A SEMI-INFINITE RANDOM WALK WITH DISCRETE STEPS

By L. R. SHENTON

Communicated by M. S. BARTLETT

Received 12 August 1954

1. A particle executes a random walk over the possible positions $x = 0, 1, 2, \ldots$, its initial position being $x = d \geqslant 0$. At the nth step it occupies the position x with probability $p_n(x \mid d)$ and is in the state (n, x). The transition from (n, x) to $(n+1, y)$ has the probability $p_{x,y}$ given by

$$\left.\begin{aligned} p_{x,x-1} = \tfrac{1}{2}K' \neq 0, \quad p_{x,x} = k, \quad p_{x,x+1} = \tfrac{1}{2}K \neq 0 \quad (x>0), \\ \text{where} \qquad \tfrac{1}{2}K' + k + \tfrac{1}{2}K = 1, \\ p_{0,x} = p_x \quad (0 \leqslant x \leqslant m), \\ p_{0,x} = 0 \quad (x > m). \end{aligned}\right\} \tag{1}$$

If the particle arrives at $x = 0$, the next step may lead to absorption with probability p_a, where $p_a = 1 - \sum_{s=0}^{m} p_s$, i.e. the particle is annihilated in this case. If $p_a = 1$, we say that $x = 0$ is an *absorbing* barrier. If $p_1 = 1$, we say that $x = 0$ is a *reflecting* barrier. If $p_0 = 1$, we say that $x = 0$ is a *retaining* barrier.

It will be seen that when $p_0 = 1$, $p_n(0 \mid d)$ represents the probability of annihilation (or absorption) in the course of $n-1$ steps.

We shall give an expression for $p_n(x \mid d)$ in the form of a contour integral. Illustrations include the gambler's ruin problem, random walk with drift as recently discussed by Kac (2), and two problems given by Lauwerier (3).

2. We introduce the g.f.

$$\Phi(x \mid d;\, t) = \sum_{n=0}^{\infty} t^n p_n(x \mid d) \quad (|t| < 1), \tag{2}$$

which will be abbreviated to Φ_x.

A consideration of the transition probabilities leads to

$$F_x(\Phi) = \delta_{x,d}, \tag{3}$$

where

$$\begin{aligned} F_0 &\equiv (1 - p_0 t)\,\Phi_0 - \tfrac{1}{2}K' t \Phi_1, \\ F_1 &\equiv -p_1 t \Phi_0 + (1 - kt)\,\Phi_1 - \tfrac{1}{2}K' t \Phi_2, \\ F_x &\equiv -p_x t \Phi_0 - \tfrac{1}{2}K t \Phi_{x-1} + (1 - kt)\,\Phi_x - \tfrac{1}{2}K' t \Phi_{x+1} \quad (x = 2, 3, \ldots), \\ \delta_{x,d} &= 1 \quad \text{if} \quad x = d, \\ &= 0 \quad \text{if} \quad x \neq d. \end{aligned}$$

It will be shown that the solution of (3) is

$$\Phi(x \mid d;\, t) = \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{1}{2\pi i t \sqrt{(KK')}} \int_C \frac{\Psi(x, d;\, z)\, 2z\, dz}{z H(z)\,(e^{\theta} - z)\,(z - e^{-\theta})}, \tag{4}$$

where

$$\text{(i)}\quad \begin{cases} H(z) = \sum_{s=0}^{m} (\lambda z)^s p_s - k - \tfrac{1}{2}\sqrt{(KK')}\,(z+z^{-1}) \quad (\lambda = \sqrt{(K'/K)}), \\ H_x(z) = \sum_{s=0}^{x} (\lambda z)^s p_s - k - \tfrac{1}{2}\sqrt{(KK')}\,(z+z^{-1}) \quad (x>0), \\ \quad\;\; = -\tfrac{1}{2}\sqrt{(KK')}\,z^{-1} \quad (x=0), \end{cases}$$

(ii) $\Psi(x,d;\,z) = H_x(z)\,z^{d-x} - H_x(z^{-1})\,z^{d+x}$,

(iii) $t^{-1} = k + \sqrt{(KK')}\cosh\theta$,

(iv) C is a contour enclosing the origin within which $\Psi/\{zH(z)\}$ has no poles except possibly $z=0$. This is equivalent to requiring $H(z)$ to be free from zeros within and on C.

