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PRINCIPAL RADICAL SYSTEMS, LEFSCHETZ PROPERTIES, AND PERFECTION OF SPECHT IDEALS OF TWO-ROWED PARTITIONS

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Abstract. We show that the Specht ideal of a two-rowed partition is perfect over an arbitrary field, provided that the characteristic is either zero or bounded below by the size of the second row of the partition, and we show this lower bound is tight. We also establish perfection and other properties of certain variants of Specht ideals, and find a surprising connection to the weak Lefschetz property. Our results, in particular, give a self-contained proof of Cohen–Macaulayness of certain h-equals sets, a result previously obtained by Etingof–Gorsky–Losev over the complex numbers using rational Cherednik algebras.

§1. Introduction

Fix an integer m, let $\mathbb{F}$ be any field, and let $R = \mathbb{F}[x_1,\dots,x_m]$ be the polynomial ring with its standard grading, and equipped with the usual action of the symmetric group $\mathfrak{S}_m$ by permuting the variables. For any partition $\lambda \vdash m$, the Specht module $V(\lambda)$ over $\mathbb{F}$ is the $\mathbb{F}$-vector space generated by the Specht polynomials of λ which are indexed by the set of tableaux T on the Young diagram of λ. If $\mathbb{F}$ has characteristic zero, then the Specht modules form a complete list of irreducible $\mathfrak{S}_m$-representations, highlighting their importance in representation theory. In this paper, we take the point of view of commutative algebra, and study the ideals generated by Specht modules called Specht ideals.

Specifically, we show that for partitions with two parts $\lambda = (\lambda_1,\lambda_2)$ (or Young diagrams with two rows), the associated Specht ideal is radical and, if the characteristic of $\mathbb{F}$ is zero or sufficiently large, perfect. Our results are stated in terms of commutative algebra, but they can be interpreted geometrically as follows:

Proposition 1.1. *Fix a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = p \geq 0$, and fix a positive integer m. For each integer h satisfying $1 \leq h \leq m$, define the h-equals set $X_{m,h} \subset \mathbb{F}^m$ as the union of $\mathfrak{S}_m$ translates of the linear subspace cut out by the $h-1$ linear equations $x_1 = \cdots = x_h$, that is,*

$$X_{m,h} = \bigcup_{\sigma\in\mathfrak{S}_m} \sigma.\{(x_1,\dots,x_m)\in\mathbb{F}^m \mid x_1 = \cdots = x_h\}.$$

Assume that $2h \geq m+2$.

1. *If $p = 0$ or $p \geq m-h+1$, then $X_{m,h}$ is Cohen–Macaulay.*[1]
2. *If $m \geq 2p+2$, then $X_{m,m-p}$ is not Cohen–Macaulay.*

Over the field $\mathbb{F} = \mathbb{C}$, Proposition 1.1(1) was obtained by Etingof–Gorsky–Losev [1, Prop. 3.11], using deep results from the representation theory of rational Cherednik algebras. Our proof of Proposition 1.1 is more elementary in that it uses only basic results

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[1] Meaning that its homogeneous coordinate ring is Cohen–Macaulay.

from commutative algebra, which we hope will appeal to those uninitiated with rational Cherednik algebras. We emphasize, however, that our elementary proof is not easy.

The study of Specht ideals seems to have been initiated by Yanagawa in his recent paper [17], although they have appeared implicitly in the earlier works of others [1, 2, 4, 9]. In particular, the connection between the h-equals set in Proposition 1.1 and Specht ideals, as first observed by Li–Li [9] and rediscovered later by Yanagawa [17], is as follows: Setting $m=n+1$ and $h=n+1-k$, if $I_{m,h}\subset R$ is the ideal cutting out the h-equals set $X_{m,h}$, and if $2h\geq m+2$, then we have

$$I_{m,h}=\bigcap_{\sigma\in\mathfrak{S}_m}\sigma.(x_1-x_2,\dots,x_1-x_h)=\sqrt{\mathfrak{a}(n+1,k+1,k+1)}, \tag{1}$$

where $\mathfrak{a}(n+1,k+1,k+1)$ is the Specht ideal associated to the two-rowed partition $\lambda=(n-k,k+1)$. In his paper [17], Yanagawa proves that two-rowed Specht ideals are radical by an ingenious but complicated argument. He then invokes the Etingof–Gorsky–Losev result [1, Prop. 3.11] to prove that the Specht ideal $\mathfrak{a}(n+1,k+1,k+1)$ (although he used different notation) is perfect if the field has characteristic zero. The present paper grew out of an attempt to understand and simplify Yanagawa's arguments and to find an elementary proof of the Etingof–Gorsky–Losev result in the two-rowed case. As the reader will surmise, the distinguishing feature of Specht ideals of two-rowed partitions is that their minimal generators are square-free, a fact which exploits throughout this paper.

Recall that an ideal $I\subset R$ in a Noetherian ring is called *perfect* if its grade is equal to its homological (or projective) dimension. In a polynomial ring R, a homogeneous ideal $I\subset R$ is perfect if and only if its quotient R/I is Cohen–Macaulay. The following is one of the main results of this paper, and is the algebraic analogue of Proposition 1.1.

THEOREM 1.2. *Let $\mathbb{F}$ be any field of characteristic $p\geq 0$, and fix positive integers n,k satisfying $n\geq 2k+1$.*

1. *If $p=0$ or $p\geq k+1$, then the Specht ideal $\mathfrak{a}(n+1,k+1,k+1)$ is perfect.*
2. *If $n\geq 2p+1$, then the Specht ideal $\mathfrak{a}(n+1,p+1,p+1)$ is not perfect.*

In his paper [17], Yanagawa has conjectured that the Specht ideal $\mathfrak{a}(n+1,k+1,k+1)$ is perfect in characteristic p *if and only if* $p=0$ or $p\geq k+1$. Theorem 1.2 proves one implication and part of the other one in Yanagawa's conjecture.

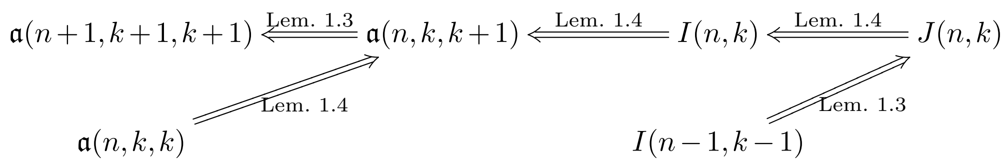
Our proof of Theorem 1.2 is inspired by the seminal paper of Hochster–Eagon [6], in which they proved perfection of generic determinantal ideals using what they termed a principal radical system. Our method, which might be more aptly described as a principal *perfect* system, is based on the following elementary facts from commutative algebra:

LEMMA 1.3. *Let $I\subset R$ be a homogeneous ideal, and let $x\in R\setminus I$ be a homogeneous polynomial of positive degree.*

1. *If $(I:x)=I$ and $I+(x)$ is perfect, then I is also perfect.*
2. *If $(I:x)\neq I$ and $I+(x)$ are both perfect of the same grade, then I is also perfect of that grade.*

LEMMA 1.4. *Suppose that ideals $I,J\subset R$ are homogeneous ideals, both perfect of the same grade g and suppose that $I+J$ has grade $g+1$. Then $I\cap J$ is perfect if and only if $I+J$ is perfect.*

Our strategy then is to start with the two-parameter family of Specht ideals $\mathfrak{a}(n+1,k+1,k+1)$, and use the constructions in Lemmas 1.3 and 1.4 to obtain new families of ideals until we arrive at one which is evidently perfect, for example, by induction on one of the parameters. Shown below are the new two-parameter families we construct, labeled $\mathfrak{a}(n,k,k+1)$, $I(n,k)$, and $J(n,k)$, together with a schematic diagram indicating the implications in our argument, where "$I \Rightarrow J$" means "perfection of I implies perfection of J" and "$I \Rightarrow J \Leftarrow K$" means "perfection of I and K implies perfection of J":

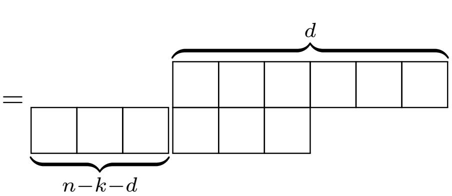


We remark that Lemma 1.3 is true almost verbatim if "perfect" is replaced by "radical," illustrating the close relationship between these two properties (see Lemma 4.1). In fact, before we show that the Specht ideals are perfect we must first show that they are radical. More generally, in order to apply Lemma 1.4 to an ideal $\mathfrak{a} = I \cap J$ we must know the ideals I and J, and these come from knowing a primary decomposition for $\mathfrak{a}$. Finding such decompositions forms the technical heart of this paper; see Theorems 1.6 and 1.8(3). We give a more detailed description of our method and these ideals listed above, together with the other results of this paper below.

Taking $I = \mathfrak{a}(n+1,k+1,k+1)$ and $x = x_{n+1}$ in Lemma 1.3, it is easy to show that $(I : x) = I$, and, with a little more work, that $I + (x) = \mathfrak{a}(n,k,k+1) + (x_{n+1})$ where $\mathfrak{a}(n,k,k+1)$ is the ideal generated by square-free products of Specht polynomials of type $\lambda = (n-k,k)$ with a linear monomial in the variables $x_1,\dots,x_n$. Generalizing, we introduce, for integers $0 \le k \le d \le n-k$, the *d-shifted Specht ideal* $\mathfrak{a}(n,k,d)$ generated by square-free products of Specht polynomials of type $\lambda = (n-k,k)$ and square-free monomials of degree $d-k$.

These shifted Specht ideals interpolate between Specht ideals $\mathfrak{a}(n,k,k)$ in case $d = k$, and square-free monomial ideals in case $k = 0$, where $\mathfrak{a}(n,0,d) = (x_1,\dots,x_n)^{\langle d\rangle}$ is the ideal generated by all square-free monomials of fixed degree d.[2] Our first step in understanding these shifted Specht ideals is to find a minimal generating set. Just as minimal generators for the Specht ideal are indexed by standard Young tableaux on λ, we show that minimal generators for the shifted Specht ideal are indexed by standard Young tableaux on a shifted version of λ.

THEOREM 1.5. *A minimal generating set for the shifted Specht ideal $\mathfrak{a}(n,k,d)$ is formed by the shifted Specht polynomials $F_T(d)$ indexed by standard tableaux T on the d-shifted shape of $\lambda = (n-k,k)$ obtained from Young diagram of λ by moving the last $n-k-d$ boxes on the top row to the first $n-k-d$ boxes on the bottom row, that is,*

$\lambda(d) =$ 

[2] The ideals $(x_1,\dots,x_n)^{\langle d\rangle}$ should not be confused with symbolic powers, which are not discussed in this paper.

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The linear span of the shifted Specht polynomials $F_T(d)$, $T \in \mathrm{Tab}(\lambda(d))$ forms an $\mathfrak{S}_n$-representation $V(n,k,d)$ that we call a *shifted Specht module.*[3] We prove that our d-shifted Specht ideals satisfy the following decomposition formula, which is crucial in our quest for perfection, and which holds if and only if our shifted Specht ideals are radical.

THEOREM 1.6. *For any integers k,d satisfying $1 \le k < d \le n-k$, we have*

$$\mathfrak{a}(n,k,d) = \mathfrak{a}(n,k,d-1) \cap (x_1,\dots,x_n)^{\langle d\rangle} = \mathfrak{a}(n,k,k) \cap (x_1,\dots,x_n)^{\langle d\rangle}. \tag{2}$$

Yanagawa [17] has proved Theorem 1.6 in the special case $d = k+1$ using a clever argument, which is described in further detail below. As it turns out, his argument goes through verbatim to prove Theorem 1.6 in the general case, and is in fact simplified by Theorem 1.5. It follows directly from Theorem 1.6 that our shifted Specht ideals are radical.

THEOREM 1.7. *Fix integers k,d satisfying $0 \le k \le d \le n-k$. Then the shifted Specht ideal $\mathfrak{a}(n,k,d)$ is radical.*

Perfection of shifted Specht ideals is more difficult to prove, and in fact, most shifted Specht ideals are not perfect. Indeed Theorem 1.6 implies that the shifted Specht ideal $\mathfrak{a}(n,k,d)$ does not have pure height and hence cannot be perfect if $d \ne k, k+1$. To show that the shifted Specht ideal $\mathfrak{a}(n,k,k+1) = \mathfrak{a}(n,k,k) \cap (x_1,\dots,x_n)^{\langle k+1\rangle}$ is perfect, we appeal to Lemma 1.4 and introduce the *Specht-monomial ideal* $I(n,k) = \mathfrak{a}(n,k,k) + (x_1,\dots,x_n)^{\langle k+1\rangle}$.

While the Specht-monomial ideal is not radical in general, it does satisfy a decomposition formula similar to (2), but it depends on the field characteristic. We were pleased to discover that this dependence on field characteristic is the same one imposed by the weak Lefschetz property of certain monomial complete intersection algebras. This is summarized in the following, which can be considered the other main result of this paper.

THEOREM 1.8. *Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = p \ge 0$, and let n and k be positive integers satisfying $n \ge 2k+1$. The following are equivalent:*

1. *$p = 0$ or $p \ge k+1$.*
2. *The quadratic monomial complete intersection*

$$C = \frac{\mathbb{F}[x_1,\dots,x_{2k}]}{(x_1^2,\dots,x_{2k}^2)}, \tag{3}$$

   has the weak Lefschetz property.
3. *The Specht-monomial ideal $I(n,k)$ satisfies the decomposition*

$$I(n,k) = I(n-1,k-1) \cap \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)\right), \tag{4}$$
$$\textit{where} \quad y_i = x_n - x_i \quad \textit{for} \quad 1 \le i \le n-1.$$

4. *The Specht-monomial ideal $I(n,k)$ is perfect.*

We shall break Theorem 1.8 into three equivalences (1) ⇔ (a) for a $= 2,3,4$. The equivalence (1) ⇔ (2) follows from a general result of Kustin–Vraciu [8], and we shall not prove it here; but we use it! Using Equation (4), which essentially amounts to computing a primary decomposition of the Specht-monomial ideal, in conjunction with Lemma 1.4 leads

[3] This is evidently the skew representation of $\mathfrak{S}_n$ associated to $\lambda(d)$ (see Remark 3.8).

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to yet another family of ideals, which remain unnamed:

$$J(n,k) = I(n-1,k-1) + (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2).$$

Finally, in order to prove equivalence (1) ⇔ (4) in Theorem 1.8, and hence also Theorem 1.2, we apply Lemma 1.3 to show that the ideal $J(n,k)$ is perfect.

Some further remarks on the connection to the weak Lefschetz property are in order here. Since (shifted) Specht polynomials are square-free they are identified with elements of the algebra $A = \mathbb{F}[x_1,\dots,x_n]/(x_1^2,\dots,x_n^2)$, which carries an $\mathfrak{sl}_2$-representation, in which the raising operator is multiplication by the sum of variables, and the lowering operator is the corresponding linear partial differential operator. Moreover, surjectivity of this lowering operator in degree k is equivalent to equality of the kernel of that lowering operator with the Specht module $V(n,k,k)$, which is in turn equivalent to the weak Lefschetz property of C in (3). Surjectivity of the lowering operator on A is key to proving decomposition (4), and in fact reveals a hidden property of the Specht-monomial ideal: in small positive characteristic, that is, $0 < p < k+1$ the ideal $I(n,k)$ has an embedded prime divisor, a phenomenon which does not occur for the Specht ideals.

The most technically difficult parts of our arguments are the decompositions in Theorem 1.6 and Theorem 1.8(3). As it turns out, the two proofs we give are strikingly similar, and are both based on that clever argument of Yanagawa mentioned above. This argument, in general terms, runs as follows: To prove that an ideal I satisfies a decomposition formula of the form $I = I' \cap J$ where J is a monomial ideal, first show that the intersection $I' \cap J$ can be generated by products of minimal generators of I' with monomials (not necessarily from J), with the additional property that if a sum of such products is in $I' \cap J$ then each of the summands is also in $I' \cap J$ (perhaps we should call such I an I'-*monomial ideal*). Next fix a monomial m and split the minimal generators $V(I')$ into two parts say $V(I') = V_m(I') \oplus V^m(I')$, determined by whether or not m appears in their monomial expansion or not. In our situation, one can show that if m is not in the support of $\nu \in V^m(I')$, then $m \cdot \nu \in I$, and, with more effort, one can also show that if $\nu \in V_m$ and $m \cdot \nu \in I' \cap J$ then $m \cdot \nu \in I$. We highlight this argument here because it seems important in the theory of two-rowed Specht ideals.

This paper is organized as follows. In §2, we define Specht polynomials, shifted Specht polynomials, and the modules and ideals they generate. We then prove Theorem 1.5 and compute the dimensions of our shifted Specht modules (Theorems 2.2 and 2.9, respectively). In §3, we draw out the connection between Lefschetz properties and Specht modules, decompose the shifted Specht module into its irreducible representations, and derive other useful consequences. In §4, we prove Theorems 1.7 and 1.6 (Theorems 4.4 and 4.3, respectively). Finally, in §5, we prove Theorems 1.8 and 1.2 (Theorems 5.5 and 5.6, respectively).

§2. Shifted Specht polynomials, modules, and ideals

Fix a positive integer n, a field $\mathbb{F}$, and let $R = \mathbb{F}[x_1,\dots,x_n]$ be the standard graded polynomial ring in n-variables, equipped with the usual action of $\mathfrak{S}_n$ which permutes the variables. A partition of n, denoted $\lambda \vdash n$ is a sequence of nonincreasing integers which sum to n, that is, $\lambda = (\lambda_1,\dots,\lambda_r)$ where $\lambda_1 \geq \cdots \geq \lambda_r$ and $\sum_{i=1}^r \lambda_i = n$. The Young diagram of λ is a left-justified array of boxes with r-rows and λ_i boxes in each row, and a filling of those boxes with distinct numbers $1,\dots n$ is called a tableau of shape λ; the set of all tableaux of shape λ will be written as $\mathrm{Tab}(\lambda)$. To each tableau $T \in \mathrm{Tab}(\lambda)$ of shape λ we associate a polynomial $F_T \in R$ as follows.

