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# Index estimates for closed minimal submanifolds of the sphere

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In this paper we are interested in comparing the spectra of two elliptic operators acting on a closed minimal submanifold of the Euclidean unit sphere. Using an approach introduced by Savo in [A Savo. Index Bounds for Minimal Hypersurfaces of the Sphere. Indiana Univ. Math. J. 59 (2010), 823-837.], we are able to compare the eigenvalues of the stability operator acting on sections of the normal bundle and the Hodge Laplacian operator acting on 1-forms. As a byproduct of the technique and under a suitable hypothesis on the Ricci curvature of the submanifold we obtain that its first Betti's number is bounded from above by a multiple of the Morse index, which provide evidence for a well-known conjecture of Schoen and Marques & Neves in the setting of higher codimension.

*Keywords:* Closed submanifolds; Eigenvalues; Morse index; Euclidean sphere

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## 1. Introduction

The study of minimal submanifolds and its characteristics are considered classical in Riemannian geometry. In particular, problems about the Morse index. Roughly speaking, if $\Sigma^n$ is a closed minimal submanifold in a Riemannian manifold $M$, its Morse index, $\mathrm{Index}(\Sigma)$, measures how far $\Sigma$ is from being a local minimum of the area functional. There are many papers highlighting the relation between the index and geometric and topological aspects of minimal submanifolds. In his celebrated work[**19**], Simons proved that all closed minimal submanifolds $\Sigma^n$ of the unit sphere $\mathbb{S}^{n+p+1}$ have $\mathrm{Index}(\Sigma) \geqslant p+1$, with equality only when $\Sigma$ is totally geodesic. After this, others papers were made describing families of minimal submanifolds with same index and providing estimates of the Morse index in other special ambient manifolds, see for instance [**7**, **13**, **15**, **16**, **20**]. We point out some results in these works. Lawson and Simons in [**13**] characterized the complex submanifolds as the only stable minimal submanifolds in the complex projective space

$\mathbb{CP}^m$. Ohnita in [**15**] completed the classification of stable minimal submanifolds in compact rank-one symmetric spaces. Futhermore, do Carmo, Ritoré and Ros obtained in [**7**] a classification for two-sided closed minimal hypersurfaces $\Sigma^n$ with index one in the real projective spaces $\mathbb{RP}^{n+1}$. Recently, Torralbo and Urbano in [**20**] classified minimal submanifolds into a product of a sphere by an arbitrary Riemannian manifold, and Perdomo in [**16**] characterized the Clifford tori as the only closed orientable minimal hypersurfaces with antipodal symmetry and index $n+3$ into the unit Euclidean sphere. We must also mention the work done by Ros in [**17**], where the author used harmonic forms in a clever way to obtain lower bounds of the Morse index as a function of the genus of minimal surfaces immersed in a flat 3-torus $T^3$ and in $\mathbb{R}^3$. Finally, we highlight that Savo in [**18**] performed an ingenious trick to obtain a comparison theorem between the spectrums of the stability operator acting on functions and the Hodge–Laplacian acting on 1-form of closed orientable minimal hypersurfaces of $\mathbb{S}^{n+p+1}$, and such comparison implies a lower estimate of the Morse index by a linear function of its first Betti's number. This technique was refined in many directions, see for instances the breakthrough done in [**2**, **11**]. Also, due to the fruitfulness of this technique, very recently such method was successfully adapted for the constant mean curvature and weighted minimal hypersurfaces settings, see for instance [**1**, **3**, **4**, **9**, **10**].

Focusing on the higher codimensional case, we apply the Savo's approach to a new family of normal sections inspired by the works of Simons [**19**] and Savo [**18**] and we thus obtain a comparison result between the spectrums of the stability operator acting on sections of the normal bundle and the Hodge–Laplacian acting on 1-forms of a closed minimal submanifold $\Sigma^n$ into $\mathbb{S}^{n+p+1}$ and thus we generalize, in some sense, the Savo's results to closed minimal submanifolds. We point out that whether the codimension is at least 2, our comparison theorem uses topological and geometrical features of the submanifold, but in codimension one we get the same estimates of Savo, which only requires topological features. As a consequence we obtain a theorem that shows that the Morse index is bounded from below by a linear function of the first Betti's number $b_1(\Sigma)$.

The main result of the paper compares the spectrums of the stability operator acting on sections of the normal bundle and the Hodge–Laplacian acting on 1-forms.