For if $x \geqslant 2$,

$$F_x(\Phi) = -p_x t\Phi_0 + \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{1}{2\pi i}\int_C \frac{\Psi(x,d;z)\,(\cosh\theta - \frac{1}{2}(z+z^{-1}))\,dz}{zH(z)\,(\cosh\theta - \frac{1}{2}(z+z^{-1}))}$$
$$+\frac{1}{2}\left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{p_x\lambda^x}{2\pi i}\int_C \frac{(z^{d+1}-z^{d-1})\,dz}{zH(z)\,(\cosh\theta - \frac{1}{2}(z+z^{-1}))},$$

i.e.
$$F_x(\Phi) = \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{1}{2\pi i}\int_C \frac{\Psi(x,d;\,z)\,dz}{zH(z)}. \tag{5}$$

Similarly, the result may be shown to hold for $0 \leqslant x < 2$. It remains to show that $F_x(\Phi) = \delta_{x,d}$. Consider the three cases (*a*) $x>m$, (*b*) $0<x\leqslant m$, (*c*) $x=0$.

(*a*) $x>m$:

$$F_x(\Phi) = \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{1}{2\pi i}\int_C \left(z^{d-x-1} - \frac{H(z^{-1})}{H(z)}\,z^{d+x-1}\right) dz$$
$$= \delta_{x,d},$$

since the residue of $\dfrac{z^m H(z^{-1})}{zH(z)}\,z^{d+x-m}$ at the origin is zero.

(*b*) $0<x\leqslant m$:

$$F_x(\Phi) = \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{1}{2\pi i}\int_C (H_x(z)\,z^{d-x} - H_x(z^{-1})\,z^{d+x})\,\frac{dz}{zH(z)}.$$

But the residue of $\dfrac{z^x H_x(z^{-1})}{zH(z)}\,z^d$ at $z=0$ is zero, and

$$\frac{z^{d-x}H_x(z)}{zH(z)} = z^{d-x-1} - z^{d-x}\sum_{s=x+1}^{m} p_s(\lambda z)^s/\{zH(z)\} \quad (0<x<m),$$
$$= z^{d-x-1} \quad (x=m),$$

and so $\qquad F_x(\Phi) = \delta_{x,d}.$

(*c*) $x=0$. Here
$$F_0(\Phi) = \left(\frac{K}{K'}\right)^{-\frac{1}{2}d} \frac{\sqrt{(KK')}}{4\pi i}\int_C \frac{(z^{d+1}-z^{d-1})\,dz}{zH(z)}$$
$$= \delta_{0,d}.$$

In general (iv) of (4) can be taken to mean that the contour C must enclose $z = 0$ and exclude any zeros of $H(z)$. In special cases this may be relaxed since

$$H_x(z)\, z^{d-x} - H_x(z^{-1})\, z^{d+x}$$

may have a factor in common with $H(z)$. Hence we have

$$p_n(x \mid d) = \frac{1}{2\pi i}\int_C J_n(x, d;\, z)\, dz, \tag{6}$$

where
$$J_n \equiv \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \frac{(H_x(z)\, z^{d-x} - H_x(z^{-1})\, z^{d+x})}{zH(z)}\, k(z)^n,$$

$$k(z) = k + \tfrac{1}{2}\sqrt{(KK')}\,(z+z^{-1}).$$

3. *C as the unit circle.* The expression (6) still holds when C is taken to be $|z| = 1$, provided $J_n(x, d;\, z)$ has no poles in or on the unit circle except $z = 0$. If $J_n(x, d;\, z)$ has poles at z_j ($j = 1, 2, \ldots, s$; $z_j \neq 0$) inside the unit circle, then (6) is replaced by

$$p_n(x \mid d) = \frac{1}{2\pi i}\int_{|z|=1} J_n(x, d;\, z)\, dz - \sum_{j=1}^{s} [\text{residue of } J_n(z)]_{z=z_j}, \tag{7}$$

the summation excluding $z = 0$. This may be written

$$p_n(x \mid d) = \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \int_{-\pi}^{\pi} \frac{S_x^*(\phi)\, S_d(\phi)\, (k + \sqrt{(KK')}\cos\phi)^n\, d\phi}{\rho(\cos\phi)} - \sum_{j=1}^{s} [\text{res}\, J_n(z)]_{z=z_j}, \tag{8}$$

