Ergod. Th. & Dynam. Sys. (2017), **37**, 2255–2275 © *Cambridge University Press, 2016*
doi:10.1017/etds.2016.4

Automorphisms of blowups of threefolds being Fano or having Picard number 1

TUYEN TRUNG TRUONG

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Republic of Korea
(e-mail: truong@kias.re.kr)

(*Received* 3 *March* 2015 *and accepted in revised form* 21 *December* 2015)

Abstract. Let X_0 be a smooth projective threefold which is Fano or which has Picard number 1. Let $\pi : X \to X_0$ be a finite composition of blowups along smooth centers. We show that for 'almost all' of such X, if $f \in \text{Aut}(X)$, then its first and second dynamical degrees are the same. We also construct many examples of blowups $X \to X_0$, on which any automorphism is of zero entropy. The main idea is that, because of the log-concavity of dynamical degrees and the invariance of Chern classes under holomorphic automorphisms, there are some constraints on the nef cohomology classes. We will also discuss a possible application of these results to a threefold constructed by Kenji Ueno.

1. *Introduction*

While there are many examples of compact complex surfaces having automorphisms of positive entropies (for example, the works of Cantat [**8**], Bedford and Kim [**4–6**], McMullen [**23–26**], Oguiso [**28**, **29**], Cantat and Dolgachev [**9**], Zhang [**40**], Diller [**15**], Déserti and Grivaux [**14**] and Reschke [**33**]), there are few interesting examples of manifolds of higher dimensions having automorphisms of positive entropies (for example, Oguiso [**30**, **31**] and Oguiso and Perroni [**27**]). In particular, for the class of smooth rational threefolds, there are currently only two known examples of manifolds with primitive automorphisms of positive entropy (see [**10**, **11**, **32**]). Here, a primitive automorphism, defined by Zhang [**40**], is one that has no non-trivial invariant fibrations over a base of dimension one or two. For general properties on automorphism groups of compact Kähler manifolds, see the recent survey paper [**16**].

It is natural to ask what happens in dimension three and higher. For example, the following question was asked by Bedford in 2011.

Question 1. Is there a finite composition of blowups at points or smooth curves $X \to \mathbb{P}^3$ starting from $\mathbb{P}^3$ and is there an automorphism $f : X \to X$ with positive entropy?

This paper aims to study Question 1 and some related questions. We give evidence that the answer to Question 1 is negative and that the examples in [**10**, **11**, **32**] cannot be obtained as smooth blowups of smooth threefolds having Picard number 1 or being Fano.

Our results and proofs are stated in terms of dynamical degrees, which we recall now. Let X be a smooth projective threefold. We denote by $\mathrm{Pic}(X)$ the Picard group of X, $\mathrm{Pic}_{\mathbb{Q}}(X) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Pic}_{\mathbb{R}}(X) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathrm{Nef}(X) \subset \mathrm{Pic}_{\mathbb{R}}(X)$ be the cone of nef classes, which is the closure of the cone of ample classes. Recall that a class in $\mathrm{Pic}_{\mathbb{R}}(X)$ is nef if and only if it has non-negative intersection with every curve on X. For later use, we denote by $c_1(X)$ and $c_2(X)$ the first and second Chern classes of X. Let $f : X \to X$ be an automorphism. Then f preserves both $\mathrm{Pic}(X)$ and $\mathrm{Nef}(X)$. Let ω be an ample class on X. We define the first and second dynamical degrees of f as

$$\lambda_1(f) = \lim_{n\to\infty} [(f^n)^*(\omega) \cdot \omega^2]^{1/n},$$
$$\lambda_2(f) = \lim_{n\to\infty} [(f^n)^*(\omega^2) \cdot \omega]^{1/n}.$$

Here are some properties of these dynamical degrees: $\lambda_1(f)^2 \geq \lambda_2(f) \geq 1$ and $\lambda_1(f^{-1}) = \lambda_2(f)$. For more on dynamical degrees, see [**19**].

Entropy of f can be computed via dynamical degrees by Gromov–Yomdin's theorem [**20**, **37**]: $h_{\mathrm{top}}(f) = \log \max\{\lambda_1(f), \lambda_2(f)\}$. Hence f has positive entropy if and only if $\lambda_1(f) > 1$.

Primitivity of f can also be detected from dynamical degrees via the following criterion (see [**32**]), which is a consequence of results in [**17**, **18**]: if $\lambda_1(f) \neq \lambda_2(f)$, then f is primitive.

The main idea behind all the results of this paper is that the existence of an automorphism f of positive entropy on X imposes some constraints on the cohomology groups of X. In fact, let $0 \neq \zeta \in \mathrm{Nef}(X)$ be such that $f^*(\zeta) = \lambda_1(f)\zeta$ (the existence of such a class is guaranteed by the Perron–Frobenius theorem). The differential df gives an isomorphism between the tangent bundle TX and its pullback $f^*(TX)$. Hence, from the properties of Chern classes, we have $f^*c_1(X) = c_1(X)$ and $f^*c_2(X) = c_2(X)$. Since $\lambda_1(f) > 1$ and X has dimension three, it follows that

$$\zeta^3 = \zeta^2 \cdot c_1(X) = \zeta \cdot c_1(X)^2 = 0.$$

In fact, stronger constraints are satisfied.

THEOREM 1. *Let X be a projective manifold of dimension three and $f : X \to X$ an automorphism.*

(1) *If f has positive entropy, there is a nef class ζ, which is not in $\mathbb{R} \cdot \mathrm{Pic}_{\mathbb{Q}}(X)$, such that $\zeta^2 = 0$, $\zeta \cdot c_1(X)^2 = 0$ and $\zeta \cdot c_2(X) = 0$.*
(2) *If $\lambda_1(f) \neq \lambda_2(f)$, there is a nef class ζ, which is not in $\mathbb{R} \cdot \mathrm{Pic}_{\mathbb{Q}}(X)$, such that $\zeta^2 = 0$, $\zeta \cdot c_1(X) = 0$ and $\zeta \cdot c_2(X) = 0$.*

Here we comment on the condition $\zeta^2 = 0$. If X has dimension two, then this condition is one homogeneous equation in m variables (here, m is the Picard number of X) and hence is very easily satisfied. In contrast, when X has dimension three or bigger, the condition $\zeta^2 = 0$ is a system of $p \geq m$ homogeneous equations in the m variables (here, p is the

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

dimension of $\bigwedge^2 \mathrm{Pic}_{\mathbb{R}}(X)$), and hence is more difficult to be satisfied. This is a heuristic argument for why it is difficult to find automorphisms of positive entropy in dimension three or larger.

Based on Theorem 1, we state some conditions on nef cohomology classes.

CONDITION 2. *Let X be a smooth projective threefold.*

(1) *Condition A: we say that X satisfies Condition A if, whenever $\zeta \in \mathrm{Nef}(X)$ is such that $\zeta^2 = 0$, $\zeta \cdot c_1(X)^2 \geq 0$ and $\zeta \cdot c_2(X) \leq 0$, $\zeta \in \mathbb{R} \cdot \mathrm{Pic}_{\mathbb{Q}}(X)$.*

(2) *Condition B: we say that X satisfies Condition B if, whenever $\zeta \in \mathrm{Nef}(X)$ is such that $\zeta^2 = 0$, $\zeta \cdot c_1(X) = 0$ and $\zeta \cdot c_2(X) \leq 0$, $\zeta \in \mathbb{R} \cdot \mathrm{Pic}_{\mathbb{Q}}(X)$.*

By Theorem 1, if X satisfies Condition A, then any automorphism on X has zero entropy and, if X satisfies Condition B, then, for any automorphism f of X, $\lambda_1(f) = \lambda_2(f)$. While requiring more than the assumptions in part (1) of Theorem 1, Condition A is very suitable for inductive arguments. A similar comment applies for Condition B.

Now we are ready to state the main results of this paper. The first result is for blowups of some special configurations of $\mathbb{P}^3$.

THEOREM 3. *Let $p_1, \ldots, p_n$ be distinct points in $X_0 = \mathbb{P}^3$ such that any four of them do not belong to the same hyperplane. Let $C_{i,j}$ be the line connecting the points p_i and p_j. Let $\pi_1 : X_1 \to \mathbb{P}^3$ be the blowup at $p_1, \ldots, p_n$. Let $D_{i,j} \subset X_1$ be the strict transforms of $C_{i,j}$, and $\pi_2 : X_2 \to X_1$ be the blowup at $D_{i,j}$. Then any automorphism of X_2 has zero entropy.*

Remark. Dolgachev and Prokhorov informed us that, in the special cases where $4 \leq n \leq 7$, the automorphism group of X_2 in Theorem 3 is finite. The conclusion of Theorem 3 can be proven for the blowups of more general configurations in $\mathbb{P}^3$. However, since the statements of these generalizations are a bit complicated, we refer to §4 for more details.

The next two main results of the paper are for threefolds having Picard number 1 or satisfying a special property on the second Chern class.

