))^{B^{\alpha}} \cong H_A^*((i_{\lambda}^{\check{L}^{\alpha}})^! \mathfrak{R}_{\check{L}^{\alpha}}^{\check{G}}(\mathcal{F})).$$

As \check{L}^{α} has semisimple rank 1, this isomorphism is constructed in § A.7 below.

One can check that the isomorphism obtained from (7.4.3) by extension of scalars to $\mathbb{C}[\mathfrak{a}_{rs}^*]$ coincides with (7.4.1). In other words, (7.4.3) is the restriction of (7.4.1) to

$$\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} (V \otimes \mathbf{M}(\lambda))^B \subset \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} (V \otimes \mathbf{M}(\lambda))^B.$$

7.5. Equivariance

Recall the operators $\Omega_w^{V,\lambda}$ defined in §4.8.

Lemma 7.5.1. *Let $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathcal{O})}(\text{Gr}_{\check{G}})$. Isomorphism (7.4.1) is W -equivariant, in the sense that for any $w \in W$ the following diagram commutes:*

$$\begin{CD} \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} (\mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(\lambda))^B @>(7.4.1)>> \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} \mathbf{H}_A^*(i_{\lambda}^! \mathcal{F}) \\ @V \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} \Omega_w^{\mathbb{S}(\mathcal{F}),\lambda} VV @VV \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} \Xi_w^{\mathcal{F},\lambda} V \\ \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} w \left((\mathbb{S}(\mathcal{F}) \otimes \mathbf{M}(w\lambda))^B \right) @>(7.4.1)>> \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} w \left(\mathbf{H}_A^*(i_{w\lambda}^! \mathcal{F}) \right). \end{CD}$$

Proof. It is enough to prove the commutativity when w is a simple reflection. So let α be a simple root. Recall that isomorphism (7.4.1) is the restriction of isomorphism (7.4.3) to \mathfrak{a}_{rs}^* . Now isomorphism (7.4.3) is deduced from the isomorphism of Theorem 2.2.4 for \check{L}^{α} , proved (directly) in §A.7 below. Moreover, by Corollary 3.3.3 (or Lemma 4.6.5) and Lemma 6.4.2, the operators Ω and Ξ for the group \check{G} can also be constructed from the similar operators for the group \check{L}^{α} . Hence the commutativity for s_{α} follows from Theorem 2.5.5 for \check{L}^{α} , which is proved (directly) in §A.9 below. \square

7.6. Proof of the main results: the quantum case

If $\alpha \in R$ is any root, we define the localization $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]$ of S_{\hbar} by the same recipe as for simple roots (see §7.3). Then for any $w \in W$ we have $w(\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]) = \mathbb{C}[\mathfrak{a}_{w(\alpha)-rs}^*]$ as subalgebras of $\mathbb{C}[\mathfrak{a}_{rs}^*]$. To finish the proofs we will need the following obvious lemma.

Lemma 7.6.1. *Let M and N be free S_{\hbar} -modules of finite rank, and let*

$$\varphi : \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} M \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}_{rs}^*] \otimes_{S_{\hbar}} N$$

be an isomorphism of $\mathbb{C}[\mathfrak{a}_{rs}^]$ -modules. Assume that, for any $\alpha \in R$, φ restricts to an isomorphism of $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*]$ -modules $\mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} M \xrightarrow{\sim} \mathbb{C}[\mathfrak{a}_{\alpha-rs}^*] \otimes_{S_{\hbar}} N$. Then φ restricts to an isomorphism $M \xrightarrow{\sim} N$.*

Proof of Theorem 2.2.4. Let $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathcal{O})}(\text{Gr}_{\check{G}})$, and set $V := \mathbb{S}(\mathcal{F})$.

Injectivity of $\kappa_{V,\lambda}^{\text{alg}}$ follows from Lemma 4.6.4, while injectivity of $\kappa_{\mathcal{F},\lambda}^{\text{top}}$ is proved in Corollary 6.2.3. By Lemma 2.2.1(1), $\mathbf{H}_A^*(i_{\lambda}^! \mathcal{F})$ is a free S_{\hbar} -module. It follows from Proposition 3.2.3 and the first isomorphism in Lemma 2.4.1 that the same is true for $(V \otimes \mathbf{M}(\lambda))^B$. In (7.4.1) we have constructed an isomorphism between the extensions

of scalars of these S_{\hbar} -modules to $\mathbb{C}[\mathfrak{a}_{\text{rs}}^*]$. By (7.4.3) this isomorphism restricts to an isomorphism between extensions of scalars to $\mathbb{C}[\mathfrak{a}_{\alpha-\text{rs}}^*]$ for any simple root α . From Lemma 7.5.1 we deduce that the same property is true for *any* root. Hence the isomorphism follows from Lemma 7.6.1. \square

Proof of Theorem 2.5.5. As our S_{\hbar} -modules are free, it is enough to prove commutativity after restriction to $\mathfrak{a}_{\text{rs}}^* \subset \mathfrak{a}^*$. In this setting the claim follows from Lemma 7.5.1. \square

7.7. Classical analogues

Proof of Theorem 2.3.1. One can prove Theorem 2.3.1 using exactly the same strategy as for Theorem 2.2.4. Alternatively, isomorphism $\bar{\zeta}_{\mathcal{F},\lambda}$ can be deduced from isomorphism $\zeta_{\mathcal{F},\lambda}$ of Theorem 2.2.4 using Lemma 3.5.2 and Lemma 6.2.4. (The statements about injectivity are easy.) \square

Proof of Theorem 2.5.7. One can prove Theorem 2.5.7 using exactly the same strategy as for Theorem 2.5.5. Alternatively, one can deduce Theorem 2.5.7 from Theorem 2.5.5 using Proposition 5.4.1, Lemma 3.5.2 and Lemma 6.2.4 (see also Remark 5.4.2). \square

8. Complementary results and applications

8.1. Convolution

In this subsection we construct the morphisms **Conv** considered in § 1.3, and prove their compatibility with the isomorphisms of Corollary 2.4.2, thereby finishing the proof of the theorem stated in § 1.5.

First, consider the “geometric” setting. Multiplication induces a morphism $\mathcal{D}_{\hbar}(\mathcal{X}) \otimes_{\mathbb{C}[\hbar]} \mathcal{D}_{\hbar}(\mathcal{X}) \rightarrow \mathcal{D}_{\hbar}(\mathcal{X})$. One can check that for $\lambda, \mu \in \mathbb{X}$ this morphism induces a morphism of graded S_{\hbar} -modules

$${}^{(\lambda)}\mathcal{D}_{\hbar}(\mathcal{X})_{\lambda} \otimes_{S_{\hbar}} {}^{(\lambda+\mu)}\mathcal{D}_{\hbar}(\mathcal{X})_{\mu} \rightarrow {}^{(\lambda+\mu)}\mathcal{D}_{\hbar}(\mathcal{X})_{\lambda+\mu}.$$

Hence for V, V' in $\text{Rep}(G)$ we obtain a morphism of graded S_{\hbar} -modules

$$\begin{aligned} \text{Conv}_{V, V', \lambda, \mu}^{\text{geom}} : & {}^{(\lambda)}(V \otimes \mathcal{D}_{\hbar}(\mathcal{X})_{\lambda})^G \langle \lambda(2\check{\rho}) \rangle \otimes_{S_{\hbar}} {}^{(\lambda+\mu)}(V' \otimes \mathcal{D}_{\hbar}(\mathcal{X})_{\mu})^G \langle \mu(2\check{\rho}) \rangle \\ & \rightarrow {}^{(\lambda+\mu)}(V \otimes V' \otimes \mathcal{D}_{\hbar}(\mathcal{X})_{\lambda+\mu})^G \langle (\lambda + \mu)(2\check{\rho}) \rangle. \end{aligned}$$

Now, consider the “algebraic” setting. Note that if $\lambda, \mu \in \mathbb{X}$ we have a canonical isomorphism of $(S_{\hbar}, U_{\hbar}(\mathfrak{g}))$ -bimodules

$$S_{\hbar}\langle\langle\lambda\rangle\rangle \otimes_{S_{\hbar}} \mathbf{M}(\mu) \cong \mathbf{M}(\lambda + \mu)$$

which sends $1 \otimes \mathbf{v}_{\mu}$ to $\mathbf{v}_{\lambda+\mu}$. In particular if V is in $\text{Rep}(G)$ and $\phi : \mathbf{M}(0) \rightarrow V \otimes \mathbf{M}(\mu)$ is a morphism of $(S_{\hbar}, U_{\hbar}(\mathfrak{g}))$ -bimodules, then we can consider $S_{\hbar}\langle\langle\lambda\rangle\rangle \otimes_{S_{\hbar}} \phi$ as a morphism of bimodules $\mathbf{M}(\lambda) \rightarrow V \otimes \mathbf{M}(\lambda + \mu)$.

Fix V, V' in $\text{Rep}(G)$ and $\lambda, \mu \in \mathbb{X}$. Then following [3, § 8.4] we define the morphism of graded S_{\hbar} -modules

$$\begin{aligned} \text{Conv}_{V, V', \lambda, \mu}^{\text{alg}} &: \text{Hom}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \langle \lambda(2\check{\rho}) \rangle \otimes_{S_{\hbar}} \langle \lambda \rangle \otimes_{S_{\hbar}} \text{Hom}(\mathbf{M}(0), V' \otimes \mathbf{M}(\mu)) \langle \mu(2\check{\rho}) \rangle \\ &\rightarrow \text{Hom}(\mathbf{M}(0), V \otimes V' \otimes \mathbf{M}(\lambda + \mu)) \langle (\lambda + \mu)(2\check{\rho}) \rangle \end{aligned}$$

(where as usual we consider morphisms of $(S_{\hbar}, U_{\hbar}(\mathfrak{g}))$ -bimodules) which sends a pair (ϕ, ψ) to the following composition:

$$\mathbf{M}(0) \xrightarrow{\phi} V \otimes \mathbf{M}(\lambda) \xrightarrow{\text{id}_V \otimes (S_{\hbar} \langle \lambda \rangle \otimes_{S_{\hbar}} \psi)} V \otimes V' \otimes \mathbf{M}(\lambda + \mu).$$

Finally we consider the “topological” setting. Here again we follow [3, § 8.7] (though we have to be more careful because we use *equivariant* cohomology). Let \mathcal{F}, \mathcal{G} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, and let $\lambda, \mu \in \mathbb{X}$. The convolution $\mathcal{F} \star \mathcal{G}$ is defined in terms of the “convolution diagram”

$$\text{mult} : \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \text{Gr} \rightarrow \text{Gr}$$

induced by left multiplication of $\check{G}(\mathbf{K})$ on Gr . More precisely, $\mathcal{F} \star \mathcal{G} = \text{mult}_*(\mathcal{F} \boxtimes \mathcal{G})$ where $\mathcal{F} \boxtimes \mathcal{G}$ is the twisted external product, as defined e.g. in [35, § 4].

Let $\nu := \lambda + \mu$. Consider the Cartesian square

$$\begin{array}{ccc} \text{mult}^{-1}(\mathbf{v}) & \xrightarrow{j_{\nu}} & \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \text{Gr} \\ \text{mult}_{\nu} \downarrow & & \downarrow \text{mult} \\ \mathbf{v} & \xrightarrow{i_{\nu}} & \text{Gr}. \end{array}$$

This diagram is A -equivariant if we consider $\check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \text{Gr}$ as an A -variety where \check{T} acts by left multiplication on $\check{G}(\mathbf{K})$, and \mathbb{C}^{\times} acts diagonally by loop rotation. By the base change theorem we have an isomorphism

$$i_{\nu}^!(\mathcal{F} \star \mathcal{G}) = i_{\nu}^! \text{mult}_*(\mathcal{F} \boxtimes \mathcal{G}) \cong (\text{mult}_{\nu})_* j_{\nu}^!(\mathcal{F} \boxtimes \mathcal{G}),$$

so we obtain an isomorphism

$$H_A^*(i_{\nu}^!(\mathcal{F} \star \mathcal{G})) \cong H_A^*(\text{mult}^{-1}(\mathbf{v}), j_{\nu}^!(\mathcal{F} \boxtimes \mathcal{G})).$$

Now let $k_{\lambda, \mu} : \{\lambda \times_{\check{G}(\mathbf{O})} \mu\} \hookrightarrow \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \text{Gr}$ be the obvious embedding. The $(i, !)$ -adjunction for the embedding $\{\lambda \times_{\check{G}(\mathbf{O})} \mu\} \hookrightarrow \text{mult}^{-1}(\mathbf{v})$ induces a morphism

$$H_A^*(k_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G})) \rightarrow H_A^*(\text{mult}^{-1}(\mathbf{v}), j_{\nu}^!(\mathcal{F} \boxtimes \mathcal{G})),$$

which can be reinterpreted as a morphism

$$H_A^*(k_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G})) \rightarrow H_A^*(i_{\lambda + \mu}^!(\mathcal{F} \star \mathcal{G})). \tag{8.1.1}$$

Lemma 8.1.2. *There exists a canonical isomorphism of graded S_{\hbar} -modules*

$$H_A^*(k_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G})) \cong H_A^*(i_{\lambda}^! \mathcal{F}) \otimes_{S_{\hbar}} \langle \lambda \rangle H_A^*(i_{\mu}^! \mathcal{G}).$$

Proof. Choose some closed finite union of $\check{G}(\mathbf{O})$ -orbits $Y \subset \mathbf{Gr}$ such that \mathcal{G} is supported on Y , and a closed normal subgroup $H \triangleleft \check{G}(\mathbf{O})$ of finite codimension c , which acts trivially on Y . Let $f : \check{G}(\mathbf{K})/H \times Y \rightarrow \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} Y$ and $g : \check{G}(\mathbf{K})/H \rightarrow \mathbf{Gr}$ be the natural projections. Then by definition of $\mathcal{F} \boxtimes \mathcal{G}$ we have a canonical isomorphism $f^*(\mathcal{F} \boxtimes \mathcal{G}) \cong (g^*\mathcal{F}) \boxtimes \mathcal{G}$ and hence (since both f and g are smooth morphisms of relative dimension c) a canonical isomorphism

$$f^!(\mathcal{F} \boxtimes \mathcal{G}) \cong (g^!\mathcal{F}) \boxtimes \mathcal{G}. \tag{8.1.3}$$

We will consider $\check{G}(\mathbf{K})/H \times Y$ as an A -variety where any $t \in \check{T}$ (resp. $a \in \mathbb{C}^\times$) acts by $t \cdot (gH, x) := (tgt^{-1}H, t \cdot x)$ (resp. $a \cdot (gH, x) := ((a \cdot g)\lambda(a)^{-1}H, \lambda(a) \cdot (a \cdot x))$).