where

$$\begin{aligned} S_d(\phi) &= \Im\{H(z^{-1})\, z^d\} \\ &= (p_0 - k - \cos\phi\,\sqrt{(KK')})\sin d\phi + p_1\lambda \sin(d-1)\,\phi + \ldots + p_m \lambda^m \sin(d-m)\,\phi, \\ S_x^*(\phi) &= \Im\{H_x(z^{-1})\, z^x\} \\ &= (p_0 - k - \cos\phi\,\sqrt{(KK')})\sin x\phi + p_1\lambda \sin(x-1)\,\phi + \ldots + p_{x-1}\lambda^{x-1}\sin\phi \quad (x>0), \\ &= -\tfrac{1}{2}\sqrt{(KK')}\sin\phi \quad (x = 0). \\ \rho(\cos\phi) &= |\, H(e^{i\phi})\,|^2, \\ z &= e^{i\phi}. \end{aligned}$$

It will be observed that $S_x^*(\phi)/\sin\phi$ is a trigonometrical polynomial of degree x. Considered as function, of $\cos\phi$, it is of interest to note these polynomials are related to those considered by Szegö (4).† If $J_n(x, d;\, z)$ has a pole on the unit circle, say at $z = e^{i\alpha}$, then the integral in (8) is to be taken as the principal value; and similarly for several such poles.

An expression for the g.f. follows from (4). In the case when $[H(z)]^{-1}$ has simple poles at z_j inside the unit circle, we find

$$\Phi(x \mid d;\, t) = \left(\frac{K}{K'}\right)^{\frac{1}{2}(x-d)} \left\{ \frac{[e^{x\theta} H_x(e^{-\theta}) - e^{-x\theta} H_x(e^{\theta})]\, e^{-d\theta}}{t\sqrt{(KK')}\sinh\theta\, H(e^{-\theta})} - \sum_{j=1}^{s} \frac{\Psi(x, d;\, z,)}{z_j H'(z_j)\,[1 - tk(z_j)]} \right\}. \tag{9}$$

If $[H(z)]^{-1}$ has simple poles on the unit circle, then the only modification in (9) is to add the contributions resulting from the necessary indentations in the unit circle.

† Szegö considers, for example, the polynomials on the interval $[-1, +1]$ orthogonal with respect to the weight function $\rho(\cos\phi)$, where $\rho(X)$ is of precise degree e and positive in $[-1, +1]$. It may be shown that there is a unique normalized representation of $\rho(\cos\phi)$, namely $|h(e^{i\phi})|^2$, such that $h(0) > 0$ and $h(z) \neq 0$ in $|z| < 1$.

4. *Duration of the walk and probability of return*

4·1. From (7) we have

$$\sum_{n=0}^{\infty} p_n(x \mid d)$$
$$= \frac{1}{2\pi i}\int_{|z|=1} \frac{2zJ_0(x,d;\,z)\,dz}{(z\sqrt{K'}-\sqrt{K})(\sqrt{K'}-z\sqrt{K})} - \sum_{j=1}^{s} \operatorname{res}\left[\frac{2zJ_0(x,d;\,z)}{(z\sqrt{K'}-\sqrt{K})(\sqrt{K'}-z\sqrt{K})}\right]_{z=z_j}. \quad (10)$$

Hence if $K' > K$, noting that $H\left(\sqrt{\frac{K}{K'}}\right) = -p_a$ we see that if $p_a > 0$ then $z_j \neq \sqrt{(K/K')}$, and so for $0 \leqslant x \leqslant d$

$$\sum_{n=0}^{\infty} p_n(x \mid d) = \frac{p_a^{-1}}{\frac{1}{2}K' - \frac{1}{2}K}\left\{\left(\frac{K}{K'}\right)^x H_x\left(\sqrt{\frac{K'}{K}}\right) - H_x\left(\sqrt{\frac{K}{K'}}\right)\right\}. \quad (11)$$

Hence in this case the probability that the particle will *revisit* the point x is

$$1 - \frac{(\frac{1}{2}K' - \frac{1}{2}K)\,p_a}{\left(\frac{K}{K'}\right)^x H_x\left(\sqrt{\frac{K'}{K}}\right) - \sum_{0}^{x} p_s + 1}. \quad (12)$$

For if it visits a given point, that point becomes the starting point of the subsequent walk, irrespective of its previous history. If $x > d$, taking account of the pole at $z = 0$, we have

$$\sum_{0}^{\infty} p_n(x \mid d) = \frac{p_a^{-1}}{\frac{1}{2}K' - \frac{1}{2}K}\left\{\left(\frac{K}{K'}\right)^x H_x\left(\sqrt{\frac{K'}{K}}\right) + \sum_{x+1}^{m} p_s + p_a\left(\frac{K}{K'}\right)^{x-d}\right\}. \quad (13)$$