THEOREM 4. *Let X_0 be a threefold with Picard number* 1. *Let $C_1, \ldots, C_t \subset X_0$ be smooth curves which are pairwise disjoint. Let $p_1, \ldots, p_s \in X_0$ be distinct points, which are allowed to belong to the curves $C_1, \ldots, C_t$. Let $\pi_1 : X_1 \to X_0$ be the blowup at $p_1, \ldots, p_s$, and $\pi_2 : X_2 \to X_1$ be the blowup at the strict transforms of $C_1, \ldots, C_t$. Then X_2 satisfies Properties A and B.*

We note that, in general, Theorem 4 does not hold for threefolds X_0 with Picard number greater than or equal to two (for example when $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$). However, the theorems below may still hold for those manifolds. See §4 for more details. We recall that a class ζ on X is movable if there is a smooth blowup $\pi : Z \to X$ such that ζ is the pushforward of some nef class on Z.

THEOREM 5. *Let X_0 be a smooth projective threefold such that $c_2(X_0) \cdot \zeta > 0$ for all non-zero movable $\zeta \in NS_{\mathbb{R}}(X_0)$. Let $p_1, \ldots, p_n \in X_0$ be distinct points. Let $\pi_1 : X_1 \to X_0$ be the blowup of X_0 at $p_1, \ldots, p_n$. Let $D_1, \ldots, D_m \subset X_1$ be disjoint smooth curves and let $\pi_2 : X_2 \to X_1$ be the blowup at $D_1, \ldots, D_n$.*

(1) *X_2 satisfies Condition B.*
(2) *Assume, moreover, that for any j, $c_1(X_1) \cdot D_j \leq 2g_j - 2$, where g_j is the genus of D_j. Then X_2 satisfies Condition A.*

Theorem 5 applies for $X_0 = \mathbb{P}^3$ or $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It also applies for complete intersection threefolds in $\mathbb{P}^N$ (see §4 for more details). We note that, here, the images in X_0 of $D_1, \ldots, D_n$ may be singular and intersect with each other, and hence Theorem 5 is not covered by Theorem 4, not even in the case $X_0 = \mathbb{P}^3$.

Finally, we state several results which are purely inductive in nature, which can also be applied to blowups of Fano threefolds. Here we recall that a threefold is Fano if $c_1(X)$ is ample.

We start with the case of point blowups.

THEOREM 6. *Let Y be a smooth projective threefold satisfying one of the Conditions A and B. Let $\pi : X \to Y$ be the blowup at a point. Then X satisfies the same condition.*

Next, we consider the case of curve blowups.

THEOREM 7. *Let Y be a smooth projective threefold satisfying Condition A or B. Let $\pi : X \to Y$ be the blowup at a smooth curve $C \subset Y$. Let g be the genus of C, and define $\gamma = c_1(Y) \cdot C + 2g - 2$. Then X also satisfies the same condition if one of the following cases occurs.*
(1) *$c_1(Y) \cdot C$ is an odd number and the normal vector bundle $N_{C/Y}$ is decomposable. The latter means that $N_{C/Y}$ is the direct sum of two line bundles over C.*
(2) *$\gamma < 0$ and C is not the only effective curve in its cohomology class.*
(3) *There is an irreducible hypersurface $S \subset Y$ such that $2\kappa < \mu\gamma$. Here, $\kappa = S \cdot C$ and μ is the multiplicity of C in S.*

We note that in (1) of Theorem 7, the condition that $N_{C/Y}$ is decomposable, may be satisfied easily. For example, if C is a smooth rational curve, then $N_{C/Y}$ is always decomposable, by a result of Grothendieck, even if C does not move in Y.

THEOREM 8. *Let Y be a smooth projective threefold satisfying Condition B. Let $\pi : X \to Y$ be the blowup at a smooth curve $C \subset Y$. Let g be the genus of C. If $c_1(Y) \cdot C \neq 2g - 2$, then X also satisfies Condition B.*

Hence, we conclude that if X_0 is a smooth threefold which is Fano or has Picard number 1, then, for almost every $X \to X_0$ which is a finite composition of points or smooth curves, every automorphism f on X has $\lambda_1(f) = \lambda_2(f)$. This is a strong indication that probably all automorphisms on such manifolds are not primitive, that is, they have invariant fibrations over a base of dimension one or two.

In §4, we will give various examples illustrating the above results. In §5 we will discuss a possible application of the above results to Ueno's threefold, considered in [**36**].

Remark. The general case of compact Kähler threefolds can be treated, similarly, by replacing the Neron–Severi group by the $(1, 1)$ cohomology group. After the appearance of a first version of this paper (see [**35**]), some generalizations to higher dimensions have been given in [**2**, **34**].

2. *Preliminaries on nef classes and blowups*

2.1. *Kähler, nef and psef classes and effective varieties.* Let X be a compact Kähler manifold. Let $\eta \in H^{1,1}(X)$. We say that η is Kähler if it can be represented by a Kähler $(1, 1)$ form. We say that η is nef if it is the limit of a sequence of Kähler classes. We say that η is psef if it can be represented by a positive closed $(1, 1)$ current. A class $\xi \in H^{p,p}(X)$ is an effective variety if there are irreducible varieties $C_1, \ldots, C_t$ of codimension p in X and non-negative real numbers $a_1, \ldots, a_t$ so that ξ is represented by $\sum_i a_i C_i$.

Demailly and Paun [**13**] gave a characterization of Kähler and nef classes, which, in the case of projective manifolds, is summarized as follows.

THEOREM 9. *Let X be a projective manifold with a Kähler $(1, 1)$ form ω. A class $\eta \in H^{1,1}(X)$ is Kähler if and only if, for any irreducible subvariety $V \subset X$, $\int_V \eta^{\dim(V)} > 0$. A class $\eta \in H^{1,1}(X)$ is nef if and only if, for any irreducible subvariety $V \subset X$, $\int_V \eta^{\dim(V)-j} \wedge \omega^j \geq 0$ for all $0 \leq j \leq \dim(V)$.*

Nef classes are preserved under pullback by holomorphic maps.

LEMMA 10. *Let $\pi : X \to Y$ be a holomorphic map between compact Kähler manifolds. Then $\pi^*(H^{1,1}_{\mathrm{nef}}(X)) \subset H^{1,1}_{\mathrm{nef}}(Y)$.*

Proof. Since nef classes are in the closure of Kähler classes, it suffices to show that if η is a Kähler class, then $\pi^*(\eta)$ is nef. Let φ be a Kähler $(1, 1)$ form representing η. Then $\pi^*(\varphi)$ is a positive smooth $(1, 1)$ form. Let ω_X be a Kähler $(1, 1)$ form on X. Then $\pi^*(\eta)$ is represented as a limit of the Kähler classes

$$\pi^*(\varphi) + \frac{1}{n}\omega_X,$$

and hence is nef. □

Remark. Similarly, it can be shown that psef classes are preserved under pushforward by holomorphic maps. However, nef classes may not be preserved under pushforwards, even when the map is a blowup.

2.2. *Blowup of a projective 3-manifold at a point.* Let $\pi : X \to Y$ be the blowup of a projective 3-manifold at a point p. Let $E = \mathbb{P}^2$ be the exceptional divisor and let $L \subset E$ be a line. Then $H^{1,1}(X)$ is generated by $\pi^*(H^{1,1}(Y))$ and E, and $H^{2,2}(X)$ is generated by $\pi^*(H^{2,2}(Y))$ and L. The intersection product on the cohomology of X is given by

$$\pi^*(\xi) \cdot E = 0, \quad E \cdot E = -L,$$
$$\pi^*(\xi) \cdot L = 0, \quad E \cdot L = -1.$$

The first and second Chern classes of X can be computed by (see, for example, the book of Griffiths and Harris [**21**, §6, Ch. 4])

$$c_1(X) = \pi^*(c_1(Y)) - 2E,$$
$$c_2(X) = \pi^*(c_2(Y)).$$

The following result concerns the relations between cycles on X and Y.

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

LEMMA 11. *For any effective curve* $V \subset Y$*, there is an effective curve* $\widetilde{V} \subset X$ *so that* $\pi_*(\widetilde{V}) = V$ *and* $\widetilde{V} \cdot E \geq 0$.

Proof. It suffices to consider the case when V is an irreducible curve. We can choose $\widetilde{V}$ to be the strict transform of V. Then $\pi_*(\widetilde{V}) = V$, and $\widetilde{V}$ is not contained in E. Therefore $\widetilde{V} \cdot E \geq 0$. □

We end this subsection showing that nef classes are preserved under pushforward by point blowups.

LEMMA 12. *Let* $\eta \in H^{1,1}_{\mathrm{nef}}(X)$*. Then* $\pi_*(\eta) \in H^{1,1}_{\mathrm{nef}}(Y)$.

Proof. It suffices to prove the conclusion when η is a Kähler class. Let φ be a Kähler (1, 1) form representing η. Then $\pi_*(\varphi)$ is a positive closed (1, 1) current, which is smooth on $X - p$.

