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CHARACTERIZATION OF THE CONDITIONAL STATIONARY DISTRIBUTION IN MARKOV CHAINS VIA SYSTEMS OF LINEAR INEQUALITIES

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Abstract

This paper considers ergodic, continuous-time Markov chains $\{X(t)\}_{t\in(-\infty,\infty)}$ on $\mathbb{Z}^+ = \{0, 1, \ldots\}$. For an arbitrarily fixed $N \in \mathbb{Z}^+$, we study the conditional stationary distribution $\boldsymbol{\pi}(N)$ given the Markov chain being in $\{0, 1, \ldots, N\}$. We first characterize $\boldsymbol{\pi}(N)$ via systems of linear inequalities and identify simplices that contain $\boldsymbol{\pi}(N)$, by examining the $(N+1)\times(N+1)$ northwest corner block of the infinitesimal generator $\boldsymbol{Q}$ and the subset of the first $N+1$ states whose members are directly reachable from at least one state in $\{N+1, N+2, \ldots\}$. These results are closely related to the augmented truncation approximation (ATA), and we provide some practical implications for the ATA. Next we consider an extension of the above results, using the $(K+1)\times(K+1)$ $(K>N)$ northwest corner block of $\boldsymbol{Q}$ and the subset of the first $K+1$ states whose members are directly reachable from at least one state in $\{K+1, K+2, \ldots\}$. Furthermore, we introduce new state transition structures called (K, N)-skip-free sets, using which we obtain the minimum convex polytope that contains $\boldsymbol{\pi}(N)$.

Keywords: Markov chains; countably infinite state space; conditional stationary distribution; linear inequalities; convex polytopes; augmented truncation approximation

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1. Introduction

We consider a time-homogeneous, continuous-time Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ on a countably infinite state space $\mathbb{Z}^+ = \{0, 1, \ldots\}$. Let $q_{i,j}$ $(i, j \in \mathbb{Z}^+, i \neq j)$ denote the transition rate from state i to state j. The infinitesimal generator $\boldsymbol{Q}$ of the Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ is then given by

$$\boldsymbol{Q} = \begin{pmatrix} -q_0 & q_{0,1} & q_{0,2} & q_{0,3} & \cdots \\ q_{1,0} & -q_1 & q_{1,2} & q_{1,3} & \cdots \\ q_{2,0} & q_{2,1} & -q_2 & q_{2,3} & \cdots \\ q_{3,0} & q_{3,1} & q_{3,2} & -q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

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where

$$q_i = \sum_{\substack{j\in\mathbb{Z}^+\\ j\neq i}} q_{i,j}, \qquad i \in \mathbb{Z}^+.$$

We assume that $q_i < \infty$ ($i \in \mathbb{Z}^+$) and $\{X(t)\}_{t\in(-\infty,\infty)}$ is stationary and ergodic. We then define $\boldsymbol{\pi}$ as the stationary distribution of $\{X(t)\}_{t\in(-\infty,\infty)}$:

$$\boldsymbol{\pi} = (\pi_0\ \pi_1\ \pi_2\ \ldots\ ),$$

where $\pi_i = \mathbb{P}(X(0) = i)$ ($i \in \mathbb{Z}^+$). Note that $\boldsymbol{\pi}$ is determined uniquely by

$$\boldsymbol{\pi}\boldsymbol{Q} = \mathbf{0}, \qquad \boldsymbol{\pi}\boldsymbol{e} = 1, \tag{1.1}$$

where $\boldsymbol{e}$ denotes a column vector of the appropriate dimension whose elements are all equal to one.

In general, it is hard to obtain the stationary distribution of the Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ because there are infinitely many unknowns π_i ($i \in \mathbb{Z}^+$) in (1.1). In the past, this problem has been tackled mainly by two approaches: the augmented truncation approximation [4, 7, 12, 16] and the matrix-analytic method [6, 10, 11]. In the augmented truncation approximation, we choose a sufficiently large $N \in \mathbb{Z}^+$ and partition $\mathbb{Z}^+$ into a finite subset $\mathbb{Z}_0^N$ and its complement $\mathbb{Z}_{N+1}^\infty$, where $\mathbb{Z}_m^\ell$ ($m, \ell \in \mathbb{Z}^+$) and $\mathbb{Z}_m^\infty$ ($m \in \mathbb{Z}^+$) are defined as

$$\mathbb{Z}_m^\ell = \begin{cases} \{m, m+1, \ldots, \ell\}, & m \le \ell, \\ \emptyset, & m > \ell, \end{cases} \qquad \mathbb{Z}_m^\infty = \{m, m+1, \ldots\}, \qquad m \in \mathbb{Z}^+.$$

We also partition the stationary distribution $\boldsymbol{\pi}$ in conformance with the partition of the state space:

$$\boldsymbol{\pi} = \begin{array}{cc} \scriptstyle\mathbb{Z}_0^N & \scriptstyle\mathbb{Z}_{N+1}^\infty \\ \Big(\boldsymbol{\pi}^{(1)}(N) & \boldsymbol{\pi}^{(2)}(N) \Big) \end{array}. \tag{1.2}$$

We then construct a *finite-state* Markov chain on $\mathbb{Z}_0^N$, using the $(N+1)\times(N+1)$ northwest corner block $\boldsymbol{Q}^{(1,1)}(N)$ of the infinitesimal generator $\boldsymbol{Q}$. This finite-state Markov chain on $\mathbb{Z}_0^N$ can be regarded as an approximation to the original Markov chain on $\mathbb{Z}^+$. Specifically, we construct an infinitesimal generator $\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)$ of a Markov chain on $\mathbb{Z}_0^N$, where $\boldsymbol{Q}_{\mathrm{A}}(N)$ is an augmentation matrix such that $\boldsymbol{Q}_{\mathrm{A}}(N) \ge \boldsymbol{O}$ and $\big[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)\big]\boldsymbol{e} = \mathbf{0}$. Usually, $[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)]$ is assumed to be irreducible. Let $\boldsymbol{\pi}(N)$ ($N \in \mathbb{Z}^+$) denote the conditional stationary distribution given $X(0) \in \mathbb{Z}_0^N$:

$$\boldsymbol{\pi}(N) = (\pi_0(N)\ \pi_1(N)\ \cdots\ \pi_N(N)),$$

where $\pi_i(N) = \mathbb{P}(X(0) = i \mid X(0) \in \mathbb{Z}_0^N)$ ($i \in \mathbb{Z}_0^N$). By definition, we have

$$\boldsymbol{\pi}(N) = \frac{\boldsymbol{\pi}^{(1)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}}, \tag{1.3}$$

where $\boldsymbol{\pi}^{(1)}(N)$ is given in (1.2). In the augmented truncation approximation, an approximation $\boldsymbol{\pi}^{\mathrm{trunc}}(N)$ to $\boldsymbol{\pi}(N)$ is determined by $\boldsymbol{\pi}^{\mathrm{trunc}}(N)[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)] = \mathbf{0}$ and $\boldsymbol{\pi}^{\mathrm{trunc}}(N)\boldsymbol{e} = 1$. We then adopt $\boldsymbol{\pi}^{\mathrm{approx},N} = (\boldsymbol{\pi}^{\mathrm{trunc}}(N)\ \ \mathbf{0})$ as an approximation to the stationary distribution $\boldsymbol{\pi}$ of

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the original Markov chain. Under some technical conditions, $\boldsymbol{\pi}^{\text{approx},N}$ is known to converge to $\boldsymbol{\pi}$ as N goes to infinity [4, 12]. Therefore, $\boldsymbol{\pi}^{\text{approx},N}$ will be a good approximation if N is sufficiently large.

On the other hand, the matrix-analytic method utilizes structures of Markov chains which appear often in applications. Specifically, the standard matrix-analytic method is applicable to block-structured Markov chains with the following features: the state space $\mathbb{Z}^+$ can be partitioned into finite and disjoint subsets $\mathcal{L}_m$ ($m \in \mathbb{Z}^+$), called *levels*, in such a way that direct transitions between levels are skip-free in one direction (or both). In particular, Markov chains are called M/G/1-type if transitions between levels are skip-free to the left [11], and they are called G/M/1-type if transitions between levels are skip-free to the right [10]. If transitions between levels are skip-free in both directions, they are called quasi-birth-and-death processes [1, 10]. In the matrix-analytic method, the exact computation of the stationary distribution is targeted.

In both of the above-mentioned approaches, the (conditional) stationary distribution is characterized by systems of linear equations. By contrast, this paper studies the conditional stationary distribution $\boldsymbol{\pi}(N)$ ($N \in \mathbb{Z}^+$) via systems of linear *inequalities*. It is known that the boundary vector in block-structured Markov chains of level-dependent M/G/1 type can be obtained by the solution of an infinite number of inequalities; based on this, computational algorithms for the conditional stationary distribution $\boldsymbol{\pi}(N)$ have been developed in [5, 15].

The purpose of this paper is to characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ via systems of linear inequalities, without assuming any regular structures. In other words, we will attempt to find minimum regions on the first orthant $\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} \geq \boldsymbol{0}\}$ of $\mathbb{R}^{N+1}$ which contain $\boldsymbol{\pi}(N)$. Specifically, we first consider $\boldsymbol{\pi}(N)$ based only on the $(N+1) \times (N+1)$ northwest corner block $\boldsymbol{Q}^{(1,1)}(N)$ of the infinitesimal generator $\boldsymbol{Q}$, and we obtain a system of $N+1$ linear inequalities that $\boldsymbol{\pi}(N)$ satisfies. It immediately follows that $\boldsymbol{\pi}(N)$ lies in an N-simplex $\mathcal{P}(N)$ on the first orthant of $\mathbb{R}^{N+1}$, where vertices of $\mathcal{P}(N)$ are determined only by $\boldsymbol{Q}^{(1,1)}(N)$. As we will see, the derivation of these results is absurdly simple. Nonetheless, the $N+1$ vertices of the N-simplex $\mathcal{P}(N)$ (i.e., $N+1$ linearly independent probability vectors that span $\mathcal{P}(N)$) are essential for $\boldsymbol{\pi}(N)$; in any ergodic, continuous-time Markov chain with $\boldsymbol{Q}^{(1,1)}(N)$, $\boldsymbol{\pi}(N)$ can be expressed as a convex combination of those vertices. We also provide a probabilistic interpretation of this result.

Next we consider the case that a subset

$$\mathcal{J}(N) = \{j \in \mathbb{Z}_0^N; q_{i,j} > 0 \text{ for some } i \in \mathbb{Z}_{N+1}^\infty\} \tag{1.4}$$

of $\mathbb{Z}_0^N$ is available, as well as $\boldsymbol{Q}^{(1,1)}(N)$. Note that $\mathcal{J}(N)$ can be regarded as a kind of structural information, because it is the subset of states in $\mathbb{Z}_0^N$ which are directly reachable from at least one state in $\mathbb{Z}_{N+1}^\infty$. We show that the role of $\mathcal{J}(N)$ is to eliminate redundant vertices of $\mathcal{P}(N)$, and we obtain a $(|\mathcal{J}(N)| - 1)$-simplex $\mathcal{P}^+(N)$ whose relative interior contains $\boldsymbol{\pi}(N)$, where for any set $\mathcal{X}$, $|\mathcal{X}|$ stands for the cardinality of $\mathcal{X}$. In other words, when $\mathcal{J}(N)$ is available, $\boldsymbol{\pi}(N)$ is given by a convex combination of $|\mathcal{J}(N)|$ vertices of $\mathcal{P}^+(N)$ with *positive* weights. These results are closely related to the augmented truncation approximation (ATA), and we provide some practical implications for it. In particular, the linear ATA has the same degree of freedom as the general ATA does, and $\mathcal{J}(N)$ is useful in choosing an augmentation matrix in the linear ATA.

Next we consider an extension of the above results, using state transitions in peripheral states of $\mathbb{Z}_0^N$. Specifically, for a fixed K ($K > N$), we obtain an N-simplex $\widetilde{\mathcal{P}}(K, N)$ ($\subseteq \mathcal{P}(N)$) that contains $\boldsymbol{\pi}(N)$, which is determined only by the $(K+1) \times (K+1)$ northwest corner block

$\boldsymbol{Q}^{(1,1)}(K)$ of $\boldsymbol{Q}$. Furthermore, when both $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available, we show that redundant vertices of $\widetilde{\mathcal{P}}(K, N)$ can be eliminated.

The common feature of all of the above results is that we characterize $\boldsymbol{\pi}(N)$ by specifying simplices on the first orthant of $\mathbb{R}^{N+1}$ which contain $\boldsymbol{\pi}(N)$. In order to refine these results further, we consider general convex polytopes, using $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$. For this purpose, we introduce a new structure called *(K, N)-skip-free sets*, and using them, we obtain the minimum convex polytope that contains $\boldsymbol{\pi}(N)$, under the condition that only $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available.

The rest of this paper is organized as follows. In Section 2, we obtain simplices that contain $\boldsymbol{\pi}(N)$, based on $\boldsymbol{Q}^{(1,1)}(N)$ (and $\mathcal{J}(N)$). We also provide some practical implications for the augmented truncation approximation. In Section 3, we extend the result in the preceding section, using state transitions in peripheral states. In Section 4, we introduce skip-free sets and obtain the minimum convex polytope that contains $\boldsymbol{\pi}(N)$. Finally, some concluding remarks are provided in Section 5.

2. The characterization of $\boldsymbol{\pi}(N)$ by linear inequalities: fundamental results

In this section, we characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ via systems of linear inequalities, based on the $(N+1)\times(N+1)$ northwest corner block of the infinitesimal generator $\boldsymbol{Q}$.

2.1. Preliminaries

We will construct a convex polytope $\mathcal{P}$, by considering the intersection of a polyhedral convex cone $\mathcal{C}$ on the first orthant $\{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{x}\geq\boldsymbol{0}\}$ of $\mathbb{R}^{N+1}$ and a hyperplane containing all probability vectors in $\mathbb{R}^{N+1}$. As we will see, $\mathcal{C}$ and $\mathcal{P}$ take the following forms:

$$\begin{aligned}\mathcal{C} &= \{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{xA}\geq\boldsymbol{0},\ \boldsymbol{xB}=\boldsymbol{0}\},\\ \mathcal{P} &= \mathcal{C}\cap\{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{xe}=1\} = \{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{xA}\geq\boldsymbol{0},\ \boldsymbol{xB}=\boldsymbol{0},\ \boldsymbol{xe}=1\},\end{aligned} \tag{2.1}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ denote appropriate matrices with $N+1$ rows. Note that we allow $\boldsymbol{B}=\boldsymbol{O}$, and in that case, we can ignore the constraint $\boldsymbol{xB}=\boldsymbol{0}$ in (2.1). In general, a convex polytope $\mathcal{P}$ can also be represented by a set of convex combinations of vertices of $\mathcal{P}$. Specifically, by using an appropriate nonnegative matrix $\overline{\boldsymbol{C}}$ with $N+1$ columns such that $\overline{\boldsymbol{C}}\boldsymbol{e}=\boldsymbol{e}$, $\mathcal{P}$ in (2.1) can also be represented as

$$\mathcal{P} = \{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{x}=\boldsymbol{\alpha}\overline{\boldsymbol{C}}, \boldsymbol{\alpha}\geq\boldsymbol{0}, \boldsymbol{\alpha e}=1\}.$$

Note that $\overline{\boldsymbol{C}}$ is composed of $1\times(N+1)$ probability vectors $\overline{\boldsymbol{c}}_i$ $(i=0, 1, \ldots, M)$ that span $\mathcal{P}$. If the M vectors $\overline{\boldsymbol{c}}_1-\overline{\boldsymbol{c}}_0, \overline{\boldsymbol{c}}_2-\overline{\boldsymbol{c}}_0, \ldots, \overline{\boldsymbol{c}}_M-\overline{\boldsymbol{c}}_0$ are linearly independent, $\mathcal{P}$ is called an M-simplex. Let $\mathrm{ri}\,\mathcal{P}$ denote the relative interior of the convex polytope $\mathcal{P}$:

$$\begin{aligned}\mathrm{ri}\,\mathcal{P} &= \{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{xA}>\boldsymbol{0},\ \boldsymbol{xB}=\boldsymbol{0},\ \boldsymbol{xe}=1\}\\ &= \{\boldsymbol{x}\in\mathbb{R}^{N+1}; \boldsymbol{x}=\boldsymbol{\alpha}\overline{\boldsymbol{C}}, \boldsymbol{\alpha}>\boldsymbol{0},\ \boldsymbol{\alpha e}=1\}.\end{aligned} \tag{2.2}$$

2.2. Characterization of $\boldsymbol{\pi}(N)$ in terms of $\boldsymbol{Q}^{(1,1)}(N)$

In this subsection, we assume that for an arbitrarily fixed $N\in\mathbb{Z}^+$, the stationary distribution $\boldsymbol{\pi}$ of an ergodic, continuous-time Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ is partitioned as in (1.2) and the infinitesimal generator $\boldsymbol{Q}$ is partitioned as follows:

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^{\infty} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^{\infty} \end{array} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) & \boldsymbol{Q}^{(1,2)}(N) \\ \boldsymbol{Q}^{(2,1)}(N) & \boldsymbol{Q}^{(2,2)}(N) \end{pmatrix} \end{array}. \tag{2.3}$$

We define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ as the collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$ whose infinitesimal generators have the specific $(N+1) \times (N+1)$ northwest corner block $\boldsymbol{Q}^{(1,1)}(N)$.

We first characterize the conditional stationary distribution $\boldsymbol{\pi}(N)$ of Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$. Owing to the ergodicity, $\boldsymbol{Q}^{(1,1)}(N)$ is nonsingular whether it is irreducible or not. We then define $\boldsymbol{H}(N)$ as

$$\boldsymbol{H}(N) = (-\boldsymbol{Q}^{(1,1)}(N))^{-1}. \tag{2.4}$$

By definition, $\boldsymbol{H}(N) \geq \boldsymbol{O}$ and $\boldsymbol{H}(N)\boldsymbol{e} > \boldsymbol{0}$. Let $\overline{\boldsymbol{H}}(N)$ denote an $(N+1) \times (N+1)$ matrix obtained by normalizing each row of $\boldsymbol{H}(N)$ in such a way that $\overline{\boldsymbol{H}}(N)\boldsymbol{e} = \boldsymbol{e}$:

$$\overline{\boldsymbol{H}}(N) = \mathrm{diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{H}(N),$$

where for any $(n+1)$-dimensional vector $\boldsymbol{x}$, $\mathrm{diag}(\boldsymbol{x})$ denotes an $(n+1)$-dimensional diagonal matrix whose ith diagonal element is given by the ith element $[\boldsymbol{x}]_i$ of $\boldsymbol{x}$. We also define $\Gamma(n)$ $(n \in \mathbb{Z}^+)$ as the set of all $(n+1)$-dimensional probability vectors:

$$\Gamma(n) = \{\boldsymbol{\alpha} \in \mathbb{R}^{n+1}; \quad \boldsymbol{\alpha} \geq \boldsymbol{0}, \boldsymbol{\alpha}\boldsymbol{e} = 1\}, \quad n \in \mathbb{Z}^+. \tag{2.5}$$

Theorem 2.1. *For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$, we have*

$$\boldsymbol{\pi}(N) \in \mathcal{P}(N), \quad N \in \mathbb{Z}^+, \tag{2.6}$$

where $\mathcal{P}(N)$ $(N \in \mathbb{Z}^+)$ denotes an N-simplex on the first orthant of $\mathbb{R}^{N+1}$ which is given by

$$\mathcal{P}(N) = \{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \tag{2.7}$$

$$= \left\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N), \boldsymbol{\alpha} \in \Gamma(N)\right\}. \tag{2.8}$$

Proof. The starting point of the proof is the global balance equations (1.1) for $\boldsymbol{\pi}^{(1)}(N)$:

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N) = \boldsymbol{0}. \tag{2.9}$$

It then follows from (1.3) and (2.9) that

$$\boldsymbol{\pi}(N)(-\boldsymbol{Q}^{(1,1)}(N)) = \frac{\boldsymbol{\pi}^{(2)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}} \cdot \boldsymbol{Q}^{(2,1)}(N). \tag{2.10}$$

Because $\boldsymbol{\pi}^{(2)}(N)/\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} > \boldsymbol{0}$ and $\boldsymbol{Q}^{(2,1)}(N) \geq \boldsymbol{O}$, $\boldsymbol{\pi}(N)$ satisfies

$$\boldsymbol{\pi}(N)(-\boldsymbol{Q}^{(1,1)}(N)) \geq \boldsymbol{0}, \qquad \boldsymbol{\pi}(N)\boldsymbol{e} = 1,$$

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from which (2.6) with (2.7) follows.