First for any subset $S \subset \{1,\dots,n\}$, define the S-Vandermonde polynomial

$$\Delta_S = \prod_{i<j\in S}(x_i - x_j)$$

with the convention that if $|S| < 2$ then $\Delta_S = 1$. Then for any tableau T of shape λ with columns $C_1,\dots,C_{\lambda_1}$, define its Specht polynomial by

$$F_T = \prod_{i=1}^{\lambda_1}\Delta_{C_i}.$$

The $\mathbb{F}$-linear span of Specht polynomials over $\mathrm{Tab}(\lambda)$ is an $\mathfrak{S}_n$-representation called the Specht module, which we denote $V(\lambda)$, that is,

$$V(\lambda) = \langle F_T \mid T \in \mathrm{Tab}(\lambda)\rangle = \mathrm{sp}_{\mathbb{F}}(F_T \mid T \in \mathrm{Tab}(\lambda));$$

it is well known, in the case $\mathrm{char}(\mathbb{F}) = 0$, that $V(\lambda)$ is irreducible, and conversely that every irreducible $\mathfrak{S}_n$-representation is isomorphic to $V(\lambda)$ for some $\lambda \vdash n$, for example, [14]. The Specht ideal of λ is the ideal in R generated by $V(\lambda)$, that is,

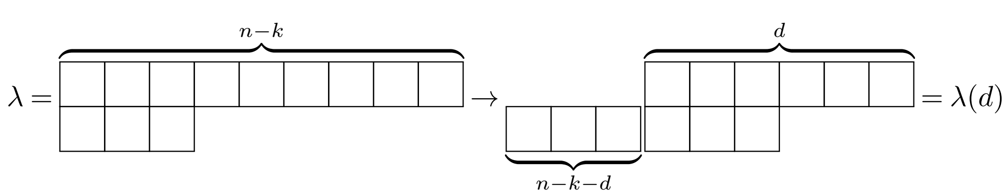
$$\mathfrak{a}(\lambda) = V(\lambda)\cdot R = (F_T \mid T \in \mathrm{Tab}(\lambda)).$$

In his paper [17, Conj. 2.9], Yanagawa has formulated the following conjecture:

CONJECTURE 2.1 ([17]). *Over any field $\mathbb{F}$, and for any partition λ, the Specht ideal $\mathfrak{a}(\lambda)$ is radical.*

This conjecture has been proved for partitions of the form $\lambda = (n-k,1,\dots,1)$ by Yanagawa–Watanabe [16], and for partitions of the form $\lambda = (n-k,k)$ and $\lambda = (d,d,1)$ by Yanagawa [17]. However, even in these simple cases, Yanagawa's proof that $\mathfrak{a}(\lambda)$ is radical is by no means easy. This paper grew out of an attempt to understand and perhaps simplify Yanagawa's proof in the case of two rowed partitions $\lambda = (n-k,k)$, which we discuss next.

Fix an integer k satisfying $1 \le k \le n-k$, and let $\lambda = (n-k,k)$ be the corresponding partition, which we regard as a left-justified two-rowed Young diagram with $n-k$ boxes in the first row and k boxes in the second. For each integer d satisfying $1 \le k \le d \le n-k$, define the d-shifted shape $\lambda(d)$ to be the Young diagram obtained by moving the rightermost $n-d-k$ boxes on the first row to the left of the first boxes in the second row:



Note that $\lambda(k)$ is λ rotated by $180°$.

A tableau T on shape $\lambda(d)$ is a labeling of the boxes of the Young diagram of $\lambda(d)$ with the numbers $\{1,\dots,n\}$ used exactly once; we say that T is standard if the rows are increasing from left to right, and the columns are increasing from top to bottom. The set of tableaux on $\lambda(d)$ we denote by $\mathrm{Tab}(n,k,d)$, and the set of standard tableaux by $\mathrm{STab}(n,k,d)$.

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Given a tableau $T \in \mathrm{Tab}(n,k,d)$ on the d-shifted shape $\lambda(d)$, such as

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & i_1 & \cdots & i_k & i_{k+1} & \cdots & i_d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_k & & & \\ \hline \end{array} \tag{5}$$

define the associated d-shifted Specht polynomial to be the homogeneous polynomial of degree d by

$$F_T(d) = (x_{i_1} - x_{j_1}) \cdots (x_{i_k} - x_{j_k}) \cdot x_{i_{k+1}} \cdots x_{i_d}.$$

Note that if $d = k$, then we recover the usual Specht polynomial:

$$F_T(k) = (x_{i_1} - x_{j_1}) \cdots (x_{i_k} - x_{j_k});$$

for the usual Specht polynomials we shall sometimes use the alternative notation F_T or F_T^k if we want to keep track of its degree. We allow $k = 0$, and in this case a tableau $S \in \mathrm{Tab}(n,0,d)$ has the form

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & & & & & & i_1 & \cdots & i_d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_k & & & \\ \hline \end{array} \tag{6}$$

and its associated shifted Specht polynomial is the monomial

$$M_S = x_{i_1} \cdots x_{i_d},$$

which we sometimes write as M_S^d to remember degree.

The *d-shifted Specht module of* $\lambda = (n-k,k)$, denoted by $V(n,k,d)$, is defined to be the $\mathbb{F}$-linear span of the d-shifted Specht polynomials, that is,

$$V(n,k,d) = \langle F_T(d) \mid T \in \mathrm{Tab}(n,k,d) \rangle;$$

like the Specht module, it is also an $\mathfrak{S}_n$-representation, although it is not irreducible for $d > k$ (cf. §3). The *d-shifted Specht ideal of* $\lambda = (n-k,k)$, denoted by $\mathfrak{a}(n,k,d)$, is the ideal in R generated by the shifted Specht module, that is,

$$\mathfrak{a}(n,k,d) = V(n,k,d) \cdot R = (F_T(d) \mid T \in \mathrm{Tab}(n,k,d)).$$

The remainder of the section is devoted to finding a basis for the shifted Specht module $V(n,k,d)$ and to computing its dimension.

2.1 Basis of a shifted Specht module

THEOREM 2.2. *A basis for the d-shifted Specht module $V(n,k,d)$, and hence a minimal generating set of the ideal $\mathfrak{a}(n,k,d)$, are indexed by the standard tableaux on the d-shifted shape $\lambda(d)$, that is,*

$$\{F_T(d) \mid T \in \mathrm{STab}(\lambda(d))\}.$$

The proof of Theorem 2.2 comes in two steps, which we state as Lemmas.

LEMMA 2.3. *The set $\{F_T(d) | T \in \mathrm{STab}(n,k,d)\}$ is linearly independent.*

Proof. By induction on $n \geq 2$. The base case, where $n = 2$ and $k = d = n-k = 1$ is trivial. For the inductive step, we assume that for every choice of k' and d' satisfying

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$1 \le k' \le d' \le (n-1)-k'$, the set $\{F_{T'}(d') | T' \in \mathrm{STab}(n-1,k',d')\}$ is linearly independent. Fix any integers $1 \le k \le d \le n-k$ and suppose we have a dependence relation

$$\sum_{T \in \mathrm{STab}(n,k,d)} c_T F_T(d) = 0. \tag{7}$$

Note that for every d-standard tableau $T \in \mathrm{STab}(n,k,d)$ as in (5), we must have either $n = i_d$ or $n = j_k$; let $I_d \subset \mathrm{STab}(n,k,d)$ denote the set of d-standard tableaux with $n = i_d$ and let $J_d \subset \mathrm{STab}(n,k,d)$ be the ones with $n = j_k$. Let $\pi\colon R \to S = \mathbb{F}[x_1,\dots,x_{n-1}]$ be the projection sending x_n to 0, and note that for every $T \in I_d$ we have $\pi(F_T(d)) = 0$. Moreover, for every $T \in J_d$, we have $\pi(F_T(d)) = F_{T'}(d)$ where

$$T' = \begin{array}{cccccccccc} & & & \boxed{i_1} & \boxed{\cdots} & \boxed{i_{k-1}} & \boxed{i_k} & \boxed{i_{k+1}} & \boxed{\cdots} & \boxed{i_d} \\ \boxed{i_{d+1}} & \boxed{\cdots} & \boxed{i_{n-k}} & \boxed{j_1} & \boxed{\cdots} & \boxed{j_{k-1}} & & & & \end{array}$$

We see that $T' \in \mathrm{STab}(n-1,k-1,d)$, hence the induction hypothesis applies. Note that since the map $J_d \to \mathrm{STab}(n-1,k-1,d)$, sending $T \mapsto T'$ is one-to-one, and since

$$\pi\left(\sum_{T \in \mathrm{STab}(n,k,d) = I_d \sqcup J_d} c_T F_T(d)\right) = \sum_{T \in J_d} c_T F_{T'}(d) = 0,$$

we deduce, by the induction hypothesis, that $c_T = 0$ for all $T \in J_d$. Then our dependence relation (7) becomes

$$\sum_{T \in I_d} c_T F_T(d) = 0 = x_n \cdot \left(\sum_{T \in I_d} c_T F_{T''}(d)\right),$$

where

$$T'' = \begin{array}{ccccccccc} & & & \boxed{i_1} & \boxed{\cdots} & \boxed{i_k} & \boxed{i_{k+1}} & \boxed{\cdots} & \boxed{i_{d-1}} \\ \boxed{i_{d+1}} & \boxed{\cdots} & \boxed{i_{n-k}} & \boxed{j_1} & \boxed{\cdots} & \boxed{j_k} & & & \end{array}$$

In this case, we see that $T'' \in \mathrm{STab}(n-1,k,d-1)$, and again our induction hypothesis applies. Since the map $I_d \to \mathrm{STab}(n-1,k,d-1)$ sending $T \mapsto T''$ is one-to-one, and since

$$\sum_{T \in I_d} c_T F_{T''}(d) = 0,$$

the induction hypothesis implies that $c_T = 0$ for every $T \in I_d$ too, and therefore the dependence relation (7) must be trivial. □

To see that the d-standard polynomials span $V(n,k,d)$ is a little more work. We follow the general method described in [14, Sec. 2.6]. For T as in (5) and any index $1 \le a \le n$, define *a-composition vector* to be the integer vector with $n-k$ components defined by

$$\gamma^a(T) = (\gamma_1^a(T),\dots,\gamma_{n-k}^a(T)), \quad \text{where} \quad \gamma_b^a(T) = \#\{c \in \mathrm{col}_b(T) \mid c \le a\}.$$

Note that for a two rowed partition $\lambda = (n-k,k)$ the a-composition vector has entries 0, 1, or 2. Define the *composition series* for T to be the n-tuple of composition vectors $\gamma(T) = (\gamma^1(T),\dots,\gamma^n(T))$; we regard $\gamma(T)$ as a matrix whose columns are the composition

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vectors of T. Given two vectors $v=(v_1,\dots,v_{n-k})$ and $w=(w_1,\dots,w_{n-k})$, we say that w *dominates* v, and write $v \triangleleft w$, if $v_1+\cdots+v_p \le w_1+\cdots+w_p$ for all $1\le p\le n-k$. Finally given two tableaux $T,T'\in \mathrm{Tab}(\lambda(d))$, we say that T' *dominates* T, and write $T\triangleleft T'$, if every composition vector of T' dominates the corresponding composition vector of T, that is,

$$T\triangleleft T' \quad\Leftrightarrow\quad \gamma^a(T)\triangleleft\gamma^a(T'),\ \forall\ 1\le a\le n.$$

This composition-dominance order is a partial order on the set of tableaux $\mathrm{Tab}(\lambda(d))$. Moreover, it is clear that the largest tableau is the one that fills the columns in order from left to right. For example, if $n=5$, $k=1$, and $d=3$ the largest tableau (with increasing columns) is the standard tableau

$$T=\begin{array}{|c|c|c|c|}\hline \multicolumn{1}{c|}{} & 2 & 4 & 5 \\ \hline 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

with composition series

$$\gamma(T)=\begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 2 & 2 & 2\\ 0 & 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}.$$

Note that if T and T' have the same columns (possibly in different orders) then they have the same composition series, and they also have the same shifted Specht polynomials (up to sign), that is, $\gamma(T)=\gamma(T')$ and $F_T(d)=\pm F_{T'}(d)$. The following lemma is useful for telling when one tableau dominates another.

LEMMA 2.4. *If $1\le a<b\le n$ and a appears in a column to the right of b, then*

$$T\triangleleft (a,b)\cdot T,$$

where $(a,b)\cdot T$ is the tableau obtained from T by transposing a and b.

Proof. Note that for $1\le i\le a-1$ and for $b\le i\le n$ we have $\gamma^i(T)=\gamma^i((a,b)\cdot T)$. Assume then that $a\le i\le b-1$, and suppose that a and b belong to columns r and q in T, respectively. Then

$$\gamma^i((a,b)\cdot T)=\gamma^i(T)\quad \begin{array}{l}\text{with } r\text{th part decreased by } 1\\ \text{and } q\text{th part increased by } 1,\end{array}$$

and since we are assuming that $q<r$, it follows that $\gamma^i(T)\triangleleft\gamma^i((a,b)\cdot T)$, and the result follows. □

We are now in a position to show the standard shifted Specht polynomials span the shifted Specht module.

LEMMA 2.5. *The set $\{F_T(d) | T\in \mathrm{STab}(n,k,d)\}$ spans $V(n,k,d)$.*

Proof. We show by downward induction on the composition-dominance order on $\mathrm{Tab}(\lambda(d))$ that for every $T\in\mathrm{Tab}(\lambda(d))$, $F_T(d)$ can be written as a linear combination of shifted Specht polynomials associated to d-standard tableaux. For the base case, note, as above, that the largest tableau of shifted shape $\lambda(d)$ is already standard. Inductively, fix a shifted tableau $T\in\mathrm{Tab}(\lambda(d))$ as in (5), and assume that for every tableau T' that

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dominates T, $F_{T'}(d)$ can be written as a linear combination of shifted Specht polynomials corresponding to standard tableaux. We may assume that the columns of T are increasing. If T has no row descents, then T must be standard and we are done. Otherwise T has some row descent, say between the a^{th} and $a+1^{st}$ column. There are several cases to consider and we claim that in all cases we can write $F_T(d)$ as a linear combination

$$F_T(d) = \sum_{T \lhd T'} c_{T'} F_{T'}(d).$$

Case 1: $1 \leq a \leq n-k-d-1$. Set $b = d+a$; in this case, we can merely swap i_b and i_{b+1} without affecting $F_T(d)$; in other words setting $T' = (i_b, i_{b+1}) \cdot T$ we have

$$F_T(d) = F_{T'}(d),$$

and since $T \lhd (i_b, i_{b+1}) \cdot T$, it follows from our inductive hypothesis that $F_T(d)$ can be written as a linear combination of d-standard Specht polynomials on the shifted shape $\lambda(d)$.

Case 2: $a = n-k-d$. Then we have

$$T = \begin{array}{cccccccccc} & & & \boxed{i_1} & \boxed{\cdots} & \boxed{i_k} & \boxed{i_{k+1}} & \boxed{\cdots} & \boxed{i_d} \\ \boxed{i_{d+1}} & \boxed{\cdots} & \boxed{i_{n-k}} & \boxed{j_1} & \boxed{\cdots} & \boxed{j_k} & & & \end{array}$$

and we have the descent $i_{n-k} > j_1 > i_1$. Then by Lemma 2.4, we see that

$$T \lhd T' = (i_{n-k}, j_1) \cdot T \text{ and } T \lhd T'' = (i_1, i_{n-k}) \cdot T.$$

Moreover, one can easily check that

$$F_T(d) = F_{T'}(d) - F_{T''}(d) = G\left((x_{i_1} - x_{i_{n-k}}) - (x_{j_1} - x_{i_{n-k}})\right),$$

and inductive hypothesis applies.

Case 3: $n-k-d+1 \leq a \leq n-d-1$. Set $b = a-n+k+d$ so we have

$$T = \begin{array}{cccccccccccc} & & & \boxed{i_1} & \boxed{\cdots} & \boxed{i_b} & \boxed{i_{b+1}} & \boxed{\cdots} & \boxed{i_k} & \boxed{i_{k+1}} & \boxed{\cdots} & \boxed{i_d} \\ \boxed{i_{d+1}} & \boxed{\cdots} & \boxed{i_{n-k}} & \boxed{j_1} & \boxed{\cdots} & \boxed{j_b} & \boxed{j_{b+1}} & \boxed{\cdots} & \boxed{j_k} & & & \end{array}$$

Sub-Case 3a: $i_{b+1} < i_b < j_b$. Then

$$T \lhd T' = (i_{b+1}, j_b) \cdot T \text{ and } T \lhd T'' = (i_{b+1}, i_b) \cdot T,$$

and again one can easily check that

$$F_T(d) = F_{T'}(d) + F_{T''}(d) = G \cdot \left((x_{i_b} - x_{i_{b+1}})(x_{j_b} - x_{j_{b+1}}) + (x_{i_{b+1}} - x_{j_b})(x_{i_b} - x_{j_{b+1}})\right),$$

and our inductive hypothesis applies.

Sub-Case 3b: $i_b < i_{b+1} < j_{b+1} < j_b$. Then

$$T \lhd T' = (i_{b+1}, j_b) \cdot T \text{ and } T \lhd T'' = (j_{b+1}, j_b) \cdot T,$$

and again one can easily check that

$$F_T(d) = F_{T'}(d) + F_{T''}(d) = G \cdot \left((x_{i_b} - x_{i_{b+1}})(x_{j_b} - x_{j_{b+1}}) + (x_{i_b} - x_{j_{b+1}})(x_{i_{b+1}} - x_{j_b})\right),$$

and our inductive hypothesis applies.

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Case 4: $a = n-d$. Then

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{4-9} \multicolumn{3}{c|}{} & i_1 & \cdots & i_k & i_{k+1} & \cdots & i_d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_k & \multicolumn{3}{c}{} \\ \cline{1-6} \end{array}$$

and we have the descent $j_k > i_k > i_{k+1}$. Then by Lemma 2.4, we have

$$T \lhd T' = (i_{k+1}, i_k) \cdot T \text{ and } T \lhd T'' = (i_{k+1}, j_k) \cdot T.$$

and again one can check that

$$F_T(d) = F_{T'}(d) + F_{T''}(d) = G\left(x_{i_k}\left(x_{i_{k+1}} - x_{j_k}\right) + x_{j_k}\left(x_{i_k} - x_{i_{k+1}}\right)\right),$$

to which the inductive hypothesis once again implies.

Case 5: $n-d+1 \leq a \leq n-k-1$. In this case, set $b = a-(n-k-d)$ so that we have

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\cline{4-12} \multicolumn{3}{c|}{} & i_1 & \cdots & i_k & i_{k+1} & \cdots & i_b & i_{b+1} & \cdots & i_d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_k & \multicolumn{6}{c}{} \\ \cline{1-6} \end{array}$$

with the descent $i_b > i_{b+1}$. Hence by Lemma 2.4, we have

$$T \lhd T' = (i_{b+1}, i_b) \cdot T,$$

and in this case we clearly have

$$F_T(d) = F_{T'}(d),$$

to which the induction hypothesis applies again.

Therefore, in all cases, we have shown that $F_T(d)$ is a linear combination of shifted Specht polynomials indexed by tableaux on the shifted shape $\lambda(d)$ which dominate T. Therefore, by induction, the d-standard shifted Specht polynomials

$$\{F_T(d) | T \in \mathrm{STab}(\lambda(d))\},$$

must span $V(n,k,d)$. ∎

Proof of Theorem 2.2. By Lemma 2.3, the polynomials in the set

$$\{F_T(d) \mid T \in \mathrm{STab}(\lambda(d))\},$$

are linearly independent, and by Lemma 2.5, they generate the shifted Specht module $V(n,k,d)$, and hence they must form a basis.

EXAMPLE 2.6. Let $n = 5$, $k = 1$, and $d = 3$. There are 9 standard tableau of shifted shape $\lambda(3)$ where $\lambda = (4,1)$:

$$\begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 2 & 4 & 5 \\ \hline 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 2 & 3 & 5 \\ \hline 1 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 2 & 3 & 4 \\ \hline 1 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 1 & 4 & 5 \\ \hline 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 1 & 3 & 4 \\ \hline 2 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 4 \\ \hline 3 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

Hence a minimal generating set for the ideal $\mathfrak{a}(5,1,3)$, and a basis for the representation $V(5,1,3)$, is given by

$$\left\{\begin{matrix}(x_2-x_3)x_4x_5, & (x_2-x_4)x_3x_5, & (x_2-x_5)x_3x_4,\\ (x_1-x_3)x_4x_5, & (x_1-x_4)x_3x_5, & (x_1-x_5)x_3x_4,\\ (x_1-x_4)x_2x_5, & (x_1-x_5)x_2x_4, & (x_1-x_5)x_2x_3\end{matrix}\right\}.$$

The utility of Theorem 2.2 is that it can transform set maps on the set of standard tableau $\mathrm{STab}(\lambda(d))$ to linear maps on the shifted Specht module $V(n,k,d)$, and sometimes the shifted Specht ideal $\mathfrak{a}(n,k,d)$. The following useful corollaries illustrate this point.

