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# On approximating minimizers of convex functionals with a convexity constraint by singular Abreu equations without uniform convexity

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We revisit the problem of approximating minimizers of certain convex functionals subject to a convexity constraint by solutions of fourth order equations of Abreu type. This approximation problem was studied in previous articles of Carlier–Radice (Approximation of variational problems with a convexity constraint by PDEs of Abreu type. *Calc. Var. Partial Differential Equations* **58** (2019), no. 5, Art. 170) and the author (Singular Abreu equations and minimizers of convex functionals with a convexity constraint, arXiv:1811.02355v3, *Comm. Pure Appl. Math.*, to appear), under the uniform convexity of both the Lagrangian and constraint barrier. By introducing a new approximating scheme, we completely remove the uniform convexity of both the Lagrangian and constraint barrier. Our analysis is applicable to variational problems motivated by the original 2D Rochet–Choné model in the monopolist's problem in Economics, and variational problems arising in the analysis of wrinkling patterns in floating elastic shells in Elasticity.

*Keywords:* Singular Abreu equation; Convex functional; Convexity constraint; Second boundary value problem; Rochet–Choné model; Wrinkling patterns

2010 *Mathematics subject classification:* 49K30; 35B40; 35J40; 35J96

## 1. Introduction

In this note, we revisit the problem of approximating minimizers of certain convex functionals subject to a convexity constraint by solutions of fourth order equations of Abreu type. This problem was investigated in previous studies by Carlier–Radice [3] and the author [6], under the uniform convexity of both the Lagrangian and constraint barrier. Here, by introducing a new approximating scheme, we completely remove the uniform convexity of both the Lagrangian and constraint barrier. We start by recalling this problem.

### 1.1. Approximating minimizers of convex functionals subject to a convexity constraint

Let $\Omega_0$ be a bounded, open, smooth and convex domain in $\mathbb{R}^n$ $(n \geqslant 2)$. Let $\Omega$ be a bounded, open, smooth uniformly convex domain containing $\overline{\Omega_0}$. Let $\varphi$ be a convex and smooth function defined in $\overline{\Omega}$. Let $F(x, z, p) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth

Lagrangian which is convex in each of the variables $z \in \mathbb{R}$ and $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$. Consider the following variational problem with a convexity constraint:

$$\inf_{u \in \bar{S}[\varphi, \Omega_0]} \int_{\Omega_0} F(x, u(x), Du(x))\,\mathrm{d}x \tag{1.1}$$

where

$$\bar{S}[\varphi, \Omega_0] = \{u : \Omega_0 \to \mathbb{R} \mid u \text{ is convex,} \\ \text{u admits a convex extension to } \Omega \text{ such that } u = \varphi \text{ on } \Omega \setminus \Omega_0\}. \tag{1.2}$$

Note that elements of $\bar{S}[\varphi, \Omega_0]$ are Lipschitz continuous with Lipschitz constants bound from above by $\|D\varphi\|_{L^\infty(\Omega)}$ and hence $\bar{S}[\varphi, \Omega_0]$ is compact in the topology of uniform convergence. Under quite general assumptions on the convexity and growth of the Lagrangian $F$, one can show that (1.1) has a minimizer in $\bar{S}[\varphi, \Omega_0]$.

Due to the intrinsic difficulty in the convexity constraint, as elucidated in [**3**, **6**], for practical purposes such as numerical computations, one wonders if minimizers of (1.1) can be well approximated in the uniform norm by solutions of some higher order equations whose global well-posedness can be established. The approximating schemes proposed in [**3**, **6**] use the second boundary value problem of fourth order equations of Abreu type which we now would like to make more precise.

Let $\psi$ be a smooth function in $\overline{\Omega}$ with $\inf_{\partial\Omega} \psi > 0$. Fix $0 \leqslant \theta < 1/n$. For each $\varepsilon > 0$, consider the following second boundary value problem for a uniform convex function $u_\varepsilon$:

$$\begin{cases} \varepsilon \displaystyle\sum_{i,j=1}^{n} U_\varepsilon^{ij}(w_\varepsilon)_{ij} = f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2 u_\varepsilon; \varphi) & \text{in } \Omega, \\ w_\varepsilon = (\det D^2 u_\varepsilon)^{\theta - 1} & \text{in } \Omega, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \\ w_\varepsilon = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here and what follows, $U_\varepsilon = (U_\varepsilon^{ij})_{1 \leqslant i,j \leqslant n}$ is the cofactor matrix of the Hessian matrix

$$D^2 u_\varepsilon = ((u_\varepsilon)_{ij})_{1 \leqslant i,j \leqslant n} \equiv \left(\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}\right)_{1 \leqslant i,j \leqslant n}$$

and

$$f_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x), D^2 u_\varepsilon(x); \varphi(x)) \\ = \begin{cases} \dfrac{\partial F}{\partial z}(x, u_\varepsilon(x), Du_\varepsilon(x)) - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}(x, u_\varepsilon(x), Du_\varepsilon(x))\right) & x \in \Omega_0, \\ \dfrac{1}{\varepsilon}(u_\varepsilon(x) - \varphi(x)) & x \in \Omega \setminus \Omega_0. \end{cases} \tag{1.4}$$

The fourth order expression $U^{ij}[(\det D^2 u)^{\theta-1}]_{ij}$ appears in several geometric contexts including Kähler geometry (such as the Abreu's equation when $\theta = 0$; see

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[1]) and affine geometry (such as the affine maximal surface equation when $\theta = 1/(n+2)$; see [9]). When the Lagrangian $F$ depends on the gradient variables, the right-hand side of (1.4) contains the Hessian $D^2u_\varepsilon$ of $u_\varepsilon$. Without further regularity for the convex function $u_\varepsilon$, the Hessian $D^2u_\varepsilon$ can be just a measure-valued matrix. Thus, as in [6], we call fourth order equations of the type (1.3)–(1.4) singular Abreu equations.

We note that the first two equations of system (1.3)–(1.4) are critical points, with respect to compactly supported variations, of the following functional:

$$J_\varepsilon(v) = \int_{\Omega_0} F(x, v(x), Dv(x))\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0} (v-\varphi)^2\,\mathrm{d}x - \varepsilon\int_\Omega \frac{(\det D^2 v)^\theta - 1}{\theta}\,\mathrm{d}x.$$

When $\theta = 0$, the integral $\int_\Omega ((\det D^2v)^\theta - 1)/\theta\,\mathrm{d}x$ is replaced by $\int_\Omega \log\det D^2 v\,\mathrm{d}x$. The requirement $0 \leqslant \theta < 1/n$ is to make $J_\varepsilon$ a convex functional.

The function $f_\varepsilon$ defined by (1.4) is not continuous in general; this is usually due to the jump discontinuity through $\partial\Omega_0$. Thus, the best global regularity one can expect for a solution to (1.3)–(1.4) is $W^{4,p}(\Omega)$ for all $p < \infty$.

The questions we would like to ask are the following:

(**Q1**) *Does system (1.3)–(1.4) have a uniformly convex solution $u_\varepsilon \in W^{4,p}(\Omega)$ (for all $p < \infty$) for each $\varepsilon > 0$ small?*

(**Q2**) *If the answer to **Q1** is yes, does $u_\varepsilon$ converge uniformly on compact subsets of $\Omega$ to a minimizer $u \in \bar{S}[\varphi, \Omega_0]$ of problem (1.1)?*

Another way to rephrase the above questions is to study limiting properties of solutions, if any, to singular Abreu equations of the type (1.3)–(1.4) when $\varepsilon \to 0$.

The positive answers to questions **Q1** and **Q2** above have been given in [3, theorem 5.3] and [6, theorem 2.3] when $F$ and $\varphi$ satisfy certain structural conditions. These studies require the uniform convexity of the Lagrangian $F(x, z, p)$ with respect to $z$ and also the uniform convexity of the barrier constraint $\varphi$. We recall these theorems here.