Next, we show the equivalence between (2.7) and (2.8). We rewrite (2.7) as

$$
\begin{aligned}
&\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{y},\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{y}\boldsymbol{H}(N),\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\}.
\end{aligned}
$$

Let $\boldsymbol{\alpha} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})$. Since $\boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N)$, we have $\boldsymbol{x}\boldsymbol{e} = \boldsymbol{\alpha}\boldsymbol{e}$. Furthermore, if $\boldsymbol{y} \geq \mathbf{0}$, we have $\boldsymbol{\alpha} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}) \geq \mathbf{0}$ and vice versa, since $\boldsymbol{H}(N)\boldsymbol{e} > \mathbf{0}$. We thus have

$$
\begin{aligned}
&\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \mathbf{0},\ \boldsymbol{x}\boldsymbol{e} = 1\} \\
&\quad = \{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N), \boldsymbol{\alpha} \geq \mathbf{0},\ \boldsymbol{\alpha}\boldsymbol{e} = 1\}.
\end{aligned}
\tag{2.11}
$$

Because $\overline{\boldsymbol{H}}(N)$ is composed of $N+1$ linearly independent probability vectors, $\mathcal{P}(N)$ is an N-simplex on the first orthant of $\mathbb{R}^{N+1}$. □

Theorem 2.1 implies that the conditional stationary distribution $\boldsymbol{\pi}(N)$ of *any* Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ is given by a convex combination of row vectors of $\overline{\boldsymbol{H}}(N)$; i.e., there exists $\boldsymbol{\alpha}^*(N)$ such that

$$
\boldsymbol{\pi}(N) = \boldsymbol{\alpha}^*(N)\overline{\boldsymbol{H}}(N), \qquad \boldsymbol{\alpha}^*(N) \in \Gamma(N). \tag{2.12}
$$

Remark 2.1. Since $\overline{\boldsymbol{H}}(N)$ is nonsingular, there is a one-to-one correspondence between $\boldsymbol{\pi}(N)$ and $\boldsymbol{\alpha}^*(N)$, i.e., $\boldsymbol{\alpha}^*(N) = \boldsymbol{\pi}(N)(-\boldsymbol{Q}^{(1,1)}(N))\mathrm{diag}(\overline{\boldsymbol{H}}(N)\boldsymbol{e})$. Consequently, obtaining the conditional stationary distribution $\boldsymbol{\pi}(N)$ in an ergodic, continuous-time Markov chain is equivalent to obtaining the nonnegative weight vector $\boldsymbol{\alpha}^*(N)$.

The following corollary can be used to evaluate the accuracy of an approximation to $\boldsymbol{\pi}(N)$.

Corollary 2.1. *For an arbitrary probability vector $\boldsymbol{x} \in \Gamma(N)$, we have*

$$
\|\boldsymbol{x} - \boldsymbol{\pi}(N)\|_1 \leq \max_{i \in \mathbb{Z}_0^N} \|\boldsymbol{x} - \overline{\boldsymbol{h}}_i(N)\|_1, \tag{2.13}
$$

where $\|\cdot\|_1$ stands for the ℓ_1 norm and $\overline{\boldsymbol{h}}_i(N)$ ($i \in \mathbb{Z}_0^N$) denotes the ith row vector of $\overline{\boldsymbol{H}}(N)$. Furthermore, if $\boldsymbol{x} \in \mathcal{P}(N)$, we have

$$
\max_{i \in \mathbb{Z}_0^N} \|\boldsymbol{x} - \overline{\boldsymbol{h}}_i(N)\|_1 \leq \max_{i,j \in \mathbb{Z}_0^N} \|\overline{\boldsymbol{h}}_j(N) - \overline{\boldsymbol{h}}_i(N)\|_1, \qquad \boldsymbol{x} \in \mathcal{P}(N). \tag{2.14}
$$

Proof. (2.13) immediately follows from Theorem 2.1 and the convexity of $\mathcal{P}(N)$. Furthermore, if $\boldsymbol{x} \in \mathcal{P}(N)$, there exists a probability vector $\boldsymbol{\beta} = (\beta_0\ \beta_1\ \cdots\ \beta_N) \in \Gamma(N)$ such that

$$
\|\boldsymbol{x} - \overline{\boldsymbol{h}}_i(N)\|_1 = \Big\| \sum_{j \in \mathbb{Z}_0^N} \beta_j \big(\overline{\boldsymbol{h}}_j(N) - \overline{\boldsymbol{h}}_i(N)\big) \Big\|_1 \leq \sum_{j \in \mathbb{Z}_0^N} \beta_j \|\overline{\boldsymbol{h}}_j(N) - \overline{\boldsymbol{h}}_i(N)\|_1,
$$

from which (2.14) follows. □

We now provide a probabilistic interpretation of (2.12). We partition the state space $\mathbb{Z}^+$ into $\mathbb{Z}_0^N$ and $\mathbb{Z}_{N+1}^\infty$ and regard $\{X(t)\}_{t \in (-\infty,\infty)}$ as an alternating Markov renewal process. For an integer n, let $T_n^{(1)}$ (resp. $T_n^{(2)}$) denote the nth time instant at which $\{X(t)\}_{t \in (-\infty,\infty)}$ enters into

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$\mathbb{Z}_0^N$ from $\mathbb{Z}_{N+1}^\infty$ (resp. into $\mathbb{Z}_{N+1}^\infty$ from $\mathbb{Z}_0^N$), where we assume $T_n^{(1)} < T_n^{(2)} < T_{n+1}^{(1)}$ without loss of generality. We define $I(t)$ ($t \in (-\infty, \infty)$) as the state at the last renewal epoch before time t, i.e.,

$$I(t) = \begin{cases} X(T_n^{(1)}), & T_n^{(1)} \le t < T_n^{(2)}, \\ X(T_n^{(2)}), & T_n^{(2)} \le t < T_{n+1}^{(1)}. \end{cases}$$

We then consider the joint process $\{(X(t), I(t))\}_{t\in(-\infty,\infty)}$, which is assumed to be stationary.

Corollary 2.2. *For any Markov chain* $\{X(t)\}_{t\in(-\infty,\infty)}$ *in* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$*, the (i,j)th* ($i, j \in \mathbb{Z}_0^N$) *element of* $\overline{\boldsymbol{H}}(N)$ *is given by*

$$[\overline{\boldsymbol{H}}(N)]_{i,j} = \mathbb{P}(X(0) = j \mid I(0) = i), \quad i, j \in \mathbb{Z}_0^N, \tag{2.15}$$

and the ith ($i \in \mathbb{Z}_0^N$) *element of* $\boldsymbol{\alpha}^*(N)$ *is given by*

$$[\boldsymbol{\alpha}^*(N)]_i = \mathbb{P}(I(0) = i \mid X(0) \in \mathbb{Z}_0^N). \tag{2.16}$$

We thus interpret (2.12) as

$$[\boldsymbol{\pi}(N)]_j = \sum_{i\in\mathbb{Z}_0^N} \mathbb{P}(I(0) = i \mid X(0) \in \mathbb{Z}_0^N) \cdot \mathbb{P}(X(0) = j \mid I(0) = i), \quad j \in \mathbb{Z}_0^N. \tag{2.17}$$

The proof of Corollary 2.2 is given in Appendix A. Note that for a given $\boldsymbol{Q}^{(1,1)}(N)$, $\mathcal{P}(N)$ in Theorem 2.1 may not be tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$. For example, $\mathcal{P}(N)$ may include $\boldsymbol{x} \in \mathbb{R}^{N+1}$ such that $[\boldsymbol{x}]_i = 0$ for some $i \in \mathbb{Z}_0^N$, whereas $\boldsymbol{\pi}(N) > \boldsymbol{0}$ since Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ are ergodic. In the next subsection, we eliminate redundancy in $\mathcal{P}(N)$ using the structural information $\mathcal{J}(N)$ in (1.4).

2.3. Characterization of $\boldsymbol{\pi}(N)$ in terms of $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$

For specific $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$, we define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ as the collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$ whose infinitesimal generators have $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$. By definition,

$$\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N)) = \bigcup_{\mathcal{J}(N)\in 2^{\mathbb{Z}_0^N}\setminus\{\emptyset\}} \mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)).$$

Note here that $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ may be empty for some pairs of $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N)$. For example, if $\boldsymbol{Q}^{(1,1)}(N)$ is a diagonal matrix, $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) = \emptyset$ unless $\mathcal{J}(N) = \mathbb{Z}_0^N$, owing to ergodicity. In the rest of this subsection, we assume that if $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) \ne \emptyset$, then $\mathcal{J}(N) = \{0, 1, \ldots, |\mathcal{J}(N)| - 1\}$ without loss of generality.

If $\mathcal{J}(N) \ne \mathbb{Z}_0^N$, we partition $\boldsymbol{Q}^{(1,1)}(N)$ and $\boldsymbol{Q}^{(2,1)}(N)$ in (2.3) into two matrices:

$$\begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) \\ \boldsymbol{Q}^{(2,1)}(N) \end{pmatrix} = \begin{array}{c} \\ \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^\infty \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathcal{J}(N) & \mathbb{Z}_0^N\setminus\mathcal{J}(N) \end{array} \\ \begin{pmatrix} \boldsymbol{Q}_+^{(1,1)}(N) & \boldsymbol{Q}_0^{(1,1)}(N) \\ \boldsymbol{Q}_+^{(2,1)}(N) & \boldsymbol{O} \end{pmatrix} \end{array}. \tag{2.18}$$

Equation (2.10) is then rewritten as

$$\boldsymbol{\pi}(N)\cdot\begin{array}{c}\begin{array}{cc}\quad\mathcal{J}(N) & \qquad\mathbb{Z}_0^N\setminus\mathcal{J}(N)\end{array}\\ \left(\begin{array}{cc}-\boldsymbol{Q}_+^{(1,1)}(N) & -\boldsymbol{Q}_0^{(1,1)}(N)\end{array}\right)\end{array}=\begin{array}{c}\begin{array}{cc}\qquad\mathcal{J}(N) & \qquad\qquad\mathbb{Z}_0^N\setminus\mathcal{J}(N)\end{array}\\ \left(\begin{array}{cc}\dfrac{\boldsymbol{\pi}^{(2)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}}\cdot\boldsymbol{Q}_+^{(2,1)}(N) & \boldsymbol{0}\end{array}\right)\end{array}. \tag{2.19}$$

We define $\Gamma^+(N)$ as

$$\Gamma^+(N)=\left\{\boldsymbol{\alpha}\in\mathbb{R}^{N+1};\,\boldsymbol{\alpha}\ge\boldsymbol{0},\,\boldsymbol{\alpha e}=1,\ [\boldsymbol{\alpha}]_i=0\,(i\in\mathbb{Z}_0^N\setminus\mathcal{J}(N))\right\}. \tag{2.20}$$

Theorem 2.2. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N),\mathcal{J}(N))\neq\emptyset$. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N),\mathcal{J}(N))$, we have*

$$\boldsymbol{\pi}(N)\in\operatorname{ri}\mathcal{P}^+(N),\qquad N\in\mathbb{Z}^+, \tag{2.21}$$

where $\mathcal{P}^+(N)$ ($N\in\mathbb{Z}^+$) denotes a $(|\mathcal{J}(N)|-1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$ which is given by $\mathcal{P}^+(N)=\mathcal{P}(N)$ if $\mathcal{J}(N)=\mathbb{Z}_0^N$, and by

$$\mathcal{P}^+(N)=\left\{\boldsymbol{x}\in\mathbb{R}^{N+1};\,\boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N))\ge\boldsymbol{0},\ \boldsymbol{x}(-\boldsymbol{Q}_0^{(1,1)}(N))=\boldsymbol{0},\ \boldsymbol{xe}=1\right\} \tag{2.22}$$

$$=\left\{\boldsymbol{x}\in\mathbb{R}^{N+1};\,\boldsymbol{x}=\boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\,\boldsymbol{\alpha}\in\Gamma^+(N)\right\} \tag{2.23}$$

otherwise.

Proof. If $\mathcal{J}(N)=\mathbb{Z}_0^N$, we have $\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)>\boldsymbol{0}$ since $\boldsymbol{\pi}^{(2)}(N)>\boldsymbol{0}$. Theorem 2.2 for $\mathcal{J}(N)=\mathbb{Z}_0^N$ then follows from (2.2), (2.10), and Theorem 2.1. On the other hand, if $\mathcal{J}(N)\neq\mathbb{Z}_0^N$, then $\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}_+^{(2,1)}(N)>\boldsymbol{0}$. In this case, (2.21) with (2.22) follows from (2.2) and (2.19). The equivalence between (2.22) and (2.23) for $\mathcal{J}(N)\neq\mathbb{Z}_0^N$ can be shown in a way similar to the proof of Theorem 2.1, as shown in Appendix B. Since $\overline{\boldsymbol{H}}(N)$ is nonnegative and nonsingular, $\mathcal{P}^+(N)$ is a $(|\mathcal{J}(N)|-1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$. □

By definition, we have $\Gamma^+(N)\subseteq\Gamma(N)$, and therefore

$$\mathcal{P}^+(N)\subseteq\mathcal{P}(N), \tag{2.24}$$

where $\mathcal{P}^+(N)=\mathcal{P}(N)$ iff $\mathcal{J}(N)=\mathbb{Z}_0^N$. Theorem 2.2 indicates the importance of the structural information $\mathcal{J}(N)$ about direct transitions from $\mathbb{Z}_{N+1}^\infty$ to $\mathbb{Z}_0^N$. For example, if $\mathcal{J}(N)$ is a singleton for a certain N, say, $\mathcal{J}(N)=\{i\}$ ($i\in\mathbb{Z}_0^N$), $\boldsymbol{\pi}(N)$ is given by the ith row vector of $\overline{\boldsymbol{H}}(N)$, as pointed out in [16, Corollary 3]. Numerical algorithms for block-structured Markov chains of level-dependent M/G/1-type in [5, 15] also utilize (2.21) with (2.22) implicitly.

The corollary below follows from Theorem 2.2; its proof is omitted because it is almost the same as that of Corollary 2.1.

Corollary 2.3. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N),\mathcal{J}(N))\neq\emptyset$. For an arbitrary probability vector $\boldsymbol{x}\in\Gamma(N)$, we have*

$$\|\boldsymbol{x}-\boldsymbol{\pi}(N)\|_1\le\max_{i\in\mathcal{J}(N)}\|\boldsymbol{x}-\overline{\boldsymbol{h}}_i(N)\|_1.$$

Furthermore, if $\boldsymbol{x}\in\mathcal{P}^+(N)$, we have

$$\max_{i\in\mathcal{J}(N)}\|\boldsymbol{x}-\overline{\boldsymbol{h}}_i(N)\|_1\le\max_{i,j\in\mathcal{J}(N)}\|\overline{\boldsymbol{h}}_j(N)-\overline{\boldsymbol{h}}_i(N)\|_1,\qquad\boldsymbol{x}\in\mathcal{P}^+(N).$$

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Example 2.1. Consider an ergodic, level-dependent $M^X/M/1$ queue with disasters, whose infinitesimal generator $\boldsymbol{Q}$ takes the following form:

$$\boldsymbol{Q} = \begin{pmatrix} -q_0 & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,4} & q_{0,5} & \cdots \\ q_{1,0} & -q_1 & q_{1,2} & q_{1,3} & q_{1,4} & q_{1,5} & \cdots \\ q_{2,0} & q_{2,1} & -q_2 & q_{2,3} & q_{2,4} & q_{2,5} & \cdots \\ q_{3,0} & 0 & q_{3,2} & -q_3 & q_{3,4} & q_{3,5} & \cdots \\ q_{4,0} & 0 & 0 & q_{4,3} & -q_4 & q_{4,5} & \cdots \\ q_{5,0} & 0 & 0 & 0 & q_{5,4} & -q_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $q_{i,0} > 0$ and $q_{i,i-1} > 0$ $(i \in \mathbb{Z}^+ \setminus \{0\})$. We then have

$$\mathcal{J}(N) = \{0, N\},$$

because the state immediately after a downward transition from state i $(i \ge N+1)$ is either state 0 or state $i-1$. It then follows that

$$\Gamma^+(N) = \left\{ \left(\alpha_0 \ \ 0 \ \ 0 \ \ \cdots \ \ 0 \ \ 1-\alpha_0 \right) \in \mathbb{R}^{N+1};\, \alpha_0 \in [0, 1] \right\},$$

and

$$\boldsymbol{\pi}(N) \in \operatorname{ri} \mathcal{P}^+(N) = \left\{ \boldsymbol{x} \in \mathbb{R}^{N+1};\, \boldsymbol{x} = \alpha_0 \overline{\boldsymbol{h}}_0(N) + (1-\alpha_0)\overline{\boldsymbol{h}}_N(N),\ \alpha_0 \in (0, 1) \right\}.$$

Note here that $\overline{\boldsymbol{h}}_i(N)$ $(i \in \mathbb{Z}_0^N)$ can be obtained by (i) solving $\boldsymbol{x}_i(-\boldsymbol{Q}^{(1,1)}(N)) = \boldsymbol{e}_i(N)$ for $\boldsymbol{x}_i \in \mathbb{R}^{N+1}$ and (ii) letting $\overline{\boldsymbol{h}}_i(N) = \boldsymbol{x}_i/\boldsymbol{x}_i\boldsymbol{e}$, where $\boldsymbol{e}_i(N) \in \mathbb{R}^{N+1}$ denotes a $1 \times (N+1)$ unit vector whose ith element is equal to one. The simplest way to obtain an approximation $\boldsymbol{\pi}^{\text{approx}}(N) \in \operatorname{ri} \mathcal{P}^+(N)$ to $\boldsymbol{\pi}(N)$ might be to solve $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}) = \beta_0\boldsymbol{e}_0(N) + (1-\beta_0)\boldsymbol{e}_N(N)$ for an arbitrary $\beta_0 \in (0, 1)$ and let $\boldsymbol{\pi}^{\text{approx}}(N) = \boldsymbol{x}/\boldsymbol{x}\boldsymbol{e}$.

We now show that $\mathcal{P}^+(N)$ is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$.