With this definition, the morphism f is A -equivariant, and the point $(\tilde{\lambda}, \mu) \in \check{G}(\mathbf{K})/H \times Y$ is A -stable. (Here $\tilde{\lambda} := \hat{\lambda}H/H$, where $\hat{\lambda}$ is λ considered as an element of $\check{G}(\mathbf{K})$). Now $k_{\lambda, \mu}$ factors as the composition

$$\{(\lambda, \mu)\} \xrightarrow{\sim} \{(\tilde{\lambda}, \mu)\} \xrightarrow{l_{\lambda, \mu}} \check{G}(\mathbf{K})/H \times Y \xrightarrow{f} \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} Y,$$

and hence using (8.1.3) we obtain an isomorphism

$$H_A^*(k_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G})) \cong H_A^*(l_{\lambda, \mu}^!(g^!\mathcal{F} \boxtimes \mathcal{G})).$$

Now we have

$$H_A^*(l_{\lambda, \mu}^!(g^!\mathcal{F} \boxtimes \mathcal{G})) \cong H_A^*(i_{\lambda}^!\mathcal{F}) \otimes_{S_h} {}^{(\lambda)}H_A^*(i_{\mu}^!\mathcal{G})$$

(where we use the fact that ${}^{(\lambda)}H_A^*(i_{\mu}^!\mathcal{G})$ is free over S_h by Lemma 2.2.1, so we have a Künneth formula in equivariant cohomology). This finishes the proof. \square

Using the isomorphism of Lemma 8.1.2 we can reinterpret (8.1.1) as a morphism

$$\mathbf{Conv}_{\mathcal{F}, \mathcal{G}, \lambda, \mu}^{\text{top}} : H_A^*(i_{\lambda}^!\mathcal{F}) \otimes_{S_h} {}^{(\lambda)}H_A^*(i_{\mu}^!\mathcal{G}) \rightarrow H_A^*(i_{\lambda+\mu}^!(\mathcal{F} \star \mathcal{G})).$$

Remark 8.1.4. In the case where λ, μ are antidominant, the morphism $\mathbf{Conv}_{\mathcal{F}, \mathcal{G}, \lambda, \mu}^{\text{top}}$ can also be described in terms of Wakimoto sheaves, as in [3, diagram on p. 652].

The following result will be proved in § 8.2

Proposition 8.1.5. *Let \mathcal{F}, \mathcal{G} be in $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ and $\lambda, \mu \in \mathbb{X}$. Under the isomorphisms of Corollary 2.4.2, the morphisms $\mathbf{Conv}_{S(\mathcal{F}), S(\mathcal{G}), \lambda, \mu}^{\text{geom}}$, $\mathbf{Conv}_{S(\mathcal{F}), S(\mathcal{G}), \lambda, \mu}^{\text{alg}}$ and $\mathbf{Conv}_{\mathcal{F}, \mathcal{G}, \lambda, \mu}^{\text{top}}$ match.*

Remark 8.1.6.

- (1) Note that, unlike most of the constructions in the paper, the morphisms \mathbf{Conv} are *not* compatible with restriction to a Levi subgroup (see § 3.3, § 4.6 or § 6.4) in the obvious strong sense. We will use a weaker compatibility result in the proof of Proposition 8.1.5.

- (2) Proposition 8.1.5 has an obvious “classical analogue”, that we do not state for simplicity, but which can be proved by the same methods.
- (3) Using similar constructions one can define, for V, V' in $\mathbf{Rep}(G)$ and $\lambda, \mu \in \mathbb{X}$, a “geometric” convolution morphism

$$(V \otimes \mathbf{M}(\lambda))^G \otimes_{S_{\hbar}}^{(\lambda)} (V'_\mu \otimes S_{\hbar})^G \rightarrow ((V \otimes V')_{\lambda+\mu} \otimes S_{\hbar})^G$$

(where for simplicity we disregard the grading) and, for \mathcal{F}, \mathcal{G} in $\mathbf{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ and $\lambda, \mu \in \mathbb{X}$, a “topological” convolution morphism

$$H_A^*(i_{\lambda}^! \mathcal{F}) \otimes_{S_{\hbar}}^{(\lambda)} H_A^*(t_{\mu}^! \mathcal{G}) \rightarrow H_A^*(t_{\lambda+\mu}^! (\mathcal{F} \star \mathcal{G})).$$

(These maps “extend” $\mathbf{Conv}_{V,V',\lambda,\mu}^{\text{alg}}$ and $\mathbf{Conv}_{\mathcal{F},\mathcal{G},\lambda,\mu}^{\text{top}}$ in the natural sense, using the inclusions $\kappa_{V',\mu}^{\text{alg}}, \kappa_{V \otimes V', \lambda+\mu}^{\text{alg}}$ ($t_{\mu}!$ and $(t_{\lambda+\mu})!$). Then the same arguments as for Proposition 8.1.5 show that the following diagram commutes:

$$\begin{CD} (V \otimes \mathbf{M}(\lambda))^G \otimes_{S_{\hbar}}^{(\lambda)} (V'_\mu \otimes S_{\hbar})^G @>>> ((V \otimes V')_{\lambda+\mu} \otimes S_{\hbar})^G \\ @V \text{Th. 2.2.4 \& Lem. 2.2.1} \wr VV @VV \text{Lem. 2.2.1} \wr V \\ H_A^*(i_{\lambda}^! \mathcal{F}) \otimes_{S_{\hbar}}^{(\lambda)} H_A^*(t_{\mu}^! \mathcal{G}) @>>> H_A^*(t_{\lambda+\mu}^! (\mathcal{F} \star \mathcal{G})). \end{CD}$$

In particular, for $V = V' = \mathbb{C}[G]$ (the regular representation of G , considered as an ind-object in $\mathbf{Rep}(G)$) and taking the direct sum over λ and μ , one can check that on composing the “algebraic” map with the morphism induced by the multiplication morphism $\mathbf{m} : \mathbb{C}[G] \otimes \mathbb{C}[G] \rightarrow \mathbb{C}[G]$, one obtains an action of the algebra $\mathcal{D}_{\hbar}(\mathcal{X})$ on the $\mathbb{C}[\hbar]$ -algebra $\mathcal{D}_{\hbar}^{\text{rel}}$ of relative asymptotic differential operators along the fibers of the projection $G \rightarrow G/T$. This action can be realized “topologically” using the composition

$$H_A^*(i_{\lambda}^! \mathcal{R}) \otimes_{S_{\hbar}}^{(\lambda)} H_A^*(t_{\mu}^! \mathcal{R}) \rightarrow H_A^*(t_{\lambda+\mu}^! (\mathcal{R} \star \mathcal{R})) \xrightarrow{\mathbb{S}(\mathbf{m})} H_A^*(t_{\lambda+\mu}^! \mathcal{R}),$$

where $\mathcal{R} := \mathbb{S}(\mathbb{C}[G])$.

In this construction one can also replace T by a Levi factor of a parabolic subgroup; the details are left to the reader.

8.2. Proof of Proposition 8.1.5

For $\lambda, \mu \in \mathbb{X}$ we set

$$\mathfrak{T}_{\lambda,\mu} := \{(n\hat{\lambda} \times_{\check{G}(\mathbf{O})} m\mu) \mid n, m \in \check{N}^-(\mathbf{K})\} \subset \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \mathbf{Gr},$$

and we denote the inclusion by $t_{\lambda,\mu} : \mathfrak{T}_{\lambda,\mu} \hookrightarrow \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \mathbf{Gr}$. (The notation $\hat{\lambda}$ is defined in the proof of Lemma 8.1.2.) Then we have decompositions

$$\check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \mathbf{Gr} = \bigsqcup_{\lambda,\mu \in \mathbb{X}} \mathfrak{T}_{\lambda,\mu}, \quad \text{mult}^{-1}(\mathfrak{T}_v) = \bigsqcup_{\lambda+\mu=v} \mathfrak{T}_{\lambda,\mu}.$$

In fact, $\mathfrak{T}_{\lambda,\mu}$ is the inverse image of $\mathfrak{T}_{\lambda} \times \mathfrak{T}_{\lambda+\mu} \subset \mathbf{Gr} \times \mathbf{Gr}$ under the isomorphism $\check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \mathbf{Gr} \xrightarrow{\sim} \mathbf{Gr} \times \mathbf{Gr}$ sending $(g_1 \times_{\check{G}(\mathbf{O})} g_2 \check{G}(\mathbf{O}))$ to $(g_1 \check{G}(\mathbf{O}), g_1 g_2 \check{G}(\mathbf{O}))$.

Lemma 8.2.1. *For any \mathcal{F}, \mathcal{G} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, there exists a natural isomorphism of graded S_{\hbar} -modules*

$$H_A^*(\mathfrak{T}_{\lambda, \mu}, t_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G})) \cong H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) \otimes_{S_{\hbar}}^{(\lambda)} H_A^*(\mathfrak{T}_{\mu}, t_{\mu}^! \mathcal{G}).$$

Proof. If we set $\tilde{\mathfrak{T}}_{\lambda} := \tilde{\lambda} \cdot \check{N}^-(\mathbf{O}^-)_1 \subset \check{G}(\mathbf{K})$ (where $\check{N}^-(\mathbf{O}^-)_1 \subset \check{N}^-(\mathbf{O}^-)$ is the kernel of the evaluation at $z = \infty$), then the composition $\tilde{\mathfrak{T}}_{\lambda} \hookrightarrow \check{G}(\mathbf{K}) \twoheadrightarrow \text{Gr}$ induces an isomorphism $\tilde{\mathfrak{T}}_{\lambda} \xrightarrow{\sim} \mathfrak{T}_{\lambda}$. Then the same arguments as in the proof of Lemma 8.1.2 prove our claim. □

Now we fix $\nu \in \mathbb{X}$ and \mathcal{F}, \mathcal{G} in $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$. For $\lambda \in \mathbb{X}$ we set

$$\mathfrak{T}_{\nu}^{\geq \lambda} := \bigsqcup_{\substack{\lambda' + \mu = \nu \\ \lambda' \geq \lambda}} \mathfrak{T}_{\lambda', \mu} \subset \text{mult}^{-1}(\mathfrak{T}_{\nu}),$$

and denote the inclusion by $t_{\nu}^{\geq \lambda} : \mathfrak{T}_{\nu}^{\geq \lambda} \hookrightarrow \check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \text{Gr}$. Then $\mathfrak{T}_{\nu}^{\geq \lambda}$ is closed in $\text{mult}^{-1}(\mathfrak{T}_{\nu})$, and $\mathfrak{T}_{\lambda, \nu - \lambda}$ is open in $\mathfrak{T}_{\nu}^{\geq \lambda}$. It follows in particular from Lemma 8.2.1 and Lemma 2.2.1 that for any λ, μ such that $\lambda + \mu = \nu$, the cohomology $H_A^*(\mathfrak{T}_{\lambda, \mu}, t_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G}))$ is concentrated in degrees of the same parity as $\nu(2\check{\rho})$. From this parity vanishing observation, one can deduce that the long exact sequence associated with the decomposition $\mathfrak{T}_{\nu}^{\geq \lambda} = \mathfrak{T}_{\nu}^{> \lambda} \sqcup \mathfrak{T}_{\lambda, \nu - \lambda}$ (where $\mathfrak{T}_{\nu}^{> \lambda}$ has the obvious definition) for the object $(t_{\nu}^{\geq \lambda})^!(\mathcal{F} \boxtimes \mathcal{G})$ breaks into a family of short exact sequences. And then (using the base change theorem) we deduce that the graded S_{\hbar} -module

$$H_A^*(\mathfrak{T}_{\nu}, t_{\nu}^!(\mathcal{F} \star \mathcal{G}))$$

admits a decreasing \mathbb{X} -filtration with the part bigger than λ isomorphic to $H_A^*(\mathfrak{T}_{\nu}^{\geq \lambda}, (t_{\nu}^{\geq \lambda})^!(\mathcal{F} \boxtimes \mathcal{G}))$, and with associated graded

$$\bigoplus_{\lambda + \mu = \nu} H_A^*(\mathfrak{T}_{\lambda, \mu}, t_{\lambda, \mu}^!(\mathcal{F} \boxtimes \mathcal{G})) \cong \bigoplus_{\lambda + \mu = \nu} H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) \otimes_{S_{\hbar}}^{(\lambda)} H_A^*(\mathfrak{T}_{\mu}, t_{\mu}^! \mathcal{G}).$$

(Here the isomorphism is provided by Lemma 8.2.1.)

Lemma 8.2.2. *Under the isomorphisms*

$$H_A^*(\mathfrak{T}_{\nu}, t_{\nu}^!(\mathcal{F} \star \mathcal{G})) \cong (\mathbb{S}(\mathcal{F}) \otimes \mathbb{S}(\mathcal{G}))_{\nu} \otimes S_{\hbar}(\nu(2\check{\rho})),$$

and

$$H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) \otimes_{S_{\hbar}}^{(\lambda)} H_A^*(\mathfrak{T}_{\mu}, t_{\mu}^! \mathcal{G}) \cong (\mathbb{S}(\mathcal{F})_{\lambda} \otimes \mathbb{S}(\mathcal{G})_{\mu}) \otimes S_{\hbar}(\nu(2\check{\rho}))$$

provided by Lemma 2.2.1, the “topological” filtration on $H_A^*(\mathfrak{T}_{\nu}, t_{\nu}^!(\mathcal{F} \star \mathcal{G}))$ considered above is induced by the filtration on $(\mathbb{S}(\mathcal{F}) \otimes \mathbb{S}(\mathcal{G}))_{\nu}$ by the subspaces

$$\bigoplus_{\lambda' \geq \lambda} \mathbb{S}(\mathcal{F})_{\lambda'} \otimes \mathbb{S}(\mathcal{G})_{\nu - \lambda'}.$$

Sketch of proof. By construction of the isomorphisms in Lemma 2.2.1, it is sufficient to prove the analogous claim for ordinary cohomology; in other words one can forget about A -equivariance. Now recall the construction of the tensor structure in [35, Proposition 6.4]; in particular let X be a smooth (algebraic) curve, and consider the local analogues of \mathbf{Gr} and the convolution diagram over X^2 as in [35, Equation (5.2)]. In the proof of [35, Proposition 6.4], the authors define a global counterpart $\mathfrak{T}_\nu(X^2) \subset \mathbf{Gr}_{X^2}$ of \mathfrak{T}_ν . Then one can consider

$$\text{mult}^{-1}(\mathfrak{T}_\nu(X^2)) \subset \mathbf{Gr}_X \widetilde{\times} \mathbf{Gr}_X$$

in the “global analogue” of $\check{G}(\mathbf{K}) \times_{\check{G}(\mathbf{O})} \mathbf{Gr}$. One can define locally closed ind-subvarieties $\mathfrak{T}_{\lambda,\mu}(X^2)$ inside this inverse image (when $\lambda + \mu = \nu$) which, over points in the diagonal copy of X in X^2 , coincide with our subvarieties $\mathfrak{T}_{\lambda,\mu}$ and which, over points outside the diagonal, coincide with $\mathfrak{T}_\lambda \times \mathfrak{T}_\mu$. Then one has a filtration as above, but this time globally over X^2 . Over points in the diagonal, this filtration coincides with the one considered above by construction. And over points outside of the diagonal, the variety $\mathfrak{T}_\nu(X^2)$ is a disjoint union $\bigsqcup_{\lambda+\mu=\nu} \mathfrak{T}_\lambda \times \mathfrak{T}_\mu$, and the filtration is obtained from the decomposition of the appropriate cohomology sheaves as a direct sum, as e.g. in [35, Equation (6.25b)]. This implies the claim. \square

Using these remarks we are now ready to give a proof of Proposition 8.1.5.

Proof of Proposition 8.1.5. First it is easy to check, by explicit computation, that the isomorphism between the left-hand side and the right-hand side of the equation in Corollary 2.4.2 is compatible with the morphisms $\mathbf{Conv}^{\text{geom}}$ and $\mathbf{Conv}^{\text{alg}}$.