THEOREM 1.1 [3, theorem 5.3]. *Let $\theta = 0$. Let $\psi$ be a smooth function in $\overline{\Omega}$ with $\inf_{\partial\Omega}\psi > 0$. Assume that $\varphi$ is uniformly convex in $\overline{\Omega}$ and that $F(x, z, p) = F^0(x, z)$ where $F^0$ is uniformly convex with respect to $z$, that is, $f^0(x, z) := \partial F^0(x, z)/\partial z$ satisfies for some $\alpha > 0$*

$$(f^0(x, z) - f^0(x, \tilde{z}))(z - \tilde{z}) \geqslant \alpha|z - \tilde{z}|^2 \quad \textit{for all } x \in \Omega_0 \textit{ and all } z, \tilde{z} \in \mathbb{R}. \tag{1.5}$$

*Assume that, for some continuous and increasing function $\eta : [0, \infty) \to [0, \infty)$, we have*

$$|f^0(x, z)| \leqslant \eta(|z|) \quad \textit{for all } x \in \Omega_0 \textit{ and all } z \in \mathbb{R}. \tag{1.6}$$

*Then, for $\varepsilon > 0$ small, system (1.3)–(1.4) has a uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ for all $q \in (n, \infty)$. Moreover, when $\varepsilon \to 0$, $u_\varepsilon$ converges uniformly on compact subsets of $\Omega$ to the unique minimizer $u \in \bar{S}[\varphi, \Omega_0]$ of (1.1).*

THEOREM 1.2 [**6**, Theorem 2.3]. *Assume $n = 2$ and $0 \leqslant \theta < 1/n$. Let $\psi$ be a smooth function in $\overline{\Omega}$ with $\inf_{\partial\Omega} \psi > 0$. Assume that $\varphi$ is uniformly convex in $\overline{\Omega}$ and that $F(x,z,p) = F^0(x,z) + F^1(x,p)$ where $F^0$ satisfies (1.5) and (1.6). Suppose that for some $M \geqslant 0$, we have for all $p \in \mathbb{R}^n$,*

$$0 \leqslant F^1_{p_ip_j}(x,p) \leqslant MI_n; |F^1_{p_ix_i}(x,p)| \leqslant M(|p|+1) \quad \textit{for all } x \in \Omega_0 \textit{ and for each } i.$$

*Then, for $\varepsilon > 0$ small and $\alpha > 0$ sufficiently large, system (1.3)–(1.4) has a uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ for all $q \in (n,\infty)$. Moreover, for $\alpha$ sufficiently large, $u_\varepsilon$ converges, when $\varepsilon \to 0$, uniformly on compact subsets of $\Omega$ to the unique minimizer $u \in \bar{S}[\varphi,\Omega_0]$ of (1.1).*

In theorem 1.2 and what follows, we use the following notation: $I_n$ is the identity $n \times n$ matrix and

$$F^1_{p_ip_j}(x,p) = \frac{\partial^2 F^1(x,p)}{\partial p_i \partial p_j}; \quad F^1_{p_ix_j}(x,p) = \frac{\partial^2 F^1(x,p)}{\partial p_i \partial x_j};$$
$$\nabla_p F^1(x,p) = \left(\frac{\partial F^1(x,p)}{\partial p_1}, \ldots, \frac{\partial F^1(x,p)}{\partial p_n}\right).$$

REMARK 1.3. *Inspecting the proof of theorem 5.3 in [**3**], we find that theorem 1.1 also holds for all $\theta \in [0,1/n)$.*

From the variational analysis and practical models in Economics and Elasticity to be described below, it would be interesting to remove the uniform convexity assumptions in theorems 1.1 and 1.2.

(**Q3**) *Can we remove the uniform convexity assumptions on $F$ and $\varphi$ in theorems 1.1 and 1.2*?

### 1.2. Examples with non-uniformly convex Lagrangians

Our examples of convex functionals subject to a convexity constraint arise in the Rochet–Choné model of the monopolist's problem in Economics and variational problems arising in the analysis of wrinkling patterns in floating elastic shells in Elasticity. In these models, the Lagrangians $F(x,z,p)$ are convex but not uniformly convex with respect to $z$.

*The Rochet–Choné model.* The analysis in [**6**] is applicable to the 2D Rochet–Choné model perturbed by a strictly convex lower order term. It is not known if the analysis in [**6**] is applicable to the original Rochet–Choné model [**7**] where

$$F(x,z,p) = \frac{1}{2}|p|^2\gamma(x) - x \cdot p\gamma(x) + z\gamma(x).$$

Rochet–Choné modelled the monopolist problem in product line design with quadratic cost using maximization of the functional

$$\Phi(u) = \int_{\Omega_0} \left\{x \cdot Du(x) - \frac{1}{2}|Du(x)|^2 - u(x)\right\}\gamma(x)\,\mathrm{d}x.$$

Here $\Phi(u)$ is the monopolist's profit; $u$ is the buyers' indirect utility function with bilinear valuation; $\Omega_0 \subset \mathbb{R}^n$ is the collection of types of agents; $\gamma$ is the relative

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frequency of different types of agents in the population. The function $\gamma$ is assumed to be nonnegative, bounded and Lipschitz continuous, that is,

$$0 \leqslant \gamma \leqslant C \quad \text{and} \quad \|D\gamma\|_{L^\infty(\Omega_0)} \leqslant C.$$

For a consumer of type $x \in \Omega_0$, the indirect utility $u(x)$ is computed via the formula

$$u(x) = \max_{q \in Q}\{x \cdot q - p(q)\}$$

where $Q \subset \mathbb{R}^n$ is the product line and $p : Q \to \mathbb{R}$ is a price schedule that the monopolist needs to both design to maximize her total profit $\Phi$. Clearly, $u$ is convex and maximizing $\Phi(u)$ is equivalent to minimizing $\int_{\Omega_0} F(x, u(x), Du(x))$ among all convex functions $u$. For economic reasons, there are other conditions for $u$ outside $\Omega_0$; see [**7**] and also [**2**] for more details.

*Thin elastic shells.* We also note that, in certain applications where $F$ is independent of the gradient variables, $F$ can be non-uniformly convex in $z$. A particular example arises in the analysis of wrinkling patterns in floating elastic shells by Tobasco [**8**]. As discussed in [**8**, § 1.2.3], describing the leading order behaviour of weakly curved floating shells lead to limiting problems which are dual to problems of the type:

Given a smooth function $q : \overline{\Omega_0} \subset \mathbb{R}^2 \to \mathbb{R}$, minimize

$$\int_{\Omega_0} \left(\frac{|x|^2}{2} - u(x)\right) \det D^2 q(x)\, \mathrm{d}x \tag{1.7}$$

over the set

$$\left\{u \text{ convex in } \mathbb{R}^2,\ u = \frac{|x|^2}{2} \text{ in } \mathbb{R}^2 \setminus \Omega_0\right\}.$$

Optimal functions in (1.7) are called optimal *Airy potential* in [**8**]. In this example,

$$F(x, z, p) = \left(\frac{|x|^2}{2} - z\right) \det D^2 q(x).$$

### 1.3. The main results

In this note, we answer question **Q3** at the end of § 1.1 by completely removing both the uniform convexity of $F$ with respect to $z$ and the uniform convexity of $\varphi$. To do this, we introduce a new approximating scheme, slightly different from (1.3)–(1.4).

As in [**6**] and motivated by the Rochet–Choné model, we consider Lagrangians of the form:

$$F(x, z, p) = F^0(x, z) + F^1(x, p).$$

Let

$$f^0(x, z) := \frac{\partial F^0(x, z)}{\partial z}.$$

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We assume the following convexity and growth assumptions on $F^0$ and $F^1$. For some *nonnegative* constant $C_*$:

$$(f^0(x,z) - f^0(x,\tilde{z}))(z-\tilde{z}) \geqslant 0; |f^0(x,z)| \leqslant \eta(|z|) \text{ for all } x \in \Omega_0 \text{ and all } z, \tilde{z} \in \mathbb{R} \tag{1.8}$$

where $\eta : [0,\infty) \to [0,\infty)$ is a continuous and increasing function. Furthermore, for all $p \in \mathbb{R}^n$

$$0 \leqslant F^1_{p_i p_j}(x,p) \leqslant C_* I_n; |F^1_{p_i x_i}(x,p)| \leqslant C_*(|p|+1) \text{ for all } x \in \Omega_0 \text{ and for each } i. \tag{1.9}$$

Let $\rho$ be a strictly convex defining function of $\Omega$, that is,

$$\Omega := \{x \in \mathbb{R}^n : \rho(x) < 0\}, \ \rho = 0 \text{ on } \partial\Omega \text{ and } D\rho \neq 0 \text{ on } \partial\Omega. \tag{1.10}$$

Let

$$C_\varphi = \begin{cases} 0 & \text{if } \varphi \text{ is uniformly convex in } \overline{\Omega}, \\ 1 & \text{otherwise.} \end{cases} \tag{1.11}$$

For $\varepsilon > 0$, consider the following second boundary value problem for a uniform convex function $u_\varepsilon$:

$$\begin{cases} \varepsilon \displaystyle\sum_{i,j=1}^n U^{ij}_\varepsilon (w_\varepsilon)_{ij} = f_\varepsilon\left(\cdot, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon; \varphi + C_\varphi \varepsilon^{1/3n^2}(\mathrm{e}^\rho - 1)\right) & \text{in } \Omega, \\ w_\varepsilon = (\det D^2 u_\varepsilon)^{\theta-1} & \text{in } \Omega, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \\ w_\varepsilon = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

Here,

$$\begin{aligned} &f_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x), D^2u_\varepsilon(x); \varphi(x) + C_\varphi \varepsilon^{1/3n^2}(\mathrm{e}^{\rho(x)} - 1)) \\ &\quad = \begin{cases} \dfrac{\partial F}{\partial z}(x, u_\varepsilon(x), Du_\varepsilon(x)) - \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial F}{\partial p_i}(x, u_\varepsilon(x), Du_\varepsilon(x))\right) & x \in \Omega_0, \\ \dfrac{1}{\varepsilon}\left(u_\varepsilon(x) - \varphi(x) - C_\varphi \varepsilon^{1/3n^2}(\mathrm{e}^{\rho(x)} - 1)\right) & x \in \Omega \setminus \Omega_0. \end{cases} \end{aligned} \tag{1.13}$$

Our main theorem states as follows.