We say that an index i, $1 \le i \le n$ is in the support of tableau $T \in \mathrm{Tab}(n,k,d)$, and write $i \in \mathrm{supp}(T)$, if i appears in or to the right of a column with more than one row, that is, if

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|}\cline{4-9} \multicolumn{3}{c|}{} & i_1 & \cdots & i_k & i_{k+1} & \cdots & i_d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_k & \multicolumn{3}{c}{} \\ \cline{1-6}\end{array}$$

then $\mathrm{supp}(T) = \{i_1, j_1, \cdots, i_k, j_k, i_{k+1}, \ldots, i_d\}$. Note every standard tableau $T \in \mathrm{STab}(n,k,k)$ has n in its support; in fact it must have the form

$$T = \begin{array}{|c|c|c|c|c|c|c|}\cline{4-7} \multicolumn{3}{c|}{} & i_1 & \cdots & i_{k-1} & i_k \\ \hline i_{k+1} & \cdots & i_{n-k} & j_1 & \cdots & j_{k-1} & n \\ \hline\end{array}$$

We get a standard tableau $T' \in \mathrm{STab}(n-1,k-1,k)$ from T by deleting the box containing n, that is,

$$T' = \begin{array}{|c|c|c|c|c|c|c|}\cline{4-7} \multicolumn{3}{c|}{} & i_1 & \cdots & i_{k-1} & i_k \\ \hline i_{k+1} & \cdots & i_{n-k} & j_1 & \cdots & j_{k-1} & \multicolumn{1}{c}{} \\ \cline{1-6}\end{array}$$

Since this map is obviously a bijection it extends to a linear isomorphism

$$V(n,k,k) \to V(n-1,k-1,k).$$

In fact it can be extended to an isomorphism of Specht ideals using a linear change of coordinates: Define the variables $y_1, \ldots, y_n$ by the formula

$$y_i = \begin{cases} x_n - x_i, & \text{if } 1 \le i \le n-1, \\ x_n, & \text{if } i = n, \end{cases}$$

and let $\Phi \colon R \to R$ be the change of coordinates map $\Phi(x_i) = y_i$.

Corollary 2.7. *Fix integers k, n satisfying $1 \le k \le n-k$. Then ring isomorphism Φ maps the Specht ideal in n-variables isomorphically onto the shifted Specht ideal in $n-1$-variables:*

$$\mathfrak{a}(n,k,k) \cong \Phi(\mathfrak{a}(n,k,k)) = \mathfrak{a}(n-1,k-1,k).$$

In fact, in adding the principal ideal (x_n), this isomorphism becomes equality:

$$\mathfrak{a}(n,k,k) + (x_n) = \mathfrak{a}(n-1,k-1,k) + (x_n).$$

Proof. With $T \in \mathrm{STab}(n,k,k)$ and $T' \in \mathrm{STab}(n-1,k-1,k)$ as above, we compute

$$F_T = \prod_{t=1}^{k}(x_{i_t} - x_{j_t}) = \prod_{t=1}^{k-1}(x_{i_t} - x_{j_t}) \cdot (x_{i_k} - x_n) = \prod_{t=1}^{k-1}(y_{j_t} - y_{i_t}) \cdot (-y_{i_k}) = (-1)^k \cdot \Phi(F_{T'}),$$

and hence by Theorem 2.2, the first statement follows. To see the second statement note that the ring automorphism $\Phi\colon R \to R$ becomes the negative identity map on the quotient $\bar{\Phi} = -I\colon R/(x_n) \to R/(x_n)$, and hence

$$\mathfrak{a}(n,k,k) + (x_n) = \mathfrak{a}(n-1,k-1,k) + (x_n),$$

which is the second statement. ∎

Fix an integer m satisfying $0 \leq m \leq d$, and define the subset $\mathrm{STab}_m(n,k,d) \subset \mathrm{STab}(n,k,d)$ consisting of standard tableaux which contain $\{1,\dots,m\}$ in its support. A standard tableau $T \in \mathrm{STab}_m(n,k,d)$ necessarily has the form

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & & & 1 & \cdots & m & m+1 & \cdots & k & k+1 & \cdots & d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_m & j_{m+1} & \cdots & j_k & & & \\ \hline \end{array}$$

and we define its image tableau as the one obtained by removing the boxes containing the numbers $1,\dots,m$:

$$T' = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|} \hline & & & & & & m+1 & \cdots & k & k+1 & \cdots & d \\ \hline i_{d+1} & \cdots & i_{n-k} & j_1 & \cdots & j_m & j_{m+1} & \cdots & j_k & & & \\ \hline \end{array}$$

Note that $T' \in \mathrm{STab}([n]_m, k-m, d-m)$ where $[n]_m$ means the tableaux are filled with numbers $\{m+1,\dots,n\}$ exactly once, and our convention is to count $k-m$ as zero if $k \leq m$. The map of sets $\mathrm{STab}_m(n,k,d) \ni T \mapsto T' \in \mathrm{STab}([n]_m, k-m, d-m)$ is evidently one-to-one and onto, and by Theorem 2.2, it induces a bijective linear map of Specht modules, which is the key to some of the main technical arguments in this paper.

COROLLARY 2.8. *The induced linear map*

$$\begin{aligned} V_m(n,k,d) &\longrightarrow V([n]_m, k-m, d-m)\ , \\ F_T(d) &\longmapsto F_{T'}(d-m) \end{aligned}$$

is bijective.

2.2 Dimension of a shifted Specht module

Using Theorem 2.2, we can compute the dimension of the shifted Specht module $V(n,k,d)$.

THEOREM 2.9. *For integers n,k,d satisfying $1 \leq k \leq d \leq n-k$, the number of standard tableaux on the shifted shape $\lambda(d)$ or equivalently the dimension of the shifted Specht module $V(n,k,d)$ is*

$$\dim_{\mathbb{F}}(V(n,k,d)) = \binom{n}{d} - \binom{n}{k-1}.$$

Proof. We prove the formula by induction on n, the base case $n=1$ ($k=0$, $d=1$) being trivial. For the inductive step, assume that the formula holds for $n-1$, that is,

$$\dim_{\mathbb{F}}(V(n-1,j,e)) = \binom{n-1}{e} - \binom{n-1}{j},$$

for all integers j,e satisfying $1 \le j \le e \le n-1-j$. Then fix integers k,d satisfying $1 \le k \le d \le n-k$. If $d > k$, then by removing the box containing n, we get a bijection of indexing sets $\mathrm{STab}(n,k,d) \to \mathrm{STab}(n-1,k,d-1) \sqcap \mathrm{STab}(n-1,k-1,d)$, and hence we find that

$$\begin{aligned}\dim_{\mathbb{F}}(V(n,k,d)) &= \dim_{\mathbb{F}}(V(n-1,k,d-1)) + \dim_{\mathbb{F}}(V(n-1,k-1,d))\\ &= \left(\binom{n-1}{d-1} - \binom{n-1}{k-1}\right) + \left(\binom{n-1}{d} - \binom{n-1}{k-2}\right)\\ &= \binom{n}{d} - \binom{n}{k-1},\end{aligned}$$

where the last equality is Pascal's identity. If $d = k$, then by Corollary 2.7, we have

$$\begin{aligned}\dim_{\mathbb{F}}(V(n,k,k)) = \dim_{\mathbb{F}}(V(n-1,k-1,k)) &= \binom{n-1}{k} - \binom{n-1}{k-2}\\ &= \left(\binom{n-1}{k} + \binom{n-1}{k-1} - \binom{n-1}{k-1} - \binom{n-1}{k-2}\right) = \binom{n}{k} - \binom{n}{k-1},\end{aligned}$$

again by Pascal's identity. ∎

§3. Lefschetz propoerties

Let $E \subset R$ be the graded vector subspace spanned by square-free monomials in the variables $x_1, \ldots, x_n$. It will be convenient to identify E with a quotient of R as well, specifically the Artinian monomial complete intersection

$$A = \frac{\mathbb{F}[x_1, \ldots, x_n]}{(x_1^2, \ldots, x_n^2)}.$$

Define the following linear operators on A: The *raising operator* is multiplication by the sum of variables:

$$L = \times(x_1 + \cdots + x_n)\colon A \to A[1],$$

the *lowering operator* is the partial derivative map corresponding to the sum of variables:

$$D = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}\colon A \to A[-1],$$

and the *semi-simple operator*:

$$H\colon A \to A, \quad H(a) = (n-2k)\cdot a, \ \forall\, a \in A_k.$$

LEMMA 3.1. *The operators $\{D, L, H\}$ forms an $\mathfrak{sl}_2$-triple on $E \cong A$, that is, they satisfy the commutator relations:*

$$[D,L] = H, \quad [H,D] = 2D, \quad [H,L] = -2L.$$

Proof. Since D, L, and H are linear, it suffices to check the relations on monomials, and by symmetry it suffices to check only the single square-free monomial $\mu = x_1 \cdots x_i$. We have

$$\begin{aligned} D\circ L(m) &= D\left(\sum_{j=i+1}^{n} x_1\cdots x_i x_j\right) \\ &= \sum_{j=i+1}^{n} D(x_1\cdots x_i\cdot x_j) = \sum_{j=i+1}^{n}\left(x_1\cdots x_i + \sum_{k=1}^{i} x_1\cdots\hat{x}_k\cdots x_i\cdot x_j\right) \\ &= (n-i)\cdot m + \sum_{j=i+1}^{n}\sum_{k=1}^{i} x_1\cdots\hat{x}_k\cdots x_i x_j, \end{aligned} \tag{8}$$

$$\begin{aligned} L\circ D(m) &= L\left(\sum_{k=1}^{i} x_1\cdots\hat{x}_k\cdots x_i\right) \\ &= \sum_{k=1}^{i}(L(x_1\cdots\hat{x}_k\cdots x_i)) = \sum_{k=1}^{i}\left(x_1\cdots x_i + \sum_{j=i+1}^{n} x_1\cdots\hat{x}_k\cdots x_i\cdot x_j\right) \\ &= i\cdot m + \sum_{k=1}^{i}\sum_{j=i+1}^{n} x_1\cdots\hat{x}_k\cdots x_i\cdot x_j. \end{aligned} \tag{9}$$

Substracting (8) and (9) yields $[D,L](m) = D\circ L(m) - L\circ D(m) = (n-2i)\cdot m = H(m)$, and hence verifies the relation $[D,L] = H$. Verifications of the other two relations are straightforward and left to the reader. ∎

For each $0 \le i \le n$, define the i^{th}-primitive subspace $P_i \subset A_i$ by

$$P_i = \ker(D)\cap A_i = \{\alpha\in A_i \mid D(\alpha)=0\}.$$

It follows from Lemma 3.1 that for any positive integer m and for any $\alpha \in P_k$, we have

$$D(L^m(\alpha)) = m\cdot(n-2k+1-m)\cdot L^{m-1}(\alpha).$$

Also note that for any Specht polynomial $F_T \in V(n,k,k) \subset A_k$, we have

$$D(F_T) = 0.$$

In particular, we have a chain of containments

$$V(n,k,k)\subseteq P_k\subseteq \ker(L^{n-2k+1})\cap A_k.$$

Lemma 3.2. *Equality $V(n,k,k) = P_k$ holds if and only if the derivative map, that is, the lowering operator*

$$D\colon A_k \to A_{k-1}$$

is surjective.

Proof. Assume that $V(n,k,k) = P_k$. Then by Theorem 2.9, we have

$$\dim(P_k) = \binom{n}{k} - \binom{n}{k-1} = \dim(A_k) - \dim(A_{k-1}).$$

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Let $I_{k-1} = D(A_k)$ be the image of the derivative map. By linear algebra $\dim(I_{k-1}) + \dim(P_k) = \dim(A_k)$ hence $\dim(I_{k-1}) - \dim(A_{k-1}) = 0$, hence the derivative map is surjective. Conversely, if $D\colon A_k \to A_{k-1}$ is surjective, then $\dim(I_{k-1}) = \dim(A_{k-1})$, hence $\dim(P_k) = \dim(A_k) - \dim(A_{k-1}) = \dim(V(n,k,k))$. Since $V(n,k,k) \subseteq P_k$, and they have the same dimension, this containment must be equality. ∎

As we shall see, surjectivity of the derivative map in Lemma 3.2 is dictated by the weak Lefschetz property.

3.1 Weak Lefschetz property

For an arbitrary graded Artinian algebra $C = R/I$, we say that C has the *weak Lefschetz property* if there is a linear form $\ell \in C_1$ such that the multiplication maps

$$\times\ell\colon C_{i-1} \to C_i \tag{10}$$

have maximum rank for every degree $i \geq 0$; in this case, we call ℓ a weak Lefschetz element for C. If C has a symmetric and unimodal Hilbert function with socle degree d, then $\ell \in C_1$ is Lefschetz if and only if the multiplication maps (10) are injective for $1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$ and surjective for $\lfloor \frac{d+3}{2} \rfloor \leq i \leq d$. If C is Gorenstein, then it suffices only to check that (10) is injective in degrees $1 \leq i \leq \lfloor \frac{d+1}{2} \rfloor$. In fact, one can show that if C is Gorenstein with the standard grading and if the multiplication map (10) is injective for some i_0, then it is injective for all $i \leq i_0$. Moreover, if the ideal I is generated by monomials then C is weak Lefschetz if and only if $x_1 + \cdots + x_n \in C_1$ is a weak Lefschetz element. For more details, especially regarding these last two facts, see [11, Props. 2.1 and 2.2] or [15, Prop. 2.5].

In our situation, A is a standard graded Artinian Gorenstein algebra with unimodal Hilbert function and cut out by a monomial ideal. In fact, in our situation, the matrix for the multiplication map $L\colon A_{k-1} \to A_k$ in the monomial basis is the transpose of the derivative map $D\colon A_k \to A_{k-1}$. Therefore, we see that A has the weak Lefschetz property if and only if the derivative maps

$$D\colon A_k \to A_{k-1}$$

are surjective for all $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. The following result is due to Kustin–Vraciu [8], and we refer the reader there for a proof. As usual, $p = \operatorname{char}(\mathbb{F}) \geq 0$.

LEMMA 3.3. *The monomial complete intersection A has the weak Lefschetz property if and only if $p = 0$ or $p \geq \lfloor \frac{n+3}{2} \rfloor$.*

From Lemma 3.3, we derive the following useful result.

LEMMA 3.4. *Fix any integer k satisfying $2k \leq n$. Then the following are equivalent:*

1. *$p = 0$ or $p \geq k+1$.*
2. *The derivative maps $D\colon A_i \to A_{i-1}$ are surjective for all $1 \leq i \leq k$.*
3. *The derivative map $D\colon A_k \to A_{k-1}$ is surjective.*

Proof. (1) ⇒ (2). Assume that $p = 0$ or $p \geq k+1$. For each index j, $2k \leq j \leq n$, define the nested chain of monomial complete intersections

$$C_{2k} \subset \cdots \subset C_n, \quad \text{where} \quad C_j = \frac{\mathbb{F}[x_1, \ldots, x_j]}{(x_1^2, \ldots, x_j^2)}.$$

By Lemma 3.3, the first monomial complete intersection C_{2k} has the weak Lefschetz property, and in particular the derivative map

$$D_{C_{2k}}\colon (C_{2k})_i \to (C_{2k})_{i-1},$$

is surjective for all $1 \le i \le k$. Inductively for $j > 2k$, note that for each $0 \le i \le k$, we have direct sum decomposition of graded vector spaces

$$(C_j)_i \cong (C_{j-1})_i \oplus x_j \cdot (C_{j-1})_{i-1},$$

and in particular the derivative map in degree k decomposes into block triangular form

$$\begin{aligned} &D_{C_j}\colon (C_j)_i \to (C_j)_{i-1} \\ &= \left(\begin{array}{c|c} D_{C_{j-1}}\colon (C_{j-1})_i \to (C_{j-1})_{i-1} & I\colon (C_{j-1})_{i-1} \to (C_{j-1})_{i-1} \\ \hline 0 & D_{C_{j-1}}\colon (C_{j-1})_{i-1} \to (C_{j-1})_{i-2} \end{array}\right). \end{aligned}$$

Since the maps

$$D_{C_{j-1}}\colon (C_{j-1})_i \to (C_{j-1})_{i-1}$$

is surjective for all $1 \le i \le k$, it follows that the maps

$$D_{C_j}\colon (C_j)_i \to (C_j)_{i-1}$$

is also surjective for all $1 \le i \le k$. In particular, this argument shows that the derivative maps $D\colon A_i \to A_{i-1}$ must be surjective for all $1 \le i \le k$ as well.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (1) Assume that $0 < p < k+1$. Then setting $i = p \le k$, we claim that $D\colon A_p\colon A_{p-1}$ cannot be surjective. Indeed, set $\alpha \in A_p$ to be the pth-elementary symmetric polynomial in the first $2p-1 \le n-1$ variables:

$$\alpha = e_p(x_1, \ldots, x_{2p-1}) \in A_p.$$

Note first that over a field of characteristic p, we have

$$D(\alpha) = p \cdot e_{p-1}(x_1, \ldots, x_{2p-1}) \equiv 0.$$

On the other hand, we have

$$\alpha(1, \ldots, 1) = \binom{2p-1}{p} = \frac{(2p-1)\cdots(p+1)}{(p-1)!} \ne 0.$$

This shows that $\alpha \in P_p = \ker(D) \cap A_p$, but $\alpha \notin V(n,p,p)$ (since every polynomial in $V(n,p,p)$ necessarily vanishes at any point in which at least $p+1$-entries are equal). Therefore, by Lemma 3.2, $D\colon A_p \to A_{p-1}$ is not surjective, and hence by [11, Prop. 2.1], $D\colon A_k \to A_{k-1}$ cannot be surjective either. □

We can also derive a useful result on specialization of Specht modules.

LEMMA 3.5. *Assume that $p = 0$ or $p \geq k+1$, fix j satisfying $2k \leq j \leq n$, and set*

$$B = \frac{\mathbb{F}[x_1,\dots,x_j]}{(x_1^2,\dots,x_j^2)}.$$

Then

$$V(j,k,k) = V(n,k,k) \cap B_k.$$

Proof. Containment $V(j,k,k) \subseteq V(n,k,k) \cap B_k$ is clear. For the reverse containment, suppose that $\alpha \in V(n,k,k) \cap B_k$. The key observation here is that the restriction of the derivative map on A, call it D_A, to the subspace $B \subset A$ is the same as D_B. Since $\alpha \in V(n,k,k) \subseteq P_{A,k} = \ker(D_A) \cap A_k$, it follows that $D_A(\alpha) = 0$, and hence also $D_B(\alpha) = 0$. This means that $\alpha \in P_{B,k}$. From the proof of Lemma 3.4, we deduce that $D_B \colon B_k \to B_{k-1}$ is surjective, which by Lemma 3.2 implies that $\alpha \in V(j,k,k)$, as desired. ∎

As we shall see, the weak Lefschetz property, or lack thereof, can be used to detect embedded primary components of our Specht-monomial ideals. Next, we shall use the strong Lefschetz property to decompose our shifted Specht modules into irreducible $\mathfrak{S}_n$-representations.