The result reads as follows:

Let $\Sigma^n$ be a closed minimal submanifold of $\mathbb{S}^{n+p+1}$, $p \geqslant 0$, and let $L$ be the stability operator acting on the space of normal sections. Then,

$$\lambda_\alpha(L) \leqslant \lambda_{m(\alpha)}(\Delta_1) - \frac{2(n-1)}{p+1} - \frac{2p}{p+1}C,$$

where $m(\alpha) = \binom{n+p+2}{2}(\alpha-1)+1$ and $C$ is a lower bound for the Ricci curvature of $\Sigma$

As a by product of the technique we obtain the following result:

Let $\Sigma^n$ be a closed minimal submanifold of $\mathbb{S}^{n+p+1}$, $p \geqslant 0$, and let $L$ be the stability operator acting on the space of normal sections. If

$$p \cdot \mathrm{Ric}_\Sigma > -(n-1),$$

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then

$$b_1(\Sigma) \leqslant \binom{n+p+2}{2}\text{Index}(\Sigma).$$

We observe that our curvature hypothesis is less restrictive than the hypotheses in [**5**, **14**], and since the spaces of closed minimal surfaces in $\mathbb{S}^3$ and in $\mathbb{S}^5$ are abundant, see for instance [**12**], we believe that our curvature hypothesis is not so restrictive, see remark 4.5.

Finally, we point out that the last result provides evidence for a positive answer to a well-known conjecture proposed by Schoen and Marques–Neves in the setting of higher codimension.

**The paper is organized in the following way**: In the second section we give a brief summary about the minimal submanifolds theory and about $p$-forms. In the third section we establish conventions that will be used later in the paper and also some technical results that will be necessary in the proof of the main results. In the last section, we provide the proof of the main results.

## 2. Background material

In this section we present some concepts and notations that will be useful throughout this manuscript.

### 2.1. Rough Laplacian

Let $E^N$ be a Riemannian bundle over a closed Riemannian manifold $\Sigma^n$ and denote by $\text{Hom}(T\Sigma, E)$ the homomorphism bundle endowed with a metric $\langle\cdot,\cdot\rangle$ and a linear connection $\nabla$ given by

$$\langle s, r\rangle = \sum_{i=1}^{n}\langle s(e_i), r(e_i)\rangle,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal frame on $\Sigma$, and

$$\nabla_X s(Y) = \nabla_X(s(Y)) - s(\nabla_X Y),$$

for all sections $s$ and $r$ in $\text{Hom}(T\Sigma, E)$ and tangent vectors $X$ and $Y$ on $\Sigma$. It is easy to show that $\langle\cdot,\cdot\rangle$ is well defined and $\text{Hom}(T\Sigma, E)$ endowed with $\nabla$ and $\langle\cdot,\cdot\rangle$ is a Riemannian bundle over $\Sigma$. For all section $s$ in $E$, we note that $\nabla s$ is a section in $\text{Hom}(T\Sigma, E)$, which is given by

$$\nabla s(X) = \nabla_X s,$$

for any vector field $X$ on $\Sigma$. Using the connection on $\text{Hom}(T\Sigma, E)$, consider $\nabla_{X,Y} s$ given by

$$\nabla_{X,Y} s = \nabla_X(\nabla s)(Y),$$

for any $X, Y \in \Gamma(T\Sigma)$.

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DEFINITION 2.1. Given $s \in \Gamma(E)$, the rough Laplacian $\nabla^2 s$ of $s$ is the trace of the bilinear form $(X, Y) \to \nabla_{X,Y} s$.

PROPOSITION 2.2. *Given an arbitrary geodesic frame $\{e_1, \ldots, e_n\}$ on $\Sigma$ at $p$, for any section $s$ in $E$, we have*

$$\nabla^2 s(p) = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} s)(p).$$

*Proof.* See [19, proposition 1.2.1]. □

### 2.2. Minimal immersions and stability operator

A closed submanifold $\Sigma$ of a Riemannian manifold $M$ is called minimal if it is a critical point of the area functional associated to admissible variations. That is, for all smooth maps $\phi : (-\varepsilon, \varepsilon) \times \Sigma \to M$, such that at each $t \in (-\varepsilon, \varepsilon)$, $\phi_t := \phi|_{\{t\}\times\Sigma}$ is an immersion of $\Sigma$ and $\phi_0$ is the inclusion map, we have that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |\Sigma_t| = 0,$$

where $\Sigma_t := \phi_t(\Sigma)$ and $|\Omega|$ means the area of $\Omega$. A straightforward computation gives us