Lemma 2.1. *If $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) \neq \emptyset$ and $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^+(N)$, we have $\boldsymbol{x} > \boldsymbol{0}$.*

Proof. We assume $[\boldsymbol{x}]_{j^*} = 0$ for some $j^* \in \mathbb{Z}_0^N$ and derive a contradiction. Since $\boldsymbol{x} = \boldsymbol{\alpha}(\boldsymbol{x})\overline{\boldsymbol{H}}(N)$ for some $\boldsymbol{\alpha}(\boldsymbol{x}) \in \operatorname{ri} \Gamma^+(N)$, $[\boldsymbol{x}]_{j^*} = 0$ is equivalent to $[\boldsymbol{\alpha}(\boldsymbol{x})\overline{\boldsymbol{H}}(N)]_{j^*} = 0$ for some $\boldsymbol{\alpha}(\boldsymbol{x}) \in \operatorname{ri} \Gamma^+(N)$. This implies that $[\overline{\boldsymbol{H}}(N)]_{i,j^*} = 0$ for all $i \in \mathcal{J}(N)$ since $[\boldsymbol{\alpha}(\boldsymbol{x})]_i > 0$ for all $i \in \mathcal{J}(N)$ and $\overline{\boldsymbol{H}}(N) \ge \boldsymbol{O}$. It then follows from (2.15) that $\mathbb{P}(X(0) = j^* \mid I(0) \in \mathcal{J}(N)) = 0$ for any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$. Furthermore, $\mathbb{P}(I(0) \in \mathbb{Z}_0^N \setminus \mathcal{J}(N)) = 0$ and $\mathbb{P}(X(0) = j^* \mid I(0) \in \mathbb{Z}_{N+1}^\infty) = 0$ by definition. We thus have

$$\begin{aligned} \mathbb{P}(X(0) = j^*) &= \mathbb{P}(I(0) \in \mathcal{J}(N))\mathbb{P}(X(0) = j^* \mid I(0) \in \mathcal{J}(N)) \\ &\qquad + \mathbb{P}(I(0) \in \mathbb{Z}_{N+1}^\infty)\mathbb{P}(X(0) = j^* \mid I(0) \in \mathbb{Z}_{N+1}^\infty) \\ &= 0, \end{aligned}$$

for any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, which contradicts the ergodicity. □

Theorem 2.3. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) \neq \emptyset$. For any $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^+(N)$, there exists a Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ whose $\boldsymbol{\pi}(N)$ is given by $\boldsymbol{x}$.*

Proof. It follows from (2.24) that if $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^{+}(N)$, we have $\boldsymbol{x} \in \mathcal{P}(N)$, i.e., $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \mathbf{0}$ and $\boldsymbol{xe} = 1$. Note also that $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \neq \mathbf{0}$, because if $\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) = \mathbf{0}$ held, we would have $\boldsymbol{x} = \mathbf{0} \cdot (-\boldsymbol{Q}^{(1,1)}(N))^{-1} = \mathbf{0}$, which contradicts $\boldsymbol{xe} = 1$. We then define $\boldsymbol{\zeta}(\boldsymbol{x})$ as

$$\boldsymbol{\zeta}(\boldsymbol{x}) = \frac{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}. \tag{2.25}$$

It is easy to see that $\boldsymbol{\zeta}(\boldsymbol{x}) \in \operatorname{ri} \Gamma^{+}(N)$ for $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^{+}(N)$. For an arbitrarily chosen $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^{+}(N)$, we consider a Markov chain whose infinitesimal generator $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{array}{c c} & \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^{\infty} \end{array} \\ \begin{array}{c} \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^{\infty} \end{array} & \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(N) & \boldsymbol{Q}^{(1,2)}(N) \\ (-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x}) & \boldsymbol{Q}^{(2,2)}(N) \end{pmatrix} \end{array}. \tag{2.26}$$

The global balance equations are then given by

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{\pi}^{(2)}(N)(-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x}) = \mathbf{0}, \tag{2.27}$$

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{Q}^{(1,2)}(N) + \boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,2)}(N) = \mathbf{0}. \tag{2.28}$$

From (2.25) and (2.27), we observe that the special form of $\boldsymbol{Q}^{(2,1)}(N) = (-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x})$ ensures $\boldsymbol{\pi}^{(1)}(N) = c_1\boldsymbol{x}$ for some positive constant c_1. We now set

$$\boldsymbol{Q}^{(1,2)}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{z}, \qquad \boldsymbol{Q}^{(2,2)}(N) = -\boldsymbol{I}, \tag{2.29}$$

where $\boldsymbol{z}$ denotes a $1 \times \infty$ positive probability vector. Equation (2.28) is then reduced to

$$\boldsymbol{\pi}^{(1)}(N)(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{z} - \boldsymbol{\pi}^{(2)}(N) = \mathbf{0}, \tag{2.30}$$

which indicates that (2.29) ensures $\boldsymbol{\pi}^{(2)}(N) = c_2\boldsymbol{z} > \mathbf{0}$ for some positive constant c_2. In fact, solving (2.27) and (2.30) with $\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e} + \boldsymbol{\pi}^{(2)}(N)\boldsymbol{e} = 1$, we obtain

$$\boldsymbol{\pi} = \begin{pmatrix} \boldsymbol{\pi}^{(1)}(N) & \boldsymbol{\pi}^{(2)}(N) \end{pmatrix} = \frac{1}{1 + \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \begin{pmatrix} \boldsymbol{x} & \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \boldsymbol{z} \end{pmatrix} > \mathbf{0}.$$

We thus conclude that the Markov chain with the infinitesimal generator $\boldsymbol{Q}$ defined by (2.26) and (2.29) is a member of $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$ and it has $\boldsymbol{\pi}(N) = \boldsymbol{x}$. □

Example 2.2. (Example 2.1 *continued.*) Consider $\boldsymbol{Q}$ in Example 2.1 of Section 2.3. Theorem 2.2 implies that there exists $\alpha_0 = \alpha_0^* \in (0, 1)$ such that $\boldsymbol{\pi}(N) = \alpha_0^*\overline{\boldsymbol{h}}_0(N) + (1 - \alpha_0^*)\overline{\boldsymbol{h}}_N(N)$. As stated in Theorem 2.3, however, $\operatorname{ri} \mathcal{P}^{+}(N)$ is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$. Therefore, we cannot identify the exact $\alpha_0^* \in (0, 1)$ if only $\boldsymbol{Q}^{(1,1)}(N)$ and $\mathcal{J}(N) = \{0, N\}$ are available.

2.4. Implications for the augmented truncation approximation

We provide some practical implications of the results in Sections 2.2 and 2.3 for the augmented truncation approximation (ATA). As mentioned in Section 1, an ATA solution $\boldsymbol{\pi}^{\text{trunc}}(N)$ to $\boldsymbol{\pi}(N)$ is obtained by

$$\boldsymbol{\pi}^{\text{trunc}}(N)[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\text{A}}(N)] = \mathbf{0}, \qquad \boldsymbol{\pi}^{\text{trunc}}(N)\boldsymbol{e} = 1, \tag{2.31}$$

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where $\boldsymbol{Q}_{\mathrm{A}}(N)$ denotes an augmentation matrix, i.e., an $(N+1)\times(N+1)$ nonnegative matrix satisfying $\boldsymbol{Q}_{\mathrm{A}}(N)\boldsymbol{e}=(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}$. The central topic in the literature on the ATA is the convergence of $\boldsymbol{\pi}^{\mathrm{approx},N}=(\boldsymbol{\pi}^{\mathrm{trunc}}(N)\ \ \mathbf{0})$ to the stationary distribution $\boldsymbol{\pi}$ as N goes to infinity. Note that our interest is different from this; that is to say, we are interested in what kind of $\boldsymbol{Q}_{\mathrm{A}}(N)$ is *reasonable* for a given N.

Although $\boldsymbol{Q}_{\mathrm{A}}(N)$ is usually chosen in such a way that $[\boldsymbol{Q}^{(1,1)}(N)+\boldsymbol{Q}_{\mathrm{A}}(N)]$ is irreducible, we allow reducible $[\boldsymbol{Q}^{(1,1)}(N)+\boldsymbol{Q}_{\mathrm{A}}(N)]$ as well. Note that the error bound for the approximation $\boldsymbol{\pi}^{\mathrm{approx},N}=(\boldsymbol{\pi}^{\mathrm{trunc}}(N)\ \ \mathbf{0})$ to the stationary distribution $\boldsymbol{\pi}$ is given by

$$2\mathbb{P}(X(0)>N)\le\|\boldsymbol{\pi}^{\mathrm{approx},N}-\boldsymbol{\pi}\|_1\le\epsilon^{\mathrm{trunc}}(N)+2\mathbb{P}(X(0)>N)$$

(see [5]), where $\epsilon^{\mathrm{trunc}}(N)$ denotes an upper bound of $\|\boldsymbol{\pi}^{\mathrm{trunc}}(N)-\boldsymbol{\pi}(N)\|_1$:

$$\|\boldsymbol{\pi}^{\mathrm{trunc}}(N)-\boldsymbol{\pi}(N)\|_1\le\epsilon^{\mathrm{trunc}}(N).$$

It is clear that $\mathbb{P}(X(0)>N)$ is a decreasing function of N and that for a given N, $\boldsymbol{\pi}^{\mathrm{approx},N}$ will be the best approximation to $\boldsymbol{\pi}$ when $\boldsymbol{\pi}^{\mathrm{trunc}}(N)=\boldsymbol{\pi}(N)$ (i.e., $\epsilon^{\mathrm{trunc}}(N)=0$) [16]. In what follows, we first consider Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$ and discuss some implications of our results in Section 2.2 for the ATA.

We define $\mathcal{T}(N)$ as the set of all possible ATA solutions satisfying (2.31):

$$\mathcal{T}(N)=\left\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\left[\boldsymbol{Q}^{(1,1)}(N)+\boldsymbol{Q}_{\mathrm{A}}(N)\right]=\mathbf{0},\ \boldsymbol{x}\boldsymbol{e}=1,\ \boldsymbol{Q}_{\mathrm{A}}(N)\in\mathcal{A}(N)\right\},$$

where $\mathcal{A}(N)$ denotes the set of all possible augmentation matrices,

$$\mathcal{A}(N)=\left\{\boldsymbol{X}\in\mathbb{R}^{(N+1)\times(N+1)};\ \boldsymbol{X}\ge\boldsymbol{O},\ \boldsymbol{X}\boldsymbol{e}=(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\right\}.$$

In the literature, the ATA is called *linear* if the rank of $\boldsymbol{Q}_{\mathrm{A}}(N)$ is equal to one [2]. We then define $\mathcal{T}_{\mathrm{L}}(N)$ as the set of all possible approximations obtained by the linear ATA:

$$\mathcal{T}_{\mathrm{L}}(N)=\left\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}\left[\boldsymbol{Q}^{(1,1)}(N)+\boldsymbol{Q}_{\mathrm{A}}(N)\right]=\mathbf{0},\ \boldsymbol{x}\boldsymbol{e}=1,\ \boldsymbol{Q}_{\mathrm{A}}(N)\in\mathcal{A}_{\mathrm{L}}(N)\right\},$$

where $\mathcal{A}_{\mathrm{L}}(N)$ denotes the set of all possible linear augmentation matrices,

$$\mathcal{A}_{\mathrm{L}}(N)=\left\{\boldsymbol{X}\in\mathbb{R}^{(N+1)\times(N+1)};\ \boldsymbol{X}=(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta},\ \boldsymbol{\zeta}\in\Gamma(N)\right\}.$$

Recall that $\Gamma(N)$ is the set of all $1\times(N+1)$ probability vectors, which is defined in (2.5).

Lemma 2.2. *For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N))$, we have*

$$\mathcal{T}(N)=\mathcal{T}_{\mathrm{L}}(N)=\mathcal{P}(N),\qquad N\in\mathbb{Z}^+,$$

where $\mathcal{P}(N)$ is given by (2.8).

The proof of Lemma 2.2 is given in Appendix C. Lemma 2.2 implies the following.

Implication 2.1. In view of Remark 2.1, we can consider that the ATA attempts to find an approximation to the weight vector $\boldsymbol{\alpha}^*(N)$ for $\overline{\boldsymbol{H}}(N)$ in (2.12) by setting the augmentation matrix $\boldsymbol{Q}_{\mathrm{A}}(N)$ appropriately. In this sense, the linear ATA has the same degree of freedom as the general ATA does.

We thus restrict our attention to the linear ATA with $\boldsymbol{Q}_{\mathrm{A}}(N)=(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}$, where $\boldsymbol{\zeta}\in\Gamma(N)$. In this case, (2.31) is reduced to

$$\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta})[\boldsymbol{Q}^{(1,1)}(N)+(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}]=\mathbf{0},\qquad\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta})\boldsymbol{e}=1.\tag{2.32}$$

Note here that $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ in (2.32) is closely related to $\boldsymbol{\pi}(N)$ in a Markov chain with the infinitesimal generator $\boldsymbol{Q}$ in (2.26). Specifically, using the cut-flow balance equation

$$\boldsymbol{\pi}^{(1)}(N)(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}=\boldsymbol{\pi}^{(2)}(N)(-\boldsymbol{Q}^{(2,2)}(N))\boldsymbol{e},$$

we can rewrite (2.27) to be

$$\boldsymbol{\pi}^{(1)}(N)\big[\boldsymbol{Q}^{(1,1)}(N)+(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x})\big]=\boldsymbol{0}.$$

We thus have $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})=\boldsymbol{\pi}(N)=\boldsymbol{x}\in\mathcal{P}(N)$ if we set $\boldsymbol{\zeta}=\boldsymbol{\zeta}(\boldsymbol{x})$ in (2.32).

Implication 2.2. The linear ATA solution $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ for a specific $\boldsymbol{\zeta}$ is identical to $\boldsymbol{\pi}(N)$ in an ergodic Markov chain whose infinitesimal generator takes the following form:

$$\boldsymbol{Q}=\begin{array}{c c} & \begin{array}{cc}\mathbb{Z}_0^N & \mathbb{Z}_{N+1}^{\infty}\end{array}\\ \begin{array}{c}\mathbb{Z}_0^N\\ \mathbb{Z}_{N+1}^{\infty}\end{array} & \begin{pmatrix}\boldsymbol{Q}^{(1,1)}(N) & \boldsymbol{Q}^{(1,2)}(N)\\ (-\boldsymbol{Q}^{(2,2)}(N)\boldsymbol{e})\boldsymbol{\zeta} & \boldsymbol{Q}^{(2,2)}(N)\end{pmatrix}\end{array}.$$

Since $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})\in\mathcal{T}_{\text{L}}(N)=\mathcal{P}(N)$, it follows from Theorem 2.1 that

$$\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})=\boldsymbol{\alpha}(N;\boldsymbol{\zeta})\overline{\boldsymbol{H}}(N) \tag{2.33}$$

for some $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})\in\Gamma(N)$. Recall that there is a one-to-one correspondence between $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ and $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$, as stated in Remark 2.1. Note also that there are one-to-one correspondences between $\boldsymbol{\zeta}$ and $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$,

$$\boldsymbol{\alpha}(N;\boldsymbol{\zeta})=\frac{\boldsymbol{\zeta}\,\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})}{\boldsymbol{\zeta}\,\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}},\qquad \boldsymbol{\zeta}=\frac{\boldsymbol{\alpha}(N;\boldsymbol{\zeta})\mathrm{diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})}{\boldsymbol{\alpha}(N;\boldsymbol{\zeta})\mathrm{diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}}, \tag{2.34}$$

and between $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ and $\boldsymbol{\zeta}$,

$$\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})=\frac{\boldsymbol{\zeta}\boldsymbol{H}(N)}{\boldsymbol{\zeta}\boldsymbol{H}(N)\boldsymbol{e}},\qquad \boldsymbol{\zeta}=\frac{\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N))}{\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}.$$

From (2.33), we also observe that $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ is given by a convex combination of the row vectors $\overline{\boldsymbol{h}}_i(N)$ $(i\in\mathbb{Z}_0^N)$ of $\overline{\boldsymbol{H}}(N)$ with the nonnegative weight vector $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$. It then follows from (2.13) that

$$\|\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})-\boldsymbol{\pi}(N)\|_1\le\max_{i\in\mathbb{Z}_0^N}\|\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})-\overline{\boldsymbol{h}}_i(N)\|_1. \tag{2.35}$$

This error bound tempts us to set $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$ in such a way that $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ is located at the *center* of the $\mathcal{P}(N)$ spanned by the $\overline{\boldsymbol{h}}_i(N)$ $(i\in\mathbb{Z}_0^N)$. For example, if we set $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})=\boldsymbol{e}^T/(N+1)$, where T stands for the transpose operator, then $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ is given by the center of gravity of $\mathcal{P}(N)$. Note here that (2.33) is equivalent to $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N))=\boldsymbol{\alpha}(N;\boldsymbol{\zeta})\mathrm{diag}^{-1}(\boldsymbol{H}(N)\boldsymbol{e})$.

Implication 2.3 If we have a desirable $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$ rather than $\boldsymbol{\zeta}$ itself, $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ for such an $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$ can be computed as follows:

(i) We first obtain $\boldsymbol{\eta}(N):=\boldsymbol{H}(N)\boldsymbol{e}$ by solving $(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{\eta}(N)=\boldsymbol{e}$.

(ii) We then obtain $\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})$ by solving

$$\boldsymbol{\pi}^{\text{trunc}}(N;\boldsymbol{\zeta})(-\boldsymbol{Q}^{(1,1)}(N))=\boldsymbol{\alpha}(N;\boldsymbol{\zeta})\mathrm{diag}^{-1}(\boldsymbol{\eta}(N)).$$

In the above procedure, we have to solve two systems of linear equations with $-\boldsymbol{Q}^{(1,1)}(N)$. Therefore, relative to the procedure for solving (2.32) directly for a specific $\boldsymbol{\zeta}$, the computational cost increases by $(N+1)^2$ in terms of the number of multiplications/divisions, if we utilize the LU decomposition. This increase can be regarded as the cost of specifying the weight vector $\boldsymbol{\alpha}(N;\boldsymbol{\zeta})$ directly, instead of specifying $\boldsymbol{\zeta}$.

Next we consider Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, where the structural information $\mathcal{J}(N)$ is assumed to be available. It follows from (2.34) that for all $i \in \mathbb{Z}_0^N$,

$$[\boldsymbol{\alpha}(N;\boldsymbol{\zeta})]_i = 0 \;\Leftrightarrow\; [\boldsymbol{\zeta}]_i = 0. \tag{2.36}$$

We then define $\mathcal{T}_{\mathrm{L}}^{+}(N)$ as the set of all possible approximations obtained by the linear ATA when $\mathcal{J}(N)$ is available:

$$\mathcal{T}_{\mathrm{L}}^{+}(N) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\, \boldsymbol{x}\big[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}\big] = \boldsymbol{0},\; \boldsymbol{x}\boldsymbol{e} = 1,\, \boldsymbol{\zeta} \in \Gamma^{+}(N)\big\},$$

where $\Gamma^{+}(N)$ is given by (2.20). By definition, we have $\mathcal{T}_{\mathrm{L}}^{+}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$.

Lemma 2.3. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N)) \neq \emptyset$. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N), \mathcal{J}(N))$, we have*

$$\mathcal{T}_{\mathrm{L}}^{+}(N) = \mathcal{P}^{+}(N), \qquad N \in \mathbb{Z}^{+},$$

where $\mathcal{P}^{+}(N)$ is given by (2.23).

The proof of Lemma 2.3 is given in Appendix D. We thus have $\boldsymbol{\pi}(N) \in \operatorname{ri} \mathcal{T}_{\mathrm{L}}^{+}(N)$ from Theorem 2.2. It also follows from (2.36) that if $[\boldsymbol{\zeta}]_i > 0$ for some $i \in \mathbb{Z}_0^N \setminus \mathcal{J}(N)$, we have $\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta}) \notin \mathcal{P}^{+}(N)$, because the $\overline{\boldsymbol{h}}_j(N)$ $(j \in \mathbb{Z}_0^N)$ are linearly independent. Moreover, it follows from Corollary 2.3 that

$$\|\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta}) - \boldsymbol{\pi}(N)\|_1 \le \max_{i \in \mathcal{J}(N)} \|\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta}) - \overline{\boldsymbol{h}}_i(N)\|_1, \quad \boldsymbol{\zeta} \in \Gamma^{+}(N), \tag{2.37}$$

which is tighter than (2.35). In summary, we have the following.

Implication 2.4. If the structural information $\mathcal{J}(N)$ is available, it is natural that we should choose $\boldsymbol{\zeta}$ from $\operatorname{ri} \Gamma^{+}(N)$ in obtaining a linear ATA solution, where $\Gamma^{+}(N)$ is given by (2.20). For example, we may choose $\boldsymbol{\zeta}$ whose ith $(i \in \mathbb{Z}_0^N)$ element is given by $1/|\mathcal{J}(N)|$ if $i \in \mathcal{J}(N)$ and by 0 otherwise.

Implication 2.4 indicates that the last-column augmentation $\boldsymbol{\zeta} = (0\;0\;\cdots\;0\;1)$, which is one of the common augmentation strategies in the literature, may not be effective unless $\mathcal{J}(N) = \{N\}$, because if $\mathcal{J}(N) \neq \{N\}$, we have $\boldsymbol{\zeta} = (0\;0\;\cdots\;0\;1) \notin \operatorname{ri} \Gamma^{+}(N)$ and therefore $\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta}) \notin \operatorname{ri} \mathcal{P}^{+}(N)$, while $\boldsymbol{\pi}(N) \in \operatorname{ri} \mathcal{P}^{+}(N)$.