THEOREM 1.4. *Let $\Omega_0$ and $\Omega$ be bounded, open, smooth and convex domains in $\mathbb{R}^n$ ($n \geqslant 2$) such that $\Omega$ is uniformly convex and contains $\overline{\Omega_0}$. Fix $0 \leqslant \theta < 1/n$. Let $\psi$ be a smooth function in $\overline{\Omega}$ with $\inf_{\partial\Omega} \psi > 0$. Let $\varphi$ be a convex and smooth function defined in $\overline{\Omega}$. Assume that (1.8) and (1.9) are satisfied. If $F^1 \not\equiv 0$ then we assume further that $n = 2$. Then the following hold.*

(i) *For $\varepsilon > 0$ small, system (1.12)–(1.13) has a uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ for all $q \in (n,\infty)$.*

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(ii) *For $\varepsilon > 0$ small, let $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ be a solution to (1.12)–(1.13). After extracting a subsequence, $u_\varepsilon$ converges uniformly on compact subsets of $\Omega$ to a minimizer $u \in \bar{S}[\varphi, \Omega_0]$ of (1.1).*

Several remarks are in order.

Remark 1.5. *Without the uniform convexity of $F$ with respect to $z$, minimizers of problem (1.1) with the convexity constraint (1.2) can be non-unique. As such, each convergent subsequence of $\{u_\varepsilon\}$ converges to a minimizer of (1.1) as stated in theorem 1.4(ii). It would be interesting to investigate whether the approximating scheme (1.12)–(1.13) selects a distinguished minimizer of problem (1.1) when it has several minimizers.*

Remark 1.6. *When $\varphi$ is not uniformly convex, the addition of $\varepsilon^{1/3n^2}(e^\rho - 1)$ to $\varphi$ is to make the new function 'sufficiently' uniformly convex. The choice of the exponent $1/3n^2$ (or any positive number not larger than this) is motivated by the need to establish uniform bounds for $u_\varepsilon$ in the a priori estimates for solutions to (1.12)–(1.13); see (3.2) and (3.6).*

Remark 1.7. *Let $G(t)$ be an antiderivative of $t^{\theta-1}$. One of the crucial information in the proof of the convergence of solutions of (1.3)–(1.4) to a minimizer of (1.1) is a variant of the estimate*

$$\liminf_{\varepsilon} \varepsilon \int_\Omega G(\det D^2 v)dx \geqslant 0 \quad \text{for all } v \in \bar{S}[\varphi, \Omega_0]. \tag{1.14}$$

*(a) When $\varphi$ is uniformly convex in $\overline{\Omega}$, for any $v \in \bar{S}[\varphi, \Omega_0]$, (1.14) was shown to be true with $v$ being replaced by $(1-\varepsilon)v + \varepsilon\varphi \in \bar{S}[\varphi, \Omega_0]$ in [**3**, proposition 3.5] and [**6**, inequality (5.15)]. When the uniform convexity of $\varphi$ is removed, unless $\theta > 0$, (1.14) might fail for all $v \in \bar{S}[\varphi, \Omega_0]$ as in the case $\varphi$ being a constant for which $\bar{S}[\varphi, \Omega_0] = \{\varphi\}$.*

*(b) On the other hand, for any convex $\varphi$, estimate (1.14) holds for $v \in \bar{S}[\varphi, \Omega_0]$ being replaced by $v + C_\varphi \varepsilon^{1/3n^2}(e^\rho - 1)$; see (3.25). This somehow indicates the advantage of our approximating scheme.*

In theorem 1.4 and in two dimensions, we can replace the convexity of $F^0$ in (1.8) by a semi-convexity condition as long as the function $F^1$ is highly uniformly convex with respect to $p$. Moreover, the whole sequence of solutions $u_\varepsilon$ to (1.12)–(1.13) converges to the unique minimizer $u \in \bar{S}[\varphi, \Omega_0]$ of (1.1). This is the content of the next theorem.

Theorem 1.8. *Let $n = 2$. Let $\Omega_0$ and $\Omega$ be bounded, open, smooth and convex domains in $\mathbb{R}^n$ such that $\Omega$ is uniformly convex and contains $\overline{\Omega_0}$. Fix $0 \leqslant \theta < 1/n$. Let $\psi$ be a smooth function in $\overline{\Omega}$ with $\inf_{\partial\Omega} \psi > 0$. Let $\varphi$ be a convex and smooth function defined in $\overline{\Omega}$. Assume that the following conditions (1.15) and (1.16) are*

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*satisfied for some positive constants* $C_b, C_l, \underline{C}, C_*$:

$$\frac{\partial^2 F^0}{\partial z^2}(x,z) = \frac{\partial f^0}{\partial z}(x,z) \geqslant -C_b,$$
$$|f^0(x,z)| \leqslant C_l(1+|z|) \quad \textit{for all } x \in \Omega_0 \textit{ and all } z \in \mathbb{R}; \tag{1.15}$$

$$\underline{C} I_2 \leqslant F^1_{p_i p_j}(x,p) \leqslant C_* I_2;$$
$$|F^1_{p_i x_i}(x,p)| \leqslant C_*(|p|+1) \forall x \in \Omega_0, \forall p \in \mathbb{R}^n \textit{ and for each } i. \tag{1.16}$$

*Then the following hold.*

(i) *For* $\varepsilon > 0$ *small, system* (1.12)–(1.13) *has a uniformly convex solution* $u_\varepsilon \in W^{4,q}(\Omega)$ *for all* $q \in (n, \infty)$.

(ii) *For* $\varepsilon > 0$ *small, let* $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ *be a solution to* (1.12)–(1.13). *Assume that* $\underline{C}$ *is large (depending only on* $C_b$ *and* $\Omega_0$*). When* $\varepsilon \to 0$, *the sequence* $\{u_\varepsilon\}$ *converges uniformly on compact subsets of* $\Omega$ *to the unique minimizer* $u \in \bar{S}[\varphi, \Omega_0]$ *of* (1.1).

Remark 1.9. *As explained in detail in [**6**, § 1.3], for a gradient-dependent Lagrangian* $F$, *nothing is known in dimensions* $n \geqslant 3$ *about the solvability of the singular Abreu equations (1.12)–(1.13) in suitable Sobolev spaces. This is the main reason why we restrict ourselves in this paper to dimensions* $n = 2$ *when the Lagrangians* $F$ *depend on the gradient variables* $p$.

Key in the proof of the existence of a uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ to system (1.12)–(1.13) is the derivation of a priori estimates. Crucial ingredients in the convergence proof of $u_\varepsilon$ are their uniform a priori estimates with respect to $\varepsilon$ small. The uniform convexity of $F$ with respect to $z$ in [**3**, **6**] allows us to control $\|u_\varepsilon\|_{L^\infty(\Omega_0)}$. Here, without the uniform convexity of $F$ with respect to $z$, our new input is that we can control $\|u_\varepsilon\|_{L^\infty(\Omega_0)}$ by $\|\varphi\|_{L^\infty(\Omega)} + 1/\varepsilon \int_{\Omega\setminus\Omega_0} |u_\varepsilon - \varphi|^2 \,\mathrm{d}x$. This follows from lemma 2.1 which is of independent interest.

The rest of the note is organized as follows. In § 2, we prove a simple but crucial convexity result stated in lemma 2.1. In § 3, we prove our main results stated in theorems 1.4 and 1.8.

## 2. A convexity lemma

Lemma 2.1. *Let* $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2$ *be bounded, convex domains in* $\mathbb{R}^n$ $(n \geqslant 2)$. *Then there is a positive constant* $C = C(n, \Omega_0, \Omega_1, \Omega_2)$ *with the following property. If* $u$ *is a continuous, convex function in* $\overline{\Omega_2}$ *with* $u \leqslant 0$ *on* $\partial\Omega_2$ *then*

$$\|u\|_{L^\infty(\Omega_1)} \leqslant C \int_{\Omega_2\setminus\Omega_0} |u| \,\mathrm{d}x.$$

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*Proof of lemma 2.1.* Suppose by contradiction that there exists a sequence of continuous, convex functions $\{u_k\}$ in $\overline{\Omega_2}$ with $u_k \leqslant 0$ on $\partial\Omega_2$ such that

$$\|u_k\|_{L^\infty(\Omega_1)} = 1 \quad \text{but} \quad \int_{\Omega_2\setminus\Omega_0} |u_k|\,\mathrm{d}x \leqslant \frac{1}{k}.$$

Thus

$$\int_{\Omega_2} |u_k|\,\mathrm{d}x = \int_{\Omega_2\setminus\Omega_0} |u_k|\,\mathrm{d}x + \int_{\Omega_0} |u_k|\,\mathrm{d}x \leqslant \frac{1}{k} + |\Omega_0|.$$

Therefore, we have (see, e.g. inequality (3.2) in [6])

$$\|u_k\|_{L^\infty(\Omega_2)} \leqslant \frac{n+1}{|\Omega_2|}\int_{\Omega_2} |u_k|\,\mathrm{d}x \leqslant C_1(n,\Omega_0,\Omega_2).$$