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |\Sigma_t| = -\int_\Sigma \langle H, X\rangle \,\mathrm{d}v_g,$$

where $X = \frac{\partial\phi}{\partial t}(0, \cdot)$ and $H$ is the trace of the second fundamental form of $\Sigma$ in $M$. From this previous formula, we have that $\Sigma$ is a minimal submanifold if and only if $H$ vanishes.

Furthermore, if $\Sigma$ is minimal, for any $Z \in \Gamma(T\Sigma^\perp)$ and $\phi(t, x) = \exp_x(tZ(x))$ for $t \in (-\epsilon, \epsilon)$, where exp is the exponential map of $M$, we have

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} |\Sigma_t| = -\int_\Sigma \langle \nabla^2 Z + \tilde{R}(Z) + \tilde{B}(Z), \quad Z\rangle \,\mathrm{d}v_g,$$

with $\tilde{R}(Z) = \sum_i \bar{R}(Z, e_i)e_i$ and $\tilde{B}(Z) = \sum_{i,j} \langle Z, B(e_i, e_j)\rangle B(e_i, e_j)$ for an orthonormal frame $\{e_1, ..., e_n\}$ on $\Sigma$, where $\bar{R}$ is the curvature tensor of $M$ and $B$ is the second fundamental form of $\Sigma$. The right-hand side of the above equation defines a quadratic form on $\Gamma(T\Sigma^\perp)$ given by

$$Q(Z) = -\int_\Sigma \langle \nabla^2 Z + \tilde{R}(Z) + \tilde{B}(Z), \quad Z\rangle \,\mathrm{d}v_g,$$

and it is known in the literature as *Index form*. The linear operator associated to the quadratic form is

$$LZ = -\nabla^2 Z - \tilde{R}(Z) - \tilde{B}(Z),$$

and it is called the *stability operator*. Simons proved in [19] that $L$ is a strongly elliptic operator on a compact manifold, its spectrum is discrete and consists of an

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increasing sequence of eigenvalues converging to infinity:

$$\lambda_1(L) \leqslant \lambda_2(L) \leqslant \cdots \leqslant \lambda_k(L) \leqslant \cdots \to +\infty,$$

which is counted with multiplicity. Notice that we can estimate the eigenvalues of $L$ using the min-max quotient associated to the quadratic form induced by the operator $L$ as follows:

$$\lambda_k(L) = \inf_{Z\neq 0} \frac{Q(Z)}{\int_\Sigma |Z|^2 \, \mathrm{d}v_g},$$

for any $Z$ orthogonal to $\mathcal{L}_{k-1}$ which is the direct sum of the first $k-1$ eigenspaces of $L$, see [**6**, chapter 4].

Finally, we recall that the Index($\Sigma$) of a minimal submanifold $\Sigma$ is the number of negative eigenvalues of $L$. Furthermore, $\Sigma$ is called stable if Index($\Sigma$) $= 0$ and unstable otherwise. The Index($\Sigma$) measures how far $\Sigma$ is from being a local minimum of the area functional.

### 2.3. The Hodge–Laplacian and harmonic forms

In a closed Riemaniann manifold $\Sigma$, the Hodge–Laplacian on the space of $p$-forms $\Omega^p(\Sigma)$ is the second order differential operator $\Delta_p$ given by

$$\Delta_p = dd^* + d^*d,$$

where $d$ is the exterior differential and $d^*$ is its formal adjoint with respect to the $L^2$-inner product. The eigenvalues of $\Delta_p$, counted with multiplicity, are

$$0 \leqslant \lambda_1(\Delta_p) \leqslant \lambda_2(\Delta_p) \leqslant \cdots \leqslant \lambda_k(\Delta_1) \leqslant \cdots \to +\infty.$$

A $p$-form $\omega \in \Omega^p(\Sigma)$ is called harmonic if $\Delta_p\omega = 0$. The Hodge's Theorem asserts that in a closed Riemannian manifold $\Sigma$, the space of harmonics $p$-forms is isomorphic to the $p$-th de Rham cohomology space of $\Sigma$. Hence, the space of $p$-harmonics forms has dimension equal to the $p$-th Betti's number $b_p(\Sigma)$ of $\Sigma$. In particular, $\lambda_i(\Delta_p) = 0$ for all $i = 1, \ldots, b_p(\Sigma)$.