Set $\nu := \lambda + \mu$, $V := \mathbb{S}(\mathcal{F})$, $V' := \mathbb{S}(\mathcal{G})$. To finish the proof we have to prove that the left square in the following diagram commutes, where the isomorphisms are as in Theorem 2.2.4:

$$\begin{CD} H_A^*(i_\lambda^! \mathcal{F}) \otimes_{S_\hbar}^{(\lambda)} H_A^*(i_\mu^! \mathcal{G}) @>{\text{Conv}_{\mathcal{F}, \mathcal{G}, \lambda, \mu}^{\text{top}}}>> H_A^*(i_\nu^! (\mathcal{F} \star \mathcal{G})) @<{(t_\nu)!}<< H_A^*(\mathfrak{T}_\nu, t_\nu^! (\mathcal{F} \star \mathcal{G})) \\ @VV\wr V @VV\wr V @VV\wr V \\ (V \otimes \mathbf{M}(\lambda))^B \otimes_{S_\hbar}^{(\lambda)} (V' \otimes \mathbf{M}(\mu))^B @>{\text{Conv}_{V, V', \lambda, \mu}^{\text{alg}}}>> (V \otimes V' \otimes \mathbf{M}(\nu))^B @<{\kappa_{V \otimes V', \nu}^{\text{alg}}}<< (V \otimes V')_\nu \otimes S_\hbar \end{CD}$$

Of course it is sufficient to prove that the outer square commutes.

Now recall the filtrations on $H_A^*(\mathfrak{T}_\nu, t_\nu^! (\mathcal{F} \star \mathcal{G}))$ and $(V \otimes V')_\nu \otimes S_\hbar$ considered above. Then each of our morphisms factors through the part of the filtration bigger than λ : for the first line this follows from the fact that $(\hat{\lambda} \times_{\check{G}(\mathbf{O})} \mu) \in \mathfrak{T}_\nu^{\geq \lambda}$, and for the second line this can be checked by explicit computation. Moreover, the restriction to the image of these morphisms of the projection to $H_A^*(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \otimes_{S_\hbar}^{(\lambda)} H_A^*(\mathfrak{T}_\mu, t_\mu^! \mathcal{G})$ (resp. $(V_\lambda \otimes V_\mu) \otimes S_\hbar$) is injective. Hence it is enough to check that the corresponding diagram:

$$\begin{CD} H_A^*(i_\lambda^! \mathcal{F}) \otimes_{S_\hbar}^{(\lambda)} H_A^*(i_\mu^! \mathcal{G}) @>>> H_A^*(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \otimes_{S_\hbar}^{(\lambda)} H_A^*(\mathfrak{T}_\mu, t_\mu^! \mathcal{G}) \\ @VV\wr V @VV\wr V \\ (V \otimes \mathbf{M}(\lambda))^B \otimes_{S_\hbar}^{(\lambda)} (V' \otimes \mathbf{M}(\mu))^B @>>> (V_\lambda \otimes V_\mu) \otimes S_\hbar \end{CD}$$

commutes. However the upper line is induced by $(t_\lambda)! \otimes (t_\mu)!$, and the bottom line is induced by $\kappa_{V,\lambda}^{\text{alg}} \otimes \kappa_{V',\mu}^{\text{alg}}$; hence this claim is clear. \square

8.3. A reminder on dynamical Weyl groups

Let us fix V in $\text{Rep}(G)$, $\lambda \in \mathbb{X}$ and $w \in W$. For any $\mu \in \mathbb{X}^-$ sufficiently large we consider the morphism of $\mathbb{C}[\hbar, \hbar^{-1}]$ -modules

$$\text{DW}_{V,\lambda,w,\mu}^{\text{alg}} : \mathbb{C}[\hbar, \hbar^{-1}] \otimes V_\lambda \rightarrow \mathbb{C}[\hbar, \hbar^{-1}] \otimes V_{w\lambda}, \quad \text{DW}_{V,\lambda,w,\mu}^{\text{alg}} := E_{w\mu}^{V,\lambda} \circ \Psi_{w,\mu}^{V,\lambda} \circ (E_\mu^{V,\lambda})^{-1}.$$

This morphism is well defined by Lemma 4.4.1(1). We will sometimes extend this morphism to a morphism of $\mathbb{C}(\hbar)$ -modules $\mathbb{C}(\hbar) \otimes V_\lambda \rightarrow \mathbb{C}(\hbar) \otimes V_{w\lambda}$ in the obvious way (and denote the extension also by $\text{DW}_{V,\lambda,w,\mu}^{\text{alg}}$).

Recall that for $\mu \in \mathfrak{t}^*$ we have defined a morphism $P \mapsto P(\mu)$ in § 4.3. We denote similarly the induced morphisms $S_\hbar \otimes V_\lambda \rightarrow \mathbb{C}[\hbar, \hbar^{-1}] \otimes V_\lambda$ or $S_\hbar \otimes V_{w\lambda} \rightarrow \mathbb{C}[\hbar, \hbar^{-1}] \otimes V_{w\lambda}$.

Lemma 8.3.1. *There exists a unique isomorphism of $\mathbb{C}(\hbar)$ -modules*

$$\text{DW}_{V,\lambda,w}^{\text{alg}} : \mathbb{Q}_\hbar \otimes V_\lambda \xrightarrow{\sim} \mathbb{Q}_\hbar \otimes V_{w\lambda}$$

such that for any $x \in \mathbb{Q}_\hbar \otimes V_\lambda$ the following property holds: for any $\mu \in \mathbb{X}^-$ sufficiently large such that $x(\mu)$ is defined, we have

$$(\text{DW}_{V,\lambda,w}^{\text{alg}}(x))(w\mu) = \text{DW}_{V,\lambda,w,\mu}^{\text{alg}}(x(\mu))$$

(in particular, the left-hand side is defined). This morphism induces an isomorphism of \mathbb{Q}_\hbar -modules

$$\text{DW}_{V,\lambda,w}^{\text{alg}} : \mathbb{Q}_\hbar \otimes V_\lambda \xrightarrow{\sim} {}^w\mathbb{Q}_\hbar \otimes V_{w\lambda}.$$

Proof. This claim is proved in [19, 39]. For later use, let us explain how it can be deduced from our constructions. Unicity is clear (see e.g. § 4.3). Let us prove existence.

It is enough to treat the case where $w = s$ is the reflection associated with a simple root α (provided we only require $\mu \in \mathbb{X}$ to satisfy $\mu(\check{\alpha}) \leq 0$, not necessarily to be antidominant); see in particular (4.4.2). Define the isomorphism of $\mathbb{C}(\hbar)$ -modules

$${}'\text{DW}_{V,\lambda,s}^{\text{alg}} : \mathbb{Q}_\hbar \otimes V_\lambda \xrightarrow{\sim} \mathbb{Q}_\hbar \otimes V_{s\lambda}$$

to be the composition

$$\begin{aligned} \mathbb{Q}_\hbar \otimes V_\lambda &\xleftarrow[\sim]{\kappa_{V,\lambda}^{\text{alg}}} \mathbb{Q}_\hbar \otimes_{S_\hbar} (V \otimes \mathbf{M}(\lambda))^B \xrightarrow[\sim]{\Omega_s^{V,\lambda}} \mathbb{Q}_\hbar \otimes_{S_\hbar} (V \otimes \mathbf{M}(s\lambda))^B \\ &\xrightarrow[\sim]{\kappa_{V,s\lambda}^{\text{alg}}} \mathbb{Q}_\hbar \otimes_{S_\hbar} {}^s(V_{s\lambda} \otimes S_\hbar) \xrightarrow[\sim]{q \otimes v \otimes p \mapsto s(q)p \otimes v} \mathbb{Q}_\hbar \otimes V_{s\lambda}. \end{aligned}$$

(The fact that the first and third arrows are invertible was proved in the course of the proof of Theorem 2.2.4; see § 7.4). Then for $x \in \mathbb{Q}_\hbar \otimes V_\lambda$ we set

$$\text{DW}_{V,\lambda,s}^{\text{alg}}(x) = \begin{cases} \frac{1}{(-\check{\alpha})(-\check{\alpha} + \hbar) \cdots (-\check{\alpha} + (\lambda(\check{\alpha}) - 1)\hbar)} \cdot {}'\text{DW}_{V,\lambda,s}^{\text{alg}}(x) & \text{if } \lambda(\check{\alpha}) \geq 0; \\ (-\check{\alpha} - \hbar)(-\check{\alpha} - 2\hbar) \cdots (-\check{\alpha} + \lambda(\check{\alpha})\hbar) \cdot {}'\text{DW}_{V,\lambda,s}^{\text{alg}}(x) & \text{if } \lambda(\check{\alpha}) \leq 0 \end{cases}$$

in $\mathbf{Q}_\hbar \otimes V_{s_\lambda}$. It follows from Proposition 4.5.1 that the morphism $\mathbf{DW}_{V,\lambda,s}^{\text{alg}}$ satisfies our requirements. \square

The operators $\mathbf{DW}_{V,\lambda,w}^{\text{alg}}$ form the *dynamical Weyl group* as considered (in the non-asymptotic case) in [19, 39] and (in the asymptotic case) in [12].

8.4. Transverse slices and semi-infinite orbits

Recall the definition of \mathfrak{W}_λ , s_λ , and n_λ in § 2.6. The following result contains in particular Lemma 2.6.1. It is essentially proved in [12]; we explain the details for the reader’s convenience.

Lemma 8.4.1. *Let \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$ and $\lambda \in \mathbb{X}$.*

(1) *There exists a canonical isomorphism of graded S_\hbar -modules*

$$H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F}) \cong (S(\mathcal{F}))_\lambda \otimes S_\hbar\langle n_\lambda \rangle.$$

(2) *Under the isomorphism of (1) and (2.2.2), the natural morphism*

$$H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F}) \rightarrow H_A^*(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F})$$

induced by the (closed) inclusion $\mathfrak{T}_\lambda \cap \mathfrak{W}_\lambda \hookrightarrow \mathfrak{T}_\lambda$ identifies with multiplication by

$$\prod_{\substack{\alpha > 0, \\ (\lambda, \check{\alpha}) < 0}} \prod_{j=0}^{-\lambda(\check{\alpha})-1} (-\check{\alpha} + j\hbar)$$

on $(S(\mathcal{F}))_\lambda \otimes S_\hbar$.

Proof. There exists a closed subvariety $V \subset \check{G}(\mathbf{O})$ isomorphic to a (finite dimensional) affine space, which is stable under conjugation by A , and which satisfies the following conditions:

- the morphism $V \rightarrow \mathbf{Gr}^\lambda$, $u \mapsto u \cdot \lambda$, is an open embedding;
- the morphism $V \times \mathfrak{W}_\lambda \rightarrow \mathbf{Gr}$ induced by the $\check{G}(\mathbf{O})$ -action on \mathbf{Gr} is an open embedding.

Indeed, by $N_{\check{G}}(\check{T})$ -equivariance it is enough to treat the case where λ is dominant. And in this case one can e.g. take as V the subvariety J^λ considered in [36, Lemme 2.2]. We will identify $V \times \mathfrak{W}_\lambda$ with its image in \mathbf{Gr} , and denote it by O .

We have canonical isomorphisms

$$H_A^*(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \cong H_A^*(t_\lambda^* t_\lambda^! \mathcal{F}) \cong H_A^*((\{\lambda\} \hookrightarrow O \cap \mathfrak{T}_\lambda)^*(O \cap \mathfrak{T}_\lambda \hookrightarrow \mathbf{Gr})^! \mathcal{F}).$$

Here the first isomorphism follows from [21, Proposition 2.3] (since λ is an attractive fixed point of A on \mathfrak{T}_λ), and the second one from the fact that O is open in \mathbf{Gr} .

As \mathcal{F} is $\check{G}(\mathbf{O})$ -equivariant, there exists a canonical isomorphism

$$(O \hookrightarrow \mathbf{Gr})^* \mathcal{F} \cong \underline{\mathbb{C}}_V[2\lambda(2\check{\rho})] \boxtimes \mathcal{F}_{(\lambda)} \quad \text{where} \quad \mathcal{F}_{(\lambda)} := (\mathfrak{W}_\lambda \hookrightarrow \mathbf{Gr})^* \mathcal{F}[-2\lambda(2\check{\rho})]. \quad (8.4.2)$$

As V is smooth of dimension $\dim(\mathbf{Gr}^\lambda) = \lambda(2\check{\rho})$, we also have canonically

$$\mathcal{F}_{(\lambda)} \cong (\mathfrak{W}_\lambda \hookrightarrow \mathbf{Gr})^! \mathcal{F}.$$

By definition we have

$$\mathfrak{T}_\lambda = \{x \in \mathbf{Gr} \mid \lim_{s \rightarrow \infty} 2\rho(s) \cdot x = \lambda\}.$$

(Here ρ is considered as a cocharacter of \check{T} .) It follows that we have an isomorphism

$$O \cap \mathfrak{T}_\lambda \cong (V \cap \mathfrak{T}_\lambda) \times (\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda). \tag{8.4.3}$$

The variety $V \cap \mathfrak{T}_\lambda$ is open in $\mathbf{Gr}^\lambda \cap \mathfrak{T}_\lambda$, and by [35, Equation (3.6)] we have

$$\mathbf{Gr}^\lambda \cap \mathfrak{T}_\lambda = \check{U}^-(\mathbf{O}) \cdot \lambda \cong \prod_{\substack{\alpha > 0, \\ (\lambda, \check{\alpha}) < 0}} \prod_{k=0}^{-(\lambda, \check{\alpha})-1} \mathbb{C}_A(-\check{\alpha} + k\hbar) \tag{8.4.4}$$

as A -varieties. In particular, $V \cap \mathfrak{T}_\lambda$ is smooth.

Using (8.4.2) we obtain (under the identification (8.4.3)) a canonical isomorphism

$$(O \cap \mathfrak{T}_\lambda \hookrightarrow \mathbf{Gr})^! \mathcal{F} \cong \mathbb{C}_{V \cap \mathfrak{T}_\lambda}[2 \dim(V \cap \mathfrak{T}_\lambda)] \boxtimes (\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda \hookrightarrow \mathfrak{W}_\lambda)^! \mathcal{F}_{(\lambda)}.$$

We deduce a canonical isomorphism

$$H_A^*(\mathfrak{T}_\lambda, t_\lambda^! \mathcal{F}) \cong H_A^*((\{\lambda\} \hookrightarrow \mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda)^* s_\lambda^! \mathcal{F})(-2 \dim(V \cap \mathfrak{T}_\lambda)).$$

Again by [21, Proposition 2.3], we have a canonical isomorphism

$$H_A^*(\mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda, s_\lambda^! \mathcal{F}) \cong H_A^*((\{\lambda\} \hookrightarrow \mathfrak{W}_\lambda \cap \mathfrak{T}_\lambda)^* s_\lambda^! \mathcal{F}).$$

Using the first isomorphism in (2.2.2), we obtain finally an isomorphism

$$H_A^*(s_\lambda^! \mathcal{F}) \cong (\mathbb{S}(\mathcal{F}))_\lambda \otimes \mathbf{S}_\hbar(\lambda(2\check{\rho}) + 2 \dim(V \cap \mathfrak{T}_\lambda)).$$

Now the dimension $\dim(V \cap \mathfrak{T}_\lambda)$ can be computed using (8.4.4), and (1) follows.

To prove (2) we have to understand the natural morphism

$$H_A^A(\{\lambda\}) \rightarrow H_A^A(V \cap \mathfrak{T}_\lambda)(-2 \dim(V \cap \mathfrak{T}_\lambda)).$$

However this morphism factorizes as the following composition:

$$H_A^A(\{\lambda\}) \rightarrow H_A^A(\mathbf{Gr}^\lambda \cap \mathfrak{T}_\lambda)(-2 \dim(\mathbf{Gr}^\lambda \cap \mathfrak{T}_\lambda)) \xrightarrow{\sim} H_A^A(V \cap \mathfrak{T}_\lambda)(-2 \dim(V \cap \mathfrak{T}_\lambda)),$$

where the first morphism is induced by the inclusion $\{\lambda\} \hookrightarrow \mathbf{Gr}^\lambda \cap \mathfrak{T}_\lambda$, and the second morphism is given by restriction to the open subvariety $V \cap \mathfrak{T}_\lambda$. Now the first morphism can be computed using (8.4.4) and the reminder in §6.2, and the result follows. \square

8.5. Geometric realization of dynamical Weyl groups

Now we are in a position to prove Proposition 2.6.2.