From $u_k \leqslant 0$ on $\partial\Omega_2$ and the gradient bound for each $x \in \Omega_2$ (see, e.g. (3.1) in [6])

$$|Du_k(x)| \leqslant \frac{\max_{\partial\Omega_2} u_k - u_k(x)}{\operatorname{dist}(x,\partial\Omega_2)} \leqslant \frac{\|u_k\|_{L^\infty(\Omega_2)}}{\operatorname{dist}(x,\partial\Omega_2)},$$

we find that, after extracting a subsequence, $\{u_k\}$ converges locally uniformly in $\Omega_2$ to a convex function $u$ in $\overline{\Omega_2}$ with $u \leqslant 0$ on $\partial\Omega_2$. Hence $\|u\|_{L^\infty(\Omega_1)} = 1$. Moreover, from

$$\int_{\Omega_1\setminus\Omega_0} |u_k|\,\mathrm{d}x \leqslant \frac{1}{k},$$

we find that $u \equiv 0$ in $\Omega_1 \setminus \Omega_0$. By the convexity of $u$, we have $u \equiv 0$ in $\Omega_1$. This contradicts $\|u\|_{L^\infty(\Omega_1)} = 1$ and hence, the lemma is proved. □

Corollary 2.2. *Let $\Omega_0 \subset\subset\subset \Omega$ be bounded, convex domains in $\mathbb{R}^n$ $(n \geqslant 2)$. If $u$ is a continuous, convex function in $\overline{\Omega}$ then*

$$\|u\|_{L^\infty(\Omega)} \leqslant C_1\left(n,\Omega_0,\Omega,\max_{\partial\Omega} u\right) + C_2(n,\Omega_0,\Omega)\int_{\Omega\setminus\Omega_0} |u|\,\mathrm{d}x. \tag{2.1}$$

*Proof.* Applying (3.2) in [6] to $u - \max_{\partial\Omega} u$, we get

$$\|u\|_{L^\infty(\Omega)} \leqslant C(n,\Omega)\int_{\Omega} |u|\,\mathrm{d}x + C\left(n,\Omega,\max_{\partial\Omega} u\right). \tag{2.2}$$

Applying lemma 2.1 to $u - \max_{\partial\Omega} u$, we find

$$\begin{aligned}
\|u\|_{L^\infty(\Omega_0)} &\leqslant \left\|u - \max_{\partial\Omega} u\right\|_{L^\infty(\Omega_0)} + \left|\max_{\partial\Omega} u\right| \\
&\leqslant C(n,\Omega_0,\Omega)\int_{\Omega\setminus\Omega_0} \left|u - \max_{\partial\Omega} u\right|\mathrm{d}x + \left|\max_{\partial\Omega} u\right| \\
&\leqslant C(n,\Omega_0,\Omega)\int_{\Omega\setminus\Omega_0} |u|\,\mathrm{d}x + C\left|\max_{\partial\Omega} u\right|.
\end{aligned}$$

It follows that

$$\int_{\Omega} |u| \,\mathrm{d}x \leqslant |\Omega_0| \|u\|_{L^\infty(\Omega_0)} + \int_{\Omega\setminus\Omega_0} |u| \,\mathrm{d}x$$
$$\leqslant C(n,\Omega_0,\Omega) \int_{\Omega\setminus\Omega_0} |u| \,\mathrm{d}x + C(n,\Omega_0,\Omega) \left|\max_{\partial\Omega} u\right|$$

and therefore (2.1) follows from (2.2). □

## 3. Proof of the main results

In this section, we prove theorems 1.4 and 1.8.

*Proof of theorem 1.4.* We divide the proof into several steps.

*Step 1: A priori estimates.* In this step, we establish the a priori $L^\infty(\Omega)$ estimates for uniformly convex solutions $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ to system (1.12)–(1.13). Recall that $\varphi \in W^{4,q}(\Omega)$ is convex. We only consider

$$0 < \varepsilon < 1.$$

For $t > 0$, let

$$G(t) = \begin{cases} \dfrac{t^\theta - 1}{\theta} & \text{if } \theta \in (0, 1/n), \\ \log t & \text{if } \theta = 0. \end{cases}$$

Then $G'(t) = t^{\theta-1}$ for all $t > 0$ and $w_\varepsilon = G'(\det D^2 u_\varepsilon)$ in $\Omega$.

In what follows, we use $C, C_0, C_1, C_2, \ldots$, etc., to denote positive constants depending only on $n$, $q$, $\Omega_0, \Omega$, $\theta$, $C_*$, $\inf_{\partial\Omega}\psi$, and $\|\varphi\|_{W^{4,q}(\Omega)}$. They are called universal constants and their values may change from line to line. However, they do not depend on $\varepsilon > 0$. *When such constants depend on* $\varepsilon$, *they will be indicated explicitly, as for* $C_d(\varphi, \varepsilon)$ *below.*

Recall from (1.10) that $\rho$ is a strictly convex defining function of $\Omega$. Then, there is $\gamma > 0$ depending only on $\Omega$ such that

$$D^2\rho \geqslant \gamma I_n \quad \text{and} \quad \rho \geqslant -\gamma^{-1} \text{ in } \Omega.$$

Recall that the constant $C_\varphi$ is defined in (1.11). For simplicity, let us denote

$$\tilde{u} = \varphi + C_\varphi \varepsilon^{1/3n^2}(\mathrm{e}^\rho - 1). \tag{3.1}$$

From the convexity of $\varphi$ and

$$D^2(\mathrm{e}^\rho - 1) = \mathrm{e}^\rho(D^2\rho + D\rho \otimes D\rho) \geqslant \mathrm{e}^{-\gamma^{-1}}\gamma I_n,$$

we find that the function $\tilde{u}$ is uniformly convex, belongs to $W^{4,q}(\Omega)$ and satisfies:

(a) $\tilde{u} = \varphi$ on $\partial\Omega$,

(b) $\|\tilde{u}\|_{C^3(\overline{\Omega})} + \|\tilde{u}\|_{W^{4,q}(\Omega)} \leqslant C$, and $\det D^2\tilde{u} \geqslant C^{-1}C_d(\varphi, \varepsilon) > 0$,

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(c) letting $\tilde{w} = G'(\det D^2\tilde{u})$, and denoting by $(\tilde{U}^{ij})$ the cofactor matrix of $(\tilde{u}_{ij})$, then

$$\|\tilde{w}\|_{L^\infty(\Omega)} \leqslant C[C_d(\varphi,\varepsilon)]^{-1}; \quad \left\|\tilde{U}^{ij}\tilde{w}_{ij}\right\|_{L^1(\Omega)} \leqslant C[C_d(\varphi,\varepsilon)]^{-3}.$$

Here, from the definition of $\tilde{u}$ in (3.1), we have the following estimate for the magnitude of $\det D^2\tilde{u}$ in terms of $\varepsilon$:

$$C_d(\varphi,\varepsilon) = \begin{cases} \min_{\overline{\Omega}} \det D^2\varphi & \text{if } \varphi \text{ is uniformly convex in } \overline{\Omega}, \\ \varepsilon^{1/3n} & \text{otherwise.} \end{cases} \tag{3.2}$$

Note that (c) follows from (b) and the following formula (see also [5, lemma 2.1]):

$$\tilde{w}_{ij} = G'''(\det D^2\tilde{u})\tilde{U}^{kl}\tilde{U}^{rs}\tilde{u}_{kli}\tilde{u}_{rsj} + G''(\det D^2\tilde{u})\tilde{U}^{kl}\tilde{u}_{klij} + G''(\det D^2\tilde{u})\tilde{U}^{kl}_j\tilde{u}_{kli}.$$

We use $\nu = (\nu_1, \ldots, \nu_n)$ to denote the unit outer normal vector field on $\partial\Omega$ and $\nu_0$ on $\partial\Omega_0$. First, from (4.5) in [6], we have

$$\begin{aligned} &\int_{\partial\Omega} (\psi U^{ij}_\varepsilon - \tilde{w}\tilde{U}^{ij})((u_\varepsilon)_j - \tilde{u}_j)\nu_i \,\mathrm{d}S + \int_\Omega U^{ij}_\varepsilon (w_\varepsilon)_{ij}(u_\varepsilon - \tilde{u})\,\mathrm{d}x \\ &\quad + \int_\Omega \tilde{U}^{ij}\tilde{w}_{ij}(\tilde{u} - u_\varepsilon)\,\mathrm{d}x \leqslant 0. \end{aligned} \tag{3.3}$$

For readers' convenience, we include the derivation of (3.3) which relies on some concavity arguments. Indeed, from the definition of $G$ and $\theta \in [0, 1/n)$, we find that the function $\tilde{G}(t) := G(t^n)$ is strictly concave on $(0,\infty)$. Using this together with $G' > 0$, and the concavity of the map $M \longmapsto (\det M)^{1/n}$ in the space of symmetric matrices $M \geqslant 0$, we obtain

$$\begin{aligned} &\tilde{G}((\det D^2\tilde{u})^{1/n}) - \tilde{G}((\det D^2u_\varepsilon)^{1/n}) \\ &\quad \leqslant \tilde{G}'((\det D^2u_\varepsilon)^{1/n})((\det D^2\tilde{u})^{1/n} - (\det D^2u_\varepsilon)^{1/n}) \\ &\quad \leqslant \tilde{G}'((\det D^2u_\varepsilon)^{1/n})\frac{1}{n}(\det D^2u_\varepsilon)^{1/n-1}U^{ij}_\varepsilon(\tilde{u} - u_\varepsilon)_{ij}. \end{aligned}$$