Let ω_Y be a Kähler (1, 1) form on Y. To show that $\pi_*(\eta)$ is a nef class, by Theorem 9, it suffices to show that, for any irreducible variety $V \subset Y$, $\pi_*(\eta)^{\dim(V)-j} \cdot V \cdot \omega_Y^j \geq 0$ for $0 \leq j \leq \dim(V)$. We let $[V]$ be the current of integration on V. Then, by the results in §4, Ch. 3 in the book of Demailly [**12**], the current $\pi_*(\varphi)^{\dim(V)-j} \wedge [V] \wedge \omega_Y^j$ is well defined and is a positive measure with mass equal to $\pi_*(\eta)^{\dim(V)-j} \cdot V \cdot \omega_Y^j$. Thus the latter quantity is non-negative. □

2.3. *Blowup of a projective* 3*-manifold along a smooth curve.* Let $\pi : X \to Y$ be the blowup of a projective 3-manifold along a smooth curve $C \subset Y$. Let g be the genus of C. Let F be the exceptional divisor and let M be a fiber of the projection $F \to C$. We can identify F with the projective bundle $\mathbb{P}(\mathcal{E}) \to C$, where $\mathcal{E} = N_{C/Y} \to C$ is the normal vector bundle of C in Y.

Then $H^{1,1}(X)$ is generated by $\pi^*(H^{1,1}(Y))$ and F, and $H^{2,2}(X)$ is generated by $\pi^*(H^{2,2}(Y))$ and M. The intersection between F and M is $F \cdot M = -1$. The first and second Chern classes of X can be computed as

$$c_1(X) = \pi^*(c_1(Y)) - F,$$
$$c_2(X) = \pi^*(c_2(Y) + C) - \pi^* c_1(Y) \cdot F.$$

Let $[F] \to X$ be the line bundle of F in X, denoted by $e = [F]|_F$. Then (see, for example, §6, Ch. 4 in the book of Griffiths and Harris [**21**]), in F,

$$e \cdot M = -1, \;\; e \cdot e = -c_1(\mathcal{E}).$$

From the short exact sequence (SES) of vector bundles on C

$$0 \to T_C \to T_Y|_C \to \mathcal{E} \to 0,$$

it follows, by the additivity of first Chern classes, that

$$c_1(\mathcal{E}) = c_1(T_Y) \cdot C - c_1(T_C) = c_1(Y) \cdot C + 2g - 2.$$

We define

$$\gamma := c_1(Y) \cdot C + 2g - 2.$$

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

We now consider an effective curve $C_0 \subset F$ with the properties

$$\begin{aligned} C_0 \cdot C_0 &= \tau, \\ C_0 \cdot M &= \mu > 0, \\ M \cdot M &= 0. \end{aligned}$$

Any divisor on F is numerically equivalent to a linear combination of C_0 and M. We now show the following lemma.

LEMMA 13.
(a)

$$F \cdot C_0 = \frac{1}{2}\left(\gamma\mu - \frac{\tau}{\mu}\right). \tag{2.1}$$

(b)

$$F \cdot F = -\frac{1}{\mu}C_0 + \frac{1}{2}\left(\frac{\tau}{\mu^2} + \gamma\right)M.$$

(c) $\pi_*(F \cdot F) = -C$.

Proof. (a) In fact,

$$F \cdot C_0 = [F]|_{C_0} = [F]|_F \cdot C_0 = e \cdot C_0,$$

where the two expressions on the right-hand side are computed in F. On F, numerically, we can write $e = aC_0 + bM$. Then, from $-1 = e \cdot M = (aC_0 + bM) \cdot M = a\mu$, we get $a = -1/\mu$. Substituting this into $e \cdot e = -\gamma$, we obtain

$$-\gamma = e \cdot e = \left(\frac{1}{\mu}C_0 - bM\right) \cdot \left(\frac{1}{\mu}C_0 - bM\right) = \frac{\tau}{\mu^2} - 2b,$$

which implies that

$$b = \frac{1}{2}\left(\frac{\tau}{\mu^2} + \gamma\right).$$

Therefore

$$e = \frac{-1}{\mu}C_0 + \frac{1}{2}\left(\frac{\tau}{\mu^2} + \gamma\right)M.$$

Thus

$$\begin{aligned} F \cdot C_0 = e \cdot C_0 &= \left[\frac{-1}{\mu}C_0 + \frac{1}{2}\left(\frac{\tau}{\mu^2} + \gamma\right)M\right]C_0 \\ &= \frac{-\tau}{\mu} + \frac{1}{2}\left(\frac{\tau}{\mu} + \gamma\mu\right) \\ &= \frac{1}{2}\left(-\frac{\tau}{\mu} + \gamma\mu\right). \end{aligned}$$

(b) From the formula for e in the proof of (a), it is not difficult to arrive at the proof of (b).

(c) Since $C_0 \cdot M = \mu$, it follows that $\pi_*(C_0) = \mu C$. Then, from (b), we obtain (c). □

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

We end this subsection with a fact about ruled surfaces to be used later. By [**22**, Proposition 2.8 in Ch. 5], there is a line bundle $\mathcal{M} \to C$ so that the vector bundle $\mathcal{E}' = \mathcal{E} \otimes \mathcal{M}$ is normalized in the following sense: $H^0(\mathcal{E}') \neq 0$ but, for all line bundles $\mathcal{L} \to C$ with $c_1(\mathcal{L}) < 0$, $H^0(\mathcal{E}' \otimes \mathcal{L}) = 0$. A canonical section $C_0 \subset F$ can be associated to such a normalized $\mathcal{E}'$. The intersection between C_0 and M is 1. Moreover, the number

$$\tau_0 = C_0 \cdot C_0 = c_1(\mathcal{E}') = c_1(\mathcal{E}) + 2c_1(\mathcal{M}),$$

is an invariant of F.

3. *Proofs of the main results*

We make use of the following result (see, for example, [**3**, **38**]).

LEMMA 14. *Let X be a smooth projective threefold and $f : X \to X$ be an automorphism. If $\lambda_1(f) > 1$, then $\lambda_1(f)$ is irrational.*

For the convenience of the readers, we reproduce the proof of this Lemma here.

Proof. Let A be the matrix of $f^* : NS_{\mathbb{R}}(X) \to NS_{\mathbb{R}}(X)$. Then A is an integer matrix and $\lambda_1(f)$ is a real eigenvalue of A. Moreover, A is invertible and its inverse A^{-1} is the matrix of the map $(f^{-1})^* : NS_{\mathbb{R}}(X) \to NS_{\mathbb{R}}(X)$ and hence is also an integer matrix. Therefore $\det(A) = \pm 1$. Thus the characteristic polynomial $P(x)$ of A is a monic polynomial of integer coefficients and $P(0) = \pm 1$. Assume that $\lambda_1(f)$ is a rational number. Since $\lambda_1(f)$ is an algebraic integer, it follows that $\lambda_1(f)$ must be an integer. Then we can write $P(x) = (x - \lambda_1(f))Q(x)$, where $Q(x)$ is a polynomial of integer coefficients. If $\lambda_1(f) > 1$, we get a contradiction, $\pm 1 = P(0) = -\lambda_1(f)Q(0)$. □

Now we give the proofs of the main results.

Proof of Theorem 1. Since f^* preserves the cone $\mathrm{Nef}(X)$, by a Perron–Frobenius type theorem, there is a non-zero nef class η so that $f^*(\eta) = \lambda_1(f)\eta$. Similarly, there is a non-zero nef class η_- so that $(f^{-1})^*(\eta_-) = \lambda_1(f^-)\eta_-$.

Assume that $\lambda_1(f) > 1$. By the log-concavity of dynamical degrees, we also have $\lambda_1(f^{-1}) > 1$. By Lemma 14, both $\lambda_1(f)$ and $\lambda_1(f^{-1})$ are irrational. Hence both ζ and ζ_- are not in $\mathbb{R} \cdot NS_{\mathbb{Q}}(X)$. It is easy to see that

$$\zeta \cdot c_1(X)^2 = \zeta \cdot c_2(X) = 0,$$
$$\zeta_- \cdot c_1(X)^2 = \zeta_- \cdot c_2(X) = 0.$$

To prove (1) it suffices to show that either $\zeta^2 = 0$ or $\zeta_-^2 = 0$. Assume otherwise. From $\zeta^2 \neq 0$ and $f^*(\zeta^2) = \lambda_1(f)^2\zeta^2$,

$$\lambda_1(f)^2 \leq \lambda_2(f) = \lambda_1(f^{-1}).$$

Similarly, from $\zeta_-^2 \neq 0$,

$$\lambda_1(f^{-1}) \geq \lambda_1(f)^2.$$

Combining these two inequalities, we conclude that $\lambda_1(f) \geq \lambda_1(f)^4$, which contradicts the fact that $\lambda_1(f) > 1$. This completes the proof of (1).

(2) The proof of (2) is similar. □

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

Proof of Theorem 3. For the proof, it suffices to show that, for any non-zero nef ζ on $X = X_2$, either $\zeta \cdot c_1(X)^2 \neq 0$ or $\zeta \cdot c_2(X) \neq 0$.

We let $E_1, \ldots, E_n$ be the exceptional divisors of the blowup $\pi_1 : X_1 \to X_0 = \mathbb{P}^3$. Let $F_{i,j}$ be the exceptional divisors of the blowup $\pi_2 : X = X_2 \to X_1$. Then we can write

$$\zeta = \pi_2^*(\xi) - \sum_{i<j} \alpha_{i,j} F_{i,j},$$
$$\xi = \pi_1^*(u) - \sum_{l} \beta_l E_l.$$

Here u is nef on $\mathbb{P}^3$ and $\alpha_{i,j}, \beta_l \geq 0$.