The Bochner–Weitzenböck formula relates the Hodge–Laplacian and the rough Laplacian. More precisely, on $\Omega^1(\Sigma)$ the Bochner–Weitzenböck formula reads as follows:

$$\Delta_1\omega = -\nabla^2\omega + \mathrm{Ric}(\omega^\sharp, \cdot), \tag{2.1}$$

for any 1-form $\omega$, where $\nabla^2$ is the rough Laplacian on 1-forms and Ric is the Ricci tensor of $\Sigma$.

## 3. Preliminary results

In this section, we will present some results and computations that will simplify the arguments and notation in the proof of the main results.

Let $\eta(x) = -x$ be a unit normal vector field on $\mathbb{S}^{n+p+1}$ and so the second fundamental form of the standard embedding of $\mathbb{S}^{n+p+1}$ in $\mathbb{R}^{n+p+2}$ is $\tilde{A}X = -D_X\eta = X$ for all $X \in \Gamma(T\mathbb{S}^{n+p+1})$, where $D$ is the standard derivative of maps on $\mathbb{R}^{n+p+2}$.

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Let $\Sigma$ be an oriented closed minimal submanifold of $\mathbb{S}^{n+p+1}$, whose second fundamental form will be indicated by $B$ and its associated shape operator in direction of $Z \in \Gamma(T\Sigma^{\perp})$ is denoted by $A_Z$. Precisely, $B(X,Y) = (\bar{\nabla}_X Y)^{\perp}$ and $\langle A_Z X, Y\rangle = \langle B(X,Y), Z\rangle$, for $X$ and $Y$ tangent vectors on $\Sigma$, where $\bar{\nabla}$ is the Levi–Civita connection on the sphere.

The next result summarizes some simple computations.

Lemma 3.1. *Let $V$ be a parallel vector field of $\mathbb{R}^{n+p+2}$. If $V^{\perp}$ and $V^{\top}$ are the normal and tangential components of $V$ along $\Sigma$ in the sphere $\mathbb{S}^{n+p+1}$, then:*

(a) *For all $X \in \Gamma(T\Sigma)$ one has $\nabla^{\perp}_X V^{\perp} = -B(X, V^{\top})$;*

(b) *For all $X \in \Gamma(T\Sigma)$ one has $\nabla_X V^{\top} = \langle V, \eta\rangle X + A_{V^{\perp}} X$;*

(c) $\nabla\langle V, \eta\rangle = -V^{\top}$;

(d) $\Delta\langle V, \eta\rangle = -n\langle V, \eta\rangle$;

(e) $\nabla^2 V^{\perp} = -\tilde{B}(V^{\perp})$,

*where $\nabla$ and $\nabla^{\perp}$ are the connections induced by the Levi–Civita connection of the sphere on the tangent and normal bundle of $\Sigma$ in $\mathbb{S}^{n+p+1}$.*

*Proof.* Noticing that $V = \langle V, \eta\rangle\eta + V^{\perp} + V^{\top}$ and differentiating this identity we obtain

$$\begin{aligned} 0 = D_X V &= -\langle V, X\rangle\eta - \langle V, \eta\rangle X + D_X V^{\perp} + D_X V^{\top} \\ &= -\langle V, X\rangle\eta - \langle V, \eta\rangle X + \langle D_X V^{\perp}, \eta\rangle\eta + \langle D_X V^{\top}, \eta\rangle\eta + \bar{\nabla}_X V^{\perp} + \bar{\nabla}_X V^{\top}. \end{aligned}$$

Taking the normal and tangential component along $\Sigma$ in $\mathbb{S}^{n+p+1}$ of the formula above we get items (a) and (b). Next, note that

$$\nabla\langle V, \eta\rangle = \sum_{i=1}^{n} e_i\langle V,\ \eta\rangle e_i,$$

for $\{e_1 \cdots, e_n\}$ a geodesic frame on $\Sigma$ at $p$. Therefore,

$$\nabla\langle V, \eta\rangle = \sum_{i=1}^{n}(\langle D_{e_i} V, \eta\rangle + \langle V, D_{e_i}\eta\rangle)e_i$$

and thus

$$\nabla\langle V, \eta\rangle = -\sum_{i=1}^{n}\langle V, e_i\rangle e_i = -V^{\top},$$

and so we get item (c).