Since $\tilde{G}'((\det D^2u_\varepsilon)^{1/n}) = nG'(\det D^2u_\varepsilon)(\det D^2u_\varepsilon)^{(n-1)/n} = nw_\varepsilon(\det D^2u_\varepsilon)^{(n-1)/n}$, we rewrite the above inequalities as

$$G(\det D^2\tilde{u}) - G(\det D^2u_\varepsilon) \leqslant w_\varepsilon U^{ij}_\varepsilon(\tilde{u} - u_\varepsilon)_{ij}.$$

Similarly, we have

$$G(\det D^2u_\varepsilon) - G(\det D^2\tilde{u}) \leqslant \tilde{w}\tilde{U}^{ij}(u_\varepsilon - \tilde{u})_{ij}.$$

Adding these two last inequalities, integrating by parts twice and using the fact that $(U_\varepsilon^{ij})$ is divergence free, we obtain

$$
\begin{aligned}
0 &\geqslant \int_\Omega \left[w_\varepsilon U_\varepsilon^{ij}(u_\varepsilon - \tilde{u})_{ij} + \tilde{w}\tilde{U}^{ij}(\tilde{u} - u_\varepsilon)_{ij}\right] \mathrm{d}x \\
&= \int_{\partial\Omega} w_\varepsilon U_\varepsilon^{ij}((u_\varepsilon)_j - \tilde{u}_j)\nu_i \,\mathrm{d}S + \int_\Omega U_\varepsilon^{ij}(w_\varepsilon)_{ij}(u_\varepsilon - \tilde{u})\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \tilde{w}\tilde{U}^{ij}(\tilde{u}_j - (u_\varepsilon)_j)\nu_i \,\mathrm{d}S + \int_\Omega \tilde{U}^{ij}\tilde{w}_{ij}(\tilde{u} - u_\varepsilon)\,\mathrm{d}x,
\end{aligned}
$$

from which (3.3) follows. Here we recall that $w_\varepsilon = \psi$ on $\Omega$.

In what follows, we will use $f_\varepsilon$ to denote $f_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon; \tilde{u})$. Then, by (1.12),

$$
U_\varepsilon^{ij}(w_\varepsilon)_{ij} = \varepsilon^{-1} f_\varepsilon.
$$

Let $K(y)$ be the Gauss curvature of $\partial\Omega$ at $y \in \partial\Omega$. We have the following assertion.

**Assertion.** For all $\varepsilon > 0$, we have

$$
\begin{aligned}
\int_{\partial\Omega} K\psi(u_\varepsilon)_\nu^n \,\mathrm{d}S \leqslant C[C_d(\varphi,\varepsilon)]^{-3} + C[C_d(\varphi,\varepsilon)]^{-3}\left(\int_{\partial\Omega}((u_\varepsilon)_\nu^+)^n \,\mathrm{d}S\right)^{(n-1)/n} \\
+ \int_\Omega -\varepsilon^{-1} f_\varepsilon(u_\varepsilon - \tilde{u})\,\mathrm{d}x.
\end{aligned}
$$

The proof of the assertion is similar to that of (4.10) in [**6**], and we include it here for readers' convenience.

We start by analysing the boundary terms in (3.3). Since $u_\varepsilon - \tilde{u} = 0$ on $\partial\Omega$, we have $(u_\varepsilon - \tilde{u})_j = (u_\varepsilon - \tilde{u})_\nu \nu_j$, and hence

$$
U_\varepsilon^{ij}(u_\varepsilon - \tilde{u})_j\nu_i = U_\varepsilon^{ij}\nu_j\nu_i(u_\varepsilon - \tilde{u})_\nu = U_\varepsilon^{\nu\nu}(u_\varepsilon - \tilde{u})_\nu
$$

where

$$
U_\varepsilon^{\nu\nu} = \det D_{x'}^2 u_\varepsilon
$$

with $x' \perp \nu$ denoting the tangential directions along $\partial\Omega$. Therefore,

$$
(\psi U_\varepsilon^{ij} - \tilde{w}\tilde{U}^{ij})((u_\varepsilon)_j - \tilde{u}_j)\nu_i = (\psi U_\varepsilon^{\nu\nu} - \tilde{w}\tilde{U}^{\nu\nu})((u_\varepsilon)_\nu - \tilde{u}_\nu).
$$

Now, using $U_\varepsilon^{\nu\nu}$ and $\tilde{U}^{\nu\nu}$, we can rewrite (3.3) as

$$
\begin{aligned}
\int_{\partial\Omega} (\psi U_\varepsilon^{\nu\nu} - \tilde{w}\tilde{U}^{\nu\nu})((u_\varepsilon)_\nu - \tilde{u}_\nu)\,\mathrm{d}S + \int_\Omega \tilde{U}^{ij}\tilde{w}_{ij}(\tilde{u} - u_\varepsilon)\,\mathrm{d}x \\
\leqslant \int_\Omega -\varepsilon^{-1} f_\varepsilon(u_\varepsilon - \tilde{u})\,\mathrm{d}x
\end{aligned}
$$

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which gives

$$\int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}(u_\varepsilon)_\nu \,\mathrm{d}S \leqslant \|\psi\tilde{u}_\nu\|_{L^\infty(\Omega)} \int_{\partial\Omega} |U_\varepsilon^{\nu\nu}| \,\mathrm{d}S + \|\tilde{U}^{\nu\nu}\tilde{u}_\nu\|_{L^\infty(\Omega)}\|\tilde{w}\|_{L^\infty(\Omega)}$$
$$+ \|\tilde{U}^{\nu\nu}\|_{L^\infty(\Omega)}\|\tilde{w}\|_{L^\infty(\Omega)} \int_{\partial\Omega} |(u_\varepsilon)_\nu| \,\mathrm{d}S$$
$$+ \|\tilde{U}^{ij}\tilde{w}_{ij}\|_{L^1(\Omega)} \left(\|\tilde{u}\|_{L^\infty(\Omega)} + \|u_\varepsilon\|_{L^\infty(\Omega)}\right)$$
$$+ \int_\Omega -\varepsilon^{-1} f_\varepsilon(u_\varepsilon - \tilde{u}) \,\mathrm{d}x. \tag{3.4}$$

Observe that:

(A) By (a)–(b), the quantities $\tilde{u}$, $\tilde{u}_\nu$ and $\tilde{U}^{\nu\nu} = \tilde{U}^{ij}\nu_i\nu_j$ are universally bounded. By (c),

$$\|\tilde{w}\|_{L^\infty(\Omega)} \leqslant C[C_d(\varphi,\varepsilon)]^{-1};\ \|\tilde{U}^{ij}\tilde{w}_{ij}\|_{L^1(\Omega)} \leqslant C[C_d(\varphi,\varepsilon)]^{-3}.$$

(B) For the convex function $u_\varepsilon \in C^2(\overline{\Omega})$ with $u_\varepsilon = \varphi$ on $\partial\Omega$, we have (see, e.g. [5, inequality (2.7)])

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant C(n,\varphi,\Omega) + C(n,\Omega)\left(\int_{\partial\Omega} ((u_\varepsilon)_\nu^+)^n \,\mathrm{d}S\right)^{1/n}.$$

(C) The Gauss curvature $K$ of $\partial\Omega$ and $U_\varepsilon^{\nu\nu}$ are related by (see, e.g. (4.9) in [6])

$$U_\varepsilon^{\nu\nu} = K((u_\varepsilon)_\nu)^{n-1} + E_\varepsilon \quad \text{where } |E_\varepsilon| \leqslant C(1 + ((u_\varepsilon)_\nu^+)^{n-2}).$$

Now, the Assertion follows from (3.4) together with the above observations.

Let us continue with the proof of *Step 1*. Since $u_\varepsilon$ is convex with boundary value $\varphi$ on $\partial\Omega$, we have

$$(u_\varepsilon)_\nu \geqslant -\|D\varphi\|_{L^\infty(\Omega)} := -C_0.$$

It follows that, for $u_\nu^+ = \max(0, u_\nu)$, we have $((u_\varepsilon)_\nu^+)^n \leqslant (u_\varepsilon)_\nu^n + C_0^n$ and therefore from the Assertion, we obtain

$$\int_{\partial\Omega} K\psi((u_\varepsilon)_\nu^+)^n \,\mathrm{d}S \leqslant C[C_d(\varphi,\varepsilon)]^{-3} + C[C_d(\varphi,\varepsilon)]^{-3}\left(\int_{\partial\Omega} ((u_\varepsilon)_\nu^+)^n \,\mathrm{d}S\right)^{(n-1)/n}$$
$$+ \int_\Omega -\varepsilon^{-1} f_\varepsilon(u_\varepsilon - \tilde{u}) \,\mathrm{d}x. \tag{3.5}$$

By the uniform convexity of $\partial\Omega$, we have $K \geqslant C(\Omega) > 0$ on $\partial\Omega$. Using this, together with $\inf_{\partial\Omega} \psi > 0$ and Young's inequality for the second term on the right-hand side

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of (3.5), we find that

$$C[C_d(\varphi,\varepsilon)]^{-3}\left(\int_{\partial\Omega}((u_\varepsilon)_\nu^+)^n\,\mathrm{d}S\right)^{(n-1)/n} \leqslant C[C_d(\varphi,\varepsilon)]^{-3n}+\frac{1}{2}\int_{\partial\Omega}K\psi((u_\varepsilon)_\nu^+)^n\,\mathrm{d}S$$

and hence

$$\int_{\partial\Omega}((u_\varepsilon)_\nu^+)^n\,\mathrm{d}S\leqslant C[C_d(\varphi,\varepsilon)]^{-3n}+\int_\Omega -\varepsilon^{-1}f_\varepsilon(u_\varepsilon-\tilde{u})\,\mathrm{d}x.$$

Therefore, multiplying both sides of the above inequality by $\varepsilon>0$ and using $\varepsilon[C_d(\varphi,\varepsilon)]^{-3n}\leqslant C$ from (3.2), we get, as in inequality (4.31) in [**6**] (which was stated for $n=2$ there)

$$\begin{aligned}\int_{\partial\Omega}\varepsilon((u_\varepsilon)_\nu^+)^n\,\mathrm{d}S&\leqslant C\varepsilon[C_d(\varphi,\varepsilon)]^{-3n}+\int_\Omega -f_\varepsilon(u_\varepsilon-\tilde{u})\,\mathrm{d}x\\&\leqslant C+\int_\Omega -f_\varepsilon(u_\varepsilon-\tilde{u})\,\mathrm{d}x.\end{aligned}\tag{3.6}$$

We will estimate the right-hand side of (3.6).