For completeness the following further results are noted:

$$\sum_{n=0}^{\infty} p_n(x \mid d) = \frac{(K'/K)^d}{\frac{1}{2}K - \frac{1}{2}K'}\left\{\left(\frac{K}{K'}\right)^x H_x\left(\sqrt{\frac{K'}{K}}\right) - H_x\left(\sqrt{\frac{K}{K'}}\right)\right\}\Big/ H\left(\sqrt{\frac{K'}{K}}\right) \quad (14)$$

$(0 \leqslant x \leqslant d,\ K > K',\ p_0 \neq 1)$,

$$= \frac{(K'/K)^d}{\frac{1}{2}K - \frac{1}{2}K'}\left\{\frac{\left(\frac{K}{K'}\right)^x H_x\left(\sqrt{\frac{K'}{K}}\right) - H_x\left(\sqrt{\frac{K}{K'}}\right)}{H\left(\sqrt{\frac{K'}{K}}\right)} - \left(\frac{K}{K'}\right)^x + \left(\frac{K}{K'}\right)^d\right\} \quad (15)$$

$(x \geqslant d,\ K > K',\ p_0 \neq 1)$.

4·2. The expected number of steps to annihilation is, for $K' > K$,

$$D(d) = 1 + p_a \sum_{s=0}^{\infty} s p_s(0 \mid d)$$
$$= 1 + p_a\left(\frac{K'}{K}\right)^{\frac{1}{2}d} \frac{1}{\pi i\sqrt{(KK')}}\int_C \frac{(z^2-1)\,z^{d+1}\,k(z)\,dz}{zH(z)\left[\left(1 - z\sqrt{\frac{K'}{K}}\right)\left(1 - z\sqrt{\frac{K}{K'}}\right)\right]^2}$$
$$- p_a \sum_{j=1}^{s} \operatorname{res}\left[\frac{J_0(z)\,k(z)}{[1-k(z)]^2}\right]_{z=z_j} \quad (16)$$

$$= \frac{1}{p_a} + \frac{d}{\frac{1}{2}K' - \frac{1}{2}K} + \frac{\sum_{0}^{m} s p_s}{p_a(\frac{1}{2}K' - \frac{1}{2}K)}. \quad (17)$$

The formula (17) gives the expected duration of the game in gambling against an infinitely rich adversary, the gambler winning at a trial with probability $\frac{1}{2}K$, tieing with probability k, and losing with probability $\frac{1}{2}K$, his initial capital being d, and the stakes a unit of capital per trial. If, however, the gambler is reduced to penury, at the next trial he may be ruined with probability p_a, win the right to spin again with probability p_0, or start again with capital of x units with probability p_x.

5. *Applications*

5·1. *Chance of ruin in gambling against infinitely rich adversary.* Suppose the gambler's chance of winning at any trial is $p = \frac{1}{2}K$, and $q = 1-p = \frac{1}{2}K'$ is his chance of losing. Then the chance of ruin in the course of n games is given by (6) with $p_0 = 1$,

$$H(z) = 1-\surd(pq)(z+z^{-1}) = z^{-1}\surd(pq)\left(\sqrt{\frac{q}{p}}-z\right)\left(z-\sqrt{\frac{p}{q}}\right)$$

and
$$p_n(0\,|\,d) = \frac{(pq)^{\frac{1}{2}(n+1)}}{2\pi i}\left(\frac{q}{p}\right)^{\frac{1}{2}d}\int_C \frac{(z-z^{-1})\,z^{d-1}\,(z+z^{-1})^n\,dz}{1-(z+z^{-1})\surd(pq)},$$

where C is a contour surrounding the origin but not $z = \surd(p/q)$ or $z = \surd(q/p)$. Hence if $q>p$, taking C as the unit circle, and thus containing $z = \surd(p/q)$ we have (subtracting the residue at $z = \surd(p/q)$)

$$p_n(0\,|\,d) = 1-\frac{[2\surd(pq)]^{n+1}}{\pi}\left(\frac{q}{p}\right)^{\frac{1}{2}d}\int_0^{\pi}\frac{\sin\phi\sin d\phi\cos^n\phi\,d\phi}{1-2\surd(pq)\cos\phi}. \tag{18}$$

If, however, $p>q$ we subtract the residue at $z = \surd(q/p)$, and this leads to (18) again except that the first term is now $(q/p)^d$. (See for example, Uspensky (5), p. 159.)