3.2 Strong Lefschetz property

The pair (A,L) is *strong Lefschetz* if the restriction of L^i to the graded components of A always has maximal rank, or equivalently if

$$L^{n-2k} \colon A_k \to A_{n-k}$$

are isomorphisms for all $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. One can show that we have containment

$$V(n,k,d) \subseteq \ker\left(L^{n-k-d+1}\right) \cap A_k.$$

LEMMA 3.6. *Let $p = \mathrm{char}(\mathbb{F})$. The following are equivalent.*

1. *$p = 0$ or $p \geq n+1$,*
2. *The pair (A,L) is strong Lefschetz.*
3. *For all integers $0 \leq k \leq d \leq n-k$, the shifted Specht modules $V(n,k,d) \subset A$ satisfy*

$$V(n,k,d) = \bigoplus_{i=k}^{d} L^{d-i}(P_i) = \ker\left(L^{n-k-d+1}\right) \cap A_d.$$

Proof. (1) ⇒ (2). This is due to Ikeda; see [5, Prop. 3.66].

(2) ⇒ (1). Note that if $p \leq n$, then we must have $L^p = 0$, and (A,L) cannot be strong Lefschetz.

(2) ⇒ (3). If (A,L) is strong Lefschetz then the map

$$L^{n-k-d+1} \colon A_d \to A_{n-k+1},$$

must have maximal rank, and hence we must have

$$\dim(\ker(L^{n-k-d+1}) \cap A_d) = \dim(A_d) - \dim(A_{n-k+1}) = \binom{n}{d} - \binom{n}{k-1} = \dim(V(n,k,d)).$$

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But since we already have containment $V(n,k,d) \subseteq \ker(L^{n-k-d+1}) \cap A_d$ it must be equality. It follows that $P_i = \ker(L^{n-2i+1}) \cap A_i$, and hence we have the containment

$$\bigoplus_{i=k}^{d} L^{d-i}(P_i) \subseteq \ker(L^{n-k-d+1}) \cap A_d.$$

Since (A,L) is strong Lefschetz it follows that for each i, $L^{d-i}(P_i) \cong P_i$, and hence a simple dimension count reveals this containment must also be equality.

(3) $\Rightarrow$ (2). Assume (3) holds, and assume that (2) does not. Fix integer k satisfying $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and suppose that $\alpha \in \ker(L^{n-2k}) \cap A_k$. Then certainly $\alpha \in \ker(A^{n-2k+1}) \cap A_k$ hence $\alpha \in P_k$ by (3). But also according to (3) we have

$$A_{n-k} = V(n,0,n-k) = \bigoplus_{i=0}^{n-k} L^{n-k-i}(P_i).$$

On the other hand, if $L^{n-2k}(\alpha) = 0$, then $\dim(L^{n-2k}(P_k)) < P_k$, and hence we must have

$$\dim(A_{n-k}) = \sum_{i=0}^{n-k} \dim\left(L^{n-k-i}(P_i)\right) < \sum_{i=0}^{n-k} P_i = \binom{n}{k},$$

which is a contradiction. Therefore, (2) must hold after all. □

If any one of the conditions in Lemma 3.6 is satisfied, one can show that $L^{d-i}(P_i) \cong P_i \cong V(n,k,k)$, and hence in this case, we get a decomposition of the shifted Specht module $V(n,k,d)$ into irreducible $\mathfrak{S}_n$-representations.

Corollary 3.7. *Let $p = \operatorname{char}(\mathbb{F})$, and assume that $p = 0$ or $p \geq n+1$. Then the primitive decomposition of the shifted Specht module is a decomposition into irreducible $\mathfrak{S}_n$-representations:*

$$V(n,k,d) \cong \bigoplus_{i=k}^{d} L^{d-i}(P_i) \cong \bigoplus_{i=k}^{d} V(n,i,i)[d-i].$$

Here, a basis for the irreducible component $L^{d-i}(P_i)$ is

$$\{e_{d-i}(T^c) \cdot F_T \mid T \in \operatorname{Tab}(n,i,i)\},$$

where $e_{d-i}(T^c)$ is the $(d-i)$th elementary symmetric polynomial in the variables which are not in the support of T.

Remark 3.8. In characteristic $p = 0$, the Littlewood–Richardson rule implies that the skew representation of $\mathfrak{S}_n$ associated to the skew diagram $\lambda(d)$ has the same $\mathfrak{S}_n$-decomposition as in Corollary 3.7, for example (see [7, Th. 5.5]). Hence in this case, it follows that our shifted Specht module $V(n,k,d)$ is isomorphic to the skew representation of $\mathfrak{S}_n$ associated to the skew shape $\lambda(d)$. We thank the referee for pointing out this connection.

§4. Radical of shifted Specht ideals

Our main idea is principal radical systems, based on the following basic facts from commutative algebra:

Lemma 4.1. *Let $I \subset R$ be a homogeneous ideal and $x \in R \setminus I$ be any homogeneous polynomial of positive degree satisfying $(I : x) = (I : x^2)$.*

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1. *If* $(I:x)=I$ *and if* $I+(x)$ *is radical, then* I *is radical too.*
2. *If* $(I:x)\neq I$ *and if* $(I:x)$ *and* $I+(x)$ *are both radical, then* I *is radical too.*

Proof. Note that $I\subset I+(x)$ and if $I+(x)$ is radical we also have

$$\sqrt{I}\subset I+(x).$$

Hence, for $g\in\sqrt{I}$, we can find $a\in I$ and $b\in R$ such that $g=a+xb$. Since $g\in\sqrt{I}$, there is some integer N for which $g^N\in I$, and we have $g^N=a'+x^Nb^N$ for some other $a'\in I$. Therefore, $x^Nb^N\in I$ and hence $b^N\in(I:x^N)=(I:x)$ and therefore, $b\in\sqrt{(I:x)}$. If $(I:x)\neq I$ and $(I:x)$ is radical, then $xb\in I$ and hence $g\in I$, and we are done. If $(I:x)=I$, then $b\in\sqrt{I}$, and hence we can find $a_1\in I$ and $b_1\in R$ for which $b=a_1+xb_1$. Looking back to g, we have $g=(a+xa_1)+x^2b_1$. Repeating this procedure a number m-times will yield $g=(a+xa_1+x^2a_2+\cdots+x^ma_m)+x^{m+1}b_m$, which implies that

$$g\in\bigcap_{m=1}^{\infty}I+(x^m).$$

Since x is homogeneous, it follows that $\bigcap_{m=1}^{\infty}I+(x^m)=I$, and the result follows. □

An easy application of principal radical systems is the ideal $(x_1,\ldots,x_n)^{\langle d\rangle}$ generated by square-free monomials of degree d.

LEMMA 4.2. *For each integer* d *satisfying* $1\leq d\leq n$ *the square-free monomial ideal*

$$(x_1,\ldots,x_n)^{\langle d\rangle}$$

is radical.

Proof. By induction on $n\geq 1$, the base case being trivial. For the inductive step, assume that $(x_1,\ldots,x_{n-1})^{\langle e\rangle}$ is radical for all $1\leq e\leq n-1$, and fix an integer d satisfying $1\leq d\leq n$. Set $I=(x_1,\ldots,x_n)^{\langle d\rangle}$ and $x=x_n$. If $d=n$ then $I=(x_1\cdots x_n)$ is principal, and clearly radical, hence we may assume that $1\leq d\leq n-1$. Then we have

$$(I:x)=(x_1,\ldots,x_{n-1})^{\langle d-1\rangle},\quad\text{and}\quad I+(x)=(x_1,\ldots,x_{n-1})^{\langle d\rangle}+(x_n),$$

which are both radical by the induction hypothesis. Also note that $(I:x^2)=(I:x)$ since I is generated by square-free monomials. It therefore follows from Lemma 4.1 that the ideal $I=(x_1,\ldots,x_n)^{\langle d\rangle}$ is radical. □

Applying principal radical systems to Specht ideals requires the following decomposition of shifted Specht ideals, and is key to the further results of this paper. This is Theorem 1.6 from §1.

THEOREM 4.3 (Theorem 1.6). *For any integers* k,d *satisfying* $0\leq k<d\leq n-k$, *we have*

$$\mathfrak{a}(n,k,d)=\mathfrak{a}(n,k,d-1)\cap(x_1,\ldots,x_n)^{\langle d\rangle}=\mathfrak{a}(n,k,k)\cap(x_1,\ldots,x_n)^{\langle d\rangle}.\tag{11}$$

Before embarking on the proof of Theorem 4.3, we show how to use Theorem 4.3 and principal radical systems to show that shifted Specht ideals are radical. This is Theorem 1.7 from §1.

THEOREM 4.4 (Theorem 1.7). *For any integers* k,d *satisfying* $0\leq k\leq d\leq n-k$, *the shifted Specht ideal* $\mathfrak{a}(n,k,d)$ *is radical.*

4.1 Shifted Specht ideals are radical

Proof of Theorem 4.4. Assuming that Theorem 4.3 holds, we prove Theorem 4.4 by induction on $n \geq 2$. For the base case $n = 2$, the only possibilities for integers k, d are $k = 0$ and $d = 1, 2$ and $k = 1 = d$. In the case $k = 0$, we have $\mathfrak{a}(2,0,d) = (x_1, x_2)^{\langle d \rangle}$ which is radical by Lemma 4.5. In the case, $k = d = 1$ we have $\mathfrak{a}(2,1,1) = ((x_1 - x_2))$ is prime, and therefore radical. For the inductive step, assume that $\mathfrak{a}(n-1, j, e)$ is radical for all integers satisfying $0 \leq j \leq e \leq n-1-j$. Fix integers k, d satisfying $1 \leq k \leq d \leq n-k$. First, we argue for the case $d = k$. Let $I = \mathfrak{a}(n,k,k)$ and $x = x_n$. Then by Corollary 2.7 it follows that we have

$$I + (x) = \mathfrak{a}(n,k,k) + (x_n) = \mathfrak{a}(n-1, k-1, k) + (x_n).$$

By induction hypothesis, $\mathfrak{a}(n-1, k-1, k)$ is radical, and since its generators are polynomials which are independent of x_n, it follows that the sum $I + (x)$ is radical. Also using the change of coordinates map $\Phi\colon x_i \mapsto y_i$ in Corollary 2.7 we find that

$$\Phi(I : x) = (\Phi(I) : \Phi(x_n)) = (\mathfrak{a}(n-1, k-1, k) : x_n) = \mathfrak{a}(n-1, k-1, k) = \Phi(I),$$

from which it follows that $(I : x) = I$. Therefore, it follows from Lemma 4.1 that $I = \mathfrak{a}(n,k,k)$ itself must be radical.

For $d > k$, we appeal again to Theorem 4.3:

$$\mathfrak{a}(n,k,d) = \mathfrak{a}(n,k,d-1) \cap (x_1, \ldots, x_n)^{\langle d \rangle} = \mathfrak{a}(n,k,k) \cap (x_1, \ldots, x_n)^{\langle d \rangle}.$$

Since $\mathfrak{a}(n,k,k)$ is radical, and $(x_1, \ldots, x_n)^{\langle d \rangle}$ is radical, it follows that $\mathfrak{a}(n,k,d)$ is radical too.

4.2 Decomposition of shifted Specht ideals

The proof of Theorem 4.3 comes in three steps, each of which we state as a lemma. First some notation: For any exponent vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we denote the associated monomial by $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$, and its radical by $\sqrt{\mathbf{x}^{\mathbf{a}}} = \prod_{a_i > 0} x_i$.

Lemma 4.5. *The ideal $\mathfrak{a}(n,k,d-1) \cap (x_1, \ldots, x_n)^{\langle d \rangle}$ is generated by products of monomials and polynomials in the shifted Specht module $V(n,k,d-1)$. In fact, if the sum of any monomials times forms in $V(n,k,d-1)$ lies in the intersection $\mathfrak{a}(n,k,d-1) \cap (x_1, \ldots, x_n)^{\langle d \rangle}$, then so do each of its summands.*

Proof. It is not difficult to see that every polynomial $P \in \mathfrak{a}(n,k,d-1)$ decomposes into a sum of terms of the form

$$P = \sum_{\mathbf{a} \in \mathbb{N}^n} \mathbf{x}^{\mathbf{a}} \cdot \ \nu_{\mathbf{a}},$$

where $\nu_{\mathbf{a}} \in V(n,k,d-1)$. We want to show that if $P \in (x_1, \ldots, x_n)^{\langle d \rangle}$, then each of its summands are too, that is, $\mathbf{x}^{\mathbf{a}} \cdot \nu_{\mathbf{a}} \in (x_1, \ldots, x_n)^{\langle d \rangle}$ for all $\mathbf{a} \in \mathbb{N}^n$. Suppose by way of contradiction that for some $\mathbf{a} \in \mathbb{N}^n$ and some $\nu_{\mathbf{a}} \in V(n,k,d-1)$ that $\mathbf{x}^{\mathbf{a}} \cdot \nu_{\mathbf{a}} \notin (x_1, \ldots, x_n)^{\langle d \rangle}$. Since $(x_1, \ldots, x_n)^{\langle d \rangle}$ is a monomial ideal, there must be some monomial $\mathbf{x}^{\mathbf{b}}$ which appears in the monomial expansion of $\mathbf{x}^{\mathbf{a}} \cdot \nu_{\mathbf{a}}$ such that $\mathbf{x}^{\mathbf{b}} \notin (x_1, \ldots, x_n)^{\langle d \rangle}$. Define the *weight* of monomial $\mathbf{x}^{\mathbf{b}}$ as $\mathrm{wt}(\mathbf{x}^{\mathbf{b}}) = \#\{b_i > 0\}$. Since $(x_1, \ldots, x_n)^{\langle d \rangle}$ consists of all monomials of weight at least d, it follows that $\mathrm{wt}(\mathbf{x}^{\mathbf{b}}) \leq d-1$. On the other hand, since $\nu_{\mathbf{a}}$ is a linear combination of shifted Specht polynomials of type $\lambda(d)$, it follows that every monomial in the monomial

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expansion of $\nu_{\mathbf{a}}$ also has weight $d-1$. This implies that $\mathrm{wt}(\mathbf{x}^{\mathbf{b}}) = d-1$, and therefore that

$$\frac{\mathbf{x}^{\mathbf{b}}}{\sqrt{\mathbf{x}^{\mathbf{b}}}} = \mathbf{x}^{\mathbf{a}}.$$

In particular, we see that the monomial $\mathbf{x}^{\mathbf{b}}$ is unique to the term $\mathbf{x}^{\mathbf{a}} \cdot \nu_{\mathbf{a}}$, and hence must occur with the same coefficient in the monomial expansion of $\mathbf{x}^{\mathbf{a}} \cdot \nu_{\mathbf{a}}$ as it does in the monomial expansion of $P = \sum_{\mathbf{a}\in\mathbb{N}^n} \mathbf{x}^{\mathbf{a}} \cdot \nu_{\mathbf{a}}$. Therefore $P \notin (x_1,\dots,x_n)^{\langle d\rangle}$, as desired. ∎

Lemma 4.5 tells us that it suffices to check Equation (11) in Theorem 4.3 on products of monomials with $V(n,k,d-1)$. So we want to show that for each $\nu \in V(n,k,d-1)$ and for each $\mathbf{a} \in \mathbb{N}^n$ the following implication holds:

$$\mathbf{x}^{\mathbf{a}} \cdot \nu \in (x_1,\dots,x_n)^{\langle d\rangle} \;\Rightarrow\; \mathbf{x}^{\mathbf{a}} \cdot \nu \in \mathfrak{a}(n,k,d).$$

Note that since $(x_1,\dots,x_n)^{\langle d\rangle}$ is generated by square-free monomials, we have

$$\mathbf{x}^{\mathbf{a}} \cdot \nu \in (x_1,\dots,x_n)^{\langle d\rangle} \;\Leftrightarrow\; \sqrt{\mathbf{x}^{\mathbf{a}}} \cdot \nu \in (x_1,\dots,x_n)^{\langle d\rangle}.$$

In particular, we may assume without loss of generality that our monomials $\mathbf{x}^{\mathbf{a}}$ are square-free. For any polynomial $F \in R$ define its *support* to be the set of square-free monomials which divide some nonzero monomial term of F. For example, given a tableau $T \in \mathrm{Tab}(n,k,d)$, the support of the shifted Specht polynomial $F_T(d-1)$ is the set of square-free monomials indexed by subsets of numbers in the support of T, no two of which lie in the same column of T.

LEMMA 4.6. *For each $T \in \mathrm{Tab}(n,k,d)$, if $\mathbf{x}^{\mathbf{a}} \notin \mathrm{supp}(F_T(d-1))$ then $\mathbf{x}^{\mathbf{a}} \cdot F_T(d-1) \in \mathfrak{a}(n,k,d)$.*

Proof. If there is a variable $x_i \in \mathrm{supp}(\mathbf{x}^{\mathbf{a}})$ which is not in $\mathrm{supp}(F_T(d-1))$, then certainly $\mathbf{x}^{\mathbf{a}} \cdot F_T(d-1) \in \mathfrak{a}(n,k,d)$. Otherwise, there must be two indices $i \neq j$ such that $x_i, x_j \in \mathrm{supp}(\mathbf{x}^{\mathbf{a}})$ and i,j in the same column of T. Choose any index $r \neq i,j$ such that $x_r \notin \mathrm{supp}(F_T(d-1))$, which exists since $0 \le k \le d-1 < d \le n-k$, and let $(i,r),(j,r) \in \mathfrak{S}_n$ be the transpositions swapping i,r and j,r, respectively. Then we have

$$F_T(d-1) = F_{(i,r).T}(d-1) + F_{(j,r).T}(d-1),$$

and since $x_i \cdot F_{(i,r).T}(d-1), x_j \cdot F_{(j,r).T}(d-1) \in \mathfrak{a}(n,k,d)$, it follows that

$$\mathbf{x}^{\mathbf{a}} \cdot F_T(d-1) \in \mathfrak{a}(n,k,d),$$

as desired. ∎

Finally, we must show what happens with square-free monomials which do lie in the support of the shifted Specht polynomials. Here, we use symmetry to make the further reduction that our square-free monomial is initial, that is,

$$\mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{m}} = x_1 \cdots x_m, \quad \text{for some } 1 \le m \le d-1.$$

Setting $V^m(n,k,d-1) \subset V(n,k,d-1)$ to be the span of shifted Specht polynomials indexed by standard tableaux $T \in \mathrm{STab}(n,k,d-1)$ for which $\{1,\dots,m\} \not\subset \mathrm{supp}(T)$. Then the shifted Specht module decomposes into a direct sum

$$V(n,k,d-1) = V_m(n,k,d-1) \oplus V^m(n,k,d-1),$$

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and Lemma 4.6 says $\mathbf{x}^{\mathbf{m}}\cdot\nu\in\mathfrak{a}(n,k,d)$ for every $\nu\in V^m(n,k,d-1)$. Recall the bijective linear map

$$\phi\colon V(n,k,d-1)\longrightarrow V([n]_m,k-m,d-m),$$
$$F_T(d)\longmapsto F_{T'}(d),$$

where

$$T=\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & & & 1 & \cdots & m & i_{m+1} & \cdots & i_k & i_{k+1} & \cdots & i_{d-1}\\ \hline i_d & \cdots & i_{n-k} & j_1 & \cdots & j_m & j_{m+1} & \cdots & j_k & & & \\ \hline\end{array}\tag{12}$$

and

$$T'=\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline & & & & & & i_{m+1} & \cdots & i_k & i_{k+1} & \cdots & i_{d-1}\\ \hline i_d & \cdots & i_{n-k} & j_1 & \cdots & j_m & j_{m+1} & \cdots & j_k & & & \\ \hline\end{array}\tag{13}$$