From the convexity of $F^0$ (see (1.8)), we can estimate

$$\begin{aligned}A_\varepsilon:=&\int_{\Omega_0}-f^0(x,u_\varepsilon(x))(u_\varepsilon-\tilde{u})\,\mathrm{d}x\\ \leqslant&\int_{\Omega_0}-f^0(x,\tilde{u}(x))(u_\varepsilon-\tilde{u})\,\mathrm{d}x\\ \leqslant&\, C+C_1\|u_\varepsilon\|_{L^\infty(\Omega_0)}.\end{aligned}\tag{3.7}$$

In what follows, we will frequently use the following inequality (see, (3.1) in [**6**])

$$|Du_\varepsilon(x)|\leqslant\frac{\max_{\partial\Omega}u_\varepsilon-u_\varepsilon(x)}{\mathrm{dist}\,(x,\partial\Omega)}\ \forall x\in\Omega.\tag{3.8}$$

By the convexity of $u_\varepsilon$ and $F^1(x,p)$ in $p$, we have $F^1_{p_ip_j}(u_\varepsilon)_{ij}\geqslant 0$. Moreover, $u_\varepsilon\leqslant\sup_{\partial\Omega}\varphi\leqslant C$ and $|\tilde{u}|\leqslant C$. Thus, recalling (1.9), we find that

$$F^1_{p_ip_j}(u_\varepsilon)_{ij}(u_\varepsilon-\tilde{u})\leqslant CF^1_{p_ip_j}(u_\varepsilon)_{ij}\leqslant CC_*\Delta u_\varepsilon.$$

By the divergence theorem and (3.8), we have

$$\begin{aligned}\int_{\Omega_0}F^1_{p_ip_j}(u_\varepsilon)_{ij}(u_\varepsilon-\tilde{u})\,\mathrm{d}x\leqslant CC_*\int_{\Omega_0}\Delta u_\varepsilon\,\mathrm{d}x=CC_*\int_{\partial\Omega_0}(u_\varepsilon)_{\nu_0}\,\mathrm{d}S\\ \leqslant C+C_2\|u_\varepsilon\|_{L^\infty(\Omega_0)}.\end{aligned}\tag{3.9}$$

On the other hand, for any $i=1,\ldots,n$, using (1.9) and (3.8), we can estimate in $\Omega_0$:

$$\begin{aligned}|F^1_{p_ix_i}(x,Du_\varepsilon(x))(u_\varepsilon(x)-\tilde{u}(x))|\leqslant C_*(|Du_\varepsilon(x)|+1)(|u_\varepsilon(x)+C)\\ \leqslant C_3(|u_\varepsilon(x)|^2+1).\end{aligned}\tag{3.10}$$

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Note that (2.1) together with $\|\tilde{u}\|_{L^\infty(\Omega)} \leqslant C$ gives

$$\|u_\varepsilon\|_{L^\infty(\Omega_0)} \leqslant C + C\int_{\Omega\setminus\Omega_0} |u_\varepsilon|\,\mathrm{d}x \leqslant C + C_4\int_{\Omega\setminus\Omega_0} |u_\varepsilon - \tilde{u}|^2\,\mathrm{d}x. \tag{3.11}$$

From (3.7), (3.9), (3.10) and (3.11), we find that

$$\begin{aligned}\int_{\Omega_0} -f_\varepsilon(u_\varepsilon - \tilde{u})\,\mathrm{d}x &= \int_{\Omega_0}\left[-f^0(x, u_\varepsilon(x)) + \frac{\partial}{\partial x_i}\left(\frac{\partial F^1}{\partial p_i}(x, Du_\varepsilon(x))\right)\right](u_\varepsilon - \tilde{u})\,\mathrm{d}x\\ &= A_\varepsilon + \int_{\Omega_0}\left[F^1_{p_i x_i}(x, Du_\varepsilon(x)) + F^1_{p_i p_j}(x, Du_\varepsilon(x))(u_\varepsilon)_{ij}\right]\\ &\quad \times (u_\varepsilon - \tilde{u})\,\mathrm{d}x\\ &\leqslant A_\varepsilon + C_3\int_{\Omega_0}(|u_\varepsilon|^2 + 1)\,\mathrm{d}x + C + C_2\|u_\varepsilon\|_{L^\infty(\Omega_0)}\\ &\leqslant C_5\|u\|^2_{L^\infty(\Omega_0)} + C \leqslant C + C_6\int_{\Omega\setminus\Omega_0}|u_\varepsilon - \tilde{u}|^2\,\mathrm{d}x. \end{aligned}\tag{3.12}$$

It follows from (3.6) and (3.11), and $f_\varepsilon = 1/\varepsilon(u_\varepsilon - \tilde{u})$ on $\Omega\setminus\Omega_0$ that

$$\begin{aligned}\int_{\partial\Omega}\varepsilon((u_\varepsilon)^+_\nu)^n\,\mathrm{d}S &\leqslant C + \int_\Omega -f_\varepsilon(u_\varepsilon - \tilde{u})\,\mathrm{d}x\\ &= C + \int_{\Omega_0} -f_\varepsilon(u_\varepsilon - \tilde{u})\,\mathrm{d}x + \int_{\Omega\setminus\Omega_0} -f_\varepsilon(u_\varepsilon - \tilde{u})\,\mathrm{d}x\\ &\leqslant C + C_6\int_{\Omega\setminus\Omega_0}|u_\varepsilon - \tilde{u}|^2\,\mathrm{d}x + \int_{\Omega\setminus\Omega_0} -\frac{1}{\varepsilon}|u_\varepsilon - \tilde{u}|^2\,\mathrm{d}x\\ &\leqslant C - \int_{\Omega\setminus\Omega_0}\frac{1}{2\varepsilon}|u_\varepsilon - \tilde{u}|^2\,\mathrm{d}x \end{aligned}\tag{3.13}$$

if $\varepsilon$ is small, say

$$\varepsilon \leqslant \frac{1}{2C_6}.$$

*From now on, we assume that $\varepsilon$ is small.* Then, we get

$$\int_{\partial\Omega}\varepsilon^2((u_\varepsilon)^+_\nu)^n\,\mathrm{d}S + \int_{\Omega\setminus\Omega_0}(u_\varepsilon - \tilde{u})^2\,\mathrm{d}x \leqslant C\varepsilon. \tag{3.14}$$

This together with (2.1) and $\|\tilde{u}\|_{L^\infty(\Omega)} \leqslant C$ gives the uniform bound for $u_\varepsilon$ on $\Omega$:

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leqslant C_7. \tag{3.15}$$

*Step 2: Existence and convergence properties of uniformly convex solutions to* (1.12)–(1.13).

(i) We consider two separate cases.

*Case 1:* $F(x, z, p) = F^0(x, z)$. In this case, from the a priori $L^\infty(\Omega)$ estimates (3.15) for uniformly convex solutions $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ to system (1.12)–(1.13),

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we can use a Leray–Schauder degree argument as in [**3**, theorem 4.2] to show the existence of a unique uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ (for all $q < \infty$) to system (1.12)–(1.13).

*Case 2: $F(x,z,p) = F^0(x,z) + F^1(x,p)$ and $n = 2$.* In this case, from the a priori $L^\infty(\Omega)$ estimates (3.15) for uniformly convex solutions $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ to system (1.12)–(1.13), we can establish the a priori $W^{4,q}(\Omega)$ estimates for $u_\varepsilon$ as in [**6**, theorem 4.1]. With these a priori estimates, we can use a Leray–Schauder degree argument as in [**6**, theorem 2.1] to show the existence of a uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ (for all $q < \infty$) to system (1.12)–(1.13).

Hence (i) is proved.