Proof of Proposition 2.6.2. Fix \mathcal{F} in $\text{Perv}_{\check{G}(\mathbf{O})}(\mathbf{Gr})$, and set $V := \mathbb{S}(\mathcal{F})$. Then it is enough to prove that for any simple root α and any $\lambda \in \mathbb{X}$ such that $\lambda(\check{\alpha}) \geq 0$ the

following diagram commutes, where the vertical isomorphisms are induced by those of Lemma 8.4.1(1) and where $s := s_\alpha$:

$$\begin{CD} \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_{\lambda} \cap \mathfrak{T}_{\lambda}, s_{\lambda}^! \mathcal{F}) @>DW_{\mathcal{F}, \lambda, s}^{\text{geom}}>> \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_{s\lambda} \cap \mathfrak{T}_{s\lambda}, s_{s\lambda}^! \mathcal{F}) \\ @V\wr VV @VV\wr V \\ \mathbb{Q}_{\hbar} \otimes V_{\lambda} @>DW_{V, \lambda, s}^{\text{alg}}>> \mathbb{Q}_{\hbar} \otimes V_{s\lambda}. \end{CD}$$

Consider the isomorphism of $\mathbb{C}(\hbar)$ -modules

$$'DW_{\mathcal{F}, \lambda, s}^{\text{geom}} : \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) \xrightarrow{\sim} \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{s\lambda}, t_{s\lambda}^! \mathcal{F})$$

defined by the composition

$$\begin{aligned} \mathbb{Q}_{\hbar} \otimes H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) &\xleftarrow{\sim} \mathbb{Q}_{\hbar} \otimes H_A^*(i_{\lambda}^! \mathcal{F}) \xrightarrow{\sim} \mathbb{Q}_{\hbar} \otimes {}^s H_A^*(i_{s\lambda}^! \mathcal{F}) \\ &\xrightarrow{\sim} \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} {}^s H_A^*(\mathfrak{T}_{s\lambda}, t_{s\lambda}^! \mathcal{F}) \xrightarrow{\sim} \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{s\lambda}, t_{s\lambda}^! \mathcal{F}). \end{aligned}$$

(Here the first and third arrows are invertible by the localization theorem in equivariant cohomology.) Using the notation of § 8.3, it follows from Theorems 2.2.4 and 2.5.5 that the following diagram commutes:

$$\begin{CD} \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F}) @>DW_{\mathcal{F}, \lambda, s}^{\text{geom}}>> \mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{s\lambda}, t_{s\lambda}^! \mathcal{F}) \\ @V\wr(2.2.2)VV @VV(2.2.2)\wr V \\ \mathbb{Q}_{\hbar} \otimes V_{\lambda} @>DW_{V, \lambda, s}^{\text{alg}}>> \mathbb{Q}_{\hbar} \otimes V_{s\lambda}. \end{CD}$$

As explained in the proof of Lemma 8.3.1, for any $x \in \mathbb{Q}_{\hbar} \otimes V_{\lambda}$ we have

$$'DW_{V, \lambda, s}^{\text{alg}}(x) = (-\check{\alpha})(-\check{\alpha} + \hbar) \cdots (-\check{\alpha} + (\lambda(\check{\alpha}) - 1)\hbar) \cdot DW_{V, \lambda, s}^{\text{alg}}(x)$$

in $\mathbb{Q}_{\hbar} \otimes V_{s\lambda}$. On the other hand, it follows from Lemma 8.4.1(2) that if we identify the \mathbb{Q}_{\hbar} -modules $\mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{\lambda}, t_{\lambda}^! \mathcal{F})$ and $\mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_{\lambda} \cap \mathfrak{T}_{\lambda}, s_{\lambda}^! \mathcal{F})$ with $\mathbb{Q}_{\hbar} \otimes V_{\lambda}$, and $\mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{T}_{s\lambda}, t_{s\lambda}^! \mathcal{F})$ and $\mathbb{Q}_{\hbar} \otimes_{S_{\hbar}} H_A^*(\mathfrak{W}_{s\lambda} \cap \mathfrak{T}_{s\lambda}, s_{s\lambda}^! \mathcal{F})$ with $\mathbb{Q}_{\hbar} \otimes V_{s\lambda}$ by the isomorphisms of Lemma 2.2.1(2) and Lemma 8.4.1(1), we have for any $x \in \mathbb{Q}_{\hbar} \otimes V_{\lambda}$

$$'DW_{V, \lambda, s}^{\text{geom}}(x) = (-\check{\alpha})(-\check{\alpha} + \hbar) \cdots (-\check{\alpha} + (\lambda(\check{\alpha}) - 1)\hbar) \cdot DW_{V, \lambda, s}^{\text{geom}}(x)$$

in $\mathbb{Q}_{\hbar} \otimes V_{s\lambda}$. The proposition follows. □

8.6. Brylinski–Kostant filtration

We have a natural isomorphism of algebras $S(\mathfrak{g}/\mathfrak{u}) = \mathbb{C}[(\mathfrak{g}/\mathfrak{u})^*]$ so for any $\varphi \in \mathfrak{t}^*$ there is a natural surjective algebra morphism

$$S(\mathfrak{g}/\mathfrak{u}) \rightarrow \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \tag{8.6.1}$$

associated with the inclusion $\varphi + (\mathfrak{g}/\mathfrak{b})^* \hookrightarrow (\mathfrak{g}/\mathfrak{u})^*$. (Here, as usual we consider φ as a linear form on \mathfrak{g} trivial on $\mathfrak{u} \oplus \mathfrak{u}^-$.)

Lemma 8.6.2. *Let V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$. For any $\varphi \in \mathfrak{t}_{\text{rs}}^*$, the morphism*

$$\mathbb{C}_\varphi \otimes_{\mathbb{S}(\mathfrak{t})} (V \otimes \mathbb{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B \rightarrow (V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B$$

induced by (8.6.1) is an isomorphism.

Proof. We have

$$\begin{aligned} \mathbb{C}_\varphi \otimes_{\mathbb{S}(\mathfrak{t})} (V \otimes \mathbb{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B &\cong \mathbb{C}_\varphi \otimes_{\mathbb{C}[\mathfrak{t}_{\text{rs}}^*]} \mathbb{C}[\mathfrak{t}_{\text{rs}}^*] \otimes_{\mathbb{S}(\mathfrak{t})} (V \otimes \mathbb{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda})^B \\ &\cong \mathbb{C}_\varphi \otimes_{\mathbb{C}[\mathfrak{t}_{\text{rs}}^*]} \left(V \otimes (\mathbb{C}[\mathfrak{t}_{\text{rs}}^*] \otimes_{\mathbb{S}(\mathfrak{t})} \mathbb{S}(\mathfrak{g}/\mathfrak{u}) \otimes \mathbb{C}_{-\lambda}) \right)^B \\ &\stackrel{(7.2.2)}{\cong} \mathbb{C}_\varphi \otimes_{\mathbb{C}[\mathfrak{t}_{\text{rs}}^*]} (V \otimes \mathbb{C}[\mathfrak{t}_{\text{rs}}^*] \otimes \text{Ind}_T^B(-\lambda))^B \\ &\cong (V \otimes \mathbb{C}_{-\lambda} \otimes \mathbb{C}[B/T])^B. \end{aligned}$$

On the other hand we have $\varphi + (\mathfrak{g}/\mathfrak{b})^* = B \cdot \varphi \cong B/T$ as a B -variety, and hence there is a natural isomorphism of B -modules $\mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \cong \mathbb{C}[B/T]$. This implies our claim. \square

Let us now fix $\lambda \in \mathbb{X}$ and \mathcal{F} in $\text{Perv}_{\check{G}(\mathcal{O})}(\text{Gr})$. To simplify notation we set $V := \mathbb{S}(\mathcal{F})$.

Recall that for $\varphi \in \mathfrak{t}^*$ we have defined the filtered vector space $H_\varphi(i_\lambda^! \mathcal{F})$ in § 2.6. On the other hand, the algebra $\mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*]$ is also naturally filtered: the filtration is defined such that for any $\psi \in \varphi + (\mathfrak{g}/\mathfrak{b})^*$, the algebra isomorphism $\mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \cong \mathbb{C}[(\mathfrak{g}/\mathfrak{b})^*]$ induced by the isomorphism

$$(\mathfrak{g}/\mathfrak{b})^* \xrightarrow{\sim} \varphi + (\mathfrak{g}/\mathfrak{b})^*, \quad f \mapsto \psi + f$$

is an isomorphism of *filtered* algebras, where the filtration on $\mathbb{C}[(\mathfrak{g}/\mathfrak{b})^*]$ is the one induced by the grading such that the vectors in $\mathfrak{g}/\mathfrak{b} \subset \mathbb{C}[(\mathfrak{g}/\mathfrak{b})^*]$ are in degree 2. Hence we have an induced filtration on the vector space $(V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B$.

Combining Theorem 2.3.1 and Lemma 8.6.2, one obtains the following.

Corollary 8.6.3. *For any $\varphi \in \mathfrak{t}_{\text{rs}}^*$, there exists a canonical isomorphism*

$$H_\varphi(i_\lambda^! \mathcal{F}) \cong (V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B.$$

This isomorphism is an isomorphism of filtered vector spaces, where the filtration on the right-hand side is shifted by $\lambda(2\check{\rho})$, i.e. for any $j \in \mathbb{Z}$ it restricts to an isomorphism

$$F_j(H_\varphi(i_\lambda^! \mathcal{F})) \cong F_{j-\lambda(2\check{\rho})} \left((V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B \right).$$

Now we are in a position to give a proof of Proposition 2.6.3. In fact, this proposition is a consequence of Corollary 8.6.3 and the following result (which is essentially proved in [15]; we reproduce the proof for the reader’s convenience).

Proposition 8.6.4. *Let $e \in \mathfrak{u}$ be a sum of non-zero simple root vectors. If $\varphi \in \mathfrak{t}^* \setminus \{0\}$ satisfies $(\text{ad}^* e)(\varphi) = 0$, then evaluation at φ induces an isomorphism*

$$(V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B \xrightarrow{\sim} V_\lambda. \tag{8.6.5}$$

For any $j \in \mathbb{Z}$, this isomorphism restricts to an isomorphism

$$F_{2j} \left((V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B \right) = F_{2j+1} \left((V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B \right) \xrightarrow{\sim} F_j^{\text{BK}}(V_\lambda).$$

Proof. Let us denote morphism (8.6.5) by Λ_λ . The fact that Λ_λ is an isomorphism follows from the proof of Lemma 8.6.2. What remains is to prove that this isomorphism is compatible with filtrations. Note that Λ_λ is the restriction to the λ -weight spaces of the isomorphism of T -modules

$$\Lambda : (V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*])^U \xrightarrow{\sim} V$$

given again by evaluation at φ . By construction, the filtration on the left-hand side of (8.6.5) has jumps only in even degrees, which justifies the equality $F_{2j} = F_{2j+1}$. For $j \in \mathbb{Z}$ we set

$$F_j^{\text{fib}}(V_\lambda) := \Lambda_\lambda \left(F_{2j} \left((V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B \right) \right).$$

Hence what we have to check is that F_j^{fib} coincides with F_j^{BK} .

It will be convenient to work in \mathfrak{g} rather than \mathfrak{g}^* . Hence we choose a G -equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$; it restricts to a B -equivariant isomorphism $(\mathfrak{g}/\mathfrak{u})^* \cong \mathfrak{b}$. Let $h \in \mathfrak{b}$ be the image of φ , so we obtain an identification $\varphi + (\mathfrak{g}/\mathfrak{b})^* \cong h + \mathfrak{u}$. Consider the isomorphisms

$$(V \otimes \mathbb{C}[\varphi + (\mathfrak{g}/\mathfrak{b})^*] \otimes \mathbb{C}_{-\lambda})^B \cong (V \otimes \mathbb{C}[h + \mathfrak{u}] \otimes \mathbb{C}_{-\lambda})^B \cong \text{Hom}^B(V^* \otimes \mathbb{C}_\lambda, \mathbb{C}[h + \mathfrak{u}]).$$

Under this identification, Λ_λ sends a morphism $f : V^* \otimes \mathbb{C}_\lambda \rightarrow \mathbb{C}[h + \mathfrak{u}]$ to the linear form on $(V_\lambda)^*$ given by $\psi \mapsto f(\psi \otimes 1)(h)$. In fact the image of $\Lambda_\lambda(f)$ in V is $\Lambda(f)$, which can be described as the linear form on V^* given by $\psi \mapsto f(\psi \otimes 1)(h)$.

Fix some $f \in \text{Hom}^B(V^* \otimes \mathbb{C}_\lambda, \mathbb{C}[h + \mathfrak{u}])$, and let $v = \Lambda_\lambda(f)$ be its image in V_λ (or in V). By definition, v is in $F_j^{\text{fib}}(V_\lambda)$ iff for any $x \in \mathfrak{u}$ and any $\psi \in V^*$, the polynomial in t given by

$$f(\psi \otimes 1)(h + tx)$$

has degree $\leq j$. Fix $\psi \in V^*$, and choose $x \in \mathfrak{u}$ such that this degree is maximal. By density of regular nilpotent elements, one can assume that x is regular. Then there exists $b \in B$ such that $x = b \cdot e$. (Indeed, it is well known that there exists $g \in G$ such that $x = g \cdot e$. Then $x \in g \cdot \mathfrak{b}$; as a regular nilpotent element is contained in only one Borel subalgebra we deduce that $\mathfrak{b} = g \cdot \mathfrak{b}$, which implies that $g \in B$.) And we have

$$f(\psi \otimes 1)(h + tx) = f(\psi \otimes 1)(b \cdot (b^{-1} \cdot h + te)) = f(b^{-1} \cdot (\psi \otimes 1))(b^{-1} \cdot h + te).$$

Now, by [15, Lemma 4.2], the degree of the polynomial on the right-hand side is the same as the degree of the polynomial $f(b^{-1} \cdot (\psi \otimes 1))(h + te)$. Moreover, we have

$$f(b^{-1} \cdot (\psi \otimes 1))(h + te) = f(b^{-1} \cdot (\psi \otimes 1))(\exp(te) \cdot h) = f(\exp(-te)b^{-1} \cdot (\psi \otimes 1))(h).$$

Hence, finally, v is in $F_j^{\text{fib}}(V_\lambda)$ iff for any $\psi \in V^*$, the polynomial

$$f(\exp(-te) \cdot (\psi \otimes 1))(h)$$

has degree $\leq j$.

Now, the linear form on V^* given by $\psi \mapsto f(\exp(-te) \cdot (\psi \otimes 1))(h)$ is $\exp(te) \cdot v$. Hence v is in $F_j^{\text{fib}}(V_\lambda)$ iff $e^{j+1} \cdot v = 0$, i.e. iff $v \in F_j^{\text{BK}}(V_\lambda)$. \square

8.7. The ordinary cohomology of cofibers

As a consequence of Corollary 2.4.5 we can prove the following result, which is equivalent to [3, Theorem 8.5.2]. (More precisely, [3, Theorem 8.5.2] also contains a claim about compatibility with convolution, which can be deduced from Proposition 8.1.5.)