LEMMA 4.7. *Fix $\nu\in V_m(n,k,d-1)$. If $\mathbf{x}^{\mathbf{m}}\cdot\nu\in(x_1,\ldots,x_n)^{\langle d\rangle}$ then $\nu=0$, and hence $\mathbf{x}^{\mathbf{m}}\cdot\nu\in\mathfrak{a}(n,k,d)$.*

Proof. We observe that

$$\mathbf{x}^{\mathbf{m}}\cdot(\nu-\mathbf{x}^{\mathbf{m}}\cdot\nu')\in(x_1,\ldots,x_n)^{\langle d\rangle},$$

where $\nu'\in V([n]_m,k-m,d-1-m)$ is the image of ν in the map above. Indeed note that for each standard tableau $T\in\mathrm{STab}_m(n,k,d-1)$ as in (12), the support of the difference $F_T(d-1)-\mathbf{x}^{\mathbf{m}}\cdot F_{T'}(d-1-m)$ does not contain the monomial $\mathbf{x}^{\mathbf{m}}$, hence the product $\mathbf{x}^{\mathbf{m}}\cdot(F_T(d)-\mathbf{x}^{\mathbf{m}}\cdot F_{T'}(d-1-m))\in(x_1,\ldots,x_n)^{\langle d\rangle}$. Hence, if $\mathbf{x}^{\mathbf{m}}\cdot\nu\in(x_1,\ldots,x_n)^{\langle d\rangle}$, it follows that $(\mathbf{x}^{\mathbf{m}})^2\cdot\nu'\in(x_1,\ldots,x_n)^{\langle d\rangle}$, and hence that $\nu'\in\left((x_1,\ldots,x_n)^{\langle d\rangle}:\mathbf{x}^{\mathbf{m}}\right)=(x_{m+1},\ldots,x_n)^{(d-m)}$. For degree reasons this implies that $\nu'=0$, and hence by Corollary 2.8, $\nu=0$ as well. ∎

Proof of Theorem 4.3. The containment $\mathfrak{a}(n,k,d)\subset\mathfrak{a}(n,k,d-1)\cap(x_1,\ldots,x_n)^{\langle d\rangle}$ is clear. For the reverse containment, Lemma 4.5 implies that it suffices to check it on products of monomials with polynomials in $V(n,k,d-1)$. Since $(x_1,\ldots,x_n)^{\langle d\rangle}$ is generated by square-free monomials we may assume that our monomials are square-free, and by symmetry, we may assume that our monomial has the form $\mathbf{x}^{\mathbf{m}}=x_1\cdots x_m$ for some integer $1\le m\le d$. Then as above we have

$$V(n,k,d-1)=V_m(n,k,d-1)\oplus V^m(n,k,d-1),$$

and by Lemma 4.7, $\mathbf{x}^{\mathbf{m}}\cdot\nu\in\mathfrak{a}(n,k,d)$ if $\nu\in V_m(n,k,d-1)$. Furthermore, Lemma 4.6 implies that for $\nu\in V^m(n,k,d-1)$ if $\mathbf{x}^{\mathbf{m}}\cdot\nu\in(x_1,\ldots,x_n)^{\langle d\rangle}$ then $\mathbf{x}^{\mathbf{m}}\cdot\nu\in\mathfrak{a}(n,k,d)$, and the result follows.

§5. Perfection of Specht and Specht-monomial ideals

Recall that an ideal $I\subset R$ in a commutative ring has projective dimension s if a minimal resolution of R/I as an R-module has length s. Its grade is the length g of a maximal R-sequence contained in I, or equivalently the smallest integer g for which the Ext group

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$\mathrm{Ext}^g_R(R/I,R)$ is nonzero. We say that the ideal $I \subset R$ is *perfect* if its projective dimension is equal to its grade. In our case, where R is polynomial (hence Cohen–Macaulay), the grade of an ideal is equal to its height, and by the Auslander–Buchsbaum formula, I is perfect if and only if the quotient R/I is Cohen–Macaulay. For more details on these matters we refer the reader to [10].

As in §4, the main idea here is to use principal radical systems.

LEMMA 5.1. *Let $I \subset R$ be any homogeneous ideal and let $x \in R \setminus I$ be any homogeneous polynomial of positive degree. Then*

1. *if $(I:x) = I$ and if $I + (x)$ is perfect, then I is perfect too.*
2. *if $(I:x) \neq I$ and if $(I:x)$ and $I + (x)$ are both perfect of the same grade g, then I is perfect of that same grade too.*

Proof. Item (1) is well known, and can be found in any commutative algebra text, for example, [10, Th. 17.3]. For (2), we use the long exact sequence for Ext-modules associated with the short exact sequence of R-modules

$$0 \longrightarrow R/(I:x) \xrightarrow{\times x} R/I \longrightarrow R/I + (x) \longrightarrow 0.$$

∎

As in §4, we give an easy application of principal radical systems to the ideal $(x_1,\ldots,x_n)^{\langle d\rangle}$ generated by square-free monomials of degree d.

LEMMA 5.2. *For every integer d satisfying $1 \le d \le n$ the square-free monomial ideal*

$$(x_1,\ldots,x_n)^{\langle d\rangle}$$

is perfect[4] *of grade $n-d+1$.*

Proof. By induction on n, where the base case $n=1$ is trivial. For the induction step, assume that $(x_1,\ldots,x_{n-1})^{\langle e\rangle}$ is perfect of grade $(n-1)-e+1$ for every $1 \le e \le n-1$, and fix an integer d satisfying $1 \le d \le n$. Set $I = (x_1,\ldots,x_n)^{\langle d\rangle}$ and $x = x_n$. Then we have

$$(I:x) = (x_1,\ldots,x_{n-1})^{\langle d-1\rangle} \qquad I + (x) = (x_1,\ldots,x_{n-1})^{\langle d\rangle} + (x_n),$$

which are both perfect of the same grade $n-d+1$ by our induction hypothesis. It follows from Lemma 5.1 that $I = (x_1,\ldots,x_n)^{\langle d\rangle}$ is perfect of grade $n-d+1$ too. ∎

It is trickier to apply Lemma 5.1 to shifted Specht ideals. For one thing, not all shifted Specht ideals are perfect. Indeed, Theorem 4.3 gives the decomposition

$$\mathfrak{a}(n,k,d) = \mathfrak{a}(n,k,k) \cap (x_1,\ldots,x_n)^{\langle d\rangle}.$$

Individually, the grades (=heights) of the ideals $\mathfrak{a}(n,k,k)$ and $(x_1,\ldots,x_n)^{\langle d\rangle}$ are $g = n-k$ and $g = n-d+1$, respectively. In particular, we see that if $d > k+1$, then the two ideals $\mathfrak{a}(n,k,k)$ and $(x_1,\ldots,x_n)^{\langle d\rangle}$ have mixed heights, and in particular, the shifted Specht ideal $\mathfrak{a}(n,k,d)$ cannot be perfect. For $d = k+1$, we must show that the intersection is perfect:

$$\mathfrak{a}(n,k,k+1) = \mathfrak{a}(n,k,k) \cap (x_1,\ldots,x_n)^{\langle d\rangle}.$$

[4] Alternatively, one could also appeal to Reisner's theorem [13] here, since $(x_1,\ldots,x_n)^{\langle d\rangle}$ is the Stanley–Reisner ideal of the $d-1$ skeleton of an $n-1$ simplex.

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To this end, we appeal to the following basic fact, which appears in the paper [6] by Hochster–Eagon, and we refer the reader there for its proof.

LEMMA 5.3 ([6, Prop. 18]). *Suppose that $I, J \subset R$ are two perfect ideals of the same grade g, and assume that $I+J$ has grade $g+1$. Then $I+J$ is perfect if and only if $I \cap J$ is perfect.*

In the spirit of Lemma 5.3, we study the following sum of ideals, which plays a key role in this paper.

DEFINITION 5.4. Fix an integer k satisfying $1 \leq k < k+1 \leq n-k$, and define the *Specht-monomial ideal* for the pair (k,n) to be the sum of ideals

$$I(n,k) = \mathfrak{a}(n,k,k) + (x_1, \dots, x_n)^{\langle k+1 \rangle}. \tag{14}$$

It turns out that perfection of Specht-monomial ideals depends on the characteristic of the field $\mathbb{F}$. What is perhaps surprising is that this dependence on characteristic is the same as the dependence on characteristic of the weak Lefschetz property of certain Artinian monomial complete intersections. This is captured in the following result which is Theorem 1.8 from §1. Recall our notational conventions: we denote by $(z_1, \dots, z_m)^{\langle q \rangle}$ the ideal generated by square-free monomials in the variables $z_1, \dots, z_m$ of degree q.

THEOREM 5.5 (Theorem 1.8). *Let $p = \mathrm{char}(\mathbb{F}) \geq 0$ and fix positive integers n, k satisfying $n \geq 2k+1$. Then the following conditions are equivalent.*

1. $p = 0$ *or* $p \geq k+1$.
2. *The quadratic monomial complete intersection*

$$C = \frac{\mathbb{F}[x_1, \dots, x_{2k}]}{(x_1^2, \dots, x_{2k}^2)}$$

has the weak Lefschetz property.

3. *The Specht monomial ideal $I(n,k)$ satisfies*

$$I(n,k) = I(n-1,k-1) \cap \left((y_1, \dots, y_{n-1})^{\langle k \rangle} + (x_n^2) \right), \tag{15}$$
$$\textit{where} \quad y_i = x_n - x_i \quad \textit{for} \quad 1 \leq i \leq n-1.$$

4. *The Specht-monomial ideal $I(n,k)$ is perfect.*

As in §1, we shall break the proof into three parts, one for each equivalence (1) $\Leftrightarrow$ (a) and refer to it as Theorem 5.5(a) for $\mathrm{a} = 2,3,4$. Recall that Theorem 5.5(2) follows from a result of Kustin–Vraciu [8], and we do not prove it here; see §3, particularly Lemmas 3.3 and 3.4. Hence to prove Theorem 5.5, it suffices to prove the two equivalences Theorems 5.5(3) and 5.5(4). Theorem 5.5(3) is the key technical result of this section, and it yields a primary decomposition of the Specht-monomial ideal. We use it to prove Theorem 5.5(4), which we shall then use in turn to prove the following result which is Theorem 1.2 from §1.

THEOREM 5.6 (Theorem 1.2). *Let $p = \mathrm{char}(\mathbb{F}) \geq 0$ and fix positive integers n, k satisfying $n \geq 2k+1$.*

1. *If $p = 0$ or $p \geq k+1$, then the Specht ideal $\mathfrak{a}(n+1, k+1, k+1)$ is perfect.*
2. *If $n \geq 2p+1$, then the Specht ideal $\mathfrak{a}(n+1, p+1, p+1)$ is not perfect.*

The proof of Theorem 5.5(3) is given at the end. The next subsection is devoted to the proof of Theorem 5.5(4), assuming the truth of Theorem 5.5(3).

5.1 Specht-monomial ideals are perfect

Assuming that Theorem 5.5(3) holds, we prove Theorem 5.5(4), focusing first on the implication (1) $\Rightarrow$ (2) first. Assuming that $p=0$ or $p\geq k+1$, Theorem 5.5(3) says that for all $1\leq k<k+1\leq n-k$ the Specht-monomial ideal satisfies (15):

$$I(n,k)=I(n-1,k-1)\cap\Big((y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)\Big).$$

We need the following Lemma, which tells us how to go between x-variables and y-variables, and provides a direct link to Lefschetz properties from §3.

LEMMA 5.7. *Let $P^j(x_1,\dots,x_{n-1})$ be any square-free polynomial of degree j in the variables $x_1,\dots,x_{n-1}$. Then modulo the principal ideal (x_n^2) we have*

$$P^j(x_1,\dots,x_{n-1})\equiv(-1)^j\left(P^j(y_1,\dots,y_{n-1})-x_n\cdot D\left(P^j(y_1,\dots,y_{n-1})\right)\right)\qquad \mathit{mod}\ (x_n^2),$$

where D is the linear partial differentiation operator $D=\dfrac{\partial}{\partial y_1}+\cdots+\dfrac{\partial}{\partial y_{n-1}}$. In particular, if $\alpha=\alpha(x_1,\dots,x_{n-1})\in V(n-1,k-1,k-1)$ is a linear combination of Specht polynomials then

$$\alpha(x_1,\dots,x_{n-1})=(-1)^{k-1}\cdot\alpha(y_1,\dots,y_{n-1}).$$

Proof. By linearity of D it suffices to assume that P^j is a square-free monomial, and by symmetry, we may assume is $P^j(x_1,\dots,x_{n-1})=x_1\cdots x_j$. Then we have

$$\begin{aligned}P^j(x_1,\dots,x_{n-1})&=x_1\cdots x_j=(x_n-y_1)\cdots(x_n-y_j)\\&=x_n^2\cdot(\text{stuff})+(-1)^{j-1}\cdot x_n\left(\sum_{i=1}^{j}y_1\cdots\hat{y}_i\cdots y_j\right)+(-1)^j\cdot y_1\cdots y_j\\&\quad\text{(where }\hat{y}\text{ means omission)}\\&=(-1)^j\left(y_1\cdots y_j-x_nD\left(y_1\cdots y_j\right)\right)+x_n^2\cdot(\text{stuff})\\&\equiv(-1)^j\left(P^j(y_1,\dots,y_{n-1})-x_nD\left(P^j(y_1,\dots,y_{n-1})\right)\right)\qquad \text{mod }(x_n^2),\end{aligned}$$

as claimed. The second statement follows from the first since $D(\alpha)=0$. ∎

Next, for each pair of positive integers n,k satisfying $n\geq 2k+1$ let us form the new ideal

$$J(n,k)=I(n-1,k-1)+(y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2). \tag{16}$$

LEMMA 5.8. *Assume that $p=0$ or $p\geq k+1$. Then the ideal $J(n,k)$ satisfies*

$$J(n,k)+(x_n)=I(n-1,k-1)+(x_n),\quad \mathit{and}\quad (J(n,k):x_n)=(x_1,\dots,x_{n-1})^{\langle k-1\rangle}+(x_n).$$

Proof. We have

$$\begin{aligned}J(n,k)+(x_n)&=I(n-1,k-1)+(y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)+(x_n)\\&=\mathfrak{a}(n-1,k-1,k-1)+(x_1,\dots,x_{n-1})^{\langle k\rangle}+(y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n)\\&=\mathfrak{a}(n-1,k-1,k-1)+(x_1,\dots,x_{n-1})^{\langle k\rangle}+(x_n)\\&=I(n-1,k-1)+(x_n).\end{aligned}$$

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For the other equality, note first that containment $(J(n,k):x_n) \supseteq (x_1,\dots,x_{n-1})^{\langle k-1\rangle} + (x_n) = (y_1,\dots,y_{n-1})^{\langle k-1\rangle} + (x_n)$ follows from Lemma 5.7. Indeed identifying the space of square-free polynomials with the monomial complete intersection $B = \mathbb{F}[x_1,\dots,x_{n-1}]/(y_1^2,\dots,y_{n-1}^2)$, Lemma 3.4 implies that the derivative map $D\colon B_k \to B_{k-1}$ is surjective, and hence for any square-free monomial of degree $k-1$ in variables $y_1,\dots,y_{n-1}$, say $P = y_1\cdots y_{k-1}$, we know by Lemma 3.4 there is square-free polynomial of degree k for which

$$D(Q(y_1,\dots,y_{n-1})) = P(y_1,\dots,y_{n-1}) = y_1\cdots y_{k-1}.$$

Then Lemma 5.7 implies that

$$\begin{aligned} P\cdot x_n = y_1\cdots y_{k-1}\cdot x_n &= x_n \cdot D\left(Q(y_1,\dots,y_{n-1})\right) \\ &\equiv Q(x_1,\dots,x_{n-1}) \pm Q(y_1,\dots,y_{n-1}) \qquad \mod (x_n^2). \end{aligned}$$

Since $Q(x_1,\dots,x_{n-1}) \in I(n-1,k-1)$ and $Q(y_1,\dots,y_{n-1}) \in (y_1,\dots,y_{n-1})^{\langle k\rangle}$, it follows that $x_n\cdot P = x_n\cdot D(Q(y_1,\dots,y_{n-1})) \in J(n,k)$, and hence $P = y_1\cdots y_{k-1} \in (J(n,k):x_n)$. For the reverse containment, fix $G \in (J(n,k):x_n)$. Then for each $S \in \mathrm{STab}(n-1,0,k)$ there exists polynomials $d_S \in R$ for which

$$x_nG - \sum_{S\in \mathrm{STab}(n-1,0,k)} d_SM_S \in I(n-1,k-1) + (x_n^2),$$

where $M_S = M_S(y_1,\dots,y_{n-1}) \in (y_1,\dots,y_{n-1})^{\langle k\rangle}$ are square-free monomials of degree k in the y-variables. By Lemma 5.7, we have for each $S \in \mathrm{STab}(n-1,0,k)$

$$M_S(y_1,\dots,y_{n-1}) \equiv \pm\left(M_S(x_1,\dots,x_{n-1}) - x_nD(M_S(x_1,\dots,x_{n-1}))\right) \qquad \mod (x_n^2),$$

and since $M_S(x_1,\dots,x_{n-1}) \in (x_1,\dots,x_{n-1})^{\langle k\rangle} \subset I(n-1,k-1)$, we see that

$$\begin{aligned} x_nG - \sum_{S\in \mathrm{STab}(n-1,0,k)} d_SM_S(y_1,\dots,y_{n-1}) &\equiv x_nG - \sum_{S\in \mathrm{STab}(n-1,0,k)} d_Sx_nD(M_S(y_1,\dots,y_{n-1})) \\ &\equiv 0 \mod I(n-1,k-1) + (x_n^2). \end{aligned}$$

Since x_n is a nonzero divisor for $I(n-1,k-1)$, it follows that

$$\begin{aligned} G - \sum_{S\in \mathrm{STab}(n-1,0,k)} & d_SD(M_S(x_1,\dots,x_{n-1})) \in I(n-1,k-1) \\ &+ (x_n) \subset (x_1,\dots,x_{n-1})^{\langle k-1\rangle} + (x_n), \end{aligned}$$

and since $D(M_S(x_1,\dots,x_{n-1})) \in (x_1,\dots,x_{n-1})^{\langle k-1\rangle}$ for all $S \in \mathrm{STab}(n-1,0,k)$, we see also that $G \in (x_1,\dots,x_{n-1})^{\langle k-1\rangle} + (x_n)$, as desired. ∎

We are now in a position to prove implication (1) ⇒ (2) in Theorem 5.5(4).

Proof of (1) ⇒ (2) in Theorem 5.5(4). Assume that $p = 0$ or $p \geq k+1$. We prove by induction on $n \geq 3$ for each integer k satisfying $1 \leq k < k+1 \leq n-k$ then the ideal $I(n,k)$ is perfect. The base case is $n = 3$ where the only possible k value is $k = 1$. Here, the assumption on p is vacuous, and we have

$$I(3,1) = \mathfrak{a}(3,1,1) + (x_1,x_2,x_3)^{\langle 2\rangle} = (x_1 - x_3, x_2 - x_3, x_1x_2, x_1x_3, x_2x_3).$$

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Note that $\mathbb{F}[x_1,x_2,x_3]/I(3,1) \cong \mathbb{F}[z]/(z^2)$ is Cohen–Macaulay, which implies that $I(3,1)$ is perfect of grade $n-k+1=3$.