Example 2.3. (Example 2.1 *continued*.) Consider $\boldsymbol{Q}$ in Example 2.1 of Section 2.3. Since $\mathcal{J}(N) = \{0, N\}$, we may set $\boldsymbol{\zeta} = (\zeta_0\;\;0\;\;0\;\;\cdots\;\;0\;\;1-\zeta_0) \in \operatorname{ri} \Gamma^{+}(N)$ for an arbitrary $\zeta_0 \in (0,1)$. The ATA solution $\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta})$ is then obtained by solving

$$\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta})[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}] = \boldsymbol{0}$$

with $\boldsymbol{\pi}^{\mathrm{trunc}}(N;\boldsymbol{\zeta})\boldsymbol{e} = 1$. If we are concerned with the error bound in (2.37), we first compute $\overline{\boldsymbol{h}}_i$ $(i = 0, N)$ by the procedure described in Example 2.1 and then adopt $(\overline{\boldsymbol{h}}_0 + \overline{\boldsymbol{h}}_N)/2$ as the ATA solution (or alternatively, we perform the procedure in Implication 2.3 with $\boldsymbol{\alpha}(N;\boldsymbol{\zeta}) =$

$(1/2\ 0\ 0\ \cdots\ 0\ 1/2)$). Note that this choice yields the minimum error bound of (2.37) in this specific example, which is half the error bound of the last-column augmentation.

Remark 2.2. As shown in (A.1), we have

$$\boldsymbol{\zeta}^* = \frac{\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)}{\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\boldsymbol{e}},$$

which satisfies $\boldsymbol{\pi}(N)[\boldsymbol{Q}^{(1,1)}(N) + (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}\boldsymbol{\zeta}^*] = \boldsymbol{0}$. Therefore, in order to obtain a good one-point approximation to the conditional stationary distribution $\boldsymbol{\pi}(N)$, we need some information about $\boldsymbol{\pi}^{(2)}(N)$, as in the Iterative Aggregation/Disaggregation methods for finite-state Markov chains [13, 14].

3. Extensions using state transitions in peripheral states

In this section, we consider extensions of the results in Sections 2.2 and 2.3 by using the $(K+1) \times (K+1)$ northwest corner block of the infinitesimal generator $\boldsymbol{Q}$, where $K > N$. For this purpose, we partition $\boldsymbol{\pi}$ and $\boldsymbol{Q}$ as follows:

$$\boldsymbol{\pi} = \begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^K & \mathbb{Z}_{K+1}^\infty \end{array} \\ \left(\boldsymbol{\pi}^{(1)}(K) \quad \boldsymbol{\pi}^{(2)}(K)\right) \end{array} = \begin{array}{c} \begin{array}{ccc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^K & \mathbb{Z}_{K+1}^\infty \end{array} \\ \left(\boldsymbol{\pi}^{(1)}(K,N) \quad \boldsymbol{\pi}^{(2)}(K,N) \;\middle|\; \boldsymbol{\pi}^{(3)}(K,N)\right) \end{array},$$

$$\begin{aligned} \boldsymbol{Q} &= \begin{array}{c|c} & \begin{array}{cc} \mathbb{Z}_0^K & \mathbb{Z}_{K+1}^\infty \end{array} \\ \begin{array}{c} \mathbb{Z}_0^K \\ \mathbb{Z}_{K+1}^\infty \end{array} & \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(K) & \boldsymbol{Q}^{(1,2)}(K) \\ \boldsymbol{Q}^{(2,1)}(K) & \boldsymbol{Q}^{(2,2)}(K) \end{pmatrix} \end{array} \\ &= \begin{array}{c|c} & \begin{array}{ccc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^K & \mathbb{Z}_{K+1}^\infty \end{array} \\ \begin{array}{c} \mathbb{Z}_0^N \\ \mathbb{Z}_{N+1}^K \\ \mathbb{Z}_{K+1}^\infty \end{array} & \left(\begin{array}{cc|c} \boldsymbol{Q}^{(1,1)}(K,N) & \boldsymbol{Q}^{(1,2)}(K,N) & \boldsymbol{Q}^{(1,3)}(K,N) \\ \boldsymbol{Q}^{(2,1)}(K,N) & \boldsymbol{Q}^{(2,2)}(K,N) & \boldsymbol{Q}^{(2,3)}(K,N) \\ \hline \boldsymbol{Q}^{(3,1)}(K,N) & \boldsymbol{Q}^{(3,2)}(K,N) & \boldsymbol{Q}^{(3,3)}(K,N) \end{array}\right) \end{array}. \end{aligned} \tag{3.1}$$

Note here that from (1.2) and (2.3),

$$\begin{aligned} \boldsymbol{\pi}^{(1)}(K,N) &= \boldsymbol{\pi}^{(1)}(N), & \boldsymbol{\pi}^{(3)}(K,N) &= \boldsymbol{\pi}^{(2)}(K), \\ \boldsymbol{Q}^{(1,1)}(K,N) &= \boldsymbol{Q}^{(1,1)}(N), & \boldsymbol{Q}^{(3,3)}(K,N) &= \boldsymbol{Q}^{(2,2)}(K). \end{aligned}$$

In order to utilize the results in the preceding section, we consider the censored Markov chain $\{\widetilde{X}(t)\}_{t\in(-\infty,\infty)}$ obtained by observing the Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ only when $X(t) \in \mathbb{Z}_0^N \cup \mathbb{Z}_{K+1}^\infty$. The infinitesimal generator of $\{\widetilde{X}(t)\}_{t\in(-\infty,\infty)}$ is then given by

$$\widetilde{\boldsymbol{Q}}(K,N) = \begin{array}{c|c} & \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{K+1}^\infty \end{array} \\ \begin{array}{c} \mathbb{Z}_0^N \\ \mathbb{Z}_{K+1}^\infty \end{array} & \begin{pmatrix} \widetilde{\boldsymbol{Q}}^{(1,1)}(K,N) & \widetilde{\boldsymbol{Q}}^{(1,3)}(K,N) \\ \widetilde{\boldsymbol{Q}}^{(3,1)}(K,N) & \widetilde{\boldsymbol{Q}}^{(3,3)}(K,N) \end{pmatrix} \end{array}, \tag{3.2}$$

where for $i, j = 1, 3$,

$$\widetilde{\boldsymbol{Q}}^{(i,j)}(K,N) = \boldsymbol{Q}^{(i,j)}(K,N) + \boldsymbol{Q}^{(i,2)}(K,N)(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,j)}(K,N). \tag{3.3}$$

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In particular, we have

$$\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N)=\boldsymbol{Q}^{(1,1)}(N)(\boldsymbol{I}-\boldsymbol{R}(K,N)), \tag{3.4}$$

where

$$\boldsymbol{R}(K,N)=(-\boldsymbol{Q}^{(1,1)}(N))^{-1}\boldsymbol{Q}^{(1,2)}(K,N)(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N). \tag{3.5}$$

Note that $(\boldsymbol{I}-\boldsymbol{R}(K,N))$ in (3.4) is nonsingular because $\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N)$ and $\boldsymbol{Q}^{(1,1)}(N)$ are nonsingular.

Clearly, Theorem 2.1 is applicable to the censored Markov chain $\{\widetilde{X}(t)\}_{t\in(-\infty,\infty)}$ on $\mathbb{Z}_0^N\cup\mathbb{Z}_{K+1}^{\infty}$. Note also that the conditional stationary distribution of $\{\widetilde{X}(t)\}_{t\in(-\infty,\infty)}$ given $\widetilde{X}(0)\in\mathbb{Z}_0^N$ is identical to $\boldsymbol{\pi}(N)$. We define $\widetilde{\Gamma}(K,N)$ $(K>N)$ as

$$\widetilde{\Gamma}(K,N)=\big\{\boldsymbol{\alpha}\in\mathbb{R}^{N+1};\ \boldsymbol{\alpha}=\boldsymbol{\beta}\,\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}),\ \boldsymbol{\beta}\in\Gamma(N)\big\}, \tag{3.6}$$

where

$$\widetilde{\boldsymbol{H}}(K,N)=(-\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N))^{-1}. \tag{3.7}$$

Remark 3.1. It can be verified that $\widetilde{\boldsymbol{H}}(K,N)$ is identical to the $(N+1)\times(N+1)$ northwest corner block of the $(K+1)\times(K+1)$ matrix $\boldsymbol{H}(K)=(-\boldsymbol{Q}^{(1,1)}(K))^{-1}$.

For a specific $(K+1)\times(K+1)$ northwest corner block $\boldsymbol{Q}^{(1,1)}(K)$, we define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K))$ as the collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$ whose infinitesimal generators have $\boldsymbol{Q}^{(1,1)}(K)$.

Theorem 3.1. *For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K))$, we have*

$$\boldsymbol{\pi}(N)\in\widetilde{\mathcal{P}}(K,N),\qquad K,N\in\mathbb{Z}^+,\ K>N,$$

where $\widetilde{\mathcal{P}}(K,N)$ denotes an N-simplex on the first orthant of $\mathbb{R}^{N+1}$ which is given by

$$\widetilde{\mathcal{P}}(K,N)=\big\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))(\boldsymbol{I}-\boldsymbol{R}(K,N))\geq\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e}=1\big\} \tag{3.8}$$

$$=\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}=\boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha}\in\widetilde{\Gamma}(K,N)\}, \tag{3.9}$$

with $\boldsymbol{R}(K,N)$ as in (3.5).

Proof. Associated with $\widetilde{\boldsymbol{H}}(K,N)$ in (3.7), we define $\overline{\widetilde{\boldsymbol{H}}}(K,N)$ as

$$\overline{\widetilde{\boldsymbol{H}}}(K,N)=\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})\widetilde{\boldsymbol{H}}(K,N). \tag{3.10}$$

Applying Theorem 2.1 to (3.2), we obtain $\boldsymbol{\pi}(N)\in\widehat{\mathcal{P}}(K,N)$, where $\widehat{\mathcal{P}}(K,N)$ has two equivalent expressions:

$$\begin{aligned}\widehat{\mathcal{P}}(K,N)&=\big\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}(-\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N))\geq\boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e}=1\big\}\\&=\big\{\boldsymbol{x}\in\mathbb{R}^{N+1};\ \boldsymbol{x}=\boldsymbol{\beta}\overline{\widetilde{\boldsymbol{H}}}(K,N),\ \boldsymbol{\beta}\in\Gamma(N)\big\}.\end{aligned} \tag{3.11}$$

Because of (3.4), $\widetilde{\mathcal{P}}(K,N)$ in (3.8) is identical to $\widehat{\mathcal{P}}(K,N)$.

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Next, we show the equivalence between (3.9) and (3.11). It follows from (3.4) and (3.7) that

$$\begin{aligned}\widetilde{\boldsymbol{H}}(K,N) = (-\widetilde{\boldsymbol{Q}}^{(1,1)}(K,N))^{-1} &= (\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}(-\boldsymbol{Q}^{(1,1)}(N))^{-1} \\ &= (\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\boldsymbol{H}(N). \end{aligned} \tag{3.12}$$

Therefore, for an arbitrary $\boldsymbol{\beta} \in \Gamma(N)$, we have from (3.10) and (3.12) that

$$\begin{aligned}\boldsymbol{\beta}\overline{\widetilde{\boldsymbol{H}}}(K,N) &= \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\boldsymbol{H}(N) \\ &= \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N) \\ &= \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N), \end{aligned} \tag{3.13}$$

where

$$\boldsymbol{\alpha} = \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K,N)\boldsymbol{e})(\boldsymbol{I}-\boldsymbol{R}(K,N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}), \tag{3.14}$$

which shows the equivalence between (3.9) and (3.11). We thus conclude that

$$\widetilde{\mathcal{P}}(K,N) = \widehat{\mathcal{P}}(K,N). \tag{3.15}$$

$\widetilde{\mathcal{P}}(K,N)$ is an N-simplex on the first orthant of $\mathbb{R}^{N+1}$ because of the nonsingularity of $\overline{\widetilde{\boldsymbol{H}}}(K,N) \ge \boldsymbol{O}$, (3.11), and (3.15). □

Corollary 3.1. *$\widetilde{\Gamma}(K,N)$ in (3.6) is a subset of $\Gamma(N)$:*

$$\widetilde{\Gamma}(K,N) \subseteq \Gamma(N), \qquad N, K \in \mathbb{Z}^+,\ N < K. \tag{3.16}$$

Furthermore,

$$\widetilde{\Gamma}(K_2,N) \subseteq \widetilde{\Gamma}(K_1,N), \qquad N, K_1, K_2 \in \mathbb{Z}^+,\ N < K_1 < K_2, \tag{3.17}$$

and therefore

$$\widetilde{\mathcal{P}}(K_2,N) \subseteq \widetilde{\mathcal{P}}(K_1,N) \subseteq \mathcal{P}(N), \tag{3.18}$$

where $\mathcal{P}(N)$ is as given in Theorem 2.1.

The proof of Corollary 3.1 is given in Appendix E.

Next, we consider an extension of the results in Section 2.3. For specific $(K+1)\times(K+1)$ northwest corner block $\boldsymbol{Q}^{(1,1)}(K)$ and the structural information $\mathcal{J}(K)$ $(K > N)$, we define $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ as the collection of ergodic, continuous-time Markov chains on $\mathbb{Z}^+$ whose infinitesimal generators have $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$. For Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, we define $\widetilde{\mathcal{J}}(K,N)$ $(K > N)$ as the set of states in $\mathbb{Z}_0^N$ which are reachable from some states in $\mathbb{Z}_{K+1}^{\infty}$ either directly or only via some states in $\mathbb{Z}_{N+1}^K$:

$$\widetilde{\mathcal{J}}(K,N) = \{j \in \mathbb{Z}_0^N;\ [\boldsymbol{e}^T\widetilde{\boldsymbol{Q}}^{(3,1)}(K,N)]_j > 0\}. \tag{3.19}$$

Note here that $\widetilde{\mathcal{J}}(K,N) \neq \emptyset$ because of the ergodicity. Note also that $\widetilde{\mathcal{J}}(K,N)$ is determined completely by $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$, because from (1.4), (3.1), and (3.3),

$$\mathcal{J}(K) = \{j \in \mathbb{Z}_0^N;\ \big[\boldsymbol{e}^T\boldsymbol{Q}^{(3,1)}(K,N)\big]_j > 0\} \cup \{j \in \mathbb{Z}_{N+1}^K;\ \big[\boldsymbol{e}^T\boldsymbol{Q}^{(3,2)}(K,N)\big]_j > 0\},$$

$$\widetilde{\boldsymbol{Q}}^{(3,1)}(K,N) = \boldsymbol{Q}^{(3,1)}(K,N) + \boldsymbol{Q}^{(3,2)}(K,N)\cdot(-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N).$$

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Furthermore, by definition,

$$\widetilde{\mathcal{J}}(K, N) \subseteq \mathcal{J}(N), \tag{3.20}$$

and states in $\mathcal{J}(N) \setminus \widetilde{\mathcal{J}}(K, N)$, if any, can be reached from $\mathbb{Z}_{K+1}^{\infty}$ only after visiting some states in $\widetilde{\mathcal{J}}(K, N)$.

Example 3.1. (Example 2.1 *continued*.) Consider $\boldsymbol{Q}$ in Example 2.1 of Section 2.3. By definition, $\mathcal{J}(K) = \{0, K\}$ and $\widetilde{\mathcal{J}}(K, N) = \{0, N\} = \mathcal{J}(N)$.

Without loss of generality, we assume $\widetilde{\mathcal{J}}(K, N) = \{0, 1, \ldots, |\widetilde{\mathcal{J}}(K, N)| - 1\}$. We then partition $\widetilde{\boldsymbol{Q}}^{(1,1)}(K, N)$ in (3.2) into two matrices,

$$\widetilde{\boldsymbol{Q}}^{(1,1)}(K, N) = \begin{array}{cc} \scriptstyle \widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K,N) \\ \Big(\widetilde{\boldsymbol{Q}}_{+}^{(1,1)}(K, N) & \widetilde{\boldsymbol{Q}}_{0}^{(1,1)}(K, N)\Big) \end{array}, \tag{3.21}$$

and define $\widetilde{\Gamma}^{+}(K, N)$ as

$$\begin{aligned} \widetilde{\Gamma}^{+}(K, N) = \big\{&\boldsymbol{\alpha} \in \mathbb{R}^{N+1}; \boldsymbol{\alpha} = \boldsymbol{\beta}\,\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}), \\ &\boldsymbol{\beta} \in \Gamma(N),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K, N))\big\}. \end{aligned} \tag{3.22}$$

Theorem 3.2. *Suppose* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$. *For any Markov chain in* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ *(*$K > N$*), we have*

$$\boldsymbol{\pi}(N) \in \mathrm{ri}\, \widetilde{\mathcal{P}}^{+}(K, N), \quad K, N \in \mathbb{Z}^{+},\ K > N, \tag{3.23}$$

where $\widetilde{\mathcal{P}}^{+}(K, N)$ *denotes a* $(|\widetilde{\mathcal{J}}(K, N)| - 1)$*-simplex on the first orthant of* $\mathbb{R}^{N+1}$ *which is given by* $\widetilde{\mathcal{P}}(K, N)$ *if* $\widetilde{\mathcal{J}}(K, N) = \mathbb{Z}_0^N$*, and by*

$$\begin{aligned} \widetilde{\mathcal{P}}^{+}(K, N) &= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\widetilde{\boldsymbol{Q}}_{+}^{(1,1)}(K, N)) \geq \boldsymbol{0}, \\ &\qquad \boldsymbol{x}(-\widetilde{\boldsymbol{Q}}_{0}^{(1,1)}(K, N)) = \boldsymbol{0},\ \boldsymbol{x}\boldsymbol{e} = 1\big\} \end{aligned} \tag{3.24}$$

$$= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \widetilde{\Gamma}^{+}(K, N)\big\} \tag{3.25}$$

otherwise.

Proof. If $\widetilde{\mathcal{J}}(K, N) = \mathbb{Z}_0^N$, Theorem 3.2 can be shown in the same way as Theorem 2.2 for $\mathcal{J}(N) = \mathbb{Z}_0^N$. We thus assume that $\widetilde{\mathcal{J}}(K, N) \neq \mathbb{Z}_0^N$. Applying Theorem 2.2 to the censored Markov chain with infinitesimal generator $\widetilde{\boldsymbol{Q}}(K, N)$ in (3.2) with partition (3.21), we obtain (3.23) with (3.24), where (3.24) can also be expressed as follows:

$$\begin{aligned} \widetilde{\mathcal{P}}^{+}(K, N) = \big\{&\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\beta}\overline{\widetilde{\boldsymbol{H}}}(K, N), \\ &\boldsymbol{\beta} \in \Gamma(N),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K, N))\big\}, \end{aligned} \tag{3.26}$$

where $\overline{\widetilde{\boldsymbol{H}}}(K, N)$ is given by (3.10). Equation (3.25) now follows from (3.13) and (3.26). Since $\overline{\widetilde{\boldsymbol{H}}}(K, N)$ is nonnegative and nonsingular, (3.26) implies that $\widetilde{\mathcal{P}}^{+}(K, N)$ is a $(|\widetilde{\mathcal{J}}(K, N)| - 1)$-simplex on the first orthant of $\mathbb{R}^{N+1}$. □

By definition,

$$\widetilde{\Gamma}^{+}(K, N) \subseteq \widetilde{\Gamma}(K, N), \tag{3.27}$$

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where $\widetilde{\Gamma}(K, N)$ is given by (3.6). We thus have

$$\widetilde{\mathcal{P}}^{+}(K, N) \subseteq \widetilde{\mathcal{P}}(K, N), \tag{3.28}$$

where $\widetilde{\mathcal{P}}(K, N)$ is given in Theorem 3.1.