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Using the definition of Laplacian of functions and items (b) and (c) we have:

$$\begin{aligned}\Delta\langle V,\eta\rangle &= -\sum_{i=1}^{n}\langle\nabla_{e_i}V^\top, e_i\rangle\\ &= -n\langle V,\eta\rangle - \sum_{i=1}^{n}\langle V, B(e_i,e_i)\rangle\\ &= -n\langle V,\eta\rangle,\end{aligned}$$

where we used the minimality of $\Sigma$ here.

Let $\{e_1,...,e_n\}$ be an orthonormal geodesic frame on $\Sigma$ at $p$. Using the proposition 2.2 and item (a) we have

$$\nabla^2 V^\perp = -\sum_i \nabla^\perp_{e_i} B(V^\top, e_i)$$

and therefore, using item (b) and Codazzi equation and omitting the summation sign, we get

$$\begin{aligned}\nabla^2 V^\perp &= -(\nabla^\perp_{e_i}B)(V^\top,e_i) - B(\nabla_{e_i}V^\top, e_i) - B(V^\top, \nabla_{e_i}e_i)\\ &= -(\nabla^\perp_{V^\top}B)(e_i,e_i) - \langle V,\eta\rangle B(e_i,e_i) - B(A_{V^\perp}e_i, e_i) - B(V^\top, \nabla_{e_i}e_i).\end{aligned}$$

Since $\Sigma$ is minimal and $\{e_1,..,e_n\}$ is an orthonormal geodesic frame, the second and last terms in equation above are equal to zero. Using minimality again and the commutativity of tracing and derivative we get

$$\nabla^2 V^\perp = -\sum_i B(A_{V^\perp}e_i, e_i) = -\sum_{i,j}\langle B(e_i,e_j), V^\perp\rangle B(e_i,e_j) = -\tilde{B}(V^\perp),$$

and thus we conclude the desired results. □

We now recall the definition of the Laplacian of a vector field on $\Sigma$. First of all, given a vector field $X$ on $\Sigma$, we can produce an 1-form by the formula

$$X^\flat(Y) = \langle Y, X\rangle,$$

for any $Y$ tangent vector to $\Sigma$. Moreover, given an 1-form $\omega$, the Riesz representation on a Hilbert space gives us a vector field $\omega^\sharp$ defined by

$$\omega(Y) = \langle Y, \omega^\sharp\rangle,$$

for any $Y$ tangent vector to $\Sigma$. Such operations $^\sharp$ and $^\flat$ are known as musical isomorphisms.

Using the Hodge–Laplacian on 1-forms and the musical isomorphisms above, we introduce the following concept.

Definition 3.2. For a vector field $X \in \Gamma(T\Sigma)$, the Laplacian of $X$ is defined by

$$\Delta X = (\Delta_1 X^\flat)^\sharp.$$

The next result provides a relation between the rough Laplacian and the Laplacian introduced above.

LEMMA 3.3. *For any vector field $X \in \Gamma(T\Sigma)$ we have*

$$\nabla^2 X = -\Delta X + (n-1)X - \sum_i A_{B(e_i,X)} e_i,$$

*where $\{e_1, ..., e_n\}$ is a local orthonormal frame on $\Sigma$. Moreover,*

$$\Delta V^\top = nV^\top.$$

*In particular,*

$$\nabla^2 V^\top = -V^\top - \sum_i A_{B(e_i,V^\top)} e_i.$$

*Proof.* Using the Bochner–Weitzenböck formula 2.1 we have

$$\langle \Delta X, Y\rangle = -\langle \nabla^2 X, Y\rangle + \mathrm{Ric}_\Sigma(X, Y),$$

for all $Y \in \Gamma(T\Sigma)$. On the other hand, the Gauss equation and minimality of $\Sigma$ yields us

$$\mathrm{Ric}_\Sigma(X, Y) = \langle (n-1)X - \sum_i A_{B(e_i,X)} e_i, Y\rangle.$$

Replacing this equality in the previous equation we deduce the first formula asserted.