Proposition 8.7.1. *For \mathcal{F} in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr})$ and $\lambda \in \mathbb{X}^+$, there exists a canonical isomorphism of graded vector spaces*

$$H^*(i_\lambda^! \mathcal{F}) \cong \left(\mathbb{S}(\mathcal{F}) \otimes \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \right)^G \langle \lambda(2\check{\rho}) \rangle.$$

Proof. By the arguments in the proof of Lemma 2.2.1 the forgetful functor induces an isomorphism

$$\mathbb{C} \otimes_{S(\mathfrak{t})} H_{\check{T}}^*(i_\lambda^! \mathcal{F}) \xrightarrow{\sim} H(i_\lambda^! \mathcal{F})$$

(where, in the left-hand side, $S(\mathfrak{t})$ acts trivially on \mathbb{C}). On the other hand, as observed in the proof of Lemma 3.6.2, restriction induces an isomorphism

$$\mathbb{C} \otimes_{S(\mathfrak{t})} \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)) \xrightarrow{\sim} \Gamma(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)).$$

Then the result follows from Corollary 2.4.5, using the fact that the functor of G -fixed points is exact. □

Remark 8.7.2. One can obtain in a similar way a description of $H_{\mathbb{C}^\times}^*(i_\lambda^! \mathcal{F})$ (in the case where λ is dominant) in terms of asymptotic \mathcal{D} -modules on \mathcal{B} . We omit the details.

8.8. The equivariant cohomology of spherical perverse sheaves

In this subsection we explain the relation between our description of the equivariant cohomology of cofibers of spherical perverse sheaves on $\text{Gr}_{\check{G}}$ and the description of the full equivariant cohomology of these perverse sheaves given in [9]. Details (and a generalization to all reductive groups) will be discussed in a future publication.

From now on, for simplicity we assume that \check{G} is quasi-simple and simply connected, so G is simple (of adjoint type). The Killing form determines an isomorphism of G -modules $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. Let us choose an element $e \in \mathfrak{u}$ which is a sum of non-zero simple root vectors. Then e is regular nilpotent, and it can be completed to an \mathfrak{sl}_2 -triple $(e, \check{\rho}, f)$. (Note that f is uniquely determined by e , and that the different choices for e are all conjugate under the adjoint action of T .) We consider the Kostant slice

$$\Sigma_e := \kappa(e + \mathfrak{g}_f) \subset \mathfrak{g}^*.$$

It is well known that Σ_e is included in \mathfrak{g}_r^* , and that the (co)adjoint quotient $\mathfrak{g}^* \rightarrow \mathfrak{g}^*/G \cong \mathfrak{t}^*/W$ restricts to an isomorphism $\Sigma_e \xrightarrow{\sim} \mathfrak{t}^*/W$. We denote by $\tilde{\Sigma}_e$ the inverse image of Σ_e under the projection $\pi : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}^*$. Then $\tilde{\Sigma}_e \subset \tilde{\mathfrak{g}}_r$, and the morphism $\tilde{\Sigma}_e \rightarrow \Sigma_e$ is W -equivariant, where W acts on $\tilde{\Sigma}_e$ by the restriction of the action on $\tilde{\mathfrak{g}}_r$, and trivially on Σ_e . It is also known that the natural morphism $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ restricts to an isomorphism of algebraic varieties $\tilde{\mathfrak{g}}_r \rightarrow \mathfrak{g}_r^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ (see [23, Remark 4.2.4(i)]). We deduce that the morphism $\delta : \tilde{\mathfrak{g}} \rightarrow \mathfrak{t}^*$ restricts to an isomorphism $\tilde{\Sigma}_e \xrightarrow{\sim} \mathfrak{t}^*$.

Lemma 8.8.1. *The projection $T^*\mathcal{X} \rightarrow \tilde{\mathfrak{g}}$ admits a canonical section $\omega_e : \tilde{\Sigma}_e \hookrightarrow T^*\mathcal{X}$ over $\tilde{\Sigma}_e$. This section is independent of the choice of e in the following sense: if $t \in T$ then the following diagram commutes:*

$$\begin{array}{ccc}
 \tilde{\Sigma}_e & \xrightarrow[\sim]{t \cdot (-)} & \tilde{\Sigma}_{t \cdot e} \\
 \downarrow \omega_e & & \downarrow \omega_{t \cdot e} \\
 T^*\mathcal{X} & \xrightarrow[\sim]{t \cdot (-)} & T^*\mathcal{X}.
 \end{array}$$

The morphism ω_e is also W -equivariant, if W acts on $(T^*\mathcal{X})_r$ via the action of § 5.5, for the choice of simple root vectors given by the components of e on each \mathfrak{g}_α .

Proof. Given a Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{g}$ with unipotent radical \mathfrak{u}_0 , the “universal Cartan subalgebra” $\mathfrak{b}_0/\mathfrak{u}_0$ acts on $(\mathfrak{u}_0/[\mathfrak{u}_0, \mathfrak{u}_0])^*$; under this action $(\mathfrak{u}_0/[\mathfrak{u}_0, \mathfrak{u}_0])^*$ decomposes as a direct sum of one-dimensional eigenspaces, and the eigenvalues can be naturally identified with the negative simple roots under the canonical isomorphism $\mathfrak{t}^* \cong (\mathfrak{b}_0/\mathfrak{u}_0)^*$. Following [24], we denote by $\mathbb{O}(\mathfrak{b}_0) \subset (\mathfrak{u}_0/[\mathfrak{u}_0, \mathfrak{u}_0])^*$ the subset of vectors whose component in each eigenspace is non-zero. We also set $\tilde{\mathcal{B}} := \{(\mathfrak{b}_0, x) \mid \mathfrak{b}_0 \in \mathcal{B}, x \in \mathbb{O}(\mathfrak{b}_0)\}$. Then our choice of simple root vectors defines a G -equivariant isomorphism $\mathcal{X} \xrightarrow{\sim} \tilde{\mathcal{B}}$ sending U/U to (\mathfrak{b}, ψ) , where $\psi := \kappa(f)|_{\mathfrak{u}} \in \mathbb{O}(\mathfrak{b})$.

To define the section we need, given some $\eta \in \Sigma_e$ and $\mathfrak{b}_0 \in \mathcal{B}$ with unipotent radical \mathfrak{u}_0 such that $\eta|_{\mathfrak{u}_0} = 0$, to define a lift of \mathfrak{b}_0 to \mathcal{X} , or equivalently to $\tilde{\mathcal{B}}$. However, by [24] the Borel subalgebras \mathfrak{b}_0 and \mathfrak{b} are in general relative position. Hence the Killing form induces a non-degenerate pairing between \mathfrak{u} and \mathfrak{u}_0 , and hence an isomorphism $\mathfrak{u} \xrightarrow{\sim} \mathfrak{u}_0^*$. If we denote by $\eta_0 \in \mathfrak{u}_0^*$ the image of e under this isomorphism, then one can check that $\eta_0 \in \mathbb{O}(\mathfrak{b}_0)$, and hence the pair (\mathfrak{b}_0, η_0) provides the desired lifting of \mathfrak{b}_0 to $\tilde{\mathcal{B}}$. \square

From now on we fix a choice for e . Let us denote by

$$i : \text{Gr}_{\tilde{T}} \hookrightarrow \text{Gr}_{\check{G}} \quad \text{and} \quad t : \text{Gr}_{\check{B}^-} \hookrightarrow \text{Gr}_{\check{G}}$$

the inclusions. We have $\text{Gr}_{\tilde{T}} = \bigsqcup_{\lambda \in \mathbb{X}} \{\lambda\}$, and i identifies with $\bigsqcup_{\lambda \in \mathbb{X}} i_\lambda$. Similarly one can identify $\text{Gr}_{\check{B}^-}$ with $\bigsqcup_{\lambda \in \mathbb{X}} \mathfrak{I}_\lambda$, and then t with $\bigsqcup_{\lambda \in \mathbb{X}} t_\lambda$. Let \mathcal{F} be in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr}_{\check{G}})$ and consider the diagram

$$H_{\tilde{T}}^*(\text{Gr}_{\check{B}^-}, t^!\mathcal{F}) \longleftarrow H_{\tilde{T}}^*(\text{Gr}_{\tilde{T}}, i^!\mathcal{F}) \longrightarrow H_{\tilde{T}}^*(\text{Gr}_{\check{G}}, \mathcal{F}).$$

Here the left-hand morphism is induced by the inclusion $\text{Gr}_{\tilde{T}} \hookrightarrow \text{Gr}_{\check{B}^-}$, and the right-hand morphism is induced by i as in § 2.2. By the localization theorem in equivariant cohomology, both morphisms become isomorphisms when we extend scalars from $S(\mathfrak{t})$ to its fraction field \mathbb{Q} , which provides a canonical isomorphism

$$\mathbb{Q} \otimes_{S(\mathfrak{t})} H_{\tilde{T}}^*(\text{Gr}_{\check{G}}, \mathcal{F}) \cong \mathbb{Q} \otimes_{S(\mathfrak{t})} H_{\tilde{T}}^*(\text{Gr}_{\tilde{T}}, t^!\mathcal{F}). \tag{8.8.2}$$

By Lemma 2.2.1 there exists a canonical isomorphism of $S(\mathfrak{t})$ -modules $H_{\tilde{T}}^*(t^!\mathcal{F}) \cong S(\mathcal{F}) \otimes S(\mathfrak{t})$; hence we obtain a canonical isomorphism

$$\mathbb{Q} \otimes H_{\tilde{T}}^*(\text{Gr}_{\check{G}}, \mathcal{F}) \xrightarrow{\sim} \mathbb{Q} \otimes S(\mathcal{F}). \tag{8.8.3}$$

On the other hand, let V in $\text{Rep}(G)$, and consider the diagram

$$V \otimes \mathbb{S}(\mathfrak{t}) \longleftarrow (V \otimes \mathbb{C}[T^*\mathcal{X}])^G \longrightarrow V \otimes \mathbb{C}[\tilde{\Sigma}_e].$$

Here the left-hand morphism is induced by an inverse image with respect to the natural morphism $G \times \mathfrak{t}^* \rightarrow T^*\mathcal{X}$, and the right-hand morphism is induced by an inverse image with respect to the morphism $\omega_e : \tilde{\Sigma}_e \hookrightarrow T^*\mathcal{X}$ of Lemma 8.8.1. One can check that both morphisms in this diagram become isomorphisms when we extend scalars from $\mathbb{S}(\mathfrak{t})$ to \mathbb{Q} , so we obtain a canonical isomorphism

$$\mathbb{Q} \otimes_{\mathbb{S}(\mathfrak{t})} (V \otimes \mathbb{C}[\tilde{\Sigma}_e]) \xrightarrow{\sim} \mathbb{Q} \otimes V. \tag{8.8.4}$$

As far as we understand, the isomorphism like our (8.8.2) which is implicitly used in [9] (see in particular the proof of Theorem 6 in *loc. cit.*) is the one that we are defining here. With this interpretation, the “quasi-classical limit” (or “classical analogue”) of [9, Theorem 6] says the following.

Proposition 8.8.5. *Let \mathcal{F} be in $\text{Perv}_{\check{G}(\mathbb{O})}(\text{Gr}_{\check{G}})$.*

The image of $H_{\check{T}}^(\text{Gr}_{\check{G}}, \mathcal{F})$ in $\mathbb{Q} \otimes \mathbb{S}(\mathcal{F})$ under isomorphism (8.8.3) coincides with the image of $\mathbb{S}(\mathcal{F}) \otimes \mathbb{C}[\tilde{\Sigma}_e]$ under isomorphism (8.8.4), which provides a canonical isomorphism*

$$H_{\check{T}}^*(\text{Gr}_{\check{G}}, \mathcal{F}) \cong \mathbb{S}(\mathcal{F}) \otimes \mathbb{C}[\tilde{\Sigma}_e]. \tag{8.8.6}$$

Combined with Lemma 2.2.1 and Corollary 2.4.5, this implies that we have the following commutative diagram:

$$\begin{array}{ccccc} H_{\check{T}}^*(\text{Gr}_{\check{B}^-}, i^! \mathcal{F}) & \longleftarrow & H_{\check{T}}^*(\text{Gr}_{\check{T}}, i^! \mathcal{F}) & \hookrightarrow & H_{\check{T}}^*(\text{Gr}_{\check{G}}, \mathcal{F}) \\ \text{Lem. 2.2.1} \downarrow \wr & & \text{Cor. 2.4.5} \downarrow \wr & & (8.8.6) \downarrow \wr \\ \mathbb{S}(\mathcal{F}) \otimes \mathbb{S}(\mathfrak{t}) & \longleftarrow & (\mathbb{S}(\mathcal{F}) \otimes \mathbb{C}[T^*\mathcal{X}])^G & \longrightarrow & \mathbb{S}(\mathcal{F}) \otimes \mathbb{C}[\tilde{\Sigma}_e]. \end{array}$$

Isomorphism (8.8.6) is W -equivariant, where the W -action on the left-hand side is defined similarly to how the isomorphisms $\Xi_w^{\mathcal{F}, \lambda}$ are defined in §2.5, and the action on the right-hand side is induced by the W -action on $\tilde{\Sigma}_e$. Hence taking fixed points we obtain the following result, also proved in [9] (see in particular [*loc. cit.*, Lemma 9]).

Corollary 8.8.7. *There exists a canonical isomorphism $H_{\check{G}(\mathbb{O})}^*(\text{Gr}_{\check{G}}, \mathcal{F}) \cong \mathbb{S}(\mathcal{F}) \otimes \mathbb{C}[\Sigma_e]$.*

A. Computations in rank 1

In §§A.1–A.5 we use the notation of Sections 3–5, assuming in addition that G has semisimple rank 1. We let α be the unique simple root, and set $s := s_\alpha$, $e := e_\alpha$, $f := f_\alpha$.

A.1. Asymptotic universal Verma modules

For any $\nu \in \mathbb{X}^+$, we denote by V^ν the corresponding simple G -module. We choose a basis $(v_\nu^\nu, v_{\nu-\alpha}^\nu, \dots, v_{\nu-\nu(\check{\alpha})\alpha}^\nu)$ of V^ν such that the following formulas are satisfied for

$k = 0, \dots, v(\check{\alpha})$:

$$\check{\alpha} \cdot v_{v-k\alpha}^v = (v(\check{\alpha}) - 2k)v_{v-k\alpha}^v, \quad e \cdot v_{v-k\alpha}^v = kv_{v-(k-1)\alpha}^v. \tag{A.1.1}$$

Such a basis exists and is unique up to a constant. We also set $v_\lambda^v = 0$ if λ is not a weight of V^v . The following commutation relation in the asymptotic enveloping algebra $U_{\hbar}(\mathfrak{g})$ is easily checked by induction:

$$f^k e = e f^k - k\hbar \cdot \check{\alpha} f^{k-1} - k(k-1)\hbar^2 \cdot f^{k-1}. \tag{A.1.2}$$

Lemma A.1.3. *Let $v \in \mathbb{X}^+$.*

- (1) *If λ is not a weight of V^v , then $(V^v \otimes \mathbf{M}(\lambda))^B = 0$.*
- (2) *If $k \in \{0, \dots, v(\check{\alpha})\}$, the S_{\hbar} -module $(V^v \otimes \mathbf{M}(v-k\alpha))^B$ is free of rank 1, and generated by*

$$\begin{aligned} x_{v-k\alpha}^v &:= v_v^v \otimes 1 \otimes f^k - \binom{k}{1} v_{v-\alpha}^v \otimes (\check{\alpha} + (v(\check{\alpha}) - k)\hbar) \otimes f^{k-1} \\ &+ \binom{k}{2} \cdot v_{v-2\alpha}^v \otimes (\check{\alpha} + (v(\check{\alpha}) - k)\hbar) (\check{\alpha} + (v(\check{\alpha}) - k - 1)\hbar) \otimes f^{k-2} + \dots \\ &+ (-1)^k \binom{k}{k} \cdot v_{v-k\alpha}^v \otimes (\check{\alpha} + (v(\check{\alpha}) - k)\hbar) \dots (\check{\alpha} + (v(\check{\alpha}) - 2k + 1)\hbar) \otimes 1. \end{aligned}$$

Proof.