For the proof of (1), it then suffices to show that $\deg(u) = 0$. From

$$c_2(X) = \pi_2^* c_2(X_1) + \sum_{i<j} \pi_2^* D_{i,j} - \sum_{i<j} \pi_2^* c_1(X_1) \cdot F_{i,j},$$

and the fact that $c_1(X_1) \cdot D_{i,j} = 0$, the condition $\zeta \cdot c_2(X) = 0$ becomes $\xi \cdot c_2(X_1) + \sum_{i<j} \xi \cdot D_{i,j} = 0$. Since $c_2(X_1) = \pi_1^*(c_2(\mathbb{P}^3))$, it follows that $\xi \cdot c_2(X_1) = 16 \deg(u)$. We also have that $\xi \cdot D_{i,j} = \deg(u) - \beta_i - \beta_j$ for every $i < j$. Therefore, we obtain

$$6 \deg(u) = - \sum_{i<j} \xi \cdot D_{i,j},$$
$$\left(6 + \frac{n(n-1)}{2}\right) \deg(u) = (n-1) \sum_{l} \beta_l.$$

From the condition $\zeta \cdot c_1(X)^2 = 0$, we obtain

$$\begin{aligned}
0 = \zeta \cdot c_1(X)^2 &= \Big(\pi_2^*(\xi) - \sum_{i<j} \alpha_{i,j} F_{i,j}\Big) \cdot \Big(\pi_2^* c_1(X_1)^2 \\
&\quad - 2 \sum_{i<j} \pi_2^* c_1(X_1) \cdot F_{i,j} + \sum_{i<j} F_{i,j}^2\Big) \\
&= \xi \cdot c_1(X_1)^2 - \sum_{i<j} \xi \cdot D_{i,j} - 2 \sum_{i<j} \alpha_{i,j} c_1(X_1) \cdot D_{i,j} \\
&\quad + \sum_{i<j} \alpha_{i,j} (c_1(X_1) \cdot D_{i,j} + 2 g_{i,j} - 2) \\
&= 22 \deg(u) - 4 \sum_{l} \beta_l + \sum_{i<j} \alpha_{i,j} (2 g_{i,j} - 2 - c_1(X_1) \cdot D_{i,j}) \\
&= 22 \deg(u) - 4 \sum_{l} \beta_l - 2 \sum_{i<j} \alpha_{i,j}.
\end{aligned}$$

In the above, $g_{i,j} = 0$ is the genus of $C_{i,j}$, and $c_1(X_1) \cdot D_{i,j} = 0$ for all $i < j$. In particular, we obtain

$$\frac{11}{2} \deg(u) \geq \sum_{l} \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u). \tag{3.1}$$

From the above inequality, we will finish by showing that $\deg(u) = 0$. We consider several cases.

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

Case 1: $n \geq 10$. From Equation (3.1), it follows, immediately, that $\deg(u) = 0$, as desired.

Case 2: $6 \leq n \leq 9$. In this case, for each of the six points $p_{i_1}, \ldots, p_{i_6}$ among n points $p_1, \ldots, p_n$, there is a unique rational normal curve $C \subset \mathbb{P}^3$ of degree three passing through the six chosen points. Let $D \subset X_1$ be the strict transform of C. Then D is different from the curves $D_{i,j}$. Therefore $\pi_2^* D$ is an effective curve, and hence

$$3 \deg(u) - \sum_{l=1}^{6} \beta_{i_l} \geq \xi \cdot D = \zeta \cdot \pi_2^*(D) \geq 0.$$

Summing over all such choices of $p_{i_1}, \ldots, p_{i_n}$ we find that

$$\frac{n}{2} \deg(u) \geq \sum_l \beta_l.$$

Combining this with

$$\sum_l \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u),$$

we obtain $\deg(u) = 0$.

Case 3: $n = 4, 5$. In this case, we use rational normal curves to obtain

$$\frac{n}{3} \deg(u) \geq \sum_l \beta_l.$$

Combining this with

$$\sum_l \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u),$$

we obtain $\deg(u) = 0$.

Case 4: $n = 1, 2, 3$. In this case, we have $n \deg(u) \geq \sum_l \beta_l$. Combining this with

$$\sum_l \beta_l = \left(\frac{6}{n-1} + \frac{n}{2}\right) \deg(u),$$

we obtain $\deg(u) = 0$. □

Proof of Theorem 4. Let $\pi_1' : X_1' \to X_0$ be the blowup at the $C_1, \ldots, C_t$. Let $F_1, \ldots, F_t$ be the exceptional divisors. Let $M_j = (\pi_1')^{-1}(p_j)$ be the preimages of the points p_j ($j = 1, \ldots, n$). These are smooth rational curves, and are among the fibers of the maps $F_1 \to C_1, \ldots, F_t \to C_t$. Let $\pi_2' : X_2' \to X_1'$ be the blowup at the curves M_j. Then X_2 is isomorphic to X_2'.

Fix a number j. Let i be such that $M_j \subset F_i$. Using

$$c_1(N_{M_j/X_1'}) = c_1(N_{M_j/F_i}) + c_1(N_{F_i/X_1}|_{M_j}) = 0 + (-1) = -1,$$

we find that

$$c_1(X_1') \cdot M_j = 1$$

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

is an odd number. Therefore, using either part (1) or part (3) of Theorem 7, for the proof of Theorem 5 it suffices to show that X_1' satisfies both Conditions A and B. To this end, we only need to show that if $\zeta \in \mathrm{Nef}(X_1')$ is such that $\zeta^2 = 0$, then $\zeta \in \mathbb{R}.NS_{\mathbb{Q}}(X_1')$.

Let $H \in NS_{\mathbb{Q}}(X_0)$ be an ample divisor. Since X_0 has Picard number 1, we can write

$$\zeta = a(\pi_1')^*(H) - \sum_j \alpha_j F_j,$$

where $a, \alpha_1, \ldots, \alpha_t \geq 0$. If $a = 0$, then, from the fact that ζ is nef, $\alpha_1 = \cdots = \alpha_t = 0$. Therefore, we may assume that $\alpha > 0$, and, after dividing by α, we may assume that $\alpha = 1$. Then, for the proof of the theorem, it suffices to show that all the numbers $\alpha_1, \ldots, \alpha_t$ are in $\mathbb{Q}$.

Since the curves C_j are pairwise disjoint, for any $i = 1, \ldots, t$,

$$\begin{aligned} 0 = \zeta^2 \cdot F_i &= \Big((\pi_1')^* H - \sum_j \alpha_j F_j\Big)^2 \cdot F_i \\ &= (\pi_1')^* H^2 \cdot F_i - 2\alpha_i (\pi_1')^* H \cdot F_i^2 + \alpha_i^2 F_i^3 \\ &= 2\alpha_i H \cdot C_i - \alpha_i^2 (c_1(X_0) \cdot C_i + 2g_i - 2), \end{aligned}$$

where g_i is the genus of C_i. We note that $H \cdot C_i$ is a positive rational number. Hence, either $\alpha_i = 0$ or

$$\alpha_i = 2H \cdot C_i / (c_1(X_0) \cdot C_i + 2g_i - 2).$$

In both cases, α_i are rational numbers as desired. □

Proof of Theorem 5. (1) Let ζ be a nef class on X_2 such that $\zeta^2 = 0$, $\zeta \cdot c_1(X) = 0$ and $\zeta \cdot c_1(X_2)^2 \leq 0$. We need to show that $\zeta \in \mathbb{R}.H^2_{alg}(X_2, \mathbb{Q})$. More strongly, we will show that ζ must be zero.

Let us denote by F_j the exceptional divisor over D_j of the blowup $\pi_2 : X_2 \to X_1$. We denote by $\pi_1 : X_1 \to X_0$ the blowup of C_0 at the points p_i.

We can write $\zeta = \pi_2^*(\xi) - \sum_j \alpha_j F_j$, where $\alpha_j \geq 0$ and ξ is a movable class on X_1. Since D_j are disjoint, by intersecting the equations $\zeta^2 = \zeta \cdot c_1(X_2) = 0$ with F_j, we find, as in [**35**], that either $\alpha_j = 0$ or

$$\xi \cdot D_j = \alpha_j c_1(X_1) \cdot D_j = \alpha_j (2g_j - 2).$$

If $\alpha_j = 0$, then

$$\xi \cdot D_j = \zeta \cdot D_j' \geq 0 = \alpha_j c_1(X_1) \cdot D_j,$$

where $D_j' \subset F_j$ is a section whose pushforward is D_j. If $\alpha_j \neq 0$, then $\xi \cdot D_j = c_1(X_1) \cdot D_j$. Therefore,

$$\begin{aligned} 0 \geq \zeta \cdot c_2(X_2) &= \Big(\pi_2^*(\xi) - \sum_j \alpha_j F_j\Big) \cdot \Big(\pi_2^* c_2(X_1) + \sum_j (\pi_2^* D_j - \pi_2^* c_1(X_j) \cdot F_j)\Big) \\ &= \xi \cdot c_2(X_1) + \sum_j (\xi \cdot D_j - \alpha_j c_1(X_1) \cdot D_j). \end{aligned}$$

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

Since each term $\xi \cdot D_j - \alpha_j c_1(X_1) \cdot D_j$ is non-negative, we find that $\xi \cdot c_2(X_1) \leq 0$. Because $c_2(X_1) = \pi_1^* c_2(X_0)$, we then get that $(\pi_1)_*(\xi) \cdot c_2(X_0) \leq 0$. Because $(\pi_1)_*(\xi)$ is movable in X_0, from the assumption on $c_2(X_0)$ we obtain $(\pi_1)_*(\xi) = 0$. From this, it follows easily that ξ and then ζ are zero.