Corollary 3.2. *$\widetilde{\Gamma}^{+}(K, N)$ in (3.22) is a subset of $\Gamma^{+}(N)$ in (2.20):*

$$\widetilde{\Gamma}^{+}(K, N) \subseteq \Gamma^{+}(N). \tag{3.29}$$

Furthermore,

$$\widetilde{\Gamma}^{+}(K_2, N) \subseteq \widetilde{\Gamma}^{+}(K_1, N), \quad N, K_1, K_2 \in \mathbb{Z}^{+},\ N < K_1 < K_2, \tag{3.30}$$

and therefore

$$\widetilde{\mathcal{P}}^{+}(K_2, N) \subseteq \widetilde{\mathcal{P}}^{+}(K_1, N) \subseteq \mathcal{P}^{+}(N). \tag{3.31}$$

The proof of Corollary 3.2 is given in Appendix F. Since $\widetilde{\mathcal{P}}(K, N)$ in Theorem 3.1 and $\widetilde{\mathcal{P}}^{+}(K, N)$ in Theorem 3.2 are compact, Corollaries 3.1 and 3.2 suggest that those simplices converge to certain sets as K goes to infinity. In fact, we have the following proposition.

Proposition 3.1. *([8, Theorem 2.3]) In any ergodic, continuous-time Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ on $\mathbb{Z}^{+}$,*

$$\lim_{K\to\infty} \overline{\overline{\boldsymbol{H}}}(K, N) = \boldsymbol{e}\boldsymbol{\pi}(N), \quad N \in \mathbb{Z}^{+}.$$

The corollary below follows immediately from (3.11), (3.15), (3.18), (3.26), (3.28), (3.31), and Proposition 3.1.

Corollary 3.3. *In any ergodic, continuous-time Markov chain $\{X(t); t \ge 0\}$ on $\mathbb{Z}^{+}$,*

$$\bigcap_{K=N+1}^{\infty} \widetilde{\mathcal{P}}(K, N) = \lim_{K\to\infty} \widetilde{\mathcal{P}}(K, N) = \{\boldsymbol{\pi}(N)\}, \quad N \in \mathbb{Z}^{+},$$

and

$$\bigcap_{K=N+1}^{\infty} \widetilde{\mathcal{P}}^{+}(K, N) = \lim_{K\to\infty} \widetilde{\mathcal{P}}^{+}(K, N) = \{\boldsymbol{\pi}(N)\}, \quad N \in \mathbb{Z}^{+}.$$

Since $\mathrm{ri}\,\widetilde{\mathcal{P}}^{+}(K, N) \subseteq \widetilde{\mathcal{P}}^{+}(K, N)$, $\mathrm{ri}\,\widetilde{\mathcal{P}}^{+}(K, N)$ also converges to $\{\boldsymbol{\pi}(N)\}$ as K goes to infinity. Note here that $\mathrm{ri}\,\widetilde{\mathcal{P}}^{+}(K, N)$ may not be tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ for a finite K. For example, we consider $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N+1), \mathcal{J}(N+1))$, where $K = N+1$, $\mathcal{J}(N+1) = \{N+1\}$, and $q_{N+1,j} > 0$ for all $j \in \mathbb{Z}_0^N$. We then have $\widetilde{\mathcal{J}}(N+1, N) = \mathbb{Z}_0^N$, and therefore $\widetilde{\mathcal{P}}^{+}(N+1, N)$ is an N-simplex spanned by $(N+1)$ row vectors of $\overline{\overline{\boldsymbol{H}}}(N+1, N)$, as shown in the proof of Theorem 3.2. On the other hand, it follows from Theorem 2.2 that $\mathcal{P}^{+}(N+1)$ is given by a singleton with the $(N+1)$th row vector $\overline{\boldsymbol{h}}_{N+1}(N+1)$ of $\overline{\boldsymbol{H}}(N+1)$. In other words, all Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(N+1), \{N+1\})$ have the same $\boldsymbol{\pi}(N)$ (i.e. the normalized vector of the first $N+1$ elements of $\overline{\boldsymbol{h}}_{N+1}(N+1)$), so that $\mathrm{ri}\,\widetilde{\mathcal{P}}^{+}(N+1, N)$ is not tight. In the next section, we find the minimum convex polytope that contains $\boldsymbol{\pi}(N)$, in the sense that it is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

4. Minimum convex polytopes for $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$

In this section, we consider minimum convex polytopes that contain $\boldsymbol{\pi}(N)$ for Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$. The outline of our approach can be summarized as follows. For convenience, we partition $\boldsymbol{H}(K) = (-\boldsymbol{Q}^{(1,1)}(K))^{-1}$ and $\boldsymbol{\pi}(K)$ as follows:

$$\boldsymbol{H}(K) = \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^K \\ \big(\boldsymbol{H}^{(*,1)}(K;N) & \boldsymbol{H}^{(*,2)}(K;N)\big), \end{array} \tag{4.1}$$

$$\boldsymbol{\pi}(K) = \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_{N+1}^K \\ \big(\boldsymbol{\pi}^{(1)}(K;N) & \boldsymbol{\pi}^{(2)}(K;N)\big). \end{array}$$

Since $\boldsymbol{\pi}(K) \in \mathrm{ri}\,\mathcal{P}^+(K)$ by Theorem 2.2, $\boldsymbol{\pi}(N) = \boldsymbol{\pi}^{(1)}(K;N)/\boldsymbol{\pi}^{(1)}(K;N)\boldsymbol{e}$ is given by a convex combination of normalized row vectors of $\boldsymbol{H}^{(*,1)}(K;N)$ corresponding to $\mathcal{J}(K)$.

Note here that in all sample paths of the first passage from $\mathbb{Z}_{K+1}^{\infty}$ to $\mathbb{Z}_0^N$, $\{X(t)\}_{t\in(-\infty,\infty)}$ must visit at least one state in $\mathcal{J}(K)$. In view of this fact, we first introduce *(K,N)-skip-free* sets. Roughly speaking, a (K, N)-skip-free set $\mathcal{X}$ is a proper subset of $\mathbb{Z}_0^K$ such that in all sample paths of the first passage from $\mathbb{Z}_{K+1}^{\infty}$ to $\mathbb{Z}_0^K$, $\{X(t)\}_{t\in(-\infty,\infty)}$ must visit at least one state in $\mathcal{X}$. We then show in Theorem 4.1 that $\boldsymbol{\pi}(N) = \boldsymbol{\pi}^{(1)}(K;N)/\boldsymbol{\pi}^{(1)}(K;N)\boldsymbol{e}$ is given by a convex combination of normalized row vectors of $\boldsymbol{H}^{(*,1)}(K;N)$ corresponding to members of $\mathcal{X}$.

To find the minimum convex polytopes that contain $\boldsymbol{\pi}(N)$, we introduce a partial order among (K, N)-skip-free sets, considering the first passage time from $\mathbb{Z}_{K+1}^{\infty}$ to them. We then show the inclusion relation among convex polytopes associated with (K, N)-skip-free sets in Lemma 4.1. Finally, in Theorem 4.2, we find the *smallest* (K, N)-skip-free set and the corresponding minimum convex polytope, which is shown to be tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

4.1. (K, N)-skip-free sets

For an arbitrary non-empty subset $\mathcal{B}$ of $\mathbb{Z}^+$, we define $F(\mathcal{B})$ as the first passage time to $\mathcal{B}$:

$$F(\mathcal{B}) = \inf\{t \ge 0; X(t) \in \mathcal{B}\}, \qquad \mathcal{B} \subseteq \mathbb{Z}^+.$$

Definition 4.1. *((K, N)-skip-free set.) For arbitrarily fixed $K, N \in \mathbb{Z}^+$ $(K > N)$, consider a Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$. We refer to a subset $\mathcal{X}$ of $\mathbb{Z}_0^K$ as a (K, N)-skip-free set if $\{X(t)\}_{t\in(-\infty,\infty)}$ starting from any state in $\mathbb{Z}_{K+1}^{\infty}$ must visit $\mathcal{X}$ by the first passage time to $\mathbb{Z}_0^N$, i.e.,*

$$\mathbb{P}(X(t) \in \mathcal{X} \text{ for some } t \in (0, F(\mathbb{Z}_0^N)] \mid X(0) \in \mathbb{Z}_{K+1}^{\infty}) = 1.$$

In particular, a (K, N)-skip-free set $\mathcal{X}$ is called proper if for any proper subset $\mathcal{Y}$ of $\mathcal{X}$,

$$\mathbb{P}(X(t) \in \mathcal{Y} \text{ for some } t \in (0, F(\mathbb{Z}_0^N)] \mid X(0) \in \mathbb{Z}_{K+1}^{\infty}) < 1.$$

Let $\mathcal{S}(K, N)$ denote the family of all (K, N)-skip-free sets and $\mathcal{S}^*(K, N)$ the family of all proper (K, N)-skip-free sets.

Remark 4.1. By definition,

(i) (K, N)-skip-free sets are determined only by $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$,

(ii) $\mathcal{S}^*(K, N) \subseteq \mathcal{S}(K, N)$,

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(iii) $\mathcal{J}(K) \in \mathcal{S}(K, N)$ and $\widetilde{\mathcal{J}}(K, N) \in \mathcal{S}^*(K, N)$, where $\widetilde{\mathcal{J}}(K, N)$ is given by (3.19), and

(iv) $\mathcal{J}(K) \cap \mathbb{Z}_0^N \subseteq \mathcal{X}$ for all $\mathcal{X} \in \mathcal{S}(K, N)$.

We can provide an equivalent definition of (K, N)-skip-free sets by considering a directed graph associated with $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$. Specifically, for arbitrarily fixed $K, N \in \mathbb{Z}^+$ $(K > N)$, we define the directed graph

$$G(K, N) = (V(K, N), E(K, N)), \tag{4.2}$$

where $V(K, N)$ and $E(K, N)$ denote sets of nodes and arcs defined as

$$\begin{aligned} V(K, N) &= \{s, t\} \cup \mathbb{Z}_0^K, \\ E(K, N) &= \{(s, j); j \in \mathcal{J}(K)\} \cup \{(i, t); i \in \mathbb{Z}_0^N\} \\ &\qquad \cup \{(i, j); i \in \mathbb{Z}_{N+1}^K,\ j \in \mathbb{Z}_0^K, q_{i,j} > 0\}. \end{aligned}$$

Node s represents the set of states in $\mathbb{Z}_{K+1}^\infty$, and node t is a virtual terminal node for the first passage to $\mathbb{Z}_0^N$. It is readily seen from Definition 4.1 that a subset $\mathcal{X}$ of $\mathbb{Z}_0^K$ is a (K, N)-skip-free set iff $\mathcal{X}$ is an *(s,t)-node cut* in $G(K, N)$, and that $\mathcal{X}$ is proper iff it is a *minimal* (s, t)-node cut.

For a given $G(K, N)$, nodes in $\mathbb{Z}_0^K$ can be classified into two classes, depending on whether they appear on the way of at least one path from node s to node t. That is,

$$\mathbb{Z}_0^K = \mathcal{N}_+(K, N) \cup \mathcal{N}_0(K, N), \qquad \mathcal{N}_+(K, N) \cap \mathcal{N}_0(K, N) = \emptyset,$$

where nodes in $\mathcal{N}_+(K, N)$ are reachable from node s and each of them has at least one path to node t. Note here that $\mathcal{N}_0(K, N) = \mathbb{Z}_0^K \setminus \mathcal{N}_+(K, N)$ is given by

$$\mathcal{N}_0(K, N) = \mathcal{N}_0^{(1)}(K, N) \cup \mathcal{N}_0^{(2)}(K, N) \cup \mathcal{N}_0^{(3)}(K, N),$$

where $\mathcal{N}_0^{(\ell)}(K, N)$ $(\ell = 1, 2, 3)$ are defined as follows:

$$\begin{aligned} \mathcal{N}_0^{(1)}(K, N) &= \mathbb{Z}_0^N \setminus \widetilde{\mathcal{J}}(K, N) = \{i \in \mathbb{Z}_0^N; \mathbb{P}(X(F(\mathbb{Z}_0^N)) = i \mid X(0) \in \mathbb{Z}_{K+1}^\infty) = 0\}, \\ \mathcal{N}_0^{(2)}(K, N) &= \{i \in \mathbb{Z}_{N+1}^K; \\ &\qquad\qquad \mathbb{P}(X(t) \in \mathbb{Z}_{N+1}^\infty \text{ for all } t \in (0, F(\{i\})] \mid X(0) \in \mathbb{Z}_{K+1}^\infty) = 0\}, \\ \mathcal{N}_0^{(3)}(K, N) &= \{i \in \mathbb{Z}_{N+1}^K; \mathbb{P}(X(t) \in \mathbb{Z}_{N+1}^K \text{ for all } t \in (0, F(\mathbb{Z}_0^N)] \mid X(0) = i) = 0\}. \end{aligned}$$

Roughly speaking, states in $\mathcal{N}_0^{(1)}(K, N) \subseteq \mathbb{Z}_0^N$ and states in $\mathcal{N}_0^{(2)}(K, N) \subseteq \mathbb{Z}_{N+1}^K$ can be reached from any states in $\mathbb{Z}_{K+1}^\infty$ only after visiting $\widetilde{\mathcal{J}}(K, N) \subseteq \mathbb{Z}_0^N$. On the other hand, states in $\mathcal{N}_0^{(3)}(K, N) \in \mathbb{Z}_{N+1}^K$ have no paths to $\widetilde{\mathcal{J}}(K, N)$ without visiting $\mathbb{Z}_{K+1}^\infty$.

By definition, $\mathcal{X} \cap \mathcal{N}_0(K, N) = \emptyset$ if $\mathcal{X} \in \mathcal{S}^*(K, N)$, whereas there may exist $\mathcal{X} \in \mathcal{S}(K, N)$ such that $\mathcal{X} \cap \mathcal{N}_0(K, N) \neq \emptyset$. As we will see, all states in $\mathcal{N}_0(K, N)$ can be excluded from consideration when we study $\boldsymbol{\pi}(N)$ for Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

Before considering minimum convex polytopes for Markov chains in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, we provide a remark on (K, N)-skip-free sets. Note that (K, N)-skip-free sets can be regarded as a natural extension of levels with the skip-free-to-the-left property in Markov chains of M/G/1-type. Suppose a Markov chain $\{X(t)\}_{t \in (-\infty,\infty)}$ is of M/G/1-type; i.e., its state space $\mathbb{Z}^+$

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is partitioned into *disjoint* levels $\{\mathcal{L}_m; m \in \mathbb{Z}^+\}$, and transitions between two levels are skip-free to the left. We now consider the first passage time from $\mathcal{L}_{K+1}$ to $\mathcal{L}_N$ $(K > N)$. It then follows that each level $\mathcal{L}_m$ $(m = N, N+1, \ldots, K)$ can be regarded as a $(\hat{K}, \hat{N})$-skip-free set, where $\hat{K}$ and $\hat{N}$ denote the total number of states in $\bigcup_{m=0}^{K} \mathcal{L}_m$ and $\bigcup_{m=0}^{N} \mathcal{L}_m$. One of the most notable differences between (K, N)-skip-free sets and levels with the skip-free-to-the-left property is that (K, N)-skip-free sets are *not* disjoint, while levels are disjoint.

Example 4.1. (Example 2.1 *continued*.) Consider $\boldsymbol{Q}$ in Example 2.1 of Section 2.3. $\mathcal{S}^*(K, N)$ is given by

$$\mathcal{S}^*(K, N) = \big\{\{0, k\}; k \in \mathbb{Z}_N^K\big\}.$$

4.2. Minimum convex polytopes

We consider convex polytopes associated with (K, N)-skip-free sets. We partition $\boldsymbol{H}(K) = (-\boldsymbol{Q}^{(1,1)}(K))^{-1}$ into two matrices as in (4.1). Note here that

$$[\boldsymbol{H}^{(*,1)}(K; N)\boldsymbol{e}]_i = 0, \qquad i \in \mathcal{N}_0^{(3)}(K, N), \tag{4.3}$$

and $[\boldsymbol{H}^{(*,1)}(K; N)\boldsymbol{e}]_i > 0$ for all $i \in \mathbb{Z}_0^K \setminus \mathcal{N}_0^{(3)}(K, N)$. For any $(K+1)$-dimensional vector $\boldsymbol{x}$, we define $\operatorname{diag}^*(\boldsymbol{x})$ as a $(K+1)$-dimensional diagonal matrix whose ith $(i \in \mathbb{Z}_0^K)$ diagonal element is given by

$$[\operatorname{diag}^*(\boldsymbol{x})]_{i,i} = \begin{cases} \dfrac{1}{[\boldsymbol{x}]_i}, & [\boldsymbol{x}]_i \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We then define $\Gamma(K, N; \mathcal{B})$ $(K, N \in \mathbb{Z}^+, K > N, \mathcal{B} \subseteq \mathbb{Z}_0^K)$ as

$$\begin{aligned} \Gamma(K, N; \mathcal{B}) = \Big\{&\boldsymbol{\alpha} \in \mathbb{R}^{N+1}; \boldsymbol{\alpha} = \boldsymbol{\beta}\boldsymbol{U}(K, N),\ \boldsymbol{\beta} \in \Gamma(K), \\ &[\boldsymbol{\beta}]_i \geq 0\ (i \in \mathcal{B}),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{B})\Big\}, \end{aligned}$$

where $\Gamma(n)$ $(n \in \mathbb{Z}^+)$ is defined in (2.5) and $\boldsymbol{U}(K, N)$ is given by

$$\begin{aligned} \boldsymbol{U}(K, N) = \operatorname{diag}^*(\boldsymbol{H}^{(*,1)}(K; N)\boldsymbol{e}) &\begin{pmatrix} \boldsymbol{I} \\ (-\boldsymbol{Q}^{(2,2)}(K, N))^{-1}\boldsymbol{Q}^{(2,1)}(K, N) \end{pmatrix} \\ &\cdot (\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1} \operatorname{diag}(\boldsymbol{H}(N)\boldsymbol{e}). \end{aligned}$$

Moreover, for $\mathcal{X} \in \mathcal{S}(K, N)$, we define $\mathcal{X}^*$ as

$$\mathcal{X}^* = \mathcal{X} \setminus \mathcal{N}_0(K, N).$$

Note that

$$\mathcal{X}^* \in \mathcal{S}(K, N) \text{ if } \mathcal{X} \in \mathcal{S}(K, N), \qquad \mathcal{X}^* = \mathcal{X} \text{ if } \mathcal{X} \in \mathcal{S}^*(K, N). \tag{4.4}$$

We then have the following theorem, whose proof is given in Appendix G.

Theorem 4.1. *Suppose* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$. *For any Markov chain in* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, *we have for* $\mathcal{X} \in \mathcal{S}(K, N)$ $(K > N)$ *that*

$$\boldsymbol{\pi}(N) \in \mathcal{P}^+(K, N; \mathcal{X}^*),$$

where $\mathcal{P}^{+}(K, N; \mathcal{X}^{*})$ *denotes a convex polytope on the first orthant of* $\mathbb{R}^{N+1}$ *which is given by*

$$\begin{aligned}\mathcal{P}^{+}(K, N; \mathcal{X}^{*}) = \big\{\boldsymbol{x} \in \mathbb{R}^{N+1};\ & \big[(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\big]_i \geq 0\ (i \in \mathcal{X}^{*}),\\ & \big[(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\big]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{X}^{*}),\\ & \boldsymbol{x}\boldsymbol{e} = 1,\ \boldsymbol{y} \geq \boldsymbol{0}\big\} \end{aligned} \tag{4.5}$$

$$= \{\boldsymbol{x} \in \mathbb{R}^{N+1};\ \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),\ \boldsymbol{\alpha} \in \Gamma(K, N; \mathcal{X}^{*})\}. \tag{4.6}$$

In particular,

$$\boldsymbol{\pi}(N) \in \operatorname{ri} \mathcal{P}^{+}(K, N; \mathcal{X}^{*}) \ \text{ if } \ \mathcal{X}^{*} \in \mathcal{S}^{*}(K, N).$$

Remark 4.2. Consider two $(K,\ N)$-skip-free sets $\mathcal{X}_A, \mathcal{X}_B \in \mathcal{S}(K, N)$. If $\mathcal{X}_A \subseteq \mathcal{X}_B$, we have $\mathcal{X}_A^{*} \subseteq \mathcal{X}_B^{*}$ by definition. Therefore, it follows from (4.5) that $\mathcal{P}^{+}(K, N; \mathcal{X}_A^{*}) \subseteq \mathcal{P}^{+}(K, N; \mathcal{X}_B^{*})$ if $\mathcal{X}_A \subseteq \mathcal{X}_B$.