For the inductive step, assume that $I(n-1,j)$ is perfect for all integers j satisfying $1 \le j < j+1 \le n-1-j$. Fix k satisfying $1 \le k < k+1 \le n-k$, and consider the Specht-monomial ideal

$$I(n,k) = \mathfrak{a}(n,k,k) + (x_1,\ldots,x_n)^{\langle k+1\rangle}.$$

Consider the sum $J(n,k)$ as in (16), that is,

$$J(n,k) = I(n-1,k-1) + (y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2).$$

Note that $I(n-1,k-1)$ is perfect by the induction hypothesis. Also $(y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ is perfect since $(y_1,\ldots,y_{n-1})^{\langle k\rangle}$ is perfect (by Lemma 5.2), and x_n^2 is $(y_1,\ldots,y_{n-1})^{\langle k\rangle}$-regular. Therefore, by Lemma 5.3, $I(n,k)$ is perfect if and only if $J(n,k)$ is perfect. But according to Lemma 5.8 $J(n,k)+(x_n) = I(n-1,k-1)+(x_n)$ and $(J(n,k):x_n) = (x_1,\ldots,x_{n-1})^{\langle k-1\rangle} + (x_n)$ which are both perfect of the same grade $g = n-k+2$, and hence by Lemma 5.2, it follows that $J(n,k)$ is also perfect (of grade $g = n-k+2$). Thus it follows that $I(n,k)$ is perfect completing the induction step, and hence the proof. ∎

Next, we prove the reverse implication (2) ⇒ (1) in Theorem 5.5(4) which also requires a bit of a set up. First note that Theorem 5.5(3) implies that if $p=0$ or $p \ge k+1$, then $I(n,k)$ has the following irredundant primary decomposition:

$$I(n,k) = \mathfrak{a}(n,k,k) + (x_1,\ldots,x_n)^{\langle k+1\rangle} = \bigcap_{\sigma\in\mathfrak{S}_n} \underbrace{\sigma.\left(x_1-x_2,\ldots,x_1-x_{n-k+1},x_1^2\right)}_{Q_\sigma}. \tag{17}$$

Note that for $\sigma = e$, we have $Q_e = \mathfrak{a}(n-k+1,1,1) + (x_1,\ldots,x_{n-k+1})^{\langle 2\rangle}$; in particular, Q_e contains all quadratic forms in the variables $x_1,\ldots,x_{n-k+1}$.

Lemma 5.9. *Over any field $\mathbb{F}$, and for any positive integers n,k satisfying $n \ge 2k+1$, if $I(n,k)$ is perfect then $I(n,k)$ must satisfy the decomposition (17).*

Proof. Assume that $I(n,k)$ is perfect, and consider the intersection of primary ideals as in (17), that is,

$$I'(n,k) = \bigcap_{\sigma\in\mathfrak{S}_n} Q_\sigma,$$

with minimal associated prime divisors given by

$$P_\sigma = \sqrt{Q_\sigma} = \sigma.(x_1,\ldots,x_{n-k+1}).$$

We would like to show that $I(n,k) = I'(n,k)$. Note first that they have the same radical:

$$\sqrt{I'(n,k)} = \bigcap_{\sigma\in\mathfrak{S}_n} P_\sigma = (x_1,\ldots,x_n)^{\langle k\rangle} = \sqrt{I(n,k)}. \tag{18}$$

For the last equality, the containment $I(n,k) \subseteq (x_1,\ldots,x_n)^{\langle k\rangle}$ is clear, and the other containment follows from the containment:

$$x_{i_1}^2\cdots x_{i_k}^2 = x_{i_1}\cdots x_{i_k}\cdot F_T + \left(\text{stuff in } (x_1,\ldots,x_n)^{\langle k+1\rangle}\right) \in I(n,k),$$

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where

$$T = \begin{array}{|c|c|c|c|c|c|} \cline{4-6} \multicolumn{3}{c|}{} & i_1 & \cdots & i_k \\ \hline i_{k+1} & \cdots & i_{n-k} & j_1 & \cdots & j_k \\ \hline \end{array}$$

It follows that $\{P_\sigma \mid \sigma \in \mathfrak{S}_n\}$ is a complete list of minimal prime divisors of $I(n,k)$, which implies Equation (18).

Set $I' = I'(n,k)$ and $I = I(n,k)$. Suppose that a primary decomposition of I is given by $I = U_1 \cap \cdots \cap U_m$. Since I is perfect, all of its associated prime divisors must be minimal, and hence the primary components must be indexed by the symmetric group and we can write

$$I = \bigcap_{\sigma \in \mathfrak{S}_n} U_\sigma,$$

where $\sqrt{U_\sigma} = P_\sigma$. Fix $\sigma \in \mathfrak{S}_n$, and set $U = U_\sigma$, $P = P_\sigma$, and $Q = Q_\sigma$. We want to show that $U_\sigma = Q_\sigma$. Let R_P be the polynomial ring $R = \mathbb{F}[x_1, \ldots, x_n]$ localized at the prime ideal P. By a theorem of Nagata [12, Th. 8.7] we have

$$U = IR_P \cap R \quad \text{and} \quad Q = JR_P \cap R.$$

We prove that in the local ring R_P, the ideals are equal $IR_P = JR_P$. Certainly because of the containment $I \subseteq J$, we have also $IR_P \subseteq JR_P$. In the other direction, we observe that $JR_P = QR_P$. Without loss of generality we may assume that $\sigma = e$, and $Q = (x_1 - x_2, \ldots, x_1 - x_{n-k+1}, x_1^2)$. For each pair $1 \leq r < s \leq n-k+1$ we may choose $1 \leq i_1 < \cdots < i_{k-1} \leq n-k+1$ such that $r, s \notin \{i_1, \ldots, i_{k-1}\}$ and setting $j_i = n-k+1+i$ define the tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|} \cline{4-7} \multicolumn{3}{c|}{} & i_1 & \cdots & i_{k-1} & r \\ \hline i_{k+1} & \cdots & i_{n-k} & j_1 & \cdots & j_{k-1} & s \\ \hline \end{array}$$

Then in the local ring R_P the Specht polynomial $F_T \in I$ has the form $F_T = u \cdot (x_r - x_s)$ where $u \in R_P$ is a unit. Also consider the monomial

$$M_S = x_{n-k+2} \cdots x_n \cdot x_r \cdot x_s \in (x_1, \ldots, x_n)^{\langle k \rangle} \subset I.$$

Then in the local ring R_P it has the form $M = w \cdot x_r \cdot x_s$ where $w \in R_P$ is a unit. Therefore, the generators of QR_P satisfy $(x_r - x_s) \in IR_P$ and $x_r \cdot x_s \in IR_P$ it follows that $QR_P \subset IR_P$ and hence that $JR_P \subset IR_P$, as desired. ∎

In particular Lemma 5.9 implies that if the Specht-monomial ideal $I(n,k)$ is not perfect then it must have an embedded prime divisor. On the other hand, note that the Specht ideal $\mathfrak{a}(n+1, k+1, k+1)$ cannot have embedded prime divisors since it is always radical. This indicates that detecting imperfection in Specht ideals is more subtle than in Specht-monomial ideals, and offers some excuse for the disparity between the statements of Theorems 5.5(4) and 5.6. We are now in a position to prove the other implication of Theorem 5.5(4).

Proof of (2) ⇒ (1) in Theorem 5.5(4). Assume that p satisfies $0<p<k+1$. We show that $I(n,k)$ is not perfect by showing it does not satisfy Decomposition (17), or equivalently that

$$\underbrace{\mathfrak{a}(n,k,k)+(x_1,\dots,x_n)^{\langle k+1\rangle}}_{I(n,k)} \neq \underbrace{\mathfrak{a}(n-1,k-1,k-1)+(x_1,\dots,x_{n-1})^{\langle k\rangle}}_{I(n-1,k-1)}\cap\Big((y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)\Big). \tag{19}$$

We identify the space of square-free x-monomials with the monomial complete intersection $B=\mathbb{F}[x_1,\dots,x_{n-1}]/(x_1^2,\dots,x_{n-1}^2)$ and $D=D_B=\frac{\partial}{\partial x_1}+\cdots+\frac{\partial}{\partial x_{n-1}}$ the associated lowering operator. From our assumptions on p, Lemma 3.4 implies that the derivative map $D\colon B_k\to B_{k-1}$ is not surjective, and hence (by Lemma 3.2), $V(n-1,k,k)=\ker(D)\cap B_k$. In particular, there must exist a square-free polynomial of degree k, say $f=f(x_1,\dots,x_{n-1})\in(x_1,\dots,x_{n-1})^{\langle k\rangle}$ with the property that $D(f)=0$ but $f\notin V(n-1,k,k)$. Since $D(f)=0$, we deduce from Lemma 5.7 that

$$f(x_1,\dots,x_{n-1})\equiv f(y_1,\dots,y_{n-1})\equiv 0 \mod (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2).$$

Therefore, $f\in I(n-1,k-1)\cap\big((y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)\big)$. On the other hand, since $f\notin V(n-1,k,k)$ it follows from Lemma 3.5 that $f\notin V(n,k,k)$ either, and it follows that $f\notin I(n,k)$. This shows that Inequality (19) holds, and hence by Lemma 5.9, the Specht-monomial ideal $I(n,k)$ is not perfect. ∎

5.2 Specht ideals are perfect

We are now in a position to prove Theorem 5.6.

Proof of Theorem 5.6. First assume that $p=0$ or $p\geq k+1$. We prove by induction on $n\geq 3$ that for each integer k satisfying $1\leq k<k+1\leq n-k$, and each i satisfying $1\leq i\leq k$ the Specht ideal $\mathfrak{a}(n+1,i+1,i+1)$ is perfect. First, recall that Corollary 2.7 says the change of coordinates map Φ gives a ring isomorphism

$$\mathfrak{a}(n+1,i+1,i+1)\cong\mathfrak{a}(n,i,i+1)=\mathfrak{a}(n,i,i)\cap(x_1,\dots,x_n)^{\langle i+1\rangle},$$

where the second equality follows from Theorem 4.3.

For the base case $n=3$ and the only possible $k=1$ gives

$$\mathfrak{a}(4,2,2)\cong\mathfrak{a}(3,1,2)=\mathfrak{a}(3,1,1)\cap(x_1,x_2,x_3)^{\langle 2\rangle}.$$

Note that $\mathfrak{a}(3,1,1)$ and $(x_1,x_2,x_3)^{\langle 2\rangle}$ are both perfect of grade $g=2$ Also note that the Specht-monomial ideal $I(3,1)=\mathfrak{a}(3,1,1)+(x_1,x_2,x_3)^{\langle 2\rangle}$ has grade $g+1=3$ and its quotient

$$\frac{\mathbb{F}[x_1,x_2,x_3]}{I(3,1)}\cong\frac{\mathbb{F}[z]}{(z^2)}$$

is Cohen–Macaulay. Therefore, $I(3,1)$ is perfect and hence by Lemma 5.3, so is $\mathfrak{a}(4,2,2)$.

For the inductive step, assume that $\mathfrak{a}(n,i,i)$ is perfect for every $1\leq i\leq k$. We have

$$\mathfrak{a}(n+1,i+1,i+1)\cong\mathfrak{a}(n,i,i)\cap(x_1,\dots,x_n)^{\langle i+1\rangle}.$$

Also note that since $1 \le i < i+1 \le k+1 \le n-k \le n-i$, Theorem 5.5(4) implies that the Specht-monomial ideal

$$I(n,i) = \mathfrak{a}(n,i,i) + (x_1,\ldots,x_n)^{\langle i+1\rangle}$$

is perfect of grade $g+1 = n-i+1$. Since $\mathfrak{a}(n,i,i)$ and $(x_1,\ldots,x_n)^{\langle i+1\rangle}$ are both perfect of grade $g = n-i$. Then it follows from Lemma 5.3 that $\mathfrak{a}(n+1,i+1,i+1) \cong \mathfrak{a}(n,i,i) \cap (x_1,\ldots,x_n)^{\langle i+1\rangle}$ is also perfect. This proves implication (1).

For (2), assume that $n \ge 2p+1$. By (1), we see that $\mathfrak{a}(n,p,p)$ is perfect. Also $(x_1,\ldots,x_n)^{\langle p\rangle}$ is perfect by Lemma 5.2. Therefore, by Lemma 5.3, it follows that the Specht ideal $\mathfrak{a}(n+1,p+1,p+1)$ and the Specht-monomial ideal $I(n,p)$ are perfect or not, alike. But Theorem 5.5(4) implies that the Specht-monomial ideal

$$I(n,p) = \mathfrak{a}(n,p,p) + (x_1,\ldots,x_n)^{\langle p+1\rangle}$$

is not perfect, and hence the Specht-monomial ideal $\mathfrak{a}(n+1,p+1,p+1)$ is not perfect either.

CONJECTURE 5.10. *If $p = \mathrm{char}(\mathbb{F})$ and n,k are positive integers satisfying $0 < p < k+1$ and $n \ge 2k+1$, then the Specht ideal $\mathfrak{a}(n+1,k+1,k+1)$ is not perfect.*

Conjecture 5.10 is true for $p = k$ by Theorem 5.6(2), but, as of the writing of this manuscript, it remains open for $0 < p < k$. Conjecture 5.10 together with Theorem 5.6 establish the following conjecture of Yanagawa [17, Conj. 5.5]:

CONJECTURE 5.11 [17]. *Fix a field $\mathbb{F}$ with $p = \mathrm{char}\,\mathbb{F}$ and fix positive integers n,k satisfying $n \ge 2k+1$. Then the following are equivalent:*

1. *$p \ge 0$ or $p \ge k+1$.*
2. *The Specht ideal $\mathfrak{a}(n+1,k+1,k+1)$ is perfect.*

EXAMPLE 5.12. Taking $p = 2$, Theorem 5.6 says that $\mathfrak{a}(n+1,3,3)$ is not perfect for every $n \ge 5$, a result also obtained by Yanagawa [17, Th. 5.3]. For example if $n = 5$, then $\mathfrak{a}(6,3,3)$ is not perfect, and one obstruction to perfection is the elementary symmetric polynomial $\alpha = e_2(x_1,\ldots,x_4)$ which lies in the intersection

$$e_2(x_1,\ldots,x_4) \in \mathfrak{a}(4,1,1) \cap \Big((y_1,\ldots,y_4)^{\langle 2\rangle} + (x_5^2)\Big) = \bigcap_{\sigma\in\mathfrak{S}_5} \sigma.(x_1-x_2, x_1-x_3, x_1-x_4, x_1^2),$$

but $e_2(x_1,\ldots,x_4) \notin I(5,2)$. This indicates that the Specht-monomial ideal $I(5,2)$ must have an embedded prime divisor in characteristic $p = 3$, which does not appear in higher characteristics. Macaulay2 [3] reveals that the primary decomposition of $I(5,2)$ over the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ is

$$I(5,2) = \bigcap_{\sigma\in\mathfrak{S}_5} \sigma.(x_1-x_2, x_1-x_3, x_1-x_4, x_1^2) \cap Q,$$

where Q is primary and satisfies $(x_1^2,\ldots,x_5^2) \subseteq Q$; in particular the maximal ideal $\mathfrak{m} = (x_1,\ldots,x_5)$ is an associated prime divisor of the Specht-monomial ideal $I(5,2) = \mathfrak{a}(5,2,2) + (x_1,\ldots,x_5)^{\langle 3\rangle}$ in characteristic $p = 2$. Since $\mathfrak{a}(5,2,2)$ and $(x_1,\ldots,x_5)^{\langle 3\rangle}$ are both perfect of grade $g = 3$ in characteristic $p = 2$ (and all characteristics), we deduce that $\mathfrak{a}(6,3,3) \cong \mathfrak{a}(5,2,3) = \mathfrak{a}(5,2,2) \cap (x_1,\ldots,x_5)^{\langle 3\rangle}$ is also not perfect by Lemma 5.3.

Similarly, for $p = 3$, Theorem 5.6 implies that $\mathfrak{a}(n+1,4,4)$ is not perfect for $n \ge 7$ in characteristic $p = 3$. For example if $n = 7$ then $\mathfrak{a}(8,4,4)$ is not perfect, and, as in the previous case, one can see that the Specht-monomial ideal $I(7,3)$ has an extra primary component that contains all the squared variables. Conjecture 5.10 would imply that, for example, $\mathfrak{a}(n+1,5,5)$ is not perfect for all $n \geq 9$ in characteristic $p=3$. Computations in Macaulay2 [3] show that $\mathfrak{a}(10,5,5)$ is indeed not perfect, supporting this claim.

5.3 Decomposition of Specht-monomial ideals

The proof of Theorem 5.5(3) is surprisingly similar to that of Theorem 4.3, and it too comes by way of several lemmas. We want to show that for every $1 \leq k < k+1 \leq n-k$ the Specht-monomial ideal $I(n,k) = \mathfrak{a}(n,k,k) + (x_1,\ldots,x_n)^{\langle k+1\rangle}$ satisfies

$$I(n,k) = I(n-1,k-1) \cap \Big((y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)\Big).$$

Of course there is an assumption about characteristic here, but we will not assume it yet, and try to point out exactly where we need it.

LEMMA 5.13. *Assume that* $1 \leq k < k+1 \leq n-k$. *Then*

$$I(n,k) \subseteq I(n-1,k-1) \cap \Big((y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)\Big).$$

Proof. Certainly, we have that $\mathfrak{a}(n,k,k) \subset \mathfrak{a}(n-1,k,k) \subset \mathfrak{a}(n-1,k-1,k-1)$, and also $(x_1,\ldots x_n)^{\langle k+1\rangle} \subset (x_1,\ldots,x_{n-1})^{\langle k\rangle}$, hence $I(n,k) \subset I(n-1,k-1)$ (if $k=1$, we should regard $I(n-1,0)$ as R). It remains to see why $I(n,k) \subset (y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. For $T \in \mathrm{STab}(n,k,k)$ we have

$$\begin{aligned} F_T^k(x_1,\ldots,x_n) &= (x_{i_1}-x_{j_1})\cdots(x_{i_k}-x_n) \\ &= (y_{j_1}-y_{i_1})\cdots(y_{j_{k-1}}-y_{i_{k-1}})\cdot(-y_i) \in (y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2). \end{aligned}$$

For $S \in \mathrm{STab}(n,0,k+1)$, if $x_n \in \mathrm{supp}(M_S^{k+1})$ then $M_S^{k+1} = x_n \cdot M_{S'}^k$ for $S' \in \mathrm{STab}(n-1,0,k)$ and by Lemma 5.7, we have

$$M_{S'}^k(x_1,\ldots,x_{n-1}) = M_{S'}^k(y_1,\ldots,y_{n-1}) + x_n \cdot D(M_{S'}^k(y_1,\ldots,y_{n-1})) \mod (y_1,\ldots,y_{n-1})^{\langle k\rangle}.$$

It follows that $M_S^{k+1} = x_n \cdot M_{S'}^k \in (y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. If $x_n \notin \mathrm{supp}(M_S^{k+1})$, then it is obvious that $M_S^{k+1} \in (y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. Hence $I(n,k) \subseteq I(n-1,k-1) \cap \big((y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)\big)$, as desired. ∎

We have added a superscript to our notation for the shifted Specht polynomials to help the reader remember degrees. The other containment

$$I(n,k) \supseteq I(n-1,k-1) \cap \Big((y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)\Big) \tag{20}$$

is harder to prove.