(ii) For $\varepsilon > 0$ small, let $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ be a solution to (1.12)–(1.13). By (3.15), the sequence $\{u_\varepsilon\}$ is uniformly bounded with respect to $\varepsilon$. By (3.8), $|Du_\varepsilon|$ is uniformly bounded on compact subsets of $\Omega$. Thus, by the Arzela–Ascoli theorem, up to extraction of a subsequence, $u_\varepsilon$ converges uniformly on compact subsets of $\Omega$, and also in $W^{1,2}(\Omega_0)$, to a convex function $u$ on $\Omega$. From (3.14) and the fact that $\lim_{\varepsilon\to 0} \tilde{u} = \varphi$, we find $u \in \bar{S}[\varphi, \Omega_0]$. Let

$$\eta_\varepsilon := \varepsilon^{1/n} \left( \int_{\partial\Omega} ((u_\varepsilon)^+_\nu)^n \, \mathrm{d}S \right)^{1/n}. \tag{3.16}$$

Then, from (3.14), we have as in [**3**, inequality (5.5)] and [**6**, inequality (4.27)]

$$\eta_\varepsilon \leqslant C. \tag{3.17}$$

Consider the functional $J$ defined over $\bar{S}[\varphi, \Omega_0]$ by

$$J(v) := \int_{\Omega_0} F(x, v(x), Dv(x)) \, \mathrm{d}x. \tag{3.18}$$

Since $u_\varepsilon$ converges uniformly to $u$ on $\overline{\Omega_0}$, by Fatou's lemma, we have

$$\liminf_{\varepsilon} \int_{\Omega_0} F^0(x, u_\varepsilon(x)) dx \geqslant \int_{\Omega_0} F^0(x, u(x)) \, \mathrm{d}x.$$

From the convexity of $F^1(x,p)$ in $p$ and the fact that $u_\varepsilon$ converges to $u$ on $W^{1,2}(\Omega_0)$, we have

$$\liminf_{\varepsilon} \int_{\Omega_0} F^1(x, Du_\varepsilon(x)) \, \mathrm{d}x \geqslant \int_{\Omega_0} F^1(x, Du(x)) \, \mathrm{d}x,$$

which is due to lower semicontinuity. Therefore

$$\liminf_{\varepsilon} J(u_\varepsilon) \geqslant J(u). \tag{3.19}$$

Our main estimate is the following.

**Claim**. If $0 \leqslant \theta < 1/n$, then for any $v \in \bar{S}[\varphi, \Omega_0]$, we have

$$J(v) \geqslant \liminf_{\varepsilon} J(u_\varepsilon) - \limsup_{\varepsilon} \left[ \varepsilon^{(n-1)/n} \eta_\varepsilon + \varepsilon^{1/n} \eta_\varepsilon^{n-1} \right]. \tag{3.20}$$

Assuming the above claim, we show that $u$ is a minimizer of (1.1). Indeed, this follows from (3.20), (3.19) and (3.17) which imply the estimate $J(v) \geqslant J(u)$ for all $v \in \bar{S}[\varphi, \Omega_0]$.

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It remains to prove the claim. The proof is similar to that of [6, theorem 2.3] where the case $n = 2$ was treated. In our context of theorem 1.4(ii), we would like to treat also the case of general dimensions $n$ when $F^1 \equiv 0$, that is, when the Lagrangian is independent of the gradient variables. For reader's convenience, we repeat the arguments there. Recall from (3.1) that

$$\tilde{u} = \varphi + C_\varphi \varepsilon^{1/3n^2}(\mathrm{e}^\rho - 1).$$

Consider the following functional $J_\varepsilon$ over the set of convex functions $v$ on $\overline{\Omega}$:

$$J_\varepsilon(v) = \int_{\Omega_0} F(x, v(x), Dv(x))\,\mathrm{d}x + \frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0} (v - \tilde{u})^2\,\mathrm{d}x - \varepsilon \int_\Omega G(\det D^2 v)\,\mathrm{d}x. \tag{3.21}$$

From the Alexandrov theorem [4, theorem 1, p. 242], $v$ is twice differentiable a.e. At those points of twice differentiability of $v$, we use $D^2 v$ to denote its Hessian matrix. Thus, in addition to setting $\log 0 = -\infty$, the functional $J_\varepsilon$ is well defined with this convention for all $\theta \geqslant 0$; it can take value $\infty$.

Let $U_\varepsilon^{\nu\nu} = U_\varepsilon^{ij}\nu_i\nu_j$. Let $K$ be the Gauss curvature of $\partial\Omega$. Then, we have (see, e.g. (4.9) in [6])

$$U_\varepsilon^{\nu\nu} = K((u_\varepsilon)_\nu)^{n-1} + E_\varepsilon \quad \text{where } |E_\varepsilon| \leqslant C(1 + ((u_\varepsilon)_\nu^+)^{n-2}). \tag{3.22}$$

First, by [6, estimate (5.6)], if $v$ is a convex function in $\overline{\Omega}$ with $v = \tilde{u}$ in a neighbourhood of $\partial\Omega$, then

$$J_\varepsilon(v) - J_\varepsilon(u_\varepsilon) \geqslant \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - \tilde{u})\,\mathrm{d}S + \int_{\partial\Omega_0}(v - u_\varepsilon)\nabla_p F^1(x, Du_\varepsilon(x))\cdot\nu_0\,\mathrm{d}S. \tag{3.23}$$

Now, we are ready to prove (3.20) for all $v \in \bar{S}[\varphi, \Omega_0]$. Indeed, applying (3.23) to

$$v_\varepsilon := v + C_\varphi \varepsilon^{1/3n^2}(e^\rho - 1),$$

which clearly satisfies $v_\varepsilon = \tilde{u}$ on $\overline{\Omega}\setminus\Omega_0$, and using the fact that the subsequential uniform limit $u \in \bar{S}[\varphi, \Omega_0]$ of $u_\varepsilon$ satisfies $u = v = \varphi$ on $\partial\Omega_0$, we conclude that the corresponding rightmost term in (3.23)

$$\int_{\partial\Omega_0}(v_\varepsilon - u_\varepsilon)\nabla_p F^1(x, Du_\varepsilon(x))\cdot\nu_0\,\mathrm{d}S \to 0 \quad \text{as } \varepsilon \to 0,$$

and hence,

$$\begin{aligned}\varepsilon\int_{\partial\Omega}\psi U_\varepsilon^{\nu\nu}\partial_\nu(u_\varepsilon - \tilde{u})\,\mathrm{d}S - o_\varepsilon(1) &\leqslant J_\varepsilon(v_\varepsilon) - J_\varepsilon(u_\varepsilon)\\ &= J(v_\varepsilon) - J(u_\varepsilon) - \frac{1}{2\varepsilon}\int_{\Omega\setminus\Omega_0}(u_\varepsilon - \tilde{u})^2\,\mathrm{d}x\\ &\quad - \varepsilon\int_\Omega [G(\det D^2 v_\varepsilon) - G(\det D^2 u_\varepsilon)]\,\mathrm{d}x.\end{aligned} \tag{3.24}$$

Here we use $o_\varepsilon(1)$ to denote a quantity that tends to 0 when $\varepsilon \to 0$.

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Since $\det D^2 v_\varepsilon \geqslant C\varepsilon^{1/3n}$, we have

$$\liminf_{\varepsilon} \varepsilon \int_\Omega G(\det D^2 v_\varepsilon)\, \mathrm{d}x \geqslant 0 \quad \text{for all } v \in \bar{S}[\varphi, \Omega_0]. \tag{3.25}$$

From the definition of $v_\varepsilon$ and the dominated convergence theorem, we have

$$\lim_{\varepsilon \to 0} J(v_\varepsilon) = J(v).$$

Combining this with (3.24) and (3.25), we get

$$\begin{aligned} J(v) &\geqslant \liminf_{\varepsilon} J(u_\varepsilon) + \liminf_{\varepsilon} \varepsilon \int_\Omega [G(\det D^2 v_\varepsilon) - G(\det D^2 u_\varepsilon)]\, \mathrm{d}x \\ &\quad + \liminf_{\varepsilon} \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \tilde{u})\, \mathrm{d}S \\ &\geqslant \liminf_{\varepsilon} J(u_\varepsilon) - \limsup_{\varepsilon} \varepsilon \int_\Omega G(\det D^2 u_\varepsilon)\, \mathrm{d}x \\ &\quad + \liminf_{\varepsilon} \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \tilde{u})\, \mathrm{d}S. \end{aligned} \tag{3.26}$$

From (3.22), $\|\tilde{u}_\nu\|_{L^\infty(\partial\Omega)} \leqslant C$ and $(u_\varepsilon)_\nu \geqslant (u_\varepsilon)_\nu^+ - C_0$ by the estimate preceding (3.5), we have

$$U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \tilde{u}) \geqslant -C((u_\varepsilon)_\nu^+)^{n-1} - C.$$

From the definition of $\eta_\varepsilon$ in (3.16), one has

$$\int_{\partial\Omega} (u_\varepsilon)_\nu^+\, \mathrm{d}S \leqslant C\varepsilon^{-1/n}\eta_\varepsilon \quad \text{and} \quad \int_{\partial\Omega} ((u_\varepsilon)_\nu^+)^{n-1}\, \mathrm{d}S \leqslant C\varepsilon^{-(n-1)/n}\eta_\varepsilon^{n-1}$$

and hence,

$$\begin{aligned} \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \tilde{u})\, \mathrm{d}S &\geqslant -C\varepsilon \int_{\partial\Omega} [1 + ((u_\varepsilon)_\nu^+)^{n-1}]\, \mathrm{d}S \\ &\geqslant -C\varepsilon^{1/n}\eta_\varepsilon^{n-1}. \end{aligned} \tag{3.27}$$