Lemma 4.1. *For Markov chains in* $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ *with* $\mathcal{X}_A, \mathcal{X}_B \in \mathcal{S}(K, N)$*, we consider the first passage times* $F(\mathcal{X}_A^{*})$*,* $F(\mathcal{X}_B^{*})$*, and* $F(\mathbb{Z}_0^N)$*. If the statements*

$$F(\mathcal{X}_A^{*}) \leq F(\mathcal{X}_B^{*}) \ \text{ given that } \ X(0) \in \mathbb{Z}_{K+1}^{\infty}$$

and

$$F(\mathcal{X}_B^{*}) \leq F(\mathbb{Z}_0^N) \ \text{ given that } \ X(0) \in \mathcal{X}_A^{*}, \tag{4.7}$$

hold sample-path-wise, we have

$$\mathcal{P}(K, N; \mathcal{X}_A^{*}) \subseteq \mathcal{P}(K, N; \mathcal{X}_B^{*}).$$

The proof of Lemma 4.1 is given in Appendix H. By definition, $\widetilde{\mathcal{J}}(K, N) \in \mathcal{S}^{*}(K, N)$, and $\mathcal{P}^{+}(K, N; \widetilde{\mathcal{J}}(K, N))$ is identical to $\widetilde{\mathcal{P}}^{+}(K, N)$ in (3.24). Recall that $\widetilde{\mathcal{J}}(K, N) \subseteq \mathbb{Z}_0^N$. Therefore, for any $\mathcal{X} \in \mathcal{S}(K, N)$, $F(\mathcal{X}) \leq F(\widetilde{\mathcal{J}}(K, N))$ holds sample-path-wise, given $X(0) \in \mathbb{Z}_{K+1}^{\infty}$. On the other hand, $\mathcal{J}(K) \in \mathcal{S}(K, N)$, and each state in $\mathcal{J}(K)$ is reachable from some states in $\mathbb{Z}_{K+1}^{\infty}$ by direct transitions. Note here that

$$\mathcal{J}^{*}(K) = \mathcal{J}(K) \setminus \mathcal{N}_0(K, N) = \mathcal{J}(K) \setminus \mathcal{N}_0^{(3)}(K, N). \tag{4.8}$$

We thus have $F(\mathcal{J}^{*}(K)) \leq F(\mathcal{X}^{*})$ sample-path-wise, given $X(0) \in \mathbb{Z}_{K+1}^{\infty}$. The corollary below follows from Theorem 4.1 and the above observations.

Corollary 4.1. *Among* $(K,\ N)$*-skip-free sets,* $\widetilde{\mathcal{J}}(K, N)$ *in (3.19) gives a maximum convex polytope, i.e.,*

$$\mathcal{P}^{+}(K, N; \mathcal{X}^{*}) \subseteq \mathcal{P}^{+}(K, N; \widetilde{\mathcal{J}}(K, N)), \qquad \mathcal{X} \in \mathcal{S}(K, N).$$

On the other hand, among $(K,\ N)$*-skip-free sets,* $\mathcal{J}^{*}(K)$ *gives a minimum convex polytope, i.e.,*

$$\mathcal{P}^{+}(K, N; \mathcal{J}^{*}(K)) \subseteq \mathcal{P}^{+}(K, N; \mathcal{X}^{*}), \qquad \mathcal{X} \in \mathcal{S}(K, N).$$

Note here that $\mathcal{J}^{*}(K)$ is not proper in general. We thus consider a *proper* (K, N)-skip-free set that gives the minimum convex polytope $\mathcal{P}^{+}(K, N; \mathcal{J}^{*}(K))$. Specifically, we introduce a subset $\mathcal{D}(K, N)$ of $\mathcal{J}(K)$, which is generated by the following procedure.

Procedure 4.1. ([9].)

(i) *We obtain a directed graph $G^*(K,N)$ from $G(K,N)$ in (4.2) by removing all incoming edges to nodes in $\mathcal{J}(K)$.*

(ii) *We then find the subset $\mathcal{D}(K,N)$ of nodes in $\mathcal{J}(K)$ from which node t is reachable in $G^*(K,N)$.*

Note that Step (ii) can be performed with $O(K)$ operations, by considering reverse paths from node t to node s in the graph $G^*(K,N)$.

Lemma 4.2. ([9].) *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ $(K > N)$, we have $\mathcal{D}(K,N) \in \mathcal{S}^*(K,N)$. Furthermore, $\mathcal{D}(K,N) \subseteq \mathcal{X}$ for all $\mathcal{X} \subseteq \mathcal{J}(K)$ such that $\mathcal{X} \in \mathcal{S}(K,N)$.*

The proof of Lemma 4.2 is given in Appendix I.

Lemma 4.3. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$. For any Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ $(K > N)$, we have*

$$\mathcal{P}^+(K,N;\mathcal{D}(K,N)) = \mathcal{P}^+(K,N;\mathcal{J}^*(K)), \qquad K,N \in \mathbb{Z}^+,\ K > N,$$

where $\mathcal{J}^*(K)$ is given by (4.8).

Proof. Since $\mathcal{D}(K,N) \in \mathcal{S}^*(K,N)$, we have $\mathcal{D}^*(K,N) = \mathcal{D}(K,N)$ from (4.4). It then follows from Remark 4.2 and $\mathcal{D}(K,N) \subseteq \mathcal{J}(K)$ that

$$\mathcal{P}^+(K,N;\mathcal{D}(K,N)) = \mathcal{P}^+(K,N;\mathcal{D}^*(K,N)) \subseteq \mathcal{P}^+(K,N;\mathcal{J}^*(K)).$$

On the other hand, it follows from Lemma 4.2 that $F(\mathcal{J}^*(K)) \le F(\mathcal{D}(K,N))$ if $X(0) \in \mathbb{Z}_{K+1}^{\infty}$ and $F(\mathcal{D}(K,N)) \le F(\mathbb{Z}_0^N)$ if $X(0) \in \mathcal{J}^*(K)$. We thus have $\mathcal{P}^+(K,N;\mathcal{J}^*(K)) \subseteq \mathcal{P}^+(K,N;\mathcal{D}(K,N))$ from Lemma 4.1, which completes the proof. □

The following theorem shows that the minimum convex polytope $\mathcal{P}^+(K,N;\mathcal{D}(K,N))$ is tight in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$.

Theorem 4.2. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$. For any $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^+(K,N;\mathcal{D}(K,N))$, there exists a Markov chain in $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ whose $\boldsymbol{\pi}(N)$ is given by $\boldsymbol{x}$.*

The proof of Theorem 4.2 is given in Appendix J. As shown in the proof, if $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$ and $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^+(K,N;\mathcal{D}(K,N))$, we have $\boldsymbol{x} > \boldsymbol{0}$. The following corollary comes from the fact that $\mathcal{P}^+(K,N;\mathcal{D}(K,N))$ is a convex polytope spanned by $\overline{\boldsymbol{h}}_i^{(*,1)}(K;N)$ for $i \in \mathcal{D}(K,N)$ (cf. (H.1)), where $\overline{\boldsymbol{h}}_i^{(*,1)}(K;N)$ $(i \in \mathbb{Z}_0^K)$ denotes the ith normalized row vector of $\boldsymbol{H}^{(*,1)}(K;N)$ in (4.1).

Corollary 4.2. *Suppose $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K)) \neq \emptyset$. For an arbitrary probability vector $\boldsymbol{x} \in \Gamma(N)$, we have*

$$\|\boldsymbol{x} - \boldsymbol{\pi}(N)\|_1 \le \max_{i \in \mathcal{D}(K,N)} \|\boldsymbol{x} - \overline{\boldsymbol{h}}_i^{(*,1)}(K;N)\|_1.$$

Furthermore, if $\boldsymbol{x} \in \mathcal{P}^+(K,N;\mathcal{D}(K,N))$, we have

$$\max_{i \in \mathcal{D}(K,N)} \|\boldsymbol{x} - \overline{\boldsymbol{h}}_i^{(*,1)}(K;N)\|_1 \le \max_{i,j \in \mathcal{D}(K,N)} \|\overline{\boldsymbol{h}}_j^{(*,1)}(K;N) - \overline{\boldsymbol{h}}_i^{(*,1)}(K;N)\|_1, \qquad \boldsymbol{x} \in \mathcal{P}^+(K,N;\mathcal{D}(K,N)).$$

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TABLE 1. The worst-case error bounds in Example 4.2 ($N = 50$).

K	worst-case error bound
50	$1.4822557975239198 \times 10^{-1}$
60	$2.8325135949980040 \times 10^{-2}$
70	$3.3500314471736218 \times 10^{-3}$
80	$2.3592892522569017 \times 10^{-4}$
90	$1.0840482894389214 \times 10^{-5}$
100	$3.5265317430057536 \times 10^{-7}$
110	$8.6146805062733978 \times 10^{-9}$
120	$1.6515206910372449 \times 10^{-10}$
130	$2.5711073894929548 \times 10^{-12}$
140	$3.2661373605691324 \times 10^{-14}$
150	$2.0140139556090730 \times 10^{-15}$

Example 4.2. (Example 2.1 *continued*.) Consider $\boldsymbol{Q}$ in Example 2.1 of Section 2.3. Note that $\mathcal{J}(N) = \{0, N\}$ and $\mathcal{J}(K) = \mathcal{J}^*(K) = \mathcal{D}(K, N) = \{0, K\}$. We set $q_{i,j} = \lambda_i b_{j-i}$ ($i \in \mathbb{Z}^+, j > i$), $q_{i,i-1} = \mu_i$ ($i \in \mathbb{Z}^+ \setminus \{0\}$), and $q_{i,0} = \gamma_i$ ($i \in \mathbb{Z}^+ \setminus \{0, 1\}$), where

$$\lambda_i = 2 - \frac{1}{i+1} \quad (i \in \mathbb{Z}^+), \qquad b_i = \frac{1}{2^i} \quad (i \in \mathbb{Z}^+ \setminus \{0\}),$$
$$\mu_i = \frac{i}{10} \quad (i \in \mathbb{Z}^+ \setminus \{0\}), \qquad \gamma_i = \frac{\lambda_i}{i} \quad (i \in \mathbb{Z}^+ \setminus \{0, 1\}).$$

Table 1 shows the worst-case error bound

$$\max_{i,j \in \mathcal{D}(K,N)} \|\overline{\boldsymbol{h}}_j^{(*,1)}(K; N) - \overline{\boldsymbol{h}}_i^{(*,1)}(K; N)\|_1 = \|\overline{\boldsymbol{h}}_0^{(*,1)}(K; N) - \overline{\boldsymbol{h}}_K^{(*,1)}(K; N)\|_1$$

of Corollary 4.2, where $N = 50$, and the result for $K = N$ indicates the worst-case error bound

$$\max_{i,j \in \mathcal{J}(N)} \|\overline{\boldsymbol{h}}_j(N) - \overline{\boldsymbol{h}}_i(N)\|_1 = \|\overline{\boldsymbol{h}}_0(N) - \overline{\boldsymbol{h}}_N(N)\|_1$$

of Corollary 2.3. We observe that the worst-case error bound steadily decreases as K increases and that if $K = 150$, any probability vector $\boldsymbol{x} \in \operatorname{ri} \mathcal{P}^+(K, N; \mathcal{D}(K, N))$ is a good approximation to $\boldsymbol{\pi}(N)$ in this specific case.

5. Concluding remarks

This paper studied the conditional stationary distribution $\boldsymbol{\pi}(N)$ in ergodic, continuous-time Markov chains on $\mathbb{Z}^+$ via systems of linear inequalities. Specifically, we first obtained an N-simplex $\mathcal{P}(N)$ containing $\boldsymbol{\pi}(N)$, assuming only the knowledge of the $(N+1) \times (N+1)$ northwest corner block $\boldsymbol{Q}^{(1,1)}(N)$ of the infinitesimal generator. We then refined this result and eliminated unnecessary vertices from $\mathcal{P}(N)$, using the structural information $\mathcal{J}(N)$. These results are closely related to the augmented truncation approximation (ATA), and we provided some practical implications for it. In particular, we observed that the linear ATA is sufficient for computing an ATA solution, and the structural information $\mathcal{J}(N)$, if available, is useful for obtaining a reasonable ATA solution.

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Next, we extended our results to the cases where the $(K+1)\times(K+1)$ $(K>N)$ northwest corner block (and $\mathcal{J}(K)$) are available. Furthermore, we introduced new state transition structures called (K, N)-skip-free sets, and we obtained a tight convex polytope that contains $\boldsymbol{\pi}(N)$, under the condition that only $\boldsymbol{Q}^{(1,1)}(K)$ and $\mathcal{J}(K)$ are available.

Note that the results in this paper are also applicable to a time-homogeneous, *discrete-time* Markov chain on $\mathbb{Z}^{+}$. Specifically, in a discrete-time Markov chain with a state transition probability matrix $\boldsymbol{P}_D$, the stationary distribution $\boldsymbol{\pi}_D$, if it exists, satisfies $\boldsymbol{\pi}_D=\boldsymbol{\pi}_D\boldsymbol{P}_D$ and $\boldsymbol{\pi}_D\boldsymbol{e}=1$. Therefore, we can study $\boldsymbol{\pi}_D$ by considering the corresponding continuous-time Markov chain with the infinitesimal generator $\boldsymbol{Q}=\boldsymbol{P}_D-\boldsymbol{I}$.

In this paper, the state space of a Markov chain $\{X(t)\}_{t\in(-\infty,\infty)}$ was assumed to be countably infinite. Except for Corollary 3.3, however, all the results in this paper also hold for Markov chains with finite state space $\mathbb{Z}_0^{K^*}$ for some $K^*\in\mathbb{Z}^+$ such that $N<K<K^*$, if we replace $\mathbb{Z}_N^{\infty}$ and $\mathbb{Z}_K^{\infty}$ by $\mathbb{Z}_N^{K^*}$ and $\mathbb{Z}_K^{K^*}$.

We scarcely ever discussed one-point approximations to the conditional stationary distribution $\boldsymbol{\pi}(N)$, except in Section 2.4. In view of the error bounds in Corollaries 2.3 and 4.2, it seems to be reasonable to choose the center of a convex polytope that contains $\boldsymbol{\pi}(N)$ as a one-point approximation to $\boldsymbol{\pi}(N)$. Note, however, that we cannot identify the exact location of $\boldsymbol{\pi}(N)$ within the convex polytopes unless further information is available. Therefore, for a specific problem, the choice of the center may or may not work well. The development of new approximation algorithms for the (un)conditional stationary distributions of Markov chains remains as future work.

Appendix A. Proof of Corollary 2.2

Substituting (2.15) and (2.16) into (2.12) yields (2.17). We thus consider (2.15) and (2.16) below. Because

$$\boldsymbol{H}(N)=(-\boldsymbol{Q}^{(1,1)}(N))^{-1}=\int_0^{\infty}\exp\,(\boldsymbol{Q}^{(1,1)}(N)t)\mathrm{d}t,$$

we have

$$[\boldsymbol{H}(N)]_{i,j}=\mathbb{E}\bigg[\int_{T_n^{(1)}}^{T_n^{(2)}}\mathbb{I}(X(t)=j)\mathrm{d}t\mid X(T_n^{(1)})=i\bigg],\qquad i,j\in\mathbb{Z}_0^N,$$

and therefore

$$[\overline{\boldsymbol{H}}(N)]_{i,j}=\frac{1}{\mathbb{E}\big[T_n^{(2)}-T_n^{(1)}\mid X(T_n^{(1)})=i\big]}\mathbb{E}\bigg[\int_{T_n^{(1)}}^{T_n^{(2)}}\mathbb{I}(X(t)=j)\mathrm{d}t\mid X(T_n^{(1)})=i\bigg],$$

from which (2.15) follows.

We also have $\boldsymbol{\pi}^{(1)}(N)=\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\boldsymbol{H}(N)$ from (2.4) and (2.9), so that

$$\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}=\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}.$$

It then follows from (2.10) and (2.12) that

$$\boldsymbol{\alpha}^*(N)=\frac{\boldsymbol{\pi}^{(2)}(N)}{\boldsymbol{\pi}^{(1)}(N)\boldsymbol{e}}\cdot\boldsymbol{Q}^{(2,1)}(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})=\frac{\boldsymbol{\zeta}^*(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})}{\boldsymbol{\zeta}^*(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\boldsymbol{e}},$$

where

$$\boldsymbol{\zeta}^*(N)=\frac{\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)}{\boldsymbol{\pi}^{(2)}(N)\boldsymbol{Q}^{(2,1)}(N)\boldsymbol{e}}.\tag{A.1}$$

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It is readily seen that

$$[\boldsymbol{\zeta}^*(N)]_i = \mathbb{P}(X(T_n^{(1)}) = i), \quad i \in \mathbb{Z}_0^N.$$

We thus have

$$[\boldsymbol{\alpha}^*(N)]_i = \frac{1}{\mathbb{E}\big[T_n^{(2)} - T_n^{(1)}\big]} \mathbb{E}\bigg[\int_{T_n^{(1)}}^{T_n^{(2)}} \mathbb{I}(I(t) = i)\mathrm{d}t\bigg], \quad i \in \mathbb{Z}_0^N,$$

from which (2.16) follows.

Appendix B. Equivalence between (2.22) and (2.23) for $\mathcal{J}(N) \neq \mathbb{Z}_0^N$

Suppose $\mathcal{J}(N) \neq \mathbb{Z}_0^N$. We partition $\overline{\boldsymbol{H}}(N)$ into two matrices:

$$\overline{\boldsymbol{H}}(N) = \begin{array}{c} \\ \mathcal{J}(N) \\ \mathbb{Z}_0^N\setminus\mathcal{J}(N) \end{array} \begin{array}{c} \mathbb{Z}_0^N \\ \begin{pmatrix} \overline{\boldsymbol{H}}_+(N) \\ \overline{\boldsymbol{H}}_0(N) \end{pmatrix} \end{array}.$$

Note that (2.23) for $\mathcal{J}(N) \neq \mathbb{Z}_0^N$ is equivalent to

$$\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}_+(N), \boldsymbol{\alpha} \geq \boldsymbol{0},\ \boldsymbol{\alpha e} = 1\}.$$

We now rewrite (2.22) as

$$\begin{aligned}
\mathcal{P}^+(N) &= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N)) \geq \boldsymbol{0},\ \boldsymbol{x}(-\boldsymbol{Q}_0^{(1,1)}(N)) = \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N)) = \boldsymbol{y},\ \boldsymbol{x}(-\boldsymbol{Q}_0^{(1,1)}(N)) = \boldsymbol{0},\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}_+^{(1,1)}(N) \quad -\boldsymbol{Q}_0^{(1,1)}(N)) = (\boldsymbol{y} \ \ \boldsymbol{0}),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) = (\boldsymbol{y} \ \ \boldsymbol{0}),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = (\boldsymbol{y} \ \ \boldsymbol{0})\boldsymbol{H}(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = (\boldsymbol{y} \ \ \boldsymbol{0})\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = (\boldsymbol{y}\mathrm{diag}(\boldsymbol{H}_+(N)\boldsymbol{e}) \ \ \boldsymbol{0})\overline{\boldsymbol{H}}(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{y}\mathrm{diag}(\boldsymbol{H}_+(N)\boldsymbol{e})\overline{\boldsymbol{H}}_+(N),\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{xe} = 1\big\} \\
&= \big\{\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}_+(N), \boldsymbol{\alpha} \geq \boldsymbol{0},\ \boldsymbol{\alpha e} = 1\big\},
\end{aligned}$$

where the last equality follows from the same reasoning as in (2.11).

Appendix C. Proof of Lemma 2.2

Since $\mathcal{A}_{\mathrm{L}}(N) \subseteq \mathcal{A}(N)$, we have $\mathcal{T}_{\mathrm{L}}(N) \subseteq \mathcal{T}(N)$. We thus show $\mathcal{T}(N) \subseteq \mathcal{P}(N)$ and $\mathcal{P}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$, from which the lemma follows.