Note that the ideal $I(n-1,k-1)$ is generated in degrees $k-1$ and k by the following subspaces of forms:

$$\begin{aligned} V &= V(n-1,k-1,k-1) = \big\langle F_T^{k-1} \mid T \in \mathrm{STab}(n-1,k-1,k-1)\big\rangle \\ U &= U(n-1,k-1,k) \\ &= \big\langle x_n F_T^{k-1} \mid T \in \mathrm{STab}(n-1,k-1,k-1)\big\rangle + \big\langle M_S^k \mid S \in \mathrm{STab}(n-1,0,k)\big\rangle \\ &= x_n \cdot V(n-1,k-1,k-1) + V(n-1,0,k). \end{aligned}$$

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We make some preliminary observations, but first some notation. Denote by $\mathbb{N}^n(m)$ the set of exponent vectors of degree, that is, $\mathbf{a}=(a_1,\dots,a_n)$ with $a_1+\cdots+a_n=m$. A monomial in the y-variables (resp., the x-variables) will be denoted by $\mathbf{y}^{\mathbf{a}}=y_1^{a_1}\cdots y_n^{a_n}$ (resp. $\mathbf{x}^{\mathbf{a}}=x_1^{a_1}\cdots x_n^{a_n}$), and its radical is the square-free monomial $\sqrt{\mathbf{y}^{\mathbf{a}}}=\prod_{a_i>0}y_i$. We also define the *weight* of a monomial to be $\mathrm{wt}(\mathbf{y}^{\mathbf{a}})=\#\{a_i>0\}$, the number of nonzero entries in its exponent vector.

LEMMA 5.14. *With U and V as above, we have*

1. $x_n\cdot V\subseteq U$,
2. $x_n\cdot U\subseteq I(n,k)$, *and*
3. *for every $P\in I(n-1,k-1)$, and for all exponent vectors $\mathbf{a}\in\mathbb{N}^{n-1}(m)$ and $\mathbf{b}\in\mathbb{N}^{n-1}(m-1)$, there exists elements $\nu_{\mathbf{a}}\in V$ and $\mu_{\mathbf{b}}\in U$ such that*

$$P\equiv\sum_{\mathbf{a}\in\mathbb{N}^{n-1}(m)}\mathbf{y}^{\mathbf{a}}\nu_{\mathbf{a}}+\sum_{\mathbf{b}\in\mathbb{N}^{n-1}(m-1)}\mathbf{y}^{\mathbf{b}}\mu_{\mathbf{b}}\qquad \mathit{mod}\ I(n,k). \tag{21}$$

Proof. (1) is obvious from the definitions. For (2), note that for $S\in\mathrm{STab}(n-1,0,k-1)$, $x_n\cdot M_S^k\in V(n,0,k+1)\subset I(n,k)$. Also for $T\in\mathrm{STab}(n-1,k-1,k-1)$ and for index $1\le i\le n-1$ such that $i\notin\mathrm{supp}(T)$, which exists because $n-1\ge 2k>2(k-1)$, we have

$$x_n^2\cdot F_T^{k-1}=x_n(x_n-x_i)\cdot F_T^{k-1}+x_nx_i\cdot F_T^{k-1},$$

and since $(x_n-x_i)\cdot F_T^{k-1}\in V(n,k,k)$ and $x_nx_i\cdot F_T^{k-1}\in V(n,0,k+1)$, it follows that $x_n^2\cdot F_T^{k-1}\in I(n,k)$, and (2) follows. Finally fix a homogeneous polynomial $P\in I(n-1,k-1)$. Then for each $T\in\mathrm{STab}(n-1,k-1,k-1)$ and each $S\in\mathrm{STab}(n-1,0,k)$, there exist polynomials, which we may take in the y-variables, $g_T(\mathbf{y})$ of degree m and $h_S(\mathbf{y})$ of degree $m-1$ for which

$$P=\sum_{T\in\mathrm{STab}(n-1,k-1,k-1)}g_T(\mathbf{y})\cdot F_T^{k-1}+\sum_{S\in\mathrm{STab}(n-1,0,k)}h_S(\mathbf{y})\cdot M_S.$$

Writing $g_T(\mathbf{y})=g_T^0(y_1,\dots,y_{n-1})+x_ng_T^1(y_1,\dots,y_{n-1})+x_n^2g_T^2(\mathbf{y})$ and also $h_S(\mathbf{y})=h_S^0(y_1,\dots,y_{n-1})+x_nh_S^1(\mathbf{y})$, it follows from (2) that $x_n^2g_T^2(\mathbf{y})\cdot F_T^{k-1}\in I(n,k)$, $x_nh_S^1(\mathbf{y})\cdot M_S^k\in I(n,k)$, and also that $x_ng_T^1(y_1,\dots,y_{n-1})\cdot F_T^{k-1}\in U$. Then taking monomial expansions, reversing orders of summations, and grouping like monomial terms, we get

$$\begin{aligned}P&\equiv\sum_{T\in\mathrm{STab}(n-1,k-1,k-1)}\sum_{\mathbf{a}\in\mathbb{N}^{n-1}(m)}c_T^0(\mathbf{a})\mathbf{y}^{\mathbf{a}}\cdot F_T^{k-1}\\&\quad+\sum_{T\in\mathrm{STab}(n-1,k-1,k-1)}\sum_{\mathbf{b}\in\mathbb{N}^{n}(m-1)}c_T^1(\mathbf{b})\mathbf{y}^{\mathbf{b}}\cdot x_nF_T^{k-1}\\&\quad+\sum_{S\in\mathrm{STab}(n-1,0,k)}\sum_{\mathbf{b}\in\mathbb{N}^{n}(m-1)}d_S^0(\mathbf{b})\mathbf{y}^{\mathbf{b}}\cdot M_S\\&=\sum_{\mathbf{a}\in\mathbb{N}^n(m)}\mathbf{y}^{\mathbf{a}}\cdot\left(\sum_{T\in\mathrm{STab}(n-1,k-1,k-1)}c_T^0(\mathbf{a})F_T^{k-1}\right)+\sum_{\mathbf{b}\in\mathbb{N}^{n-1}(m-1)}\mathbf{y}^{\mathbf{b}}\\&\quad\cdot\left(\sum_{T\in\mathrm{STab}(n-1,k-1,k-1)}c_T^1(\mathbf{b})F_T^{k-1}+\sum_{S\in\mathrm{STab}(n-1,0,k)}d_S^0(\mathbf{b})M_S^k\right)\quad\mathrm{mod}\ I(n,k),\end{aligned}$$

and (3) follows. ∎

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The following Lemma is analogous to Lemma 4.5 in §4.

LEMMA 5.15. *If* $P \in I(n-1,k-1) \cap \big((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)\big)$ *is expressed as in (21), then each of its monomial summands must lie in* $(y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ *is too, that is,* $\mathbf{y}^{\mathbf{a}} \cdot \mu_{\mathbf{a}} \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ *and* $\mathbf{y}^{\mathbf{b}} \cdot \nu_{\mathbf{b}} \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ *for all* $\mathbf{a} \in \mathbb{N}^{n-1}(m)$ *and* $\mathbf{b} \in \mathbb{N}^{n-1}(m-1)$.

Proof. By Lemma 5.14, we may write

$$P = \underbrace{\sum_{\mathbf{a}\in\mathbb{N}^{n-1}(m)} \mathbf{y}^{\mathbf{a}}\nu_{\mathbf{a}}}_{P_1} + \underbrace{\sum_{\mathbf{b}\in\mathbb{N}^{n-1}(m-1)} \mathbf{y}^{\mathbf{b}}\mu_{\mathbf{b}}}_{P_2} + P_3$$

for some $\nu_{\mathbf{a}} \in V$, some $\mu_{\mathbf{b}} \in U$, and some $P_3 \in I(n,k)$. Next note that if $P \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ then both $P_1 \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ and $P_2 \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. Indeed, note that in their respective y-monomial expansions, those monomials in P_1 are all independent of y_n, whereas all monomials in the y-monomial expansion of P_2 are either divisible by $y_n = x_n$, or in $(y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ already, by Lemma 5.7.

For the monomial products in P_1 we assume by way of contradiction that $\mathbf{y}^{\mathbf{a}} \cdot \nu_{\mathbf{a}} \notin (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ for some $\mathbf{a} \in \mathbb{N}^{n-1}$. Then since $(y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ is a monomial ideal, it follows that in the y-monomial expansion of $\mathbf{y}^{\mathbf{a}}\nu_{\mathbf{a}}$, there must be some monomial say $\mathbf{y}^{\mathbf{d}}$ which is not in $(y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. Since $\mathbf{y}^{\mathbf{a}}\nu_{\mathbf{a}}$ is independent of x_n, so is $\mathbf{y}^{\mathbf{d}}$. Since $(y_1,\dots,y_{n-1})^{\langle k\rangle}$ consists of all y-monomials of weight at least k, it follows that $\mathrm{wt}(\mathbf{y}^{\mathbf{d}}) \le k-1$. On the other hand, every y-monomial in the monomial expansion of $\nu_{\mathbf{a}}$ has weight equal to $k-1$, hence it follows that $\mathrm{wt}(\mathbf{y}^{\mathbf{d}}) = k-1$. But then we can deduce, as in the proof of Lemma 4.5, that

$$\frac{\mathbf{y}^{\mathbf{d}}}{\sqrt{\mathbf{y}^{\mathbf{d}}}} = \mathbf{y}^{\mathbf{a}},$$

and hence the exponent vector $\mathbf{d}$ uniquely determines the exponent vector $\mathbf{a}$. This implies that $\mathbf{y}^{\mathbf{d}}$ occurs with the same coefficient in the monomial expansion of the term $\mathbf{y}^{\mathbf{a}}\nu_{\mathbf{a}}$ as it does in the entire sum

$$P_1 = \sum_{\mathbf{a}\in\mathbb{N}^{n-1}(m)} \mathbf{y}^{\mathbf{a}}\nu_{\mathbf{a}},$$

contradicting the fact that $P_1 \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. The argument for P_2 is similar. □

Lemma 5.15 says that to check the containment

$$I(n,k) \supseteq I(n-1,k-1) \cap \Big((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)\Big),$$

it suffices to check on products of monomials and forms in either V or U, that is, for any $\mathbf{a},\mathbf{b} \in \mathbb{N}^{n-1}$ and for any $\nu \in V$ and any $\mu \in U$

$$\mathbf{y}^{\mathbf{a}}\nu \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2) \Rightarrow \qquad \mathbf{y}^{\mathbf{a}}\nu \in I(n,k), \tag{22}$$

$$\mathbf{y}^{\mathbf{b}}\mu \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2) \Rightarrow \qquad \mathbf{y}^{\mathbf{b}}\mu \in I(n,k). \tag{23}$$

The next lemma verifies implications (22) and (23) in the special case where $\mathbf{a},\mathbf{b} = \mathbf{0}$. This seems to be where we need our assumptions on p.

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Lemma 5.16. *Fix* $\nu \in V(n-1,k-1,k-1)$ *and* $\mu \in U(n-1,k-1,k)$. *If* $\nu \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$ *then* $\nu = 0$. *If* $p=0$ *or* $p \geq k+1$ *and if* $\mu \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$ *then* $\mu \in \mathfrak{a}(n,k,k) \subset I(n,k)$.

Proof. The first statement for ν is obvious for degree reasons. For the second statement, we can write $\mu = \mu(x_1,\dots,x_n) = x_n\alpha+\beta$ for polynomials $\alpha = \alpha(x_1,\dots,x_{n-1}) \in V_{\mathbf{x}}(n-1,k-1,k-1)$ and $\beta = \beta(x_1,\dots,x_{n-1}) \in V_{\mathbf{x}}(n-1,0,k)$. Here, it is important to distinguish between polynomials in x-variables and those in the y-variables, hence we adopt the notation $V_{\mathbf{x}}(m,j,j)$ and $V_{\mathbf{y}}(m,j,j)$ to denote the $\mathbb{F}$-span of Specht polynomials in the x-variables and y-variables, respectively; note that if $m \leq n-1$ these two subspaces coincide.

Consider inclusions of monomial complete intersections, $B \subset A$ defined in the y-variables by:

$$B = \frac{\mathbb{F}[y_1,\dots,y_{n-1}]}{(y_1^2,\dots,y_{n-1}^2)} \hookrightarrow A = \frac{\mathbb{F}[y_1,\dots,y_n]}{(y_1^2,\dots,y_n^2)},$$

and let L_B, D_B, H_B, and L_A, D_A, H_A be their respective raising, lowering, and semi-simple operators, respectively, as in §3. Then since $p=0$ or $p \geq k+1$, Lemma 3.4 implies that the lowering maps for B and A,

$$D_B\colon B_k \to B_{k-1}, \quad \text{and} \quad D_A\colon A_k \to A_{k-1}$$

are both surjective, which by Lemma 3.2 is equivalent to their primitive subspaces $P_{B,k} = \ker(D_B)\cap B_k$ and $P_{A,k} = \ker(D_A)\cap A_k$ satisfying

$$P_{B,k} = V_{\mathbf{y}}(n-1,k,k), \quad \text{and} \quad P_{A,k} = V_{\mathbf{y}}(n,k,k).$$

We identify B (resp. A) with the subspace spanned by square-free monomials in variables $y_1,\dots,y_{n-1}$ (resp. $y_1,\dots,y_n$).

Then if $\mu = x_n\alpha(x_1,\dots,x_{n-1})+\beta(x_1,\dots,x_{n-1}) \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$, by Lemma 5.7, we have

$$\mu \equiv (-1)^{k-1}x_n\left(\alpha(y_1,\dots,y_{n-1})+D_B(\beta(y_1,\dots,y_{n-1}))\right) \equiv 0 \mod (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2), \tag{24}$$

(note that $D_B(\alpha(y_1,\dots,y_{n-1}))=0$ and $\beta(y_1,\dots,y_{n-1}) \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$ automatically). Dividing by x_n in (24), we see that

$$\alpha(y_1,\dots,y_{n-1})+D_B(\beta(y_1,\dots,y_{n-1})) \in ((y_1,\dots,y_{n-1})^{\langle k\rangle}+x_n^2) : x_n = (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n),$$

which, for degree reasons, implies that

$$\alpha(y_1,\dots,y_{n-1})+D_B(\beta(y_1,\dots,y_{n-1})) = 0.$$

Therefore, $\beta_{\mathbf{y}} = \beta(y_1,\dots,y_{n-1}) \in B_k$ is a square-free polynomial in $y_1,\dots,y_{n-1}$ such that $D_B(\beta_{\mathbf{y}}) = -\alpha_{\mathbf{y}} \in V(n-1,k-1,k-1) = P_{B,k-1}$. Applying the commutator relation

$$[D_B, L_B] = H_B,$$

to $\alpha_{\mathbf{y}} = \alpha(y_1,\dots,y_{n-1})$, we find that

$$D_B \circ L_B(\alpha_{\mathbf{y}}) = H(\alpha_{\mathbf{y}}) = (n-1-2(k-1))\cdot\alpha_{\mathbf{y}},$$

which implies that

$$\mu_B(y_1,\dots,y_{n-1}) := L_B(\alpha_{\mathbf{y}}) + (n-2k+1)\cdot\beta_{\mathbf{y}} \in P_{B,k} = V_{\mathbf{y}}(n-1,k,k). \tag{25}$$

Note that Equation (25) also implies that $\mu_B := \mu_B(x_1,\dots,x_{n-1}) \in V_{\mathbf{x}}(n-1,k,k)$. We can also apply the commutator relations for those operators on A. Note that the restriction of $D_A = D_B + \partial/\partial y_n$ to B is D_B, and that by Lemma 3.5, we have

$$V_{\mathbf{y}}(n-1,k,k) = V_{\mathbf{y}}(n,k,k)\cap B_k.$$

It follows that $D_A(\beta_{\mathbf{y}}) = D_B(\beta_{\mathbf{y}}) = -\alpha_{\mathbf{y}} \in V_{\mathbf{y}}(n-1,k-1,k-1) \subset V_{\mathbf{y}}(n,k-1,k-1)$, and hence applying the commutator relation

$$[D_A, L_A] = H_A,$$

to $D_A(\beta_{\mathbf{y}}) = -\alpha_{\mathbf{y}}$ we also find that

$$\mu_A(y_1,\dots,y_{n-1},y_n) := L_A(\alpha_{\mathbf{y}}) + (n-2k+2)\cdot\beta_{\mathbf{y}} \in P_{A,k} = V_{\mathbf{y}}(n,k,k), \tag{26}$$

and hence $\mu_A = \mu_A(x_1,\dots,x_n) \in V_{\mathbf{x}}(n,k,k)$. Noting that $L_A = L_B + x_n$ we see that adding μ to μ_B yields μ_A, that is,

$$\underbrace{(x_n\alpha+\beta)}_{\mu} + \underbrace{((x_1+\cdots+x_{n-1})\cdot\alpha + (n-2k+1)\beta)}_{\mu_B} = \underbrace{(x_1+\cdots+x_n)\cdot\alpha + (n-2k+2)\beta}_{\mu_A}.$$

It follows from (25) and (26) that $\mu = x_n\cdot\alpha+\beta = \mu_A - \mu_B \in V_{\mathbf{x}}(n,k,k) = V(n,k,k) \subset \mathfrak{a}(n,k,k)$, as claimed. ∎

Next, we check containment (20) for monomials which are not contained in the support of ν and μ. The following Lemma is analogous to Lemma 4.6 in §4. Recall that the support of the (shifted) Specht polynomial for T is the set of square-free monomials indexed by subsets of numbers in the support of T, no two of which lie in the same column of T.