If the games are equitable, then the probability of ruin in the course of n games is

$$\frac{1}{2\pi i}\int_C \frac{1+z}{1-z}\left(\frac{z+z^{-1}}{2}\right)^n z^d\,dz, \tag{19}$$

and using $|z| = 1$ indented at $z = 1$ as C we find

$$p_n(0\,|\,d) = 1-\frac{1}{\pi}\int_0^{\pi}\sin d\phi\cot(\tfrac{1}{2}\phi)\cos^n\phi\,d\phi$$
$$= 1-\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin d\phi}{\sin\phi}\cos^{n+1}\phi\,d\phi, \tag{20}$$

if n and d have the same parity. (See (5), p. 159.)

5·2. *Particle with drift in the presence of a reflecting barrier.* The boundary conditions in this case are $p_0 = 0$, $p_1 = 1$, so that $H(z) = \frac{1}{2}z^{-1}(K'z^2-K)\surd(K'/K)$ with $k = 0$. In (8) we have

$$S_d(\phi) = \frac{1}{2}\sqrt{\frac{K'}{K}}\{K'\sin(d-1)\phi-K\sin(d+1)\phi\},$$

$$S_x^*(\phi) = S_x(\phi)\quad (x>0),$$
$$= -\tfrac{1}{2}\surd(KK')\sin\phi\quad (x=0),$$

https://doi.org/10.1017/S0305004100030449 Published online by Cambridge University Press

and the residues of $J(z)$ occur at $z = \pm\sqrt{(K/K')}$ for $K' > K$. Writing $K' = 2p$, $K = 2q$, we find

$$p_n(x\,|\,d) = \frac{q^*}{2pq}(p-q)\,(q/p)^x\,(1+(-1)^{x+d+n}) + \frac{2}{\pi}q^*\left(\frac{q}{p}\right)^{\frac{1}{2}(x-d)}\int_0^\pi \frac{f_x(\phi)f_d(\phi)\tan^2\phi(2\sqrt{(pq)}\cos\phi)^n\,d\phi}{(p-q)^2+\tan^2\phi}, \tag{21}$$

where (i) $f_x(\phi) = \cos x\phi - (p-q)\sin x\phi\cos\phi/\sin\phi$,

(ii) $q^* = q$ if $x = 0$,
$= 1$ if $x > 0$,

(iii) $p > q$.

If† $p < q$ the first term in (21) is zero for in this case $z = \pm\sqrt{(q/p)}$ is outside $|z| = 1$.

5·3. (i) Lauwerier (3) discusses the random walk where a particle starts at $z = 0$ in the presence of an elastic barrier at $z = m$. In our notation the boundary conditions are $p_0 = 0$, $p_1 = 1$ and $K' = K = 1$. Lauwerier does not give an expression for $p_n(x\,|\,d)$. We have $H(z) = \frac{1}{2}z^{-1}((q-p)z^2-1)$, and the zeros $z = \dfrac{\pm 1}{\sqrt{(q-p)}}$ are outside $|z| = 1$ for $0 < q < 1$. Hence from (8) we have

$$p_n(x\,|\,d) = \left(\frac{1}{2q}\right)^*\frac{2}{\pi}\int_0^\pi \frac{(p\sin x\theta\cos\theta + q\cos x\theta\sin\theta)\,(p\sin d\theta\cos\theta + q\cos d\theta\sin\theta)\cos^n\theta\,d\theta}{p^2\cos^2\theta + q^2\sin^2\theta}, \tag{22}$$

where

$$\left(\frac{1}{2q}\right)^* = 1 \quad (x \neq 0),$$
$$= \frac{1}{2q} \quad (x = 0).$$

The expression (22) corresponds to Lauwerier's $h_n(z)$ (see (3), p. 298), where $m - z = x$, the boundary being at $z = m$.