We first prove $\mathcal{T}(N) \subseteq \mathcal{P}(N)$. Suppose $\boldsymbol{x} \in \mathcal{T}(N)$; i.e., $\boldsymbol{x}[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}(N)] = \boldsymbol{0}$ for some $\boldsymbol{Q}_{\mathrm{A}}(N) \in \mathcal{A}(N)$. Post-multiplying both sides of this equation by $\boldsymbol{H}(N) = (-\boldsymbol{Q}^{(1,1)}(N))^{-1}$ and rearranging terms, we obtain

$$\boldsymbol{x} = \boldsymbol{x}\boldsymbol{Q}_{\mathrm{A}}(N)\boldsymbol{H}(N) = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N),$$

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where $\boldsymbol{\alpha} = \boldsymbol{x}\boldsymbol{Q}_{\mathrm{A}}(N)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e})$. Note here that $\boldsymbol{\alpha} \in \Gamma(N)$ because $\boldsymbol{\alpha} \geq \boldsymbol{0}$, $\boldsymbol{xe} = 1$, and $\boldsymbol{xe} = \boldsymbol{\alpha}\overline{\boldsymbol{H}}(N)\boldsymbol{e} = \boldsymbol{\alpha e}$. We thus obtain $\boldsymbol{x} \in \mathcal{P}(N)$ from (2.8), so that $\mathcal{T}(N) \subseteq \mathcal{P}(N)$.

Next, we prove $\mathcal{P}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$. For an arbitrarily fixed $\boldsymbol{x} \in \mathcal{P}(N)$, let $\boldsymbol{y} = \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N)) \geq \boldsymbol{0}$ (cf. (2.7)). Since $\boldsymbol{xe} = 1$ and $\boldsymbol{Q}^{(1,1)}(N)$ is nonsingular, we have $\boldsymbol{y} \neq \boldsymbol{0}$. We then consider a linear augmentation matrix $\boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N)$ given by

$$\boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{\boldsymbol{y}}{\boldsymbol{ye}} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{\boldsymbol{y}}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}.$$

It is clear that $\boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N) \in \mathcal{A}_{\mathrm{L}}(N)$ and

$$\boldsymbol{x}[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N)] = -\boldsymbol{y} + \boldsymbol{x} \cdot \frac{(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \cdot \boldsymbol{y} = \boldsymbol{0},$$

which implies $\boldsymbol{x} \in \mathcal{T}_{\mathrm{L}}(N)$, so that $\mathcal{P}(N) \subseteq \mathcal{T}_{\mathrm{L}}(N)$.

Appendix D. Proof of Lemma 2.3

If $\mathcal{J}(N) = \mathbb{Z}_0^N$, Lemma 2.3 immediately follows from Lemma 2.2. We thus assume $\mathcal{J}(N) \neq \mathbb{Z}_0^N$. Implication 2.2 with $\boldsymbol{\zeta} \in \Gamma^{+}(N)$ implies $\mathcal{T}_{\mathrm{L}}^{+}(N) \subseteq \mathcal{P}^{+}(N)$. We thus show $\mathcal{P}^{+}(N) \subseteq \mathcal{T}_{\mathrm{L}}^{+}(N)$ below. For an arbitrarily fixed $\boldsymbol{x} \in \mathcal{P}^{+}(N)$, let $\boldsymbol{y} = \boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))$. Since $\boldsymbol{xe} = 1$ and $\boldsymbol{Q}^{(1,1)}(N)$ is nonsingular, $\boldsymbol{y} \neq \boldsymbol{0}$. Furthermore, by definition, $\boldsymbol{x}(-\boldsymbol{Q}_{+}^{(1,1)}(N)) \geq \boldsymbol{0}$ and $\boldsymbol{x}(-\boldsymbol{Q}_{0}^{(1,1)}(N)) = \boldsymbol{0}$, where $\boldsymbol{Q}_{+}^{(1,1)}(N)$ and $\boldsymbol{Q}_{0}^{(1,1)}(N)$ are given in (2.18). We then partition $\boldsymbol{y}$ into two parts:

$$\boldsymbol{y} = \begin{array}{c} \begin{array}{cc} \mathcal{J}(N) & \mathbb{Z}_0^N \setminus \mathcal{J}(N) \end{array} \\ \begin{pmatrix} \boldsymbol{y}_{+} & \qquad \boldsymbol{0} \qquad \end{pmatrix} \end{array},$$

where $\boldsymbol{y}+ \geq \boldsymbol{0}$ and $\boldsymbol{y}_{+}\boldsymbol{e} > 0$. We now consider a linear augmentation matrix $\boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N)$ given by

$$\boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N) = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{\boldsymbol{y}}{\boldsymbol{ye}} = (-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e} \cdot \frac{(\boldsymbol{y}_{+} \ \ \boldsymbol{0})}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}.$$

It is clear that

$$\frac{(\boldsymbol{y}_{+} \ \ \boldsymbol{0})}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \in \Gamma^{+}(N),$$

and

$$\boldsymbol{x}[\boldsymbol{Q}^{(1,1)}(N) + \boldsymbol{Q}_{\mathrm{A}}^{\dagger}(N)] = -\boldsymbol{y} + \boldsymbol{x} \cdot \frac{(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}}{\boldsymbol{x}(-\boldsymbol{Q}^{(1,1)}(N))\boldsymbol{e}} \cdot \boldsymbol{y} = \boldsymbol{0},$$

which implies $\boldsymbol{x} \in \mathcal{T}_{\mathrm{L}}^{+}(N)$, so that $\mathcal{P}^{+}(N) \subseteq \mathcal{T}_{\mathrm{L}}^{+}(N)$.

Appendix E. Proof of Corollary 3.1

We first prove (3.16) by showing that $\boldsymbol{\beta} \in \Gamma(N) \Rightarrow \boldsymbol{\alpha} \in \Gamma(N)$ in (3.14), i.e., that

$$\boldsymbol{\beta} \geq \boldsymbol{0}, \ \ \boldsymbol{\beta e} = 1 \Rightarrow \ \boldsymbol{\alpha} \geq \boldsymbol{0}, \ \ \boldsymbol{\alpha e} = \boldsymbol{\beta e} \ (=1).$$

For this purpose, we consider (3.5). By definition, we have for $i, j \in \mathbb{Z}_0^N$,

$$[\boldsymbol{R}(K,N)]_{i,j} = \mathbb{P}(X(T_{n+1}^{(1)}) = j, \ \ X(t) \in \mathbb{Z}_{N+1}^{K} \ (T_n^{(2)} \leq t < T_{n+1}^{(1)}) \mid X(T_n^{(1)}) = i),$$

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and therefore $\boldsymbol{R}(K, N)$ is substochastic; i.e.,

$$\boldsymbol{R}(K, N) \geq \boldsymbol{O}, \qquad \boldsymbol{R}(K, N)\boldsymbol{e} \leq \boldsymbol{e}, \qquad \boldsymbol{R}(K, N)\boldsymbol{e} \neq \boldsymbol{e}.$$

It follows that

$$(\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1} = \sum_{n=0}^{\infty} \big(\boldsymbol{R}(K, N)\big)^n \geq \boldsymbol{O}.$$

Recall that $\boldsymbol{H}(N) \geq \boldsymbol{O}$ and $\widetilde{\boldsymbol{H}}(K, N) \geq \boldsymbol{O}$. We thus conclude $\boldsymbol{\beta} \geq \boldsymbol{0} \Rightarrow \boldsymbol{\alpha} \geq \boldsymbol{0}$ from (3.14). Furthermore, we have

$$\begin{aligned}
\boldsymbol{\alpha}\boldsymbol{e} &= \boldsymbol{\beta}\,\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1}\boldsymbol{H}(N)\boldsymbol{e} \\
&= \boldsymbol{\beta}\,\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e})\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e} \\
&= \boldsymbol{\beta}\boldsymbol{e}.
\end{aligned}$$

Next we show (3.17). For this purpose, we consider the censored process on $\mathbb{Z}_0^N \cup \mathbb{Z}_{K_1+1}^{\infty}$ whose generator is given by $\widetilde{\boldsymbol{Q}}(K_1, N)$. We then apply Theorem 2.1 with $N = K = K_1$ and obtain $\widetilde{\Gamma}(K_1, N)$. Moreover, we apply Theorem 3.1 to the above censored process on $\mathbb{Z}_0^N \cup \mathbb{Z}_{K_1+1}^{\infty}$ with $K = K_2$ and obtain $\widetilde{\Gamma}(K_2, N)$. Now (3.17) follows from (3.16), with $\widetilde{\Gamma}(K_1, N)$ and $\widetilde{\Gamma}(K_2, N)$ corresponding to $\Gamma(N)$ and $\widetilde{\Gamma}(K, N)$ in (3.16).

Appendix F. Proof of Corollary 3.2

We first consider (3.29). If $\mathcal{J}(N) = \mathbb{Z}_0^N$, (3.29) is immediate because

$$\widetilde{\Gamma}^{+}(K, N) \subseteq \widetilde{\Gamma}(K, N) \subseteq \Gamma(N) = \Gamma^{+}(N).$$

We thus assume $\mathcal{J}(N) \neq \mathbb{Z}_0^N$. Noting (3.20), we partition $\mathbb{Z}_0^N$ into the three subsets $\widetilde{\mathcal{J}}(K, N)$, $\mathcal{J}(N) \setminus \widetilde{\mathcal{J}}(K, N)$, and $\mathbb{Z}_0^N \setminus \mathcal{J}(N)$:

$$\boldsymbol{Q}^{(2,1)}(K, N) = \begin{array}{c} \begin{array}{ccc} \scriptstyle \widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathcal{J}(N)\setminus\widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathbb{Z}_0^N\setminus\mathcal{J}(N) \end{array} \\ \begin{pmatrix} \boldsymbol{Q}_{+}^{(2,1)}(K, N) & \boldsymbol{Q}_{++}^{(2,1)}(K, N) & \boldsymbol{O} \end{pmatrix} \end{array}. \tag{F.1}$$

Note here that $\mathcal{J}(N) \setminus \widetilde{\mathcal{J}}(K, N)$ may be empty; if this is the case, we simply ignore the corresponding terms. It follows from (3.5) that

$$(\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1} = \boldsymbol{I} + (\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1}\boldsymbol{R}(K, N) = \boldsymbol{I} + \boldsymbol{B}\boldsymbol{Q}^{(2,1)}(K, N), \tag{F.2}$$

where

$$\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1}(-\boldsymbol{Q}^{(1,1)}(K, N))^{-1}\boldsymbol{Q}^{(1,2)}(K, N)(-\boldsymbol{Q}^{(2,2)}(K, N))^{-1} \geq \boldsymbol{O}.$$

Suppose $\boldsymbol{\alpha} \in \widetilde{\Gamma}^{+}(K, N)$, i.e., there exists a nonnegative vector $\boldsymbol{\beta}$ such that

$$\boldsymbol{\beta} = \begin{array}{c} \begin{array}{ccc} \scriptstyle \widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathcal{J}(N)\setminus\widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathbb{Z}_0^N\setminus\mathcal{J}(N) \end{array} \\ (\begin{array}{ccc} \boldsymbol{\beta}_1 & \qquad \boldsymbol{0} \qquad & \boldsymbol{0} \end{array}) \end{array}, \qquad \boldsymbol{\beta}_1 \geq \boldsymbol{0}, \qquad \boldsymbol{\beta}_1\boldsymbol{e} = 1,$$

and

$$\begin{aligned}
\boldsymbol{\alpha} &= \begin{array}{c} \begin{array}{ccc} \scriptstyle \widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathcal{J}(N)\setminus\widetilde{\mathcal{J}}(K,N) & \scriptstyle \mathbb{Z}_0^N\setminus\mathcal{J}(N) \end{array} \\ \begin{pmatrix} \boldsymbol{\alpha}_1 & \qquad \boldsymbol{\alpha}_2 \qquad & \boldsymbol{\alpha}_3 \end{pmatrix} \end{array} \\
&= (\boldsymbol{\beta}_1 \;\; \boldsymbol{0} \;\; \boldsymbol{0})\,\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e})(\boldsymbol{I} - \boldsymbol{R}(K, N))^{-1}\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}).
\end{aligned}$$

It follows from (3.16) and (3.27) that $\widetilde{\Gamma}^{+}(K, N) \subseteq \Gamma(N)$, so that $\boldsymbol{\alpha}\boldsymbol{e} = 1$ and $\boldsymbol{\alpha} \geq \boldsymbol{0}$. Moreover, using (F.1) and (F.2), we can rewrite $\boldsymbol{\alpha}$ as

$$\begin{aligned}\boldsymbol{\alpha} &= (\boldsymbol{\beta}_1 \;\; \boldsymbol{0} \;\; \boldsymbol{0})\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e})\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}) \\ &\quad + \boldsymbol{\beta}\mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, N)\boldsymbol{e})\boldsymbol{B}\big(\boldsymbol{Q}_{+}^{(2,1)}(K, N) \;\; \boldsymbol{Q}_{++}^{(2,1)}(K, N) \;\; \boldsymbol{O}\big)\mathrm{diag}(\boldsymbol{H}(N)\boldsymbol{e}),\end{aligned}$$

and therefore $\boldsymbol{\alpha}_3 = \boldsymbol{0}$, which completes the proof of (3.29).

We omit the proof of (3.30) because it is almost the same as the proof of (3.17) in Corollary 3.1, except that the former uses Theorem 2.2, Theorem 3.2, and (3.29) where the latter uses Theorem 2.1, Theorem 3.1, and (3.16), respectively.

Appendix G. Proof of Theorem 4.1

Without loss of generality, we assume $\mathbb{Z}_0^N \cup \mathcal{X}^* = \mathbb{Z}_0^M$, where $M = |\mathbb{Z}_0^N \cup \mathcal{X}^*| - 1$. It then follows from Theorem 3.1 with $N := M$ that

$$\boldsymbol{\pi}(M) \in \widetilde{\mathcal{P}}(K, M),$$

where from (3.11) and (3.15),

$$\widetilde{\mathcal{P}}(K, M) = \widehat{\mathcal{P}}(K, M) = \big\{\boldsymbol{x} \in \mathbb{R}^{M+1};\, \boldsymbol{x} = \boldsymbol{\beta}_M \overline{\widetilde{\boldsymbol{H}}}(K, M),\, \boldsymbol{\beta}_M \in \Gamma(M)\big\}.$$

Note here that if $X(0) \in \mathbb{Z}_{K+1}^{\infty}$, the first passage time to $\mathbb{Z}_0^M$ ends on $\mathcal{X}^*$. Therefore, following a discussion similar to the proof of Theorem 3.2, we obtain

$$\boldsymbol{\pi}(M) \in \widehat{\mathcal{P}}^{+}(K, M; \mathcal{X}^*) \quad \text{for} \quad \mathcal{X} \in S(K, N), \tag{G.1}$$

where

$$\begin{aligned}\widehat{\mathcal{P}}^{+}(K, M; \mathcal{X}^*) = \big\{&\boldsymbol{x} \in \mathbb{R}^{M+1};\, \boldsymbol{x} = \boldsymbol{\beta}_M \overline{\widetilde{\boldsymbol{H}}}(K, M), \\ &\boldsymbol{\beta}_M \in \Gamma(M),\, [\boldsymbol{\beta}_M]_i = 0\ (i \in \mathbb{Z}_0^M \setminus \mathcal{X}^*)\big\}.\end{aligned} \tag{G.2}$$

For a proper $\mathcal{X}^*$, we have $\mathcal{X}^* \cap \mathcal{N}_0(K, N) = \emptyset$; i.e., the probability that the first passage time from $\mathbb{Z}_{K+1}^{\infty}$ to $\mathbb{Z}_0^M$ ends on state i is strictly positive for all $i \in \mathcal{X}^*$. We thus have $\mathcal{X}^* = \widetilde{\mathcal{J}}(K, M)$ for $\mathcal{X}^* \in \mathcal{S}^*(K, N)$, and therefore from Theorem 3.2,

$$\boldsymbol{\pi}(M) \in \mathrm{ri}\, \widetilde{\mathcal{P}}^{+}(K, M; \mathcal{X}^*) \quad \text{for} \quad \mathcal{X}^* \in S^*(K, N).$$

Since $\mathbb{Z}_0^N \cup \mathcal{X}^* = \mathbb{Z}_0^M$, $\boldsymbol{\pi}(N)$ can be expressed in terms of $\boldsymbol{\pi}(M)$. Specifically, we first partition $\boldsymbol{\pi}(M)$ and $\widetilde{\boldsymbol{H}}(K, M)$ as follows:

$$\begin{aligned}\boldsymbol{\pi}(M) &= \big(\overset{\mathbb{Z}_0^N}{\boldsymbol{\pi}^{(1)}(M; N)} \quad \overset{\mathbb{Z}_{N+1}^M}{\boldsymbol{\pi}^{(2)}(M; N)}\big), \\ \widetilde{\boldsymbol{H}}(K, M) &= \Big(\overset{\mathbb{Z}_0^N}{\widetilde{\boldsymbol{H}}^{(*,1)}(K, M; N)} \quad \overset{\mathbb{Z}_{N+1}^M}{\widetilde{\boldsymbol{H}}^{(*,2)}(K, M; N)}\Big).\end{aligned}$$

It then follows from (G.1) and (G.2) that $\boldsymbol{\pi}^{(1)}(M; N) \in \widehat{\mathcal{P}}_M^{+(1)}(K, M; \mathcal{X}^*)$, where

$$\begin{aligned}\widehat{\mathcal{P}}_M^{+(1)}(K, M; \mathcal{X}^*) = \big\{&\boldsymbol{x} \in \mathbb{R}^{N+1};\, \boldsymbol{x} = \boldsymbol{\beta}_M \mathrm{diag}^{-1}(\widetilde{\boldsymbol{H}}(K, M)\boldsymbol{e})\widetilde{\boldsymbol{H}}^{(*,1)}(K, M; N), \\ &\boldsymbol{\beta}_M \in \Gamma(M),\;\; [\boldsymbol{\beta}_M]_i = 0\ (i \in \mathbb{Z}_0^M \setminus \mathcal{X}^*)\big\}.\end{aligned} \tag{G.3}$$

By definition, $\boldsymbol{\pi}(N) = \boldsymbol{\pi}^{(1)}(M;N)/\boldsymbol{\pi}^{(1)}(M;N)\boldsymbol{e}$. It follows from (G.3) that $\boldsymbol{\pi}^{(1)}(M;N)$ is given by a weighted sum of row vectors of $\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;N)$. On the other hand, it follows from Remark 3.1 that if $\mathbb{Z}_0^N \cup \mathcal{X}^* = \mathbb{Z}_0^M$, then $\widetilde{\boldsymbol{H}}(K,M)$ is identical to the $(M+1)\times(M+1)$ northwest corner block of $\boldsymbol{H}(K)$. Therefore, the row vectors corresponding to state $i \in \mathcal{X}^*$ in $\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;N)$ and in $\boldsymbol{H}^{(*,1)}(K;N)$ are identical. Therefore, replacing $\widetilde{\boldsymbol{H}}^{(*,1)}(K,M;N)$ by $\boldsymbol{H}^{(*,1)}(K;N)$ and normalizing $\boldsymbol{\pi}^{(1)}(M;N)$, we obtain

$$\boldsymbol{\pi}(N) \in \widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*) \text{ for } \mathcal{X} \in \mathcal{S}(K,N),$$

where

$$\begin{aligned}\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*) = \big\{&\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\beta}\,\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K;N),\\ &\boldsymbol{\beta} \in \Gamma(K),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{X}^*)\big\}.\end{aligned} \tag{G.4}$$

In particular, for a proper $\mathcal{X}^*$, we have

$$\boldsymbol{\pi}(N) \in \mathrm{ri}\,\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}) \text{ for } \mathcal{X}^* \in \mathcal{S}^*(K,N).$$

Note that $\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*)$ in (G.4) is a convex polytope on the first orthant of $\mathbb{R}^{N+1}$.