- (1) follows from the injectivity of $\kappa_{V^v, \lambda}^{\text{alg}}$; see Lemma 4.6.4.
- (2) We have to decide when an element of the form

$$v_{v-k\alpha}^v \otimes P_k(\hbar, \check{\alpha}) \otimes 1 + \dots + v_v^v \otimes P_0(\hbar, \check{\alpha}) \otimes f^k$$

is annihilated by $e \in \mathfrak{b} \subset U(\mathfrak{b})$. (Here the action of $U(\mathfrak{b})$ is the differential of the B -action.) However, the image by e of such an element is given by

$$\begin{aligned} e \cdot v_{v-k\alpha}^v \otimes P_k \otimes 1 + \sum_{i=1}^{k-1} \left(e \cdot v_{v-i\alpha}^v \otimes P_i \otimes f^{k-i} - \frac{1}{\hbar} v_{v-i\alpha}^v \otimes P_i \otimes f^{k-i} e \right) \\ - \frac{1}{\hbar} v_v^v \otimes P_0 \otimes f^k e, \end{aligned}$$

i.e. (using (A.1.1) and (A.1.2)) by

$$\begin{aligned} kv_{v-(k-1)\alpha}^v \otimes P_k \otimes 1 + \sum_{i=1}^{k-1} \left((i v_{v-(i-1)\alpha}^v \otimes P_i \otimes \right. \\ \left. f^{k-i} + (k-i)v_{v-i\alpha}^v \otimes P_i (\check{\alpha} + (v(\check{\alpha}) - k - i)\hbar) \otimes f^{k-i-1} \right) \\ + kv_v^v \otimes P_0 (\check{\alpha} + (v(\check{\alpha}) - k)\hbar) \otimes f^{k-1}. \end{aligned}$$

Hence the fact that this element is zero amounts to the following equations:

$$\begin{aligned}
 k \cdot P_k &= -(\check{\alpha} + (v(\check{\alpha}) - 2k + 1)\hbar) \cdot P_{k-1}, \\
 &\vdots \\
 i \cdot P_i &= -(k - i + 1)(\check{\alpha} + (v(\check{\alpha}) - k - i + 1)\hbar) \cdot P_{i-1}, \\
 &\vdots \\
 P_1 &= -k(\check{\alpha} + (v(\check{\alpha}) - k)\hbar) \cdot P_0.
 \end{aligned}$$

The result follows. □

If $\lambda \in \mathbb{X}$ is not a weight of V^ν , we set $x_\lambda^\nu = 0$. We obtain as an immediate consequence of Lemma A.1.3 the following result.

Corollary A.1.4. *For $\nu \in \mathbb{X}^+$ and $k \in \{0, \dots, \nu(\check{\alpha})\}$, the image of the morphism*

$$\kappa_{V^\nu, \nu - k\alpha}^{\text{alg}} : (V^\nu \otimes \mathbf{M}(\nu - k\alpha))^B \rightarrow V_{\nu - k\alpha}^\nu \otimes \mathbf{S}_\hbar \cong \mathbf{S}_\hbar$$

is generated by $(\check{\alpha} + (v(\check{\alpha}) - k)\hbar) \cdots (\check{\alpha} + (v(\check{\alpha}) - 2k + 1)\hbar)$.

A.2. Operators Φ

In this subsection we consider the constructions of Section 3, with the choice of root vector f . In particular, for $\nu \in \mathbb{X}^+$ and $\lambda \in \mathbb{X}$, we have an isomorphism

$$\Phi_s^{V^\nu, \lambda} : (V^\nu \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \xrightarrow{\sim} {}^s(V^\nu \otimes {}^{(s\lambda)}\mathcal{D}_\hbar(\mathcal{X})_{s\lambda})^G.$$

Recall that, by the first isomorphism in Lemma 2.4.1, x_λ^ν defines an element

$$y_\lambda^\nu \in (V^\nu \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G.$$

Lemma A.2.1. *For $\nu \in \mathbb{X}^+$ and $\lambda \in \mathbb{X}$ we have*

$$\Phi_s^{V^\nu, \lambda}(y_\lambda^\nu) = y_{s\lambda}^\nu.$$

Proof. If λ is not a weight of V^ν , then $s\lambda$ is not either. Hence $y_\lambda^\nu = 0 = y_{s\lambda}^\nu$, and the result is clear.

Now assume that $\lambda = \nu - k\alpha$ for some $k \in \{0, \dots, \nu(\check{\alpha})\}$. As $\Phi_s^{V^\nu, \lambda}$ is an isomorphism of \mathbf{S}_\hbar -modules, we know that $\Phi_s^{V^\nu, \lambda}(y_\lambda^\nu) = c \cdot y_{s\lambda}^\nu$ for some $c \in \mathbb{C}^\times$, and we have to prove that $c = 1$. Let z_λ^ν be the image of y_λ^ν under the composition

$$(V^\nu \otimes {}^{(\lambda)}\mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \hookrightarrow V^\nu \otimes \mathcal{D}_\hbar(\mathcal{X}) \rightarrow V^\nu \otimes \mathcal{A}(\mathcal{X}),$$

and similarly for $z_{s\lambda}^\nu$. Then by definition (see § 3.5) we have $(\text{id}_{V^\nu} \otimes \mathcal{F}_\alpha)(z_\lambda^\nu) = c \cdot z_{s\lambda}^\nu$.

Since, in the notation of § 3.2, one can take as G^{sc} a product of $\text{SL}(2, \mathbb{C})$ and a torus, it suffices to prove the claim in the case $G = \text{SL}(2, \mathbb{C})$, with B the subgroup of upper

triangular matrices, T the subgroup of diagonal matrices, and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Let v_1, v_2 be the natural basis of \mathbb{C}^2 and η_1, η_2 the dual basis. Then the morphism $gU \mapsto g \cdot v_1$ induces an isomorphism $G/U \xrightarrow{\sim} \mathbb{C}^2 \setminus \{0\}$, and $\mathcal{V}_\alpha = \mathbb{C}^2$, with the symplectic form ω defined by $\omega(v_1, v_2) = 1$. Hence we have a canonical isomorphism $\mathcal{A}(\mathcal{X}) \cong \mathbb{C}[T^*(\mathbb{C}^2)] = \mathbb{C}[\mathbb{C}^2 \times (\mathbb{C}^2)^*]$, and the automorphism \mathcal{F}_α can be computed using Example B.2.1 below: it is defined by

$$\eta_1 \mapsto v_2, \quad \eta_2 \mapsto -v_1, \quad v_1 \mapsto -\eta_2, \quad v_2 \mapsto \eta_1.$$

Consider the morphism $\text{ev} : \mathcal{A}(\mathcal{X}) \rightarrow \mathbb{C}$ given by the evaluation of functions at $(v_1, \eta_2) \in T^*(\mathbb{C}^2)$. The formulas above imply that $\text{ev} \circ \mathcal{F}_\alpha = \text{ev}$. Hence to conclude, we only have to check that

$$(\text{id}_{V^\nu} \otimes \text{ev})(z_\lambda^\nu) = (\text{id}_{V^\nu} \otimes \text{ev})(z_{s\lambda}^\nu).$$

However we have

$$(\text{id}_{V^\nu} \otimes \text{ev})(z_\lambda^\nu) = v_\nu^\nu \otimes 1 = (\text{id}_{V^\nu} \otimes \text{ev})(z_{s\lambda}^\nu),$$

which finishes the proof. □

A.3. Operators Θ

In this subsection we consider the constructions of §4, with the choice $f_\alpha := f$. In particular, for $\nu \in \mathbb{X}^+$ and $\lambda \in \mathbb{X}$, we have a morphism

$$\Theta_s^{V^\nu, \lambda} : \text{Hom}_{(\mathfrak{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V^\nu \otimes \mathbf{M}(\lambda)) \rightarrow {}^s\text{Hom}_{(\mathfrak{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V^\nu \otimes \mathbf{M}(s\lambda)).$$

By Lemma 4.3.1, x_λ^ν defines a morphism of $(\mathfrak{S}_\hbar, U_\hbar(\mathfrak{g}))$ -bimodules

$$\varphi_\lambda^\nu : \mathbf{M}(0) \rightarrow V^\nu \otimes \mathbf{M}(\lambda),$$

which is a generator of the \mathfrak{S}_\hbar -module $\text{Hom}_{(\mathfrak{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V^\nu \otimes \mathbf{M}(\lambda))$.

Lemma A.3.1. *For $\nu \in \mathbb{X}^+$ and $\lambda \in \mathbb{X}$ we have*

$$\Theta_s^{V^\nu, \lambda}(\varphi_\lambda^\nu) = \varphi_{s\lambda}^\nu.$$

Proof. If λ is not a weight of V^ν , then $s\lambda$ is not either. Hence $\varphi_\lambda^\nu = 0 = \varphi_{s\lambda}^\nu$, and the result is clear.

Now assume that $\lambda = \nu - k\alpha$ for some $k \in \{0, \dots, \nu(\check{\alpha})\}$. Then $s\lambda = \nu - (\nu(\check{\alpha}) - k)\alpha$. By Lemma A.1.3 we know that $\Theta_s^{V^\nu, \nu - k\alpha}(\varphi_{\nu - k\alpha}^\nu)$ is a multiple of $\varphi_{\nu - (\nu(\check{\alpha}) - k)\alpha}^\nu$ (by an element of \mathfrak{S}_\hbar). Hence to prove the lemma it is enough to check that the coefficient of $v_\nu^\nu \otimes 1 \otimes f^{\nu(\check{\alpha}) - k}$ in the element of $(V^\nu \otimes \mathbf{M}(\nu - (\nu(\check{\alpha}) - k)\alpha))^B$ corresponding to $\Theta_s^{V^\nu, \nu - k\alpha}(\varphi_{\nu - k\alpha}^\nu)$ is the same as the coefficient of $v_\nu^\nu \otimes 1 \otimes f^{\nu(\check{\alpha}) - k}$ in $x_{\nu - (\nu(\check{\alpha}) - k)\alpha}^\nu$, i.e. 1. However this coefficient can be computed using formula (4.5.4), and the result is 1 as expected. This concludes the proof. □

Corollary A.3.2. *For any V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$ we have*

$${}^s(\Theta_s^{V, s\lambda}) \circ \Theta_s^{V, \lambda} = \text{id}$$

as endomorphisms of $\text{Hom}_{(\mathfrak{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda))$. In particular, each $\Theta_s^{V, \lambda}$ is an isomorphism.

Proof. By complete reducibility it is enough to prove the result when $V = V^\nu$ for some $\nu \in \mathbb{X}^+$. In this case, by Lemma A.3.1 we have

$$s(\Theta_s^{V^\nu, s\lambda}) \circ \Theta_s^{V^\nu, \lambda}(\varphi_\lambda^\nu) = \varphi_\lambda^\nu,$$

which proves the claim, since φ_λ^ν is a generator of $\text{Hom}_{(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V^\nu \otimes \mathbf{M}(\lambda))$ over \mathcal{S}_\hbar by Lemma A.1.3. □

A.4. Comparison of Θ and Φ

Lemma A.4.1. *For any V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$, the following diagram commutes:*

$$\begin{CD} (V \otimes^{(\lambda)} \mathcal{D}_\hbar(\mathcal{X})_\lambda)^G @>{(4.8.1)}>> \text{Hom}_{(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(\lambda)) \\ @V{\Phi_s^{V, \lambda}}VV @VV{\Theta_s^{V, \lambda}}V \\ s(V \otimes^{(s\lambda)} \mathcal{D}_\hbar(\mathcal{X})_{s\lambda})^G @>{(4.8.1)}>> s\text{Hom}_{(\mathcal{S}_\hbar, U_\hbar(\mathfrak{g}))}(\mathbf{M}(0), V \otimes \mathbf{M}(s\lambda)). \end{CD}$$

Proof. By complete reducibility it is enough to prove the lemma when $V = V^\nu$ for some $\nu \in \mathbb{X}^+$. In this case it follows from Lemma A.2.1 and Lemma A.3.1. □

A.5. Operators σ

If $\nu \in \mathbb{X}^+$ and $\lambda \in \mathbb{X}$, we denote by

$$z_\lambda^\nu \in (V^\nu \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G$$

the image of y_λ^ν under the natural morphism

$$(V^\nu \otimes^{(\lambda)} \mathcal{D}_\hbar(\mathcal{X})_\lambda)^G \rightarrow (V^\nu \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G$$

sending \hbar to 0. These elements can be naturally identified with the ones denoted similarly in the proof of Lemma A.2.1. By Lemma 3.5.2 and Lemma A.1.3, z_λ^ν is a generator of the $\mathcal{S}(\mathfrak{t})$ -module $(V^\nu \otimes \Gamma(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)))^G$, and $z_\lambda^\nu \neq 0$ iff λ is a weight of V^ν .

Lemma A.5.1. *For $\nu \in \mathbb{X}^+$ and $\lambda \in \mathbb{X}$ we have*

$$\sigma_s^{V^\nu, \lambda}(z_\lambda^\nu) = z_{s\lambda}^\nu.$$

Proof. The statement is clear if λ is not a weight of V^ν . Now assume that $\lambda = \nu - k\alpha$ for some $k \in \{0, \dots, \nu(\check{\alpha})\}$. As $\sigma_s^{V^\nu, \lambda}$ is an isomorphism of $\mathcal{S}(\mathfrak{t})$ -modules we must have $\sigma_s^{V^\nu, \lambda}(z_\lambda^\nu) = c \cdot z_{s\lambda}^\nu$ for some $c \in \mathbb{C}^\times$. Now recall the commutative diagram (5.2.4). We have

$$(\text{id}_{V^\nu} \otimes \text{ev}_{\eta_0}^\lambda)(z_\lambda^\nu) = v_\nu^\nu = (\text{id}_{V^\nu} \otimes \text{ev}_{\eta_0}^{s\lambda})(z_{s\lambda}^\nu).$$

This proves that $c = 1$, and finishes the proof. □

Corollary A.5.2. *Let V in $\text{Rep}(G)$ and $\lambda \in \mathbb{X}$. The following diagram commutes, where vertical maps are the natural morphisms sending \hbar to 0:*

$$\begin{CD} (V \otimes {}^{(\lambda)}\mathcal{D}_{\hbar}(\mathcal{X})_{\lambda})^G @>\Phi_s^{V,\lambda}>> (V \otimes {}^{(s\lambda)}\mathcal{D}_{\hbar}(\mathcal{X})_{s\lambda})^G \\ @VVV @VVV \\ (V \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(\lambda)))^G @>\sigma_s^{V,\lambda}>> (V \otimes \Gamma(\check{\mathfrak{g}}, \mathcal{O}_{\check{\mathfrak{g}}}(s\lambda)))^G \end{CD}$$

Proof. By complete reducibility it is enough to prove the claim when $V = V^{\nu}$ for some $\nu \in \mathbb{X}^+$. In this case it follows by comparing Lemma A.2.1 and Lemma A.5.1. \square

A.6. Satake equivalence

From now on we use the notation of § 6–8, assuming in addition that \check{G} has semisimple rank 1. We denote by α the unique positive coroot of \check{G} . Note that $\check{\alpha}$ and \hbar can be considered either as characters of $A = \check{T} \times \mathbb{C}^{\times}$ or as elements of $\mathfrak{a}^* = \mathfrak{t} \oplus \mathbb{C}$.