From $0 \leqslant \theta < 1/n$ and the convexity of $u_\varepsilon$, we can find $C > 0$ depending only on $\theta$ and $n$ such that

$$G(\det D^2 u_\varepsilon) \leqslant C[1 + (\det D^2 u_\varepsilon)^{1/n}] \leqslant C(1 + \Delta u_\varepsilon).$$

Therefore, from the divergence theorem, we obtain

$$\begin{aligned} \int_\Omega G(\det D^2 u_\varepsilon)\, \mathrm{d}x &\leqslant C \int_\Omega (1 + \Delta u_\varepsilon)\, \mathrm{d}x \\ &= C|\Omega| + C \int_{\partial\Omega} (u_\varepsilon)_\nu\, \mathrm{d}S \leqslant C(1 + \varepsilon^{-1/n}\eta_\varepsilon). \end{aligned} \tag{3.28}$$

Combining (3.26)–(3.28), we get

$$J(v) \geqslant \liminf_{\varepsilon} J(u_\varepsilon) - \limsup_{\varepsilon} \left[\varepsilon^{(n-1)/n}\eta_\varepsilon + \varepsilon^{1/n}\eta_\varepsilon^{n-1}\right]$$

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which implies (3.20). The Claim is proved and the proof of the theorem is complete. □

*Sketch of proof of theorem 1.8.* The proof is parallel to that of theorem 1.4 with some minor modifications. We briefly indicate these. Recall that $n = 2$.

(i) *Existence result.* The key is still the a priori estimate (3.15) for a uniformly convex solution $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ to (1.12)–(1.13). Assume that there holds the linear growth condition of $f^0(x, z)$ with respect to $z$ in (1.15). In this case, the quantity $A_\varepsilon$ defined in (3.7) can be estimated from above by

$$A_\varepsilon \leqslant C + C_1 \|u_\varepsilon\|^2_{L^\infty(\Omega_0)}.$$

Thus, the final estimate in (3.12) holds. Therefore (3.15) holds if $\varepsilon$ is small; the constant $C_7$ now depends also on $C_l$. As a consequence, the existence result of theorem 1.8(i) follows as that of theorem 1.4(i).

(ii) *Convergence result.* The new input here is the following well-known trace inequality. There is a constant $C_t = C_t(\Omega_0) > 0$ depending only on $\Omega_0$ such that

$$\int_{\Omega_0} |v - u|^2 \,\mathrm{d}x \leqslant C_t \int_{\Omega_0} |Dv - Du|^2 \,\mathrm{d}x + C_t \int_{\partial\Omega_0} |v - u|^2 \,\mathrm{d}S \quad \text{for all } u, v \in W^{1,2}(\Omega_0). \tag{3.29}$$

Assume that (1.15) and (1.16) hold. With (1.15), we have

$$F^0(x, \tilde{z}) - F^0(x, z) \geqslant f^0(x, z)(\tilde{z} - z) - \frac{C_b}{2}|\tilde{z} - z|^2 \quad \text{for all } x \in \Omega_0 \text{ and all } z, \tilde{z} \in \mathbb{R}. \tag{3.30}$$

With (1.16), we have

$$F^1(x, \tilde{p}) - F^1(x, p) \geqslant \nabla_p F^1(x, p) \cdot (\tilde{p} - p) + \frac{\mathrm{C}}{2}|\tilde{p} - p|^2 \quad \text{for all } x \in \Omega_0 \text{ and all } p, \tilde{p} \in \mathbb{R}^2. \tag{3.31}$$

For $\varepsilon > 0$ small, let $u_\varepsilon \in W^{4,q}(\Omega)$ $(q > n)$ be a solution to (1.12)–(1.13).

*Step 1: Convergence of a subsequence of $\{u_\varepsilon\}$ to a minimizer of (1.1).* As in the proof of theorem 1.4(ii), up to extraction of a subsequence, $u_\varepsilon$ converges uniformly on compact subsets of $\Omega$, and also in $W^{1,2}(\Omega_0)$, to a convex function $u \in \overline{S}[\varphi, \Omega_0]$.

We show that $u$ is a minimizer of (1.1). The proof is similar to that of theorem 1.4(ii) except that (3.23) is replaced by

$$J_\varepsilon(v) - J_\varepsilon(u_\varepsilon) \geqslant \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \tilde{u}) \,\mathrm{d}S + \int_{\partial\Omega_0} (v - u_\varepsilon) \nabla_p F^1(x, Du_\varepsilon(x)) \cdot \nu_0 \,\mathrm{d}S - \frac{\mathrm{C}}{2} \int_{\partial\Omega_0} |v - u_\varepsilon|^2 \,\mathrm{d}S \tag{3.32}$$

for all convex functions $v$ in $\overline{\Omega}$ with $v = \tilde{u}$ in a neighbourhood of $\partial\Omega$ and provided that

$$\underline{C} \geqslant C_t C_b + 1. \tag{3.33}$$

In (3.32), the function $\tilde{u}$ is defined as in (3.1). Clearly, when $v \in \overline{S}[\varphi, \Omega_0]$, the extra boundary term in (3.32) disappears in the limit $\varepsilon \to 0$.

Now, we explain how to obtain (3.32) from (1.15), (1.16) and (3.33). Again, let $v$ be a convex function in $\overline{\Omega}$ with $v = \tilde{u}$ in a neighbourhood of $\partial\Omega$. In the derivation of (3.23) in [**6**, estimate (5.6)], we used (3.30) with $C_b = 0$ (and $\tilde{z}$ a mollification $v_h$ of $v$ and $z$ the function $u_\varepsilon$) and (3.31) with $\underline{C} = 0$ (and $\tilde{p}$ the gradient $Dv_h$ and $p$ the gradient $Du_\varepsilon$); see [**6**, estimate (5.10)]. With $C_b > 0, \underline{C} > 0$, instead of (3.23), we have the following:

$$\begin{aligned} J_\varepsilon(v) - J_\varepsilon(u_\varepsilon) \geqslant \varepsilon \int_{\partial\Omega} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \tilde{u}) \, \mathrm{d}S + \int_{\partial\Omega_0} (v - u_\varepsilon) \nabla_p F^1(x, Du_\varepsilon(x)) \cdot \nu_0 \, \mathrm{d}S \\ - \frac{C_b}{2} \int_{\Omega_0} |v - u_\varepsilon|^2 \, \mathrm{d}x + \frac{\underline{C}}{2} \int_{\Omega_0} |Dv - Du_\varepsilon|^2 \, \mathrm{d}x. \end{aligned} \tag{3.34}$$

Thus, provided (3.33) holds, (3.32) follows from (3.34) and (3.29).

*Step 2: The whole sequence $\{u_\varepsilon\}$ converges to the unique minimizer in $\bar{S}[\varphi, \Omega_0]$ of (1.1) when (3.33) holds.* To show this, in view of *Step 1*, it suffices to show that (1.1) has unique minimizer in $\bar{S}[\varphi, \Omega_0]$.

Suppose that $u, v \in \bar{S}[\varphi, \Omega_0]$ are two minimizers of the functional $J$ defined by

$$J(u) := \int_{\Omega_0} F(x, u(x), Du(x)) \, \mathrm{d}x$$

where we recall $F(x, z, p) = F^0(x, z) + F^1(x, p)$.

Note that $(u + v)/2 \in \bar{S}[\varphi, \Omega_0]$. From (3.30) and (3.31), we find

$$F^0(x, u(x)) + F^0(x, v(x)) \geqslant 2F^0\left(x, \frac{u(x) + v(x)}{2}\right) - \frac{C_b}{4}|u(x) - v(x)|^2 \ \forall x \in \Omega_0$$

and

$$\begin{aligned} F^1(x, Du(x)) + F^1(x, Dv(x)) \geqslant 2F^1\left(x, \frac{Du(x) + Dv(x)}{2}\right) \\ + \frac{\underline{C}}{4}|Du(x) - Dv(x)|^2 \ \forall x \in \Omega_0. \end{aligned}$$

Adding these inequalities and integrating over $\Omega_0$, we find that

$$\begin{aligned} J(u) + J(v) &\geqslant 2J\left(\frac{u+v}{2}\right) + \frac{\underline{C}}{4} \int_{\Omega_0} |Du(x) - Dv(x)|^2 \, \mathrm{d}x \\ &\quad - \frac{C_b}{4} \int_{\Omega_0} |u(x) - v(x)|^2 \, \mathrm{d}x \\ &\geqslant 2J\left(\frac{u+v}{2}\right) + \frac{1}{4} \int_{\Omega_0} |Du(x) - Dv(x)|^2 \, \mathrm{d}x. \end{aligned} \tag{3.35}$$

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In the last inequality of (3.35), we used (3.29) while recalling (3.33) and $u = v$ on $\partial\Omega_0$. By the minimality of $u$ and $v$, we deduce from (3.35) that $u \equiv v$. Therefore, (1.1) has unique minimizer in $\bar{S}[\varphi, \Omega_0]$ as asserted. □

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