(2) The proof is similar to that of (1). The difference is that, here, for each j, either $\alpha_j = 0$ or

$$\xi \cdot D_j - \alpha_j c_1(X_1) \cdot D_j = \frac{\alpha_j}{2}[(2g_j - 2) - c_1(X_1) \cdot D_j].$$

In the first case,

$$\xi \cdot D_j - \alpha_j c_1(X_1) \cdot D_j = \xi \cdot D_j = \zeta \cdot D'_j \geq 0,$$

where $D'_j \subset F_j$ is a section. In the second case, by the assumption $(2g_j - 2) - c_1(X_1) \cdot D_j \geq 0$, we also have $\xi \cdot D_j - \alpha_j c_1(X_1) \cdot D_j \geq 0$.

Hence,

$$0 \geq -\sum_j (\xi \cdot D_j - \alpha_j c_1(X_1) \cdot D_j) \geq \xi \cdot c_2(X_1).$$

Then we can proceed as before. □

Proof of Theorem 6. Let F be the exceptional divisor of the blowup π. Let ζ be a nef class on X. Then we can write $\zeta = \pi^*(\xi) - \alpha F$ for some $\alpha \geq 0$ and for some movable class $\xi = \pi_*(\zeta)$ on Y.

Assume that $\zeta^2 = 0$. Then,

$$0 = \zeta^2 = (\pi^*(\xi) - \alpha F)^2 = \pi^*(\xi^2) + \alpha^2 F^2.$$

Here we used that $\pi^*(\xi) \cdot F = 0$. Because the classes of $\pi^*(\xi^2)$ and F^2 are linearly independent in the (2, 2) cohomology group of X, from the above, we have that $\alpha = 0$. Then it follows that ξ is nef on Y and $\xi^2 = 0$. Moreover, since $c_1(X) = \pi^* c_1(Y) - 2F$ and $c_2(X) = \pi^* c_2(Y)$ (see [**21**, Ch. 4]),

$$\begin{aligned}
\xi \cdot c_1(Y) &= \pi_*(\pi^*(\xi) \cdot \pi^*(c_1(Y))) = \pi_*(\pi^*(\xi).(\pi^* c_1(Y) - 2F)) = \pi_*(\zeta \cdot c_1(X)),\\
\xi \cdot c_1(Y)^2 &= \zeta \cdot c_1(X)^2,\\
\xi \cdot c_2(Y) &= \zeta \cdot c_2(X).
\end{aligned}$$

Then it follows easily that, if Y satisfies one of the Conditions A and B, X also satisfies the same condition. □

Proof of Theorem 7. We will show that if Y satisfies Condition A, then X also satisfies Condition A. The proof for Condition B is similar.

Let ζ be a nef class on X. We need to show that if

$$\begin{aligned}
\zeta^2 &= 0,\\
\zeta \cdot c_1(X)^2 &\geq 0,\\
\zeta \cdot c_2(X) &\leq 0,
\end{aligned}$$

then $\zeta \in \mathbb{R} \cdot NS_{\mathbb{Q}}(X)$.

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

Let F be the exceptional divisor of the blowup $\pi : X \to Y$. We can write $\zeta = \pi^*(\xi) - \alpha F$ for some $\alpha \geq 0$. Also (see §2),

$$\begin{aligned} c_1(X) &= \pi^* c_1(Y) - F, \\ c_2(X) &= \pi^* c_2(Y) + \pi^* C - \pi^* c_1(Y) \cdot F, \\ \pi_*(F \cdot F) &= -C. \end{aligned}$$

We first consider the case $\alpha = 0$. Then, ξ is nef on Y and, moreover, $\xi^2 = 0$. In this case,

$$\begin{aligned} \pi_*(\zeta \cdot c_1(X)) &= \pi_*(\pi^*(\xi) \cdot (\pi^* c_1(Y) - F)) = \xi \cdot c_1(Y), \\ \zeta \cdot c_1(X)^2 &= \pi^*(\xi) \cdot (\pi^* c_1(Y) - F)^2 = \pi^*(\xi) \cdot (\pi^* c_1(Y)^2 - 2\pi^* c_1(Y) \cdot F + F^2) \\ &= \xi \cdot c_1(Y)^2 - \xi \cdot C, \\ \zeta \cdot c_2(X) &= \pi^*(\xi) \cdot (\pi^* c_2(Y) + \pi^* C - \pi^* c_1(Y) \cdot F) \\ &= \xi \cdot c_2(Y) + \xi \cdot C. \end{aligned}$$

Since ξ is nef and C is an effective curve, $\xi \cdot C \geq 0$. Therefore, from the assumptions $\zeta \cdot c_1(X)^2 \geq 0$ and $\zeta \cdot c_2(Y) \leq 0$, we obtain

$$\begin{aligned} \xi^2 &= 0, \\ \xi \cdot c_1(Y)^2 &= \zeta \cdot c_1(X)^2 + \xi \cdot C \geq 0, \\ \xi \cdot c_2(Y) &= \zeta \cdot c_2(X) - \xi \cdot C \leq 0. \end{aligned}$$

Since Y satisfies Condition A, by assumption, it follows that $\xi \in \mathbb{R} \cdot NS_{\mathbb{Q}}(Y)$. Then $\zeta = \pi^*(\xi) \in \mathbb{R} \cdot NS_{\mathbb{Q}}(X)$. Hence X also satisfies Condition A.

Now we show that, under the assumptions of Theorem 6, actually α must be zero. Assume otherwise, that is, that $\alpha > 0$: we will obtain a contradiction. We recall that $\gamma = c_1(Y) \cdot C + 2g - 2$. From the assumption that $\zeta^2 = 0$,

$$\begin{aligned} 0 = \zeta^2 \cdot F &= (\pi^*(\xi) - \alpha F)^2 \cdot F \\ &= \pi^*(\xi^2) \cdot F - 2\alpha\pi^*(\xi) \cdot F^2 + F^3 \\ &= \alpha\xi \cdot C - 2\alpha^2 \cdot \gamma. \end{aligned}$$

In the fourth equality, we used the results in §2. The assumption that $\alpha > 0$ implies that

$$\xi \cdot C = \alpha \cdot \gamma/2.$$

We now proceed according to parts (1)–(3) of the theorem.

(1) In this case, $c_1(Y) \cdot C$ is an odd number and $N_{C/Y}$ is decomposable. We have a SES of vector bundles over C: that is,

$$0 \to T_C \to T_Y|_C \to N_{C/Y} \to 0.$$

From this, it follows that

$$c_1(N_{C/Y}) = c_1(Y) \cdot C + 2g - 2 = \gamma.$$

Recall that F is the exceptional divisor of the blowup π. Then $F = \mathbb{P}(N_{C/X}) \to C$ is a ruled surface over C. Hence (see [**22**, Proposition 2.8 in Ch. 5]) there is a line bundle

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

$\mathcal{M}$ over C such that $\mathcal{E} = N_{C/Y} \otimes \mathcal{M}$ is normalized, in the sense that $H^0(\mathcal{E}) \neq 0$, but, for every line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) < 0$, $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$.

Let f be a fiber of the fibration $F \to C$. Then (see [**22**, Proposition 2.9 in Ch. 5]), there is a so-called zero section $C_0 \subset F$ with the properties

$$\tau := C_0 \cdot C_0 = c_1(\mathcal{E}),$$
$$C_0 \cdot f = 1.$$

Because $N_{C/Y}$ is decomposable, $\mathcal{E}$ is also decomposable. By part (a) of [**22**, Theorem 2.12 in §5], $c_1(\mathcal{E}) \leq 0$. Moreover, from

$$c_1(\mathcal{E}) = c_1(\mathcal{N}_{\mathcal{C}/\mathcal{Y}}) + 2c_1(\mathcal{M}) = c_1(Y) \cdot C + 2g - 2 + 2c_1(\mathcal{M}),$$

and the assumption that $c_1(Y) \cdot C$ is an odd number, we get that $c_1(\mathcal{E}) < 0$. Hence $\tau < 0$.

From Lemma 13,

$$C_0 = -F \cdot F + \tfrac{1}{2}(\tau + \gamma) f.$$

Now we obtain the desired contradiction. Since ζ is nef and C_0 is an effective curve, $\zeta \cdot C_0 \geq 0$. Hence,

$$\begin{aligned} 0 &\leq (\pi^*(\xi) - \alpha F) \cdot \left(-F \cdot F + \frac{1}{2}(\tau + \gamma) f\right) \\ &= \xi \cdot \pi_*(-F \cdot F) + \alpha F \cdot F \cdot F - \frac{1}{2}\alpha(\tau + \gamma) F \cdot f \\ &= \xi \cdot C - \alpha\gamma + \frac{1}{2}\alpha(\tau + \gamma) = \frac{\alpha\tau}{2} < 0. \end{aligned}$$

In the above, we used that $\pi_*(-F \cdot F) = \pi_*(C_0) = C$ (see, for example, [**35**, Lemma 4]), $F \cdot f = -1$, $F \cdot F \cdot F = -\gamma$, $\xi \cdot C = \alpha\gamma/2$, $\alpha > 0$ and $\tau = C_0 \cdot C_0 < 0$.