(ii) Lauwerier ((3), p. 296) also discusses the random walk with start at $z = 0$ and a barrier at $z = m$, such that the particle arriving at the barrier may be absorbed with probability p, or move to $m-1$ or $m+1$ with probabilities $\frac{1}{2}q = \frac{1}{2}(1-p)$. In our present notation the start is at $x = d$, $K = K' = 1$, and at the barrier $p_a = p$, $p_1 = \frac{1}{2}q$. The expression for $p_n(x\,|\,d)$ can easily be found by using Kelvin's method of images. Thus the source at $x = d$ is equivalent to sources $\frac{1}{2}$ and $\frac{1}{2}$ at $+d$ and $-d$, and a source $\frac{1}{2}$ at $x = d$ and sink $-\frac{1}{2}$ at $x = -d$. The solution to the former is exactly the same as for a source $\frac{1}{2}$ at $x = d$ in the presence of a barrier with properties $p_0 = p$, $p_1 = q = 1-p$, provided we double the value of the probability at the origin. The latter is equivalent to an absorbing barrier (as defined by Feller (1)) at $x = 0$ with a source $\frac{1}{2}$ at $x = d$. Hence

$$p_n(x\,|\,d) = \frac{1}{\pi}\int_0^\pi \sin x\theta\sin d\theta\cos^n\theta\,d\theta + \left(\frac{1}{q}\right)^*\frac{1}{\pi}\int_0^\pi \frac{(p\sin|x|\theta\cos\theta + q\cos x\theta\sin\theta)\,(p\sin d\theta\cos\theta + q\cos d\theta\sin\theta)\cos^n\theta\,d\theta}{p^2\cos^2\theta + q^2\sin^2\theta},$$

† The expression (21) differs from that given by Kac (2) for this problem, the $\cos\phi$ factor not appearing in $f_x(\phi)$. In correspondence Mr Kac informs me that this was omitted from his article by a misprint.

where

$$\left(\frac{1}{q}\right)^* = 1 \quad (x \neq 0),$$
$$= \frac{1}{q} \quad (x = 0).$$

5·4 Some further special cases are given below:†

(i) $p_0 = 1;\ K = K' = 1$:

$$p_n(x \mid d) = \frac{2}{\pi}\int_0^\pi \sin x\phi \sin d\phi \cos^n \phi \, d\phi \quad (x > 0),$$
$$= 1 - \frac{1}{\pi}\int_0^\pi \sin d\phi \cot(\tfrac{1}{2}\phi) \cos^n \phi \, d\phi \quad (x = 0). \tag{23}$$

(ii) $p_1 = 1;\ K = K' = 1$:

$$p_n(x \mid d) = \frac{2^*}{\pi}\int_0^\pi \cos x\phi \cos d\phi \cos^n \phi \, d\phi. \tag{24}$$

(iii) In (21) put $x - d = x'$, and let x and $d \to \infty$, and using well-known properties of oscillatory integrals we have

$$p_n(x' \mid 0) = \frac{2^n}{\pi} q^{\frac{1}{2}(n+x)} p^{\frac{1}{2}(n-x)} \int_0^\pi \cos x'\phi \cos^n \phi \, d\phi. \tag{25}$$

(iv) $p_0 = 1-p,\ p_1 = p;\ \tfrac{1}{2}K = p,\ k = 1-2p,\ \tfrac{1}{2}K' = p \quad (0 < p \leqslant \tfrac{1}{2})$:

$$p_n(x \mid d) = \frac{2}{\pi}\int_0^\pi \cos(x+\tfrac{1}{2})\phi \cos(d+\tfrac{1}{2})\phi (1 - 4p\sin^2(\tfrac{1}{2}\phi))^n \, d\phi. \tag{26}$$

(v) $p_0 = 1-p,\ p_1 = p;\ \tfrac{1}{2}K = \tfrac{1}{2}K' = \tfrac{1}{2}p,\ k = 1-p$:

$$p_n(x \mid d) = \frac{2^*}{\pi}\int_0^\pi \cos x\phi \cos d\phi (q + p\cos\phi)^n \, d\phi. \tag{27}$$

I am indebted to Dr J. A. Storrow of the Chemical Engineering Department, Manchester College of Technology, for drawing my attention to a problem similar to the one considered here.

REFERENCES

(1) FELLER, W. *An introduction to probability theory and its applications*, vol. 1 (New York, 1950).
(2) KAC, MARK. Random walk and the theory of Brownian motion. *Amer. math. Mon.* 54 (1947), 369–91.
(3) LAUWERIER, H. A. A linear random walk with a barrier. *Appl. sci. Res., Hague*, B, 2 (1951), 294–300.
(4) SZEGÖ, G. *Orthogonal polynomials* (Colloq. Publ. Amer. math. Soc. 1939).
(5) USPENSKY, J. V. *Introduction to mathematical probability* (New York, 1937).

COLLEGE OF TECHNOLOGY
MANCHESTER 1

† $2^* = 1$ if $x \neq 0$, $2^* = 2$ if $x = 0$.