The proof of (4.5) will be complete if we can show that

$$\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*) = \mathcal{P}^{+}(K,N;\mathcal{X}^*); \tag{G.5}$$

this will be done below. Note that $\boldsymbol{H}^{(*,1)}(K;N)$ is given in terms of $\widetilde{\boldsymbol{H}}(K,N)$ in (3.7):

$$\boldsymbol{H}^{(*,1)}(K;N) = \begin{array}{c}\mathbb{Z}_0^N\\ \mathbb{Z}_{N+1}^K\end{array}\begin{pmatrix}\boldsymbol{I}\\ (-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N)\end{pmatrix}\widetilde{\boldsymbol{H}}(K,N). \tag{G.6}$$

It then follows from (3.12) and (G.6) that

$$\begin{aligned}&\boldsymbol{\beta}\,\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K;N)\\ &= \boldsymbol{\beta}\,\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;N)\boldsymbol{e})\begin{pmatrix}\boldsymbol{I}\\ (-\boldsymbol{Q}^{(2,2)}(K,N))^{-1}\boldsymbol{Q}^{(2,1)}(K,N)\end{pmatrix}\widetilde{\boldsymbol{H}}(K,N)\\ &= \boldsymbol{\beta}\boldsymbol{U}(K,N)\overline{\boldsymbol{H}}(N),\end{aligned}$$

so that $\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*)$ is identical to $\mathcal{P}^{+}(K,N;\mathcal{X}^*)$ in (4.6). The equivalence between (4.5) and (4.6) can be shown in the same way as in the proof of Theorem 2.2, so that we omit it.

Appendix H. Proof of Lemma 4.1

If $\mathcal{X}_A^* = \mathcal{X}_B^*$, Lemma 4.1 holds. We thus assume $\mathcal{X}_A^* \neq \mathcal{X}_B^*$. Note that for $\mathcal{X} \in S(K,N)$, $\mathcal{P}^{+}(K,N;\mathcal{X}^*)$ is equivalent to $\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*)$ in (G.4); i.e.,

$$\begin{aligned}\widehat{\mathcal{P}}^{+}(K,N;\mathcal{X}^*) = \big\{&\boldsymbol{x} \in \mathbb{R}^{N+1}; \boldsymbol{x} = \boldsymbol{\beta}\overline{\boldsymbol{H}}^{(*,1)}(K;N),\\ &\boldsymbol{\beta} \in \Gamma(K),\ [\boldsymbol{\beta}]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{X}^*)\big\},\end{aligned} \tag{H.1}$$

where

$$\overline{\boldsymbol{H}}^{(*,1)}(K;N) = \mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K;N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K;N).$$

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Let $\boldsymbol{h}_j^{(*,1)}(K;N)$ and $\overline{\boldsymbol{h}}_j^{(*,1)}(K;N)$ $(j \in \mathbb{Z}_0^K)$ denote the jth row vectors of $\boldsymbol{H}^{(*,1)}(K;N)$ and $\overline{\boldsymbol{H}}^{(*,1)}(K;N)$. Note that the lemma is proven if

$$\overline{\boldsymbol{h}}_i^{(*,1)}(K;N) \in \widehat{\mathcal{P}}(K,N;\mathcal{X}_B^*), \qquad i \in \mathcal{X}_A^*. \tag{H.2}$$

It is clear that (H.2) holds for $i \in \mathcal{X}_A^* \cap \mathcal{X}_B^*$. We thus show (H.2) for $i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*$ below.

For a fixed $i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*$, we consider a censored Markov chain $\{\widetilde{X}(t)\}_{t\in(-\infty,\infty)}$ by observing $\{X(t)\}_{t\in(-\infty,\infty)}$ only when $X(t) \in \mathbb{Z}_0^N \cup \mathcal{X}_B^* \cup \{i\} \cup \mathbb{Z}_{K+1}^\infty$. The infinitesimal generator $\widetilde{\boldsymbol{Q}} := \widetilde{\boldsymbol{Q}}(K,N,\mathcal{X}_B^*,i)$ of the censored Markov chain $\{\widetilde{X}(t)\}_{t\in(-\infty,\infty)}$ takes the following form:

$$\widetilde{\boldsymbol{Q}} = \begin{array}{c} \\ \mathbb{Z}_0^N\cup\mathcal{X}_B^*\cup\{i\} \\ \mathbb{Z}_{K+1}^\infty \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N\cup\mathcal{X}_B^*\cup\{i\} & \mathbb{Z}_{K+1}^\infty \end{array} \\ \begin{pmatrix} \widetilde{\boldsymbol{Q}}^{(1,1)} & \widetilde{\boldsymbol{Q}}^{(1,2)} \\ \widetilde{\boldsymbol{Q}}^{(2,1)} & \widetilde{\boldsymbol{Q}}^{(2,2)} \end{pmatrix} \end{array},$$

where, with $\mathbb{Z}_B = \mathbb{Z}_0^N \cup \mathcal{X}_B^*$,

$$\widetilde{\boldsymbol{Q}}^{(1,1)} = \begin{array}{c} \\ \mathbb{Z}_B \\ \{i\} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{Z}_B & \{i\} \end{array} \\ \begin{pmatrix} \widetilde{\boldsymbol{Q}}^{(1,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{\boldsymbol{q}}^{(1,1)}_{\mathbb{Z}_B,i} \\ \widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B} & -\widetilde{q}^{(1,1)}_{i,i} \end{pmatrix} \end{array}.$$

Note here that

$$\widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B} = \begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N\setminus\mathcal{X}_B^* & \mathcal{X}_B^* \end{array} \\ \begin{pmatrix} \mathbf{0} & \widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B,+} \end{pmatrix} \end{array}, \tag{H.3}$$

since $i \in \mathcal{X}_A^* \setminus \mathcal{X}_B^*$ and (4.7) holds sample-path-wise. We define $\widetilde{\boldsymbol{H}} := \widetilde{\boldsymbol{H}}(K,N,\mathcal{X}_B^*,i)$ as

$$\begin{aligned} \widetilde{\boldsymbol{H}} = (-\widetilde{\boldsymbol{Q}}^{(1,1)})^{-1} &= \begin{array}{c} \\ \mathbb{Z}_B \\ \{i\} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{Z}_B & \{i\} \end{array} \\ \begin{pmatrix} \widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{\boldsymbol{h}}_{\mathbb{Z}_B,i} \\ \widetilde{\boldsymbol{h}}_{i,\mathbb{Z}_B} & \widetilde{h}_{i,i} \end{pmatrix} \end{array} \\ &= \begin{array}{c} \\ \mathbb{Z}_B \\ \{i\} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{Z}_B & \{i\} \end{array} \\ \begin{pmatrix} \widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{\boldsymbol{h}}_{\mathbb{Z}_B,i} \\ (\widetilde{q}^{(1,1)}_{i,i})^{-1}\widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B}\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{h}_{i,i} \end{pmatrix} \end{array}. \end{aligned}$$

Note here that if we partition $\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B}$ as

$$\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} = \begin{array}{c} \\ \mathbb{Z}_0^N\setminus\mathcal{X}_B^* \\ \mathcal{X}_B^* \end{array} \!\!\begin{array}{c} \begin{array}{cc} \mathbb{Z}_0^N & \mathbb{Z}_B\setminus\mathbb{Z}_0^N \end{array} \\ \begin{pmatrix} \widetilde{\boldsymbol{H}}^{(1,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{\boldsymbol{H}}^{(1,2)}_{\mathbb{Z}_B,\mathbb{Z}_B} \\ \widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} & \widetilde{\boldsymbol{H}}^{(2,2)}_{\mathbb{Z}_B,\mathbb{Z}_B} \end{pmatrix} \end{array},$$

the row vector of $\widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}$ corresponding to $j \in \mathcal{X}_B^*$ is identical to $\boldsymbol{h}_j^{(*,1)}(K;N)$. Furthermore, noting (H.3), we have

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$$\widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B}\widetilde{\boldsymbol{H}}_{\mathbb{Z}_B,\mathbb{Z}_B} = \left(\widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B}\begin{pmatrix}\widetilde{\boldsymbol{H}}^{(1,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}\\ \widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}\end{pmatrix} \widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B}\begin{pmatrix}\widetilde{\boldsymbol{H}}^{(1,2)}_{\mathbb{Z}_B,\mathbb{Z}_B}\\ \widetilde{\boldsymbol{H}}^{(2,2)}_{\mathbb{Z}_B,\mathbb{Z}_B}\end{pmatrix}\right)$$

$$= \left(\widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B,+}\widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B} \quad \widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B,+}\widetilde{\boldsymbol{H}}^{(2,2)}_{\mathbb{Z}_B,\mathbb{Z}_B}\right),$$

and

$$\boldsymbol{h}^{(*,1)}_i(K;N) = (\widetilde{q}^{(1,1)}_{i,i})^{-1}\widetilde{\boldsymbol{q}}^{(1,1)}_{i,\mathbb{Z}_B,+}\widetilde{\boldsymbol{H}}^{(2,1)}_{\mathbb{Z}_B,\mathbb{Z}_B}.$$

Therefore, $\boldsymbol{h}^{(*,1)}_i(K;N)$ ($i \in \mathcal{X}^*_A \setminus \mathcal{X}^*_B$) is given by a linear combination of the $\boldsymbol{h}^{(*,1)}_j(K;N)$ ($j \in \mathcal{X}^*_B$) with nonnegative weights $\alpha_{i,j}$:

$$\boldsymbol{h}^{(*,1)}_i(K;N) = \sum_{j\in\mathcal{X}^*_B} \alpha_{i,j}\boldsymbol{h}^{(*,1)}_j(K;N).$$

We thus conclude that $\overline{\boldsymbol{h}}^{(*,1)}_i(K;N)$ ($i \in \mathcal{X}^*_A \setminus \mathcal{X}^*_B$) is given by a convex combination of $\overline{\boldsymbol{h}}^{(*,1)}_j(K;N)$ ($j \in \mathcal{X}^*_B$); i.e.,

$$\overline{\boldsymbol{h}}^{(*,1)}_i(K;N) = \sum_{j\in\mathcal{X}^*_B} \beta_{i,j}\overline{\boldsymbol{h}}^{(*,1)}_j(K;N),$$

where

$$\beta_{i,j} = \frac{\alpha_{i,j}\boldsymbol{h}^{(*,1)}_j(K;N)\boldsymbol{e}}{\boldsymbol{h}^{(*,1)}_i(K;N)\boldsymbol{e}}, \quad j \in \mathcal{X}^*_B.$$

Note that $\beta_{i,j} \geq 0$ and $\sum_{j\in\mathcal{X}^*_B}\beta_{i,j} = 1$, which completes the proof.

Appendix I. Proof of Lemma 4.2

We first prove that $\mathcal{D}(K, N)$ is an (s, t)-node cut in $G(K, N)$, and then we show its minimality. To prove the former, we show that the graph $G(K, N; -\mathcal{D}(K, N))$, which is obtained from $G(K, N)$ by removing all nodes in $\mathcal{D}(K, N)$, has no directed paths from s to t. To prove this, we assume that $G(K, N; -\mathcal{D}(K, N))$ has a directed path P from s to t and derive a contradiction. Since $\mathcal{J}(K)$ is the set of all neighboring nodes of s, P must contain a node $v \in \mathcal{J}(K) \setminus \mathcal{D}(K, N)$, where we assume that if P contains more than one node in $\mathcal{J}(K) \setminus \mathcal{D}(K, N)$, then v is the node closest to t. Because the directed path from v to t has no edges incoming to any nodes in $\mathcal{J}(K)$, $G^*(K, N)$ should contain the subpath $P_{v,t}$ of P from v to t. This, however, contradicts the fact that $\mathcal{D}(K, N)$ contains all nodes in $\mathcal{J}(K)$ from which t is reachable in $G^*(K, N)$.

Next we show the minimality of $\mathcal{D}(K, N)$. For each node $v \in \mathcal{D}(K, N)$, $G^*(K, N)$ contains a directed path $P_{v,t}$ from v that visits t without passing through any other nodes in $\mathcal{J}(K)$. As a result, the edge $(s, v) \in E(K, N)$ and $P_{v,t}$ form a directed path from s to t in $G(K, N)$, and therefore $v \in \mathcal{D}(K, N)$ must be included in any (s,t)-node cut $\mathcal{X} \subseteq \mathcal{J}(K)$ in $G(K, N)$. We thus conclude that $\mathcal{D}(K, N) \subseteq \mathcal{X}$ for any $\mathcal{X}$ such that $\mathcal{X} \subseteq \mathcal{J}(K)$ and $\mathcal{X} \in S(K, N)$, which completes the proof.

Appendix J. Proof of Theorem 4.2

Note that $\boldsymbol{x} \in \operatorname{ri}\mathcal{P}^+(K, N; \mathcal{J}^*(K))$ from Lemma 4.3 and $\mathcal{P}^+(K, N; \mathcal{J}^*(K)) = \widehat{\mathcal{P}}^+(K, N; \mathcal{J}^*(K))$ from (G.5), where $\widehat{\mathcal{P}}^+(K, N; \mathcal{J}^*(K))$ is given in (G.5). We thus have

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$\boldsymbol{x} \in \mathrm{ri}\,\widehat{\mathcal{P}}^{+}(K, N; \mathcal{J}^*(K))$; therefore there exists $\boldsymbol{\beta} \in \Gamma(K)$ such that $[\boldsymbol{\beta}]_i > 0$ $(i \in \mathcal{J}^*(K))$, $[\boldsymbol{\beta}]_i = 0$ $(i \in \mathbb{Z}_0^K \setminus \mathcal{J}^*(K))$, and

$$\boldsymbol{x} = \boldsymbol{\beta}\,\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K; N)\boldsymbol{e})\boldsymbol{H}^{(*,1)}(K; N).$$

On the other hand, it follows from (4.8) that $\mathcal{J}(K) \setminus \mathcal{J}^*(K) = \mathcal{J}(K) \cap \mathcal{N}_0^{(3)}(K, N)$. Therefore, we have from (4.3) that

$$\boldsymbol{h}_i^{(*,1)}(K; N) = \boldsymbol{0}, \qquad i \in \mathcal{J}(K) \setminus \mathcal{J}^*(K),$$

where $\boldsymbol{h}_i^{(*,1)}(K; N)$ $(i \in \mathbb{Z}_0^K)$ denotes the ith row vector of $\boldsymbol{H}^{(*,1)}(K; N)$. Let $\boldsymbol{\gamma} \in \mathbb{R}^{K+1}$ be defined by

$$\boldsymbol{\gamma} = \boldsymbol{\beta}\,\mathrm{diag}^*(\boldsymbol{H}^{(*,1)}(K; N)\boldsymbol{e}) + \sum_{i \in \mathcal{J}(K) \setminus \mathcal{J}^*(K)} \boldsymbol{e}_i,$$

where $\boldsymbol{e}_i \in \Gamma(K)$ $(i \in \mathbb{Z}_0^K)$ denotes the unit vector whose ith element is equal to one. Note here that

$$[\boldsymbol{\gamma}]_i > 0, \qquad i \in \mathcal{J}(K), \tag{J.1}$$

$$[\boldsymbol{\gamma}]_i = 0, \qquad i \in \mathbb{Z}_0^K \setminus \mathcal{J}(K). \tag{J.2}$$

We thus have

$$\boldsymbol{\gamma}\boldsymbol{H}^{(*,1)}(K; N) = \boldsymbol{x} + \sum_{i \in \mathcal{J}(K) \setminus \mathcal{J}^*(K)} \boldsymbol{h}_i^{(*,1)}(K; N) = \boldsymbol{x}.$$

Let $\boldsymbol{y} = \boldsymbol{\gamma}\boldsymbol{H}^{(*,2)}(K; N)$. We then have

$$(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K)) = \boldsymbol{\gamma}.$$

It follows from (J.1) and (J.2) that

$$\begin{aligned} &[(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{Q}^{(1,1)}(K)]_i > 0, \qquad i \in \mathcal{J}(K), \\ &[(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{Q}^{(1,1)}(K)]_i = 0, \qquad i \in \mathbb{Z}_0^K \setminus \mathcal{J}(K). \end{aligned}$$

Furthermore, $(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{e} = \boldsymbol{\gamma}\boldsymbol{H}(K)\boldsymbol{e} > \boldsymbol{0}$ since $\boldsymbol{H}(K)\boldsymbol{e} > \boldsymbol{0}$. We thus have $(\boldsymbol{x}\ \boldsymbol{y})/((\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{e}) \in \mathrm{ri}\,\mathcal{P}^{+}(K)$, and therefore

$$(\boldsymbol{x}\ \boldsymbol{y}) > \boldsymbol{0},$$

from Lemma 2.1. We then define $\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y})$ as

$$\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y}) = \frac{(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))}{(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}} = \frac{\boldsymbol{\gamma}}{\boldsymbol{\gamma}\boldsymbol{e}}.$$

Note that $\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y}) \in \mathrm{ri}\,\Gamma^{+}(K)$, where

$$\Gamma^{+}(K) = \left\{\boldsymbol{\alpha} \in \mathbb{R}^{K+1}; \boldsymbol{\alpha} \geq \boldsymbol{0}, \boldsymbol{\alpha}\boldsymbol{e} = 1,\ [\boldsymbol{\alpha}]_i = 0\ (i \in \mathbb{Z}_0^K \setminus \mathcal{J}(K))\right\}.$$

We now consider a Markov chain whose infinitesimal generator $\boldsymbol{Q}$ is given by

$$\boldsymbol{Q} = \begin{array}{c} \\ \mathbb{Z}_0^K \\ \mathbb{Z}_{K+1}^{\infty} \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad\mathbb{Z}_0^K \quad & \quad\quad \mathbb{Z}_{K+1}^{\infty} \end{array} \\ \begin{pmatrix} \boldsymbol{Q}^{(1,1)}(K) & (-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}\boldsymbol{z} \\ \boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x}, \boldsymbol{y}) & -\boldsymbol{I}, \end{pmatrix} \end{array}, \tag{J.3}$$

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where z denotes a $1\times\infty$ positive probability vector. Note here that the Markov chain with $\boldsymbol{Q}$ given by (J.3) is a member of $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$ if it is ergodic, i.e., if the global balance equation $\boldsymbol{\pi}\boldsymbol{Q}=\boldsymbol{0}$ has a unique positive solution $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi}\boldsymbol{e}=1$, where $\boldsymbol{\pi}\boldsymbol{Q}=\boldsymbol{0}$ is written as

$$\boldsymbol{\pi}^{(1)}(K)\boldsymbol{Q}^{(1,1)}(K)+\boldsymbol{\pi}^{(2)}(K)\boldsymbol{e}\boldsymbol{\zeta}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0},$$
$$\boldsymbol{\pi}^{(1)}(N)(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}\boldsymbol{z}-\boldsymbol{\pi}^{(2)}(K)=\boldsymbol{0}.$$

Solving the above with $\boldsymbol{\pi}^{(1)}(K)\boldsymbol{e}+\boldsymbol{\pi}^{(2)}(K)\boldsymbol{e}=1$, we obtain

$$\begin{aligned}\boldsymbol{\pi}&=\big(\boldsymbol{\pi}^{(1)}(K)\boldsymbol{\pi}^{(2)}(K)\big)\\ &=\frac{1}{(\boldsymbol{x}\ \boldsymbol{y})\boldsymbol{e}+(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}}\big((\boldsymbol{x}\ \boldsymbol{y})\quad(\boldsymbol{x}\ \boldsymbol{y})(-\boldsymbol{Q}^{(1,1)}(K))\boldsymbol{e}\cdot z\big)\\ &=\frac{1}{1+\boldsymbol{y}\boldsymbol{e}+\boldsymbol{\gamma}\boldsymbol{e}}\big((\boldsymbol{x}\ \boldsymbol{y})\quad \boldsymbol{\gamma}\boldsymbol{e}\cdot z\big)>\boldsymbol{0};\end{aligned}$$

in particular,

$$\boldsymbol{\pi}^{(1)}(N)=\boldsymbol{\pi}^{(1)}(K,N)=\frac{1}{1+\boldsymbol{y}\boldsymbol{e}+\boldsymbol{\gamma}\boldsymbol{e}}\cdot\boldsymbol{x}.$$

We thus conclude that the Markov chain with $\boldsymbol{Q}$ in (J.3) is a member of $\mathcal{M}(\boldsymbol{Q}^{(1,1)}(K), \mathcal{J}(K))$, and it has $\boldsymbol{\pi}(N)=\boldsymbol{x}$.

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