(2) In this case, $\gamma < 0$ and C is not the only effective curve in its cohomology class. Let D be another curve in the cohomology class of C. Since C is irreducible, we can assume that C is not contained in the support of D. Then $\pi^*(D)$ is an effective curve in X. Since ζ is nef, we obtain a contradiction

$$0 \leq \pi^*(D) \cdot \zeta = D \cdot \pi_*(\zeta) = D \cdot \xi = C \cdot \xi = \alpha\gamma/2 < 0.$$

(3) In this case, there is an irreducible hypersurface $S \subset Y$ such that $2\kappa < \mu\gamma$. Here $\kappa = S \cdot C$ and μ is the multiplicity of C in S. We now construct an effective curve $C_0 \subset F$ and use it to derive a contradiction.

The strict transform $\widetilde{S}$ of S is given by $\widetilde{S} = \pi^*(S) - \mu F$ and is an irreducible hypersurface of X. Since $\widetilde{S}$ and F are different irreducible hypersurfaces, their intersection $C_0 = \widetilde{S} \cdot F = (\pi^*(S) - \mu F) \cdot F$ is an effective curve of F. We now compute the numbers $C_0 \cdot C_0$ and $C_0 \cdot M$.

$$\begin{aligned} C_0 \cdot C_0 &= \widetilde{S}|_F \cdot \widetilde{S}|_F = \widetilde{S} \cdot \widetilde{S} \cdot F \\ &= (\pi^*(S) - \mu F) \cdot (\pi^*(S) - \mu F) \cdot F = -2\mu\pi^*(S) \cdot F \cdot F + \mu^2 F \cdot F \cdot F \\ &= 2\mu S \cdot C - \mu^2\gamma = 2\mu\kappa - \mu^2\gamma. \end{aligned}$$

Define $\tau = C_0 \cdot C_0$ and $\mu_0 = C_0 \cdot M$. Note that $\mu_0 \neq 0$, otherwise C_0 is a multiplicity of M, and hence $\pi_*(C_0) = 0$. But, from the definition of C_0, we can see that $\pi_*(C_0) = \mu C \neq 0$. Then, by Lemma 13,

$$F \cdot F = -\frac{1}{\mu_0} C_0 + \frac{1}{2}\left(\frac{\tau}{\mu_0^2} + \gamma\right) M.$$

Pushforward this by the map π and, using that $\pi_*(F \cdot F) = -C$ and $\pi_*(C_0) = \mu C$, we have that $\mu_0 = \mu$.

From the above computation, $\tau = 2\mu\kappa - \mu^2\gamma$, we obtain

$$F \cdot C_0 = \frac{1}{2}\left(\gamma\mu - \frac{\tau}{\mu}\right) = \gamma\mu - \kappa.$$

Because ζ is nef, it follows that

$$\begin{aligned} 0 \leq \zeta \cdot C_0 = (\pi^*(\xi) - \alpha F) \cdot C_0 &= \mu\xi \cdot C - \frac{\alpha}{2}\left(\gamma\mu - \frac{\tau}{\mu}\right), \\ = \frac{\alpha}{2}\gamma\mu - \frac{\alpha}{2}\left(\gamma\mu - \frac{\tau}{\mu}\right) &= \frac{\alpha}{2}\frac{\tau}{\mu} = \alpha\left(\kappa - \frac{1}{2}\gamma\mu\right). \end{aligned}$$

This contradicts the assumptions that $2\kappa < \gamma\mu$ and $\alpha > 0$. □

Proof of Theorem 8. Let ζ be a nef class on X. We need to show that if

$$\begin{aligned} \zeta^2 &= 0, \\ \zeta \cdot c_1(X) &= 0, \\ \zeta \cdot c_2(X) &\leq 0, \end{aligned}$$

then $\zeta \in \mathbb{R} \cdot NS_{\mathbb{Q}}(X)$.

Let F be the exceptional divisor of the blowup $\pi : X \to Y$. We can write $\zeta = \pi^*(\xi) - \alpha F$ for some $\alpha \geq 0$. As in the proof of Theorem 7, it suffices to show that $\alpha = 0$. We assume, otherwise, that $\alpha > 0$. Let $f \subset F$ be a fiber of the projection $F \to C$. Then

$$\begin{aligned} 0 = \zeta.\zeta &= (\pi^*(\xi) - \alpha F) \cdot (\pi^*(\xi) - \alpha F) \\ &= \pi^*(\xi \cdot \xi) - 2\alpha\pi^*(\xi) \cdot F + \alpha^2 F \cdot F, \\ 0 = \zeta \cdot c_1(X) &= (p^*(\xi) - \alpha F) \cdot (\pi^* c_1(Y) - F) \\ &= \pi^*(\xi \cdot c_1(Y)) - \pi^*(\xi) \cdot F - \pi^* c_1(Y) \cdot F + \alpha F^2. \end{aligned}$$

Intersecting both of these equations with F, and using $F \cdot F \cdot F = -\gamma$ and $\pi_*(F \cdot F) = -C$, we obtain

$$\begin{aligned} 2\alpha\xi \cdot C - \alpha^2\gamma &= 0, \\ \alpha c_1(Y) \cdot C + \xi \cdot C - \alpha\gamma &= 0. \end{aligned}$$

Then we must have $\alpha = 0$. Otherwise, dividing 2α from the first equation we would have $\xi \cdot C = \alpha\gamma/2$. Substituting this into the second equation and dividing by α we get $2c_1(Y) \cdot C = \gamma$. Hence $c_1(Y) \cdot C = 2g - 2$, which is a contradiction. □

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

4. *Examples*

4.1. *The case* $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$. The Picard number of X_0 is 2. By Künneth's formula, $H^{1,1}(X_0)$ is generated by the classes of $\mathbb{P}^2 \times \{pt\}$ and $\mathbb{P}^1 \times \mathbb{P}^1$ (here, $\{pt\}$ means a point), and $H^{2,2}(X_0)$ is generated by $\mathbb{P}^1 \times \{pt\}$ and $\{pt\} \times \mathbb{P}^1$. By Whitney's formula,

$$c_1(X_0) = 2\mathbb{P}^2 \times \{pt\} + 3\mathbb{P}^1 \times \mathbb{P}^1,$$
$$c_2(X_0) = 6\mathbb{P}^1 \times \{pt\} + 3\{pt\} \times \mathbb{P}^1.$$

Therefore, we can check that X_0 satisfies all the conditions of part (1) of Theorem 5. In particular, if $D_1, \ldots, D_n \subset X_0$ are pairwise disjoint smooth curves, and $\pi_1 : X_1 \to X_0$ is the blowup at $D_1, \ldots, D_n$, then, for any automorphism f of X_1, $\lambda_1(f) = \lambda_2(f)$. However, X_0 does not satisfy the conditions of Theorem 4: its Picard number is 2 which is greater than 1. For an appropriate choice of curves $D_1, \ldots, D_n$, the threefold X_1 has automorphisms of positive entropy. In fact, there is a rational surface S obtained from $\mathbb{P}^2$ by blowing up distinct points $p_1, \ldots, p_n \in \mathbb{P}^2$ such that S has an automorphism of positive entropy. If we choose $D_j = p_j \times \mathbb{P}^1$, then D_j are smooth rational curves which are disjoint, and X_1 has an automorphism of positive entropy.

4.2. *The case* $X_0 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This case is very similar to the case $X_0 = \mathbb{P}^2 \times \mathbb{P}^1$ above. The readers can easily redo all the (analogs of) computations and constructions in the previous section.

4.3. *The case* $X_0 =$ *a complete intersection in* $\mathbb{P}^N$. Let X_0 be a smooth projective threefold which is a complete intersection in $\mathbb{P}^N$. This means that X_0 is the intersection of smooth hypersurfaces $D_1, \ldots, D_{N-3}$ of $\mathbb{P}^N$. By Lefschetz's hyperplane theorem, X_0 has Picard number 1. We now show that X_0 satisfies the conditions of Theorem 5.