To simplify the statements of the next results we introduce the following notation. For $\lambda \in \mathbb{X}$ and $k \geq 1$ we define the A -module

$$V_k^{\lambda} := \mathbb{C}_{-\check{\alpha} - (\lambda(\check{\alpha})+1)\hbar} \oplus \mathbb{C}_{-\check{\alpha} - (\lambda(\check{\alpha})+2)\hbar} \oplus \cdots \oplus \mathbb{C}_{-\check{\alpha} - (\lambda(\check{\alpha})+k)\hbar}.$$

We also set $V_0^{\lambda} = \{0\}$. Note that $\dim(V_k^{\lambda}) = k$, and that there are natural inclusions $V_k^{\lambda} \subset V_{k+1}^{\lambda}$.

Lemma A.6.1. *Let $\lambda \in \mathbb{X}$.*

(1) *Assume that $\lambda(\check{\alpha}) \geq 0$. For $\nu \in \mathbb{X}^+$ we have isomorphisms of A -varieties*

$$\mathfrak{T}_{\lambda} \cap \text{Gr}_{\check{G}}^{\nu} \cong \begin{cases} \{\lambda\} & \text{if } \nu = \lambda; \\ V_k^{\lambda} \setminus V_{k-1}^{\lambda} & \text{if } \nu = \lambda + k\alpha \text{ for some } k \in \mathbb{Z}_{>0}; \\ \emptyset & \text{otherwise.} \end{cases}$$

(2) *Assume that $\lambda(\check{\alpha}) < 0$. For $\nu \in \mathbb{X}^+$ we have isomorphisms of A -varieties*

$$\mathfrak{T}_{\lambda} \cap \text{Gr}_{\check{G}}^{\nu} \cong \begin{cases} V_{-\lambda(\check{\alpha})}^{\lambda} & \text{if } \nu = \lambda + (-\lambda(\check{\alpha}))\alpha; \\ V_k^{\lambda} \setminus V_{k-1}^{\lambda} & \text{if } \nu = \lambda + k\alpha \text{ for some } k \in \mathbb{Z}_{>-\lambda(\check{\alpha})}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Each connected component of $\text{Gr}_{\check{G}}$ is isomorphic to a connected component of $\text{Gr}_{\text{PGL}(2, \mathbb{C})}$, and the action of $\ker(\check{\alpha}) = Z(\check{G}) \subset \check{T}$ is trivial. Hence it is enough to prove the isomorphism when $\check{G} = \text{PGL}(2, \mathbb{C})$. In this case we can identify \mathbb{X} with \mathbb{Z} through $\mu \mapsto \mu(\check{\alpha})$, so ν and λ can be considered as integers.

The first case: even weights. Write $\lambda = 2\ell$ with $\ell \in \mathbb{Z}$. Then λ is the class of the matrix

$$\begin{pmatrix} z^{\ell} & 0 \\ 0 & z^{-\ell} \end{pmatrix},$$

and \mathfrak{T}_λ is given by the classes of matrices of the form

$$M(Q) = \begin{pmatrix} z^\ell & 0 \\ Q(z) & z^{-\ell} \end{pmatrix}$$

where $Q(z) \in z^{-\ell-1}\mathbb{C}[z^{-1}]$. If $Q(z) = 0$, then this point is in \mathbf{Gr}^λ . Otherwise, write $Q(z) = az^{-m} + \dots$, where $a \neq 0$, $m > \ell$, and “ \dots ” means terms of degree between $-m + 1$ and $-\ell - 1$. Assume first that $m + \ell \geq 0$. (This condition is always satisfied when $\ell \geq 0$. Note also that it implies that $m \geq 0$.) Then we have

$$\begin{pmatrix} z^\ell & 0 \\ Q(z) & z^{-\ell} \end{pmatrix} \cdot \begin{pmatrix} z^{m-\ell} & R(z) \\ -z^m Q(z) & 0 \end{pmatrix} = \begin{pmatrix} 1 & z^{m+\ell} R(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z^m & 0 \\ 0 & z^{-m} \end{pmatrix}$$

where $R(z) \in \mathbf{O}$ is the inverse to $z^m Q(z)$. This equality implies that $M(Q) \in \mathbf{Gr}^{2m}$.

If $\ell < 0$, then we also have to consider the case where $\ell < m < -\ell$. However, in this case $M(Q)$ is in $\check{\mathbf{G}}(\mathbf{O}) \cdot \lambda = \mathbf{Gr}^{\lambda+(-\lambda(\check{\alpha}))\alpha}$. This settles the first case.

The second case: odd weights. Write $\lambda = 2\ell + 1$ with $\ell \in \mathbb{Z}$. Then λ is the class of the matrix

$$\begin{pmatrix} z^{\ell+1} & 0 \\ 0 & z^{-\ell} \end{pmatrix},$$

and \mathfrak{T}_λ is given by the classes of matrices of the form

$$N(Q) = \begin{pmatrix} z^{\ell+1} & 0 \\ Q(z) & z^{-\ell} \end{pmatrix}$$

where $Q(z) \in x^{-\ell-1}\mathbb{C}[z^{-1}]$. If $Q(z) = 0$, then this point is in \mathbf{Gr}^λ . Otherwise, write as above $Q(z) = az^{-m} + \dots$, where $a \neq 0$ and $m > \ell$. We have the following equality:

$$\begin{pmatrix} z^{\ell+1} & 0 \\ Q(z) & z^{-\ell} \end{pmatrix} \cdot \begin{pmatrix} z^{m-\ell} & R(z) \\ -z^m Q(z) & 0 \end{pmatrix} = \begin{pmatrix} 1 & z^{\ell+m+1} R(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z^{m+1} & 0 \\ 0 & z^{-m} \end{pmatrix}$$

where as above $R(z) \in \mathbf{O}$ is the inverse to $z^m Q(z)$. This equality implies that $N(Q) \in \mathbf{Gr}^{2m+1}$ if $\ell + m + 1 \geq 0$.

If $\ell < 0$ and $\ell + m + 1 < 0$, then $N(Q)$ is in $\check{\mathbf{G}}(\mathbf{O}) \cdot \lambda = \mathbf{Gr}^{\lambda+(-\lambda(\check{\alpha}))\alpha}$. This settles the second case, and finishes the proof. □

The following result is a direct consequence of Lemma A.6.1. It is stated without proof in [9, 12].

Corollary A.6.2. *Let $v \in \mathbb{X}^+$.*

- (1) If $\lambda \in \mathbb{X}$ is not of the form $\nu - k\alpha$ for some $k \in \{0, \dots, \nu(\check{\alpha})\}$, then $\mathfrak{T}_\lambda \cap \overline{\text{Gr}}_G^\nu = \emptyset$.
- (2) If $k \in \{0, \dots, \nu(\check{\alpha})\}$ and $\lambda = \nu - k\alpha$, then there exists an isomorphism of A -varieties

$$\mathfrak{T}_\lambda \cap \overline{\text{Gr}}_G^\nu \cong V_k^{\nu-k\alpha}$$

sending λ to 0.

By the constructions of § 6.1 we have a dual group G (which is also of semisimple rank 1) and a maximal torus $T \subset G$ such that $\mathbb{X} = X^*(T)$. If $\nu \in \mathbb{X}^+$, we set $\text{IC}_\nu := \text{IC}(\overline{\text{Gr}}_G^\nu, \mathbb{C}_{\overline{\text{Gr}}_G^\nu})$.

Corollary A.6.3. For $\nu \in \mathbb{X}^+$ and $k \in \{0, \dots, \nu(\check{\alpha})\}$, the image of the morphism

$$\kappa_{\text{IC}_\nu, \nu-k\alpha}^{\text{top}} : \mathbf{H}_A^!(i_{\nu-k\alpha}^! \text{IC}_\nu) \rightarrow (\mathbb{S}_{\check{G}}(\text{IC}_\nu))_{\nu-k\alpha} \otimes \mathbf{S}_\hbar \cong \mathbf{S}_\hbar$$

is generated by $(\check{\alpha} + (\nu(\check{\alpha}) - k)\hbar) \cdots (\check{\alpha} + (\nu(\check{\alpha}) - 2k + 1)\hbar)$.

Proof. It is well known that $\overline{\text{Gr}}_G^\nu$ is rationally smooth. This implies that

$$\text{IC}_\nu \cong \mathbb{C}_{\overline{\text{Gr}}_G^\nu}[\nu(\check{\alpha})] \cong \mathbb{D}_{\overline{\text{Gr}}_G^\nu}[-\nu(\check{\alpha})].$$

We deduce isomorphisms

$$\mathbf{H}_A^!(i_{\nu-k\alpha}^! \text{IC}_\nu) \cong \mathbf{H}_{A, -\nu(\check{\alpha})}^A(\{\lambda\}), \quad \mathbf{H}_A^!(i_{\nu-k\alpha}^! \text{IC}_\nu) \cong \mathbf{H}_{A, -\nu(\check{\alpha})+2k}^A(\mathfrak{T}_{\nu-k\alpha} \cap \overline{\text{Gr}}_G^\nu),$$

and the morphism $\mathbf{H}_A^!(i_{\nu-k\alpha}^! \text{IC}_\nu) \rightarrow \mathbf{H}_A^!(i_{\nu-k\alpha}^! \text{IC}_\nu)$ identifies with the morphism

$$\mathbf{H}_{A, -\nu(\check{\alpha})}^A(\{\lambda\}) \rightarrow \mathbf{H}_{A, -\nu(\check{\alpha})+2k}^A(\mathfrak{T}_{\nu-k\alpha} \cap \overline{\text{Gr}}_G^\nu)$$

given by the proper push-forward in the equivariant Borel–Moore homology. Hence we deduce the result from the description of $\mathfrak{T}_{\nu-k\alpha} \cap \overline{\text{Gr}}_G^\nu$ in Corollary A.6.2 and the considerations on equivariant (co)homology in § 6.2. \square

A.7. Proof of Theorem 2.2.4 for \check{G}

By semisimplicity of the category $\text{Perv}_{\check{G}(\mathbf{O})}(\text{Gr})$, it is enough to prove the theorem when $\mathcal{F} = \text{IC}_\nu$ for some $\nu \in \mathbb{X}^+$. Then $V^\nu := \mathbb{S}_{\check{G}}(\text{IC}_\nu)$ is the G -module with highest weight ν . Comparing Corollary A.1.4 and Corollary A.6.3 we observe that the images of $\kappa_{\text{IC}_\nu, \lambda}^{\text{top}}$ and $\kappa_{V^\nu, \lambda}^{\text{alg}}$ do indeed coincide, which implies the existence of the isomorphism $\zeta_{\text{IC}_\nu, \lambda}$.

A.8. The root vector and the Mirković–Vilonen basis

We denote by $e \in \mathfrak{g}_\alpha$ the vector constructed in § 6.5.

Let $\nu \in \mathbb{X}^+$ and $k \in \{0, \dots, \nu(\check{\alpha})\}$. Let $V^\nu := \mathbb{S}_{\check{G}}(\text{IC}_\nu)$, a simple G -module with highest weight ν . As in the proof of Corollary A.6.3, we have canonical isomorphisms

$$V_{\nu-k\alpha}^\nu \stackrel{(6.1.2)}{\cong} \mathbf{H}^{\nu(\check{\alpha})-2k}(i_{\nu-k\alpha}^! \text{IC}_\nu) \cong \mathbf{H}_0^A(\mathfrak{T}_{\nu-k\alpha} \cap \overline{\text{Gr}}_G^\nu).$$

The right-hand side is one-dimensional, and has a canonical generator, namely the fundamental class $[\mathfrak{T}_{\nu-k\alpha} \cap \overline{\text{Gr}}_G^\nu]$. We denote by $v_{\nu-k\alpha}^\nu \in V_{\nu-k\alpha}^\nu$ the vector corresponding to this generator.

Remark A.8.1. Be aware that the basis that we consider here is not the same as the one used in [9, § 5.2] or in [12, § 2.2], but rather the basis which is dual in the sense of Poincaré duality.

The following lemma is a special case of [5, Théorème 2].

Lemma A.8.2. For any $\nu \in \mathbb{X}^+$ and $k \in \{0, \dots, \nu(\check{\alpha})\}$ we have

$$e \cdot v_{\nu-k\alpha}^\nu = kv_{\nu-(k-1)\alpha}^\nu.$$

It follows from this lemma that, for the choice of root vector $e \in \mathfrak{g}_\alpha$, the basis just constructed satisfies conditions (A.1.1). Hence we can use the results and notation of §§ A.1–A.5 for this choice.

We can now give a more precise version of Theorem 2.2.4 for \check{G} . As in the proof of Corollary A.6.3 there exists a canonical isomorphism

$$H_A^*(i_{\nu-k\alpha}^! \mathbf{IC}_\nu) \cong H_{\bullet, -\nu(\check{\alpha})}^A(\{\lambda\}).$$

The right-hand side has a canonical generator, namely the unique element

$$c_{\nu-k\alpha}^\nu \in H_A^{\nu(\check{\alpha})}(i_{\nu-k\alpha}^! \mathbf{IC}_\nu)$$

whose image in $H^{\nu(\check{\alpha})}(i_{\nu-k\alpha}^! \mathbf{IC}_\nu) \cong H_0(\{\lambda\})$ is the fundamental class $[\{\lambda\}]$. Then (see § 6.2) we have

$$\zeta_{\mathbf{IC}_\nu, \nu-k\alpha}(x_{\nu-k\alpha}^\nu) = c_{\nu-k\alpha}^\nu. \tag{A.8.3}$$

A.9. Proof of Theorem 2.5.5 for \check{G}

We have already proved in Lemma A.4.1 that the operators Φ and Θ match under the natural isomorphisms. Hence we only have to compare them with operators Ξ . Moreover, it is enough to prove the theorem in the case $\mathcal{F} = \mathbf{IC}_\nu$ for some $\nu \in \mathbb{X}^+$. By construction (see § 2.5), we have $\Xi_s^{\mathbf{IC}_\nu, \nu-k\alpha}(c_{\nu-k\alpha}^\nu) = c_{\nu-(\nu(\check{\alpha})-k)\alpha}^\nu$. Hence the claim follows from (A.8.3) and Lemma A.2.1 or Lemma A.3.1.

B. The Fourier transform for differential operators

In this appendix we briefly explain how to adapt some classical constructions of Fourier transform for \mathcal{D} -modules (see e.g. [16]) to the asymptotic setting.

B.1. The partial Fourier transform

Let X be a smooth complex algebraic variety, and let $p : E \rightarrow X$ be a rank r algebraic vector bundle. Let $\check{p} : E^* \rightarrow X$ be the dual vector bundle, and let $\mathfrak{E} := E \times_X E^*$ be the total space of the direct sum $E \oplus E^*$, a vector bundle on X of rank $2r$. The canonical pairing of E and E^* gives a regular function $\mathbf{f} : E \times_X E^* \rightarrow \mathbb{C}$. We define a connection $\nabla : \mathcal{O}_{\mathfrak{E}} \rightarrow \Omega_{\mathfrak{E}}^1$ by $\nabla = d - d\mathbf{f}$. This connection is flat and makes $\mathcal{P} = (\mathcal{O}_{\mathfrak{E}}, \nabla)$ a holonomic left $\mathcal{D}_{\mathfrak{E}}$ -module, with *irregular singularities*. Explicitly, we have $\mathcal{P} = \mathcal{D}_{\mathfrak{E}}/\mathcal{J}$ where $\mathcal{J} \subset \mathcal{D}_{\mathfrak{E}}$ is the left ideal generated by the elements $\xi - \xi(\mathbf{f})$ for all vector fields ξ on \mathfrak{E} .

(Note that this ideal is the annihilator of the function $\exp \mathbf{f}$ on \mathfrak{E} , the classical kernel for the Fourier transform.)