LEMMA 5.17. *Fix tableaux $T \in \mathrm{STab}(n-1,k-1,k-1)$ and $S \in \mathrm{STab}(n-1,0,k)$, and exponent vectors $\mathbf{a},\mathbf{b} \in \mathbb{N}^{n-1}$.*

1. *If $\sqrt{\mathbf{x}^{\mathbf{a}}} \notin \mathrm{supp}(F_T^{k-1})$ then $\mathbf{y}^{\mathbf{a}}\cdot F_T^{k-1} \in \mathfrak{a}(n,k,k) \subset I(n,k)$.*
2. *If $\sqrt{\mathbf{x}^{\mathbf{b}}} \notin \mathrm{supp}(M_S^k)$ then $\mathbf{y}^{\mathbf{b}}\cdot M_S^k \in (x_1,\dots,x_n)^{\langle k+1\rangle} \subset I(n,k)$.*

Proof. For (1) assume $\sqrt{\mathbf{x}^{\mathbf{a}}} \notin \mathrm{supp}(F_T^{k-1})$. If some variable $x_i \in \mathrm{supp}(\mathbf{y}^{\mathbf{a}})$ but $x_i \notin \mathrm{supp}(F_T^{k-1})$, then clearly

$$\mathbf{y}^{\mathbf{a}}\cdot F_T^{k-1} = \mathbf{y}^{\mathbf{a}'}\cdot y_i\cdot F_T^{k-1} = \mathbf{y}^{\mathbf{a}'}\cdot(x_n-x_i)\cdot F_T^{k-1} \in V(n,k,k).$$

We may therefore assume that every variable of $\mathbf{x}^{\mathbf{a}}$ lies in the support of F_T^{k-1}. But since $\sqrt{\mathbf{x}^{\mathbf{a}}} \notin \mathrm{supp}(F_T^{k-1})$ there must be two distinct indices i,j for which $y_i,y_j \in \mathrm{supp}(\mathbf{y}^{\mathbf{a}})$ and i,j lie in the same column in T. Choose any index $r \neq i,j$ such that $x_r \notin \mathrm{supp}(F_T^{k-1})$–presumably there is such an r since we are assuming that $k+1 \le n-k$–and let $(i,r),(j,r) \in \mathfrak{S}_n$ be the transpositions swapping i,r and j,r, respectively. Then we have

$$\mathbf{y}^{\mathbf{a}}\cdot F_T^{k-1} = \mathbf{y}^{\mathbf{a}}\cdot\left(F_{(i,r).T}^{k-1} + F_{(j,r).T}^{k-1}\right),$$

and since $x_i \notin \mathrm{supp}(F_{(i,r).T}^{k-1})$, we must have

$$\mathbf{y}^{\mathbf{a}}\cdot F_{(i,r).T}^{k-1} = \mathbf{y}^{\mathbf{a}'}\cdot y_i\cdot F_{(i,r).T}^{k-1} = \mathbf{y}^{\mathbf{a}'}\cdot(x_n-x_i)\cdot F_{(i,r).T}^{k-1} \in \mathfrak{a}(n,k,k) \subset I(n,k),$$

and similarly for $F^{k-1}_{(j,r).T}$. This proves that $\mathbf{y}^{\mathbf{a}} \cdot F^{k-1}_T \in I(n,k)$. Item (2) is easier and left to the reader. ∎

Finally, we need to check what containment (20) for monomials which are contained in the support of ν or μ. Since the ideal $(y_1,\ldots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ is generated by square-free monomials in $(y_1,\ldots,y_{n-1})$ and the monomial x_n^2, it suffices to prove implications (22) and (23) for square free $\mathbf{y}^{\mathbf{a}}$. Also, by symmetry, it will suffice to assume that $\mathbf{y}^{\mathbf{a}} = \mathbf{y}^{\mathbf{m}} = y_1\cdots y_m$. As in Corollary 2.8, we define the subset of standard tableau on a shape λ as those which contain the set of integers $\{1,\ldots,m\}$ in their support, denoted by $\mathrm{STab}_m(\lambda) \subset \mathrm{STab}(\lambda)$, and define the subspaces

$$\begin{aligned} V_m &= V_m(n-1,k-1,k-1) = \left\langle F^{k-1}_T \mid T \in \mathrm{STab}_m(n-1,k-1,k-1)\right\rangle,\\ U_m &= U_m(n-1,k-1,k)\\ &= \left\langle x_nF^{k-1}_T \mid T \in \mathrm{STab}_m(n-1,k-1,k-1)\right\rangle + \left\langle M^k_S \mid S \in \mathrm{STab}_m(n-1,0,k)\right\rangle. \end{aligned}$$

We also define their complimentary subspaces

$$\begin{aligned} V^m &= V^m(n-1,k-1,k-1)\\ &= \left\langle F^{k-1}_T \mid T \in \mathrm{STab}(n-1,k-1,k-1)\setminus \mathrm{STab}_m(n-1,k-1,k-1)\right\rangle,\\ U^m &= U^m(n-1,k-1,k)\\ &= \left\langle x_nF^{k-1}_T \mid T \in \mathrm{STab}(n-1,k-1,k-1)\setminus \mathrm{STab}_m(n-1,k-1,k-1)\right\rangle,\\ &\quad + \left\langle M^k_S \mid S \in \mathrm{STab}(n-1,0,k)\setminus \mathrm{STab}_m(n-1,0,k)\right\rangle. \end{aligned}$$

Then we have vector space decompositions $V = V_m \oplus V^m$ and $U = U_m + U^m$, and Lemma 5.17 implies that $\mathbf{y}^{\mathbf{m}} \cdot \alpha \in I(n,k)$ for $\alpha \in V^m \sqcup U^m$. Hence, it only remains to check $\alpha \in V_m \sqcup U_m$. As in Corollary 2.8, we have the following bijective map of vector spaces:

$$\begin{aligned} V_m(n-1,k-1,k-1) &\longrightarrow V([n-1]_m, k-1-m, k-1-m),\\ F^{k-1}_T &\longmapsto F^{k-1-m}_{T'}, \end{aligned}$$

where

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|} \cline{4-9} \multicolumn{3}{c|}{} & 1 & \cdots & m & i_{m+1} & \cdots & i_k \\ \hline i_{k+1} & \cdots & i_{n-k} & j_1 & \cdots & j_m & j_{m+1} & \cdots & j_k \\ \hline \end{array} \tag{27}$$

and

$$T' = \begin{array}{|c|c|c|c|c|c|c|c|c|} \cline{7-9} \multicolumn{6}{c|}{} & i_{m+1} & \cdots & i_k \\ \hline i_{k+1} & \cdots & i_{n-k} & j_1 & \cdots & j_m & j_{m+1} & \cdots & j_k \\ \hline \end{array} \tag{28}$$

We also have the (possibly noninjective) linear map

$$\begin{aligned} U_m(n-1,k-1,k-1) &\longrightarrow U([n-1]_m, k-1-m, k-m),\\ x_nF^{k-1}_T &\longmapsto x_nF^{k-1-m}_{T'},\\ M^k_S &\longmapsto M^{k-m}_{S'}, \end{aligned}$$

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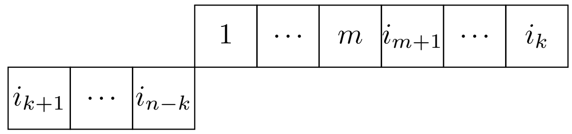
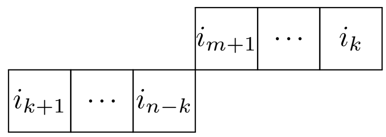
where $T \in \mathrm{STab}_m(n-1,k-1,k-1)$ and $T' \in \mathrm{STab}([n-1]_m, k-1-m, k-1-m)$ are the tableau in (27) and (28), respectively, and where where $S \in \mathrm{STab}(n-1,0,k)$ and $S'\,\mathrm{STab}([n-1]_m, 0, k-m)$ are given by

$S =$  (29)

and

$S' =$  (30)

The following two lemmas are useful.

Lemma 5.18. *For any $\beta \in V([n-m], k-m, k-m)$ where $[n-m] = \{m+1, \ldots, n\}$, we have*

$$(\mathbf{x}^{\mathbf{m}})^2 \cdot \beta \in I(n,k).$$

Proof. Fix a tableau $T' \in \mathrm{STab}([n-m], k-m, k-m)$ as in (28) and let $T \in \mathrm{STab}_m(n-1, k-1, k-1)$ be the corresponding tableau as in (27). Then it suffices to show that we have

$$(\mathbf{x}^{\mathbf{m}})^2 \cdot F_{T'}^{k-m} = x_1^2 \cdots x_m^2 \cdot F_{T'}^{k-m} \in I(n,k) = \mathfrak{a}(n,k,k) + (x_1, \ldots, x_n)^{\langle k+1 \rangle}.$$

On the other hand, we clearly have

$$x_1 \cdots x_m \cdot F_T^k = x_1^2 \cdots x_m^2 \cdot F_{T'}^{k-m} + \Big(\text{stuff in } (x_1, \ldots, x_n)^{\langle k+1 \rangle}\Big) \in \mathfrak{a}(n,k,k)$$

and the result follows. ∎

Lemma 5.19. *Fix elements $\nu \in V_m(n-1,k-1,k-1)$ and $\mu \in U_m(n-1,k-1,k)$, and let $\nu' \in V([n-1]_m, k-1-m, k-1-m)$ and $\mu' \in U([n-1]_m, k-1-m, k-m)$ their respective images under the maps above. Then we have*

1. $\mathbf{y}^{\mathbf{m}}(\nu - (-1)^m \mathbf{y}^{\mathbf{m}} \nu') \in (y_1, \ldots, y_{n-1})^{\langle k \rangle} + (x_n^2)$,
2. $\mathbf{y}^{\mathbf{m}}(\mu - (-1)^m \mathbf{y}^{\mathbf{m}} \mu') \in (y_1, \ldots, y_{n-1})^{\langle k \rangle} + (x_n^2)$,
3. $\mathbf{x}^{\mathbf{m}}(\mu - \mathbf{x}^{\mathbf{m}} \mu') \in I(n,k)$, *and*
4. $\mathbf{y}^{\mathbf{m}} \cdot \mu - (-1)^m \mathbf{x}^{\mathbf{m}} \cdot \mu \in I(n,k)$.

Proof. For (1) note that it suffices to see that for $T \in \mathrm{STab}_m(n-1,k-1,k-1)$ and $T' \in \mathrm{STab}([n-1]_m, k-1-m, k-1-m)$ as in (27) and (28), we have

$$\mathbf{y}^{\mathbf{m}}\left(F_T^{k-1} - (-1)^m \mathbf{y}^{\mathbf{m}} \cdot F_{T'}^{k-1}\right) \in (y_1, \ldots, y_{n-1})^{\langle k \rangle} + (x_n^2).$$

By Lemma 5.7, we have $F_{T,\mathbf{y}}^{k-1} = F_T^{k-1}(y_1, \ldots, y_{n-1}) = (-1)^{k-1} \cdot F_T^{k-1}(x_1, \ldots, x_{n-1}) = F_T^{k-1}$ and $F_{T',\mathbf{y}}^{k-1-m} = (-1)^{k-1-m} \cdot F_{T'}^{k-1}$. Since the difference $F_{T,\mathbf{y}}^{k-1} - \mathbf{y}^{\mathbf{m}} \cdot F_{T',\mathbf{y}}^{k-1}$ is a combination of y-monomials which do not contain $\mathbf{y}^{\mathbf{m}}$ in their support, it follows that the product

$$\mathbf{y}^{\mathbf{m}}\left(F_T^{k-1} - (-1)^m \mathbf{y}^m \cdot F_{T'}^{k-1}\right) \in (y_1, \ldots, y_{n-1})^{\langle k \rangle} + (x_n^2).$$

For (2), write $\mu = x_n\cdot\alpha+\beta$ where $\alpha \in V_m(n-1,k-1,k-1)$ and $\beta \in V_m(n-1,0,k)$. By (1) we know that

$$\mathbf{y}^{\mathbf{m}}(x_n\cdot\alpha-(-1)^m x_n\cdot\alpha') \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2),$$

hence, it suffices to take $\mu=\beta=M_S^k \in V_m(n-1,0,k)$ and show that

$$\mathbf{y}^{\mathbf{m}}\left(M_S^k-(-1)^m\mathbf{y}^{\mathbf{m}}M_{S'}^{k-m}\right) \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2),$$

for $S \in \mathrm{STab}_m(n-1,0,k)$ and $S' \in \mathrm{STab}([n-1]_m,0,k-m)$ as in (29) and (30). In this case, we have

$$M_S^k = x_1\cdots x_m\cdot M_{S'}^{k-m},$$

and Lemma 5.7 implies that the difference satisfies

$$\begin{aligned}
&M_S^k-(-1)^m\mathbf{y}^{\mathbf{m}}\cdot M_{S'}^{k-m}\\
&=(-1)^k M_{S,\mathbf{y}}^k+(-1)^{k-1}x_nD(M_{S,\mathbf{y}}^k)\\
&\quad-(-1)^m\mathbf{y}^{\mathbf{m}}\left((-1)^{k-m}M_{S',\mathbf{y}}^{k-m}+(-1)^{k-1-m}x_nD(M_{S',\mathbf{y}}^{k-m})\right) && \bmod (x_n^2)\\
&=(-1)^{k-1}x_n\cdot\left(D(M_{S,\mathbf{y}}^k)-\mathbf{y}^{\mathbf{m}}\cdot D(M_{S'}^{k-m})\right) && \bmod (x_n^2)\\
&=(-1)^{k-1}x_n\cdot\left(\sum_{i=1}^m y_1\cdots\hat{y}_i\cdots y_m\cdot M_{S',\mathbf{y}}^{k-m}\right) && \bmod (x_n^2).
\end{aligned}$$

Therefore, it follows that the product satisfies

$$\mathbf{y}^m\left(M_S^k-(-1)^m\mathbf{y}^{\mathbf{m}}\cdot M_{S'}^{k-m}\right) \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2).$$

For (3), we write $\mu = x_n\alpha+\beta$ where $\alpha\in V_m(n-1,k-1,k-1)$ and $\beta \in V_m(n-1,0,k)$, and also $\mu' = x_n\alpha'+\beta'$. Then we have

$$\mathbf{x}^{\mathbf{m}}(x_n\alpha-\mathbf{x}^{\mathbf{m}}x_n\alpha') \in (x_1,\dots,x_n)^{\langle k+1\rangle} \subset I(n,k),$$

and also

$$\mathbf{x}^m(\beta-\mathbf{x}^{\mathbf{m}}\beta') = 0 \in I(n,k),$$

and the result follows.

For (4), note that

$$\mathbf{y}^{\mathbf{m}}-(-1)^m\mathbf{x}^{\mathbf{m}} = (x_n-x_1)\cdots(x_n-x_m)-(-1)^m x_1\cdots x_m = x_n\,(\text{stuff}),$$

and since $x_n\cdot\mu \in I(n,k)$ by Lemma 5.14, the result follows. ∎

The following is an analogue of Lemma 4.7 in §4.

LEMMA 5.20. *If $\mathbf{y}^{\mathbf{m}}\cdot\nu \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$ for some $\nu\in V_m$, then $\nu=0$, and in particular, $\mathbf{y}^{\mathbf{m}}\cdot\nu\in I(n,k)$. If $p=0$ or $p\geq k+1$, then if $\mathbf{y}^{\mathbf{m}}\cdot\mu \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$ for some $\mu\in U_m$, then $\mathbf{y}^{\mathbf{m}}\cdot\mu\in I(n,k)$.*

Proof. First assume that for some $\nu\in V_m$ we have $\mathbf{y}^{\mathbf{m}}\cdot\nu \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2)$. By Lemma 5.19(1), we have

$$\mathbf{y}^{\mathbf{m}}(\nu-\mathbf{y}^{\mathbf{m}}\cdot\nu') \in (y_1,\dots,y_{n-1})^{\langle k\rangle}+(x_n^2),$$

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and it follows also that

$$(\mathbf{y^m})^2 \cdot \nu' \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2).$$

Therefore, we have

$$\begin{aligned}\nu' \in \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2) : (\mathbf{y^m})^2\right) &= \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2) : \mathbf{y^m}\right)\\ &= (y_{m+1},\dots,y_{n-1})^{k-m} + (x_n^2),\end{aligned}$$

and hence that $\nu' \in V([n-1]_m, k-1-m, k-1-m) \cap ((y_{m+1},\dots,y_{n-1})^{(k-m)} + (x_n^2)$. By Lemma 5.16, it follows that $\nu' = 0$, and therefore also $\nu = 0$, which proves the first statement.

Next, assume that $p = 0$ or $p \geq k+1$, and assume that for some $\mu = \mu(x_1,\dots,x_{n-1}) \in U_m$, we have $\mathbf{y^m} \cdot \mu \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$. Then by Lemma 5.19(2), we must also have $(\mathbf{y^m})^2 \mu' \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$, and hence also we must have

$$\begin{aligned}\mu' \in \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2) : (\mathbf{y^m})^2\right) &= \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2) : \mathbf{y^m}\right)\\ &= (y_{m+1},\dots,y_{n-1})^{k-m} + (x_n^2).\end{aligned}$$

Therefore, we have $\mu' \in U([n-1]_m, k-1-m, k-m) \cap (y_{m+1},\dots,y_{n-1})^{k-m} + (x_n^2)$, and it follows from Lemma 5.16 that $\mu' \in V([n]_m, k, k)$. Therefore, by Lemma 5.18, it follows that $(\mathbf{x^m})^2 \cdot \mu' \in I(n,k)$. Then by Lemma 5.19(3), we must also have $\mathbf{x}^m \cdot \mu \in I(n,k)$, and from Lemma 5.19(4) it follows that $\mathbf{y^m} \cdot \mu \in I(n,k)$, as desired. ∎

Finally, we are in a position to prove Theorem 5.5(3):

Proof of Theorem 5.5(3). Assume that $p = 0$ or $p \geq k+1$. By Lemma 5.13, we have

$$I(n,k) \subseteq I(n-1,k-1) \cap \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)\right).$$

For the reverse containment, Lemma 5.15 implies that we only have to check on products of monomials in $(y_1,\dots,y_{n-1})$ and forms in the subspaces $V = V(n-1,k-1,k-1)$ and $U = U(n-1,k-1,k)$. Since $(y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$ is generated by square-free monomials in $(y_1,\dots,y_{n-1})$ it follows that we may assume our monomials are square-free, and by symmetry, we may assume that our monomial is $\mathbf{y^m} = y_1 \cdots y_m$. Write $V = V_m \oplus V^m$ and $U = U_m + U^m$ as above. Then Lemma 5.17 implies that $\mathbf{y^m} \cdot \alpha \in I(n,k)$ for any $\alpha \in V^m \sqcup U^m$. Also if $\beta \in V_m \sqcup U_m$, then Lemma 5.20 implies that if $\mathbf{y^m} \cdot \beta \in (y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)$, then $\mathbf{y^m} \cdot \beta \in I(n,k)$, and the result follows.

Conversely, assume that $0 < p < k+1$. Then as in the proof of Theorem 5.5(4) and in particular (19), we have $I(n,k) \neq I(n-1,k-1) \cap \left((y_1,\dots,y_{n-1})^{\langle k\rangle} + (x_n^2)\right)$.

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References

[1] P. Etingof, E. Gorsky, and I. Losev, *Representations of Cherednik algebras with minimal support and torus knots*, Adv. Math. **277** (2015), 124–180.

[2] R. Fröberg and B. Shapiro, *On Vandermonde varieties*, Math. Scand. **119** (2016), no. 1, 76–91.

[3] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry.*

[4] S. Griffeth, S. V. Sam, and C. B. Zamaere, *Jack polynomials as fractional quantum hall states and the Betti numbers of the (k + 1)-equals ideal*, Comm. Math. Phys. **330** (2014), 415–434.

[5] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe, *The Lefschetz Properties*, Lecture Notes in Math. **2080**, Springer, Berlin-Heidelberg, 2013.

[6] M. Hochster and J. Eagon, *Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), no. 4, 1020–1058.

[7] G. D. James and M. H. Peel, *Specht series for skew representations of symmetric groups*, J. Algebra **56** (1979), no. 2, 343–364.

[8] A. Kustin and A. Vraciu, *The weak Lefschetz property for monomial complete intersections in positive characteristic*, Trans. Amer. Math. Soc. **366** (2014), 4571–4601.

[9] S. Li and W. Li, *Independence numbers of graphs and generators of ideals*, Combinatorica **1** (1981), no. 1, 55–61.

[10] H. Matsumura, *Commutative Ring Theory*, Cambridge Stud. Adv. Math. **8**, Cambridge University Press, New York, 2006.

[11] J. Migliore, R. Miró-Roig, and U. Nagel, *Monomial ideals, almost complete intersections and the weak Lefschetz property*, Trans. Amer. Math. Soc. **363** (2011), 229–257.

[12] M. Nagata, *Local Rings*, Interscience Publishers, New York, 1962.

[13] G. A. Reisner, *Cohen–Macaulay quotients of polynomial rings*, Adv. Math. **21** (1976), 30–49.

[14] B. Sagan, *The Symmetric Group Representations, Combinatorial Algorithms, and Symmetric Functions*, 2nd ed. , Grad. Texts in Math. **203**, Springer, New York–Heidelberg, 2001.

[15] J. Watanabe, "*The Dilworth number of Artinian rings and finite posets with rank function*" in *Commutative Algebra and Combinatorics, Kyoto, 1985*, (eds. M. Nagata and H. Matsumura), Adv. Stud. Pure Math. **11**, North-Holland, Amsterdam; Kinokuniya, 1987, 303–312.

[16] J. Watanabe and K. Yanagawa, *Vandermonde determinantal ideals*, Math. Scand. **125** (2019), no. 2, 179–184.

[17] K. Yanagawa, When is a Specht ideal Cohen–Macaulay? J. Commut. Algebra, to appear.

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