LEMMA 15. *Let ζ be a non-zero movable class in X_0. Then $\zeta \cdot c_2(X_0) > 0$.*

Proof. Let $d_1, \ldots, d_{N-3}$ be the degrees of $V_1, \ldots, V_{N-3}$. Let h be the class of a hyperplane on X. The Chern classes of the normal bundle $N_{X_0/\mathbb{P}^n}$ is given by the formula

$$c(N_{X_0/\mathbb{P}^n}) = \prod_{j=1}^{N-3} (1 + d_j h).$$

In particular,

$$c_1(N_{X_0/\mathbb{P}^n}) = \left(\sum_j d_j\right) h,$$
$$c_2(N_{X_0/\mathbb{P}^n}) = \left(\sum_{i<j} d_i d_j\right) h^2.$$

From the exact sequence

$$0 \to T_{X_0} \to T_{\mathbb{P}^4}|_{X_0} \to N_{X_0/\mathbb{P}^3} \to 0,$$

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

and, from the splitting principle for Chern classes, it follows that

$$c_1(X_0) = \Big((n+1) - \sum_j d_j\Big)h,$$
$$c_2(X_0) = \bigg(\frac{(n+1)n}{2} - \sum_{i<j} d_i d_j - (n+1)\sum_j d_j + \Big(\sum_j d_j\Big)^2\bigg)h^2.$$

We have

$$\begin{aligned}
&\frac{(n+1)n}{2} - \sum_{i<j} d_i d_j - (n+1)\sum_j d_j + \Big(\sum_j d_j\Big)^2 \\
&= \bigg[\frac{n-4}{2(n-3)}\Big(\sum_j d_j\Big)^2 - \sum_{i<j} d_i d_j\bigg] \\
&\quad + \bigg[\frac{n(n+1)}{2} + \frac{n-2}{2(n-3)}\Big(\sum_j d_j\Big)^2 - (n+1)\sum_j d_j\bigg].
\end{aligned}$$

By the Cauchy–Schwarz inequality, the first bracket on the right-hand side of the above expression is non-negative. We now show that the second bracket is positive. We define $x = \sum_j d_j$. Then x is a positive integer which is $\geq n-3$, and the second bracket is quadratic in x: that is,

$$\begin{aligned}
&\frac{n(n+1)}{2} + \frac{n-2}{2(n-3)}\Big(\sum_j d_j\Big)^2 - (n+1)\sum_j d_j \\
&= \frac{n(n+1)}{2} - (n+1)x + \frac{(n-2)}{2(n-3)}x^2 =: g(x).
\end{aligned}$$

The critical point of g is $x_0 = (n+1)(n-3)/(n-2) < n$. Hence, to show that $g(x) > 0$ for all positive integers $x \geq n-3$, it suffices to show that $g(n-3)$, $g(n-2)$, $g(n-1)$, $g(n) > 0$ for any positive integer $n \geq 4$. The latter claim can be checked by direct computation.

A movable class is, in particular, psef: that is, it can be represented by a positive closed current. Hence, if ζ is a non-zero movable class on X_0 then $\zeta \cdot c_2(X_0) > 0$. Therefore, part (1) of Theorem 5 can be applied for such a X_0. □

4.4. *A generalization of Theorem 3.* The proof of Theorem 6 shows that the conclusion is still valid in the following, more general, setting. Let $\pi_1 : X_1 \to X_0 = \mathbb{P}^3$ be the blowup at n points $p_1, \ldots, p_n$. Let $E_1, \ldots, E_n$ be the exceptional divisors. Let $D_1, \ldots, D_m \subset X_1$ be pairwise disjoint smooth curves. Let $X = X_2$ be the blowup of X_1 at $D_1, \ldots, D_m$. We define

$$\gamma := \sum_j \deg(\pi_1)_*(D_j).$$

Assume that there is $\lambda > 0$ such that, for any l,

$$\sum_j E_l \cdot D_j \leq \lambda,$$

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

and, moreover,

$$\frac{6+\gamma}{\lambda} > \frac{11}{2}.$$

In addition, assume that, for any j,

$$\left(\frac{1}{2}+\frac{1}{\lambda}\right)c_1(X_1)\cdot D_j \geq \frac{g_j-1}{2},$$

where g_j is the genus of D_j.

5. *A possible application to the Ueno's threefold*

Let $E_{\sqrt{-1}}$ be an elliptic curve with an automorphism of order four, which we denote by $\sqrt{-1}$. In [**36**], Ueno asked whether the quotient variety $E^3_{\sqrt{-1}}/\sqrt{-1}$ is rational. Campana [**7**] showed that the variety is rationally connected. Then, by a combination of the two papers [**10**, **11**], it follows that $E^3_{\sqrt{-1}}/\sqrt{-1}$ is rational. Previously, a similar construction, using, instead, an elliptic curve with an automorphism of order six, has been shown to be rational (see [**32**]).

The automorphism $\sqrt{-1}$ on $E^3_{\sqrt{-1}}$ has 8 fixed points and $64-8$ points of period two. Therefore, $E^3_{\sqrt{-1}}/\sqrt{-1}$ has $8+28=36$ singular points. Let X_4 be the minimal resolution of $E^3_{\sqrt{-1}}/\sqrt{-1}$: that is, X_4 is the blowup of $E^3_{\sqrt{-1}}/\sqrt{-1}$ at the 36 singular points. The analog of X_4 in the case of elliptic curves with an automorphism of order six is denoted by X_6.

Since X_4 is birationally equivalent to $\mathbb{P}^3$, by the weak factorization theorem (see [**1**]), X_4 can be obtained from $\mathbb{P}^3$ by a combination of smooth blowups and blowdowns. It is then natural to ask the following question.

Question 2. Can X_4 be obtained from $\mathbb{P}^3$ or $\mathbb{P}^2\times\mathbb{P}^1$ or $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ by a finite composition of smooth blowups only?

This question is interesting in several aspects. First, the two dimensional analog, that is, the minimal resolution of $E^2_{\sqrt{-1}}/\sqrt{-1}$, has been shown to be a finite composition of point blowups starting from $\mathbb{P}^1\times\mathbb{P}^1$ in [**7**]. Second, the final proof that X_4 is rational in [**10**] is rather abstract. Hence, if the answer to Question 2 is affirmative, it will give an explicit proof that X_4 is rational.

We note that the smooth threefold X_4 has automorphisms f coming from the complex torus $E^3_{\sqrt{-1}}$ with $\lambda_1(f)\neq\lambda_2(f)$. Therefore, from the discussion in the introduction of this paper, it is plausible to conclude that the answer to Question 2 is negative. The purpose of this section is to give more weight to this speculation.

We first show that, if the answer for Question 2 is affirmative, then centers of the individual blowups must be points or smooth rational curves. In the following, for any quasi-projective variety Z we will denote the Euler characteristic with compact support by $\chi(Z)$. For a smooth projective manifold Z, we denote the Picard number of Z by $\rho(Z)$.

THEOREM 16. *Let X_0 be any smooth projective threefold. Assume that X_4 can be obtained from X_0 by a finite composition of smooth blowups. Then the curves which are centers of the blowups must be smooth rational curves. The same conclusion holds if we replace X_4 by X_6.*

Proof. We give the proof for X_4 only. We divide the proof into several steps.

Step 1. We claim that $\chi(X_4) = 92$. In fact, first we consider the quotient map $\sigma : E^3_{\sqrt{-1}} \to E^3_{\sqrt{-1}}/\sqrt{-1}$. Let $A \subset E^3_{\sqrt{-1}}$ be the set of fixed points of $\sqrt{-1}$, and $B \subset E^3_{\sqrt{-1}}$ be the set of points of period two of $\sqrt{-1}$. As mentioned before, the cardinals of $|A|$ and $|B|$ are 8 and 56, and the cardinals of $\sigma(A)$ and $\sigma(B)$ are 8 and 28. Since the map $\pi : E^3_{\sqrt{-1}} - (A \cup B) \to E^3_{\sqrt{-1}}/\sqrt{-1} - (\sigma(A) \cup \sigma(B))$ is a $4:1$ map, by the excision property,

$$\begin{aligned}\chi(E^3_{\sqrt{-1}}/\sqrt{-1} - (\sigma(A) \cup \sigma(B))) &= \chi(E^3_{\sqrt{-1}} - (A \cup B))/4 \\ &= (\chi(E^3_{\sqrt{-1}}) - \chi(A \cup B))/4 \\ &= (0 - 64)/4 = -16.\end{aligned}$$

Hence, by the excision property, $\chi(E^3_{\sqrt{-1}}/\sqrt{-1}) = -16 + 36 = 20$.

Next, we consider the blowup $\pi : X_4 \to E^3_{\sqrt{-1}}/\sqrt{-1}$. This map has 36 exceptional divisors, and each is a $\mathbb{P}^2$. Since $\chi(\mathbb{P}^2) = 3$ and the blowup map is $1:1$ outside exceptional divisors, arguing as above, we obtain

$$\chi(X_4) = (\chi(E^3_{\sqrt{-1}}/\sqrt{-1}) - 36\chi(pt)) + 36 \times \chi(\mathbb{P}^2) = 20 - 36 + 36 \times 3 = 92.$$

Here pt denotes a point.

Step 2. We claim that $\rho(X_4) = 45$ and $h^3(X_4) = 0$. To see this, we can proceed as follows. The complex torus $E^3_{\sqrt{-1}}$ has Picard number 9. We can construct X_4, alternatively, as follows. We let Y be the blowup of $E^3_{\sqrt{-1}}$ at the 64 points of order one or two. Then the automorphism $\sqrt{-1}$ lifts to an automorphism of Y, which we still denote by $\sqrt{-1}$. Then $X_4 = Y/\sqrt{-1}$. It follows that the Picard group of X_4 is generated by the pushforward of the Picard group of Y. For generators of the Picard group of Y, we can take the generators of $E^3_{\sqrt{-1}}$ plus the exceptional divisors of the blowup $Y \to E^3_{\sqrt{-1}}$. The pushforward to X_4 of these generators are 45 hypersurfaces of X_4. Therefore, $\rho(X_4) \leq 45$. Moreover, from Step 1 and Serre's result mentioned above,

$$92 = \chi(X_4) = 2 + 2\rho(X_4) - h^3(X_4) \leq 92.$$

Since equality occurs, we conclude that $\rho(X_4) = 45$ and $h^3(X_4) = 0$.