Now, let $\epsilon : \mathfrak{E} = E \times_X E^* \hookrightarrow E \times E^*$ be the natural closed embedding and let $\epsilon_* \mathcal{P}$ be a direct image of the \mathcal{D} -module \mathcal{P} . Thus, $\epsilon_* \mathcal{P}$ is a holonomic left $\mathcal{D}_{E \times E^*}$ -module supported on the subvariety \mathfrak{E} . Write \mathcal{K}_Y for the canonical bundle on a smooth variety Y and let

$$\mathcal{Q} := (\mathcal{O}_E \boxtimes \check{p}^* \mathcal{K}_X) \otimes_{\mathcal{O}_{E \times E^*}} \epsilon_* \mathcal{P}.$$

The sheaf \mathcal{Q} has the structure of a module of the algebra

$$\mathcal{D}_E \boxtimes (\check{p}^* \mathcal{K}_X \otimes_{\mathcal{O}_{E^*}} \mathcal{D}_{E^*} \otimes_{\mathcal{O}_{E^*}} \check{p}^* \mathcal{K}_X^{-1}),$$

and this module has a canonical section $1_{\mathcal{Q}} \in \mathcal{Q}$ that corresponds to the section $1 \bmod \mathcal{J} \in \mathcal{D}_{\mathfrak{E}}/\mathcal{J}$. Furthermore, a local computation shows that \mathcal{Q} is a rank 1 free module (with generator $1_{\mathcal{Q}}$) over the ring $\mathcal{D}_E \boxtimes 1$, as well as over the ring $1 \boxtimes (\check{p}^* \mathcal{K}_X \otimes \mathcal{D}_{E^*} \otimes \check{p}^* \mathcal{K}_X^{-1})$.

Let $\bar{p} : \mathfrak{E} \rightarrow X$ be the natural projection. Then $\bar{p}_* \mathcal{Q}$ is a rank 1 free module both over $p_* \mathcal{D}_E$ and over $\mathcal{K}_X \otimes_{\mathcal{O}_X} \check{p}_* \mathcal{D}_{E^*} \otimes_{\mathcal{O}_X} \mathcal{K}_X^{-1}$, with a canonical generator $1_{\bar{p}_* \mathcal{Q}}$. Therefore, there is a uniquely determined morphism $F : p_* \mathcal{D}_E \rightarrow \mathcal{K}_X \otimes_{\mathcal{O}_X} \check{p}_* \mathcal{D}_{E^*} \otimes_{\mathcal{O}_X} \mathcal{K}_X^{-1}$ such that one has $u \cdot 1_{\bar{p}_* \mathcal{Q}} = F(u) \cdot 1_{\bar{p}_* \mathcal{Q}}$. It is immediately checked that this morphism is an anti-isomorphism of rings, i.e. it induces a ring isomorphism

$$p_* \mathcal{D}_E \xrightarrow{\sim} (\mathcal{K}_X \otimes_{\mathcal{O}_X} \check{p}_* \mathcal{D}_{E^*} \otimes_{\mathcal{O}_X} \mathcal{K}_X^{-1})^{\text{op}}. \tag{B.1.1}$$

On the other hand, we have $\mathcal{K}_{E^*} = \check{p}^*(\det(E) \otimes_{\mathcal{O}_X} \mathcal{K}_X)$, where $\det(E)$ denotes the sheaf of sections of the line bundle $\wedge^r E$ on X . Hence, using the well-known isomorphism $\mathcal{D}_{E^*}^{\text{op}} \cong \mathcal{K}_{E^*} \otimes_{\mathcal{O}_{E^*}} \mathcal{D}_{E^*} \otimes_{\mathcal{O}_{E^*}} \mathcal{K}_{E^*}^{-1}$, we compute

$$\begin{aligned} (\mathcal{K}_X \otimes_{\mathcal{O}_X} \check{p}_* \mathcal{D}_{E^*} \otimes_{\mathcal{O}_X} \mathcal{K}_X^{-1})^{\text{op}} &\cong \mathcal{K}_X^{-1} \otimes_{\mathcal{O}_X} \check{p}_*(\mathcal{D}_{E^*}^{\text{op}}) \otimes_{\mathcal{O}_X} \mathcal{K}_X \\ &\cong \mathcal{K}_X^{-1} \otimes_{\mathcal{O}_X} \check{p}_*(\mathcal{K}_{E^*} \otimes_{\mathcal{O}_{E^*}} \mathcal{D}_{E^*} \otimes_{\mathcal{O}_{E^*}} \mathcal{K}_{E^*}^{-1}) \otimes_{\mathcal{O}_X} \mathcal{K}_X \\ &\cong \det(E) \otimes_{\mathcal{O}_X} \check{p}_* \mathcal{D}_{E^*} \otimes_{\mathcal{O}_X} \det(E)^{-1}. \end{aligned}$$

Thus, from (B.1.1) we deduce a canonical isomorphism of sheaves of algebras on X , called the *Fourier isomorphism*:

$$\mathbf{F} : p_* \mathcal{D}_E \xrightarrow{\sim} \det(E) \otimes_{\mathcal{O}_X} (\check{p}_* \mathcal{D}_{E^*}) \otimes_{\mathcal{O}_X} \det(E)^{-1}.$$

Now, in a Rees algebra setting, we define \mathcal{J}_{\hbar} to be the left ideal of $\mathcal{D}_{\hbar, \mathfrak{E}}$ generated by the elements $\xi - \xi(\mathbf{f})$ for all vector fields ξ on \mathfrak{E} , viewed as degree 2 homogeneous elements of the graded algebra $\mathcal{D}_{\hbar, \mathfrak{E}}$. (Here we use the notational conventions of § 2.4. Note that the ideal \mathcal{J}_{\hbar} is the annihilator of the “function” $\exp(\frac{1}{\hbar} \mathbf{f})$, considered as an element of some completion of $\mathcal{O}_X[\hbar, \hbar^{-1}]$.) We put $\mathcal{P}_{\hbar} := \mathcal{D}_{\hbar, \mathfrak{E}}/\mathcal{J}_{\hbar}$, a $\mathcal{D}_{\hbar, \mathfrak{E}}$ -module. Note that the ideal \mathcal{J}_{\hbar} is not homogeneous, so the module \mathcal{P}_{\hbar} has no natural grading.

Under the specialization $\hbar = 0$, we have $\mathcal{D}_{\hbar, \mathfrak{E}}/(\hbar) = (p_{\mathfrak{E}})_* \mathcal{O}_{T^* \mathfrak{E}}$, where $p_{\mathfrak{E}} : T^* \mathfrak{E} \rightarrow \mathfrak{E}$ is the cotangent bundle. The differential of the function \mathbf{f} gives a section $d\mathbf{f} : \mathfrak{E} \rightarrow T^* \mathfrak{E}$. The image of this section is a smooth closed Lagrangian subvariety $\Lambda \subset T^* \mathfrak{E}$, so the sheaf $(p_{\mathfrak{E}})_* \mathcal{O}_{\Lambda}$ has a natural structure of a $(p_{\mathfrak{E}})_* \mathcal{O}_{T^* \mathfrak{E}}$ -module. Then, it follows from definitions that the projection $\mathcal{D}_{\hbar, \mathfrak{E}} \rightarrow \mathcal{P}_{\hbar}$ induces an isomorphism of $(p_{\mathfrak{E}})_* \mathcal{O}_{T^* \mathfrak{E}}$ -modules:

$$\mathcal{P}_{\hbar}/(\hbar) \cong (p_{\mathfrak{E}})_* \mathcal{O}_{\Lambda}.$$

We also consider the cotangent bundle $\mathbf{q} : T^*(E \times E^*) \rightarrow E \times E^*$ and let $T^*(E \times E^*)|_{\mathfrak{E}}$ denote the total space of the restriction of the cotangent bundle to $\mathfrak{E} \subset E \times E^*$, a closed

subvariety. One has a natural diagram

$$T^*E \times T^*(E^*) \xlongequal{\quad} T^*(E \times E^*) \xleftarrow{\quad \varepsilon \quad} T^*(E \times E^*)|_{\mathfrak{E}} \xrightarrow{\quad \text{pr} \quad} T^*\mathfrak{E}.$$

Here, the isomorphism on the left involves a sign and the map pr on the right is a smooth morphism.

The following result is easily verified by a local computation.

Lemma B.1.2. *The variety $Z := \varepsilon(\text{pr}^{-1}(\Lambda))$ is a smooth Lagrangian subvariety of $T^*(E \times E^*)$. Furthermore, this subvariety is the graph of an isomorphism $T^*E \xrightarrow{\sim} T^*(E^*)$, of algebraic varieties over X .*

To proceed further we observe that, for any smooth variety Y , the sheaf $\mathcal{K}_Y[\hbar]$ has a canonical right $\mathcal{D}_{\hbar,Y}$ -action such that a vector field $\xi \in \mathcal{T}_Y$ acts on \mathcal{K}_Y by $\beta \mapsto -\hbar \cdot L_{\xi} \beta$, where L_{ξ} stands for the Lie derivative. We write \mathcal{K}_Y^{\hbar} for the resulting right $\mathcal{D}_{\hbar,Y}$ -module. Then, one has a canonical isomorphism

$$\mathcal{D}_{\hbar,Y}^{op} \cong \mathcal{K}_Y^{\hbar} \otimes_{\mathcal{O}_Y[\hbar]} \mathcal{D}_{\hbar,Y} \otimes_{\mathcal{O}_Y[\hbar]} (\mathcal{K}_Y^{\hbar})^{-1}.$$

Note that this isomorphism specializes at $\hbar = 0$ to the identity map $(p_{\mathfrak{E}})_* \mathcal{O}_{T^*Y} \rightarrow (p_{\mathfrak{E}})_* \mathcal{O}_{T^*Y}$.

Next, mimicking the corresponding constructions for \mathcal{D} -modules, one can define a direct image $\epsilon_* \mathcal{P}_{\hbar}$, a left $\mathcal{D}_{\hbar,E \times E^*}$ -module. Further, we put $\mathcal{Q}_{\hbar} := (\mathcal{O}_E \boxtimes \check{p}^* \mathcal{K}_X^{\hbar}) \otimes_{\mathcal{O}_{E \times E^*}[\hbar]} \epsilon_* \mathcal{P}_{\hbar}$. Then, one checks that there is a natural isomorphism

$$\mathcal{Q}_{\hbar}/(\hbar) \cong \mathbf{q}_* \mathcal{O}_Z,$$

of $\mathbf{q}_* \mathcal{O}_{T^*(E \times E^*)}$ -modules. Furthermore, repeating earlier constructions, one obtains a canonical isomorphism

$$\mathbf{F}_{\hbar} : p_* \mathcal{D}_{\hbar,E} \xrightarrow{\sim} \det(E)[\hbar] \otimes_{\mathcal{O}_X[\hbar]} (\check{p}_* \mathcal{D}_{\hbar,E^*}) \otimes_{\mathcal{O}_X[\hbar]} \det(E)^{-1}[\hbar]. \tag{B.1.3}$$

This is an isomorphism of sheaves of $\mathbb{C}[\hbar]$ -algebras on X . This isomorphism does not respect the natural gradings on each side in (B.1.3) unless $r = 0$ and it specializes, at $\hbar = 0$, to the isomorphism $p_* \mathcal{O}_{T^*E} \xrightarrow{\sim} \check{p}_* \mathcal{O}_{T^*(E^*)}$ that results from Lemma B.1.2.

The above isomorphism can be described locally as follows. Let $U \subset X$ be an open subvariety over which E is trivializable, and let us choose an isomorphism of vector bundles $E|_U \cong \mathbb{C}^r \times U$. Then for $i = 1, \dots, r$ we have a function x_i on $E|_U$ given by the projection on the i th copy of \mathbb{C} , and the corresponding vector fields ∂_{x_i} , so we have an isomorphism of sheaves of $\mathbb{C}[\hbar]$ -algebras

$$(p_* \mathcal{D}_{\hbar,E})|_U \cong \mathcal{D}_{\hbar,U} \otimes_{\mathbb{C}[\hbar]} (\mathbb{C}\langle x_i, \partial_{x_i} \rangle / [\partial_{x_i}, x_i] = \hbar). \tag{B.1.4}$$

(Here i runs over $\{1, \dots, r\}$.) Our isomorphism $E|_U \cong \mathbb{C}^r \times U$ also defines a canonical section τ of $\det(E)$ over U , the dual section τ^{\vee} of $\det(E)^{-1}$, and an isomorphism $(E^*)|_U \cong \mathbb{C}^r \times U$, so we obtain functions ξ_1, \dots, ξ_r and vector fields $\partial_{\xi_1}, \dots, \partial_{\xi_r}$ on $(E^*)|_U$, and an isomorphism

$$(p_* \mathcal{D}_{\hbar,E^*})|_U \cong \mathcal{D}_{\hbar,U} \otimes_{\mathbb{C}[\hbar]} (\mathbb{C}\langle \xi_i, \partial_{\xi_i} \rangle / [\partial_{\xi_i}, \xi_i] = \hbar). \tag{B.1.5}$$

Then using isomorphisms (B.1.4) and (B.1.5), the restriction of isomorphism (B.1.3) to U can be described as follows: it sends any $P \in \mathcal{D}_{\hbar,U}$ to $\tau \otimes P \otimes \tau^{\vee}$, x_i to $\tau \otimes -\partial_{\xi_i} \otimes \tau^{\vee}$, and ∂_{x_i} to $\tau \otimes \xi_i \otimes \tau^{\vee}$.

B.2. The “symplectic” partial Fourier transform

Now we assume that E is a *symplectic* vector bundle with symplectic form ω over a smooth complex algebraic variety X . Then we have an isomorphism of vector bundles

$$E \xrightarrow{\sim} E^*, \quad v \mapsto \omega(v, -)$$

over X , and hence an induced isomorphism $p_*\mathcal{D}_{\hbar,E} \cong \check{p}_*\mathcal{D}_{\hbar,E^*}$. Moreover, ω defines a trivialization of $\det(E)$. Hence isomorphism (B.1.3) provides an automorphism of $p_*\mathcal{D}_{\hbar,E}$. We denote the induced automorphism by

$$\mathbf{F}_E : \mathcal{D}_{\hbar}(E) \xrightarrow{\sim} \mathcal{D}_{\hbar}(E).$$

One can easily check that \mathbf{F}_E is equivariant under the natural action of the group of symplectic automorphisms of E , and that we have $\mathbf{F}_E \circ \mathbf{F}_E = \text{id}_{\mathcal{D}_{\hbar}(E)}$.

Example B.2.1. If $X = \text{pt}$, then E is simply a symplectic vector space. For instance, assume that $E = \mathbb{C}^2 = \mathbb{C}v_1 \oplus \mathbb{C}v_2$, equipped with the symplectic form such that $\omega(v_1, v_2) = 1$. Let (η_1, η_2) be the basis of E^* dual to (v_1, v_2) . Then $\mathcal{D}_{\hbar}(E)$ is generated by η_1, η_2 (considered as functions on E) and v_1, v_2 (considered as vector fields on E), and \mathbf{F}_E is defined by

$$\eta_1 \mapsto v_2, \quad \eta_2 \mapsto -v_1, \quad v_1 \mapsto -\eta_2, \quad v_2 \mapsto \eta_1.$$

The following result (which can easily be checked using local trivializations) is used in §3.4.

Lemma B.2.2. *Let $f \in \mathbb{C}[X]$ be an invertible function, which we consider as a function on E via the projection $E \rightarrow X$. Then the automorphism of $\mathcal{D}_{\hbar}(E)$ given by $D \mapsto f^{-1} \cdot D \cdot f$ commutes with \mathbf{F}_E .*

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