From the fact that $h^3(X_4) = 0$, the conclusion of the Theorem follows immediately, since, if we blow up a smooth threefold at an irrational smooth curve, h^3 increases. □

Now we show how Theorem 16 and Theorems 6–8 almost give the proof that the answer to Question 2 is negative. In fact, let X_0 be $\mathbb{P}^3$, $\mathbb{P}^2 \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X_0 satisfies Condition B, while X_4 does not satisfy Condition B. Assume that X_4 is a finite composition of smooth blowups starting from X_0. Let $\pi_j : Z_{j+1} \to Z_j$ be an individual blowup in the sequence, where Z_j satisfies Condition B. If π_j is a point blowup, then, by Theorem 6, Z_{j+1} also satisfies Condition B. If π_j is the blowup of a smooth curve $C \subset Z_j$, then, by Theorem 16, C must be a smooth rational curve. If $c_1(Z_j) \cdot C \neq 2g - 2 = -2$, then, by Theorem 8, Z_{j+1} also satisfies Condition B. The remaining case is when $c_1(Z_j) \cdot C = -2$. But, in this case, half of the conditions of part (2) of Theorem 7 are satisfied. The only

https://doi.org/10.1017/etds.2016.4 Published online by Cambridge University Press

condition that is missing is that C is not the only effective curve in its cohomology class. Using part (1) of Theorem 6, we can also show that if the normal vector bundle N_{C/Z_j} is not isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, then Z_{j+1} also satisfies Condition B.

Acknowledgements. The author is grateful to Tien-Cuong Dinh for his suggestion that the answer to Question 1 is negative. The author has benefited from helpful discussions and correspondence with Ekaterina Amerik, Turgay Bayraktar, Eric Bedford, Frederic Campana, Fabrizio Catanese, Igor Dolgachev, Mattias Jonsson, Jan-Li Lin, Viet-Anh Nguyen, Keiji Oguiso, Yuri Prokhorov, Roland Roeder and Konstantin Shramov. The author also would like to thank the referee for useful suggestions which helped to improve the paper.

REFERENCES

[**1**] D. Abramovich, K. Karu, K. Matsuki and J. Wlodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.* **15**(3) (2002), 531–572.
[**2**] T. Bayraktar and S. Cantat. Constraints on automorphism groups of higher dimensional manifolds. *J. Math. Anal. Appl.* **405** (2013), 209–213.
[**3**] E. Bedford. The dynamical degrees of a mapping. *Proceedings of the Workshop Future Directions in Difference Equations (Colección Congresos, 69).* Servizo de Publicacións da Universidade de Vigo, Vigo, 2011, pp. 3–13.
[**4**] E. Bedford and K.-H. Kim. Dynamics of rational surface automorphisms: rotation domains. *Amer. J. Math.* **134**(2) (2012), 379–405.
[**5**] E. Bedford and K.-H. Kim. Continuous families of rational surface automorphisms with positive entropy. *Math. Ann.* **348**(3) (2010), 667–688.
[**6**] E. Bedford and K.-H. Kim. Dynamics of rational surface automorphisms: linear fractional recurrences. *J. Geom. Anal.* **19**(3) (2009), 553–583.
[**7**] F. Campana. Remarks on an example of K. Ueno. *Series of Congress Reports, Classification of Algebraic Varieties*. Eds. C. Faber, G. van der Geer and E. Looijenga. European Mathematical Society, Zürich, 2011, pp. 115–121.
[**8**] S. Cantat. Dynamique des automorphismes des surfaces K3. *Acta Math.* **187**(1) (2001), 1–57.
[**9**] S. Cantat and I. Dolgachev. Rational surfaces with a large group of automorphisms. *J. Amer. Math. Soc.* **25**(3) (2012), 863–905.
[**10**] J.-L. Colliot-Thélène. Rationalité d'un fibré en coniques. *Manuscripta Math.* **147**(3–4) (2015), 305–310.
[**11**] F. Catanese, K. Oguiso and T. T. Truong. Unirationality of Ueno-Campana's threefold. *Manuscripta Math.* **145**(3–4) (2014), 399–406.
[**12**] J.-P. Demailly. Complex analytic and differential geometry, Online book, version of Thursday 10 September 2009.
[**13**] J.-P. Demailly and M. Paun. Numerical characterization of the Kähler cone of a compact Kähler manifold. *Ann. of Math. (2)* **159**(3) (2004), 1247–1274.
[**14**] J. Déserti and J. Grivaux. Automorphisms of rational surfaces with positive entropy. *Indiana Univ. Math. J.* **60**(5) (2011), 1589–1622.
[**15**] J. Diller. Cremona transformations, surface automorphisms, and plane cubics. *Michigan Math. J.* **60**(2) (2011), 409–440 . With an appendix by Igor Dolgachev.
[**16**] T.-C. Dinh. Tits alternative for automorphism groups of compact Kähler manifolds. *Acta Math. Vietnam.* **37**(4) (2012), 513–529.
[**17**] T.-C. Dinh and V.-A. Nguyen. Comparison of dynamical degrees for semi-conjugate meromorphic maps. *Comment. Math. Helv.* **86**(4) (2011), 817–840.
[**18**] T.-C. Dinh, V.-A. Nguyen and T. T. Truong. On the dynamical degrees of meromorphic maps preserving a fibration. *Comm. Contemp. Math.* **14**(6) (2012),1250042 ,18 pp.
[**19**] T.-C. Dinh and N. Sibony. Une borne supérieure pour l'entropie topologique d'une application rationnelle. *Ann. of Math. (2)* **161**(3) (2005), 1637–1644.
[**20**] M. Gromov. On the entropy of holomorphic maps. *Enseign. Math.* **49** (2003), 217–235, Manuscript (1977).

[**21**] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley, New York, 1978.
[**22**] R. Hartshorne. *Algebraic Geometry (Graduate Texts in Mathematics, 52)*. Springer, New York, 1977.
[**23**] C. McMullen. Dynamics with small entropy on projective K3 surfaces. *Preprint*.
[**24**] C. McMullen. K3 surfaces, entropy and glue. *J. Reine Angew. Math.* **658** (2011), 1–25.
[**25**] C. McMullen. Dynamics on blowups of the projective plane. *Publ. Math. Inst. Hautes Études Sci.* **105** (2007), 49–89.
[**26**] C. McMullen. Dynamics on K3 surfaces: Salem numbers and Siegel disks. *J. Reine Angew. Math.* **545** (2002), 201–233.
[**27**] K. Oguiso and F. Perroni. Automorphisms of rational manifolds of positive entropy with Siegel disks. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **22**(4) (2011), 487–504.
[**28**] K. Oguiso. Free automorphisms of positive entropy on smooth Kähler surfaces. *Algebraic Geometry in East Asia, Taipei 2011 (Advanced Studies in Pure Mathematics, 65)*. Mathematical Society of Japan, Tokyo, 2015, pp. 187–199.
[**29**] K. Oguiso. The third smallest Salem number in automorphisms of K3 surfaces. *Algebraic Geometry in East Asia, Seoul 2008 (Advanced Studies in Pure Mathematics, 60)*. Mathematical Society, Japan, Tokyo, 2010, pp. 331–360.
[**30**] K. Oguiso. A remark on dynamical degrees of automorphisms of hyperkähler manifolds. *Manuscripta Math.* **130**(1) (2009), 101–111.
[**31**] K. Oguiso. Automorphisms of hyperkähler manifolds in the view of topological entropy. *Algebraic Geometry (Contemporary Mathematics, 422)*. American Mathematical Society, Providence, RI, 2007, pp. 173–185.
[**32**] K. Oguiso and T. T. Truong. Explicit examples of rational and Calabi-Yau threefolds with primitive automorphisms of positive entropy. *J. Math. Sci. Univ. Tokyo* **22**(1) (2015), 361–385 (Kodaira centennial issue).
[**33**] P. Reschke. Salem numbers and automorphisms of complex surfaces. *Math. Res. Lett.* **19**(2) (2012), 475–482.
[**34**] T. T. Truong. On automorphisms of blowups of projective manifolds. *Preprint*, arXiv:1301.4957.
[**35**] T. T. Truong. On automorphisms of blowups of $\mathbb{P}^3$. *Preprint*, arXiv:1202.4224.
[**36**] K. Ueno. Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack. *(Lecture Notes in Mathematics, 439)*. Springer, Berlin, 1975.
[**37**] Y. Yomdin. Volume growth and entropy. *Israel J. Math.* **57** (1987), 285–300.
[**38**] D.-Q. Zhang. The g-periodic subvarieties for an automorphism g of positive entropy on a compact Kähler manifold. *Adv. Math.* **223**(2) (2010), 405–415.
[**39**] D.-Q. Zhang. Dynamics of automorphisms on projective complex manifolds. *J. Differential Geom.* **82**(3) (2009), 691–722.
[**40**] D.-Q. Zhang. Automorphism groups and anti-pluricanonical curves. *Math. Res. Lett.* **15**(1) (2008), 163–183.