Since the Hodge–Laplacian $\Delta_1$ and exterior differential $d$ commutes, we have from items (c) and (d) of lemma 3.1 that

$$\Delta V^\top = -(\Delta_1(d\langle V, \eta\rangle))^\sharp = -(-d\Delta\langle V, \eta\rangle)^\sharp = nV^\top,$$

and so we conclude the proof. □

LEMMA 3.4. *For any vector $V$ of $\mathbb{R}^{n+p+2}$ and any vector field $X \in \Gamma(T\Sigma)$ we have*:

*(a)* $\nabla\langle V, X\rangle = \langle V, \eta\rangle X + A_{V^\perp} X + \sum_i \langle V, \nabla_{e_i} X\rangle e_i$;

*(b)* $\Delta\langle V, X\rangle = (n-2)\langle V, X\rangle - \langle V, \Delta X\rangle + 2\mathrm{div}(X)\langle V, \eta\rangle - 2\sum_i \langle B(e_i, V^\top), B(e_i, X)\rangle + 2\sum_i \langle A_{V^\perp} e_i, \nabla_{e_i} X\rangle$,

*where $\{e_1, ..., e_n\}$ is a local orthonormal frame on $\Sigma$ at $q$.*

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*Proof.* Let $\{e_1, \cdots, e_n\}$ be an orthonormal geodesic frame on $\Sigma$ at $q$. Using the definition of Laplacian and rough Laplacian we obtain

$$\Delta\langle V, X\rangle = \langle \nabla^2 V^\top, X\rangle + \langle \nabla^2 X, V^\top\rangle + 2\sum_i \langle \nabla_{e_i} V^\top, \nabla_{e_i} X\rangle,$$

and

$$\nabla\langle V, X\rangle = \sum_i e_i\langle V, X\rangle e_i = \sum_i \langle \nabla_{e_i} V^\top, X\rangle e_i + \sum_i \langle V, \nabla_{e_i} X\rangle e_i.$$

Using the equalities above, lemma 3.3 and lemma 3.1 we get the desired formulas. □

Next we have a simple and useful formula that will be used many times in our computations. Its expression is the following:

Lemma 3.5. *For all pair of vector $X, Y$ of $\mathbb{R}^{n+p+2}$, we have*

$$\int_{\mathbb{S}^{n+p+1}} \langle U, X\rangle\langle U, Y\rangle \, d\mathcal{H}^{n+p+1}(U) = C(n,p)\langle X, Y\rangle,$$

*where $C(n,p) = \frac{\omega_{n+p+1}}{n+p+2}$ and $\omega_k$ is the area of the $k$-dimensional unit sphere.*

*Proof.* Using the Divergence Theorem for the vector field $V(q) = \langle q, Y\rangle X$ on the unit ball $\mathbb{B}^{n+p+2}$ and noticing that $\mathbb{S}^{n+p+1}$ is its boundary, we deduce the equality directly. □

Now we introduce a new family of normal vector fields on $\Sigma$ which depends on a given $X \in \Gamma(T\mathbb{R}^{n+p+2})$ and given vectors $V, W$ of $\mathbb{R}^{n+p+2}$. Such family of vector fields $Z_X \in \Gamma(T\mathbb{S}^{n+p+1})$ are given by:

$$Z_X = ((V \wedge W)X)^\perp = \langle W, X\rangle V^\perp - \langle V, X\rangle W^\perp \in \Gamma(T\Sigma^\perp).$$

We point out that such vector field $Z$ can be seen as tri-linear function from the space $\Gamma(T\mathbb{S}^{n+p+1}) \times \Gamma(\mathbb{R}^{n+p+2}) \times \Gamma(\mathbb{R}^{n+p+2})$ to $\Gamma(T\Sigma^\perp)$ and it is skew-symmetric in the variables $V$ and $W$, and this fact will be very useful in this manuscript.

Using the notation established above we are able to prove the following result:

Proposition 3.6. *For any $X \in \Gamma(T\Sigma)$ and $L$ the stability operator, we have*

$$L(Z_X) = -2(n-1)Z_X + Z_{\Delta X} + 2N,$$

*and here*

$$N = -\mathrm{div}(X)Z_\eta + Z_{\sum_i A_{B(X,e_i)}e_i} - Z_{\sum_i B(e_i, \nabla_{e_i} X)} + (B(\nabla\langle W, X\rangle, V^\top) - B(\nabla\langle V, X\rangle, W^\top)),$$

*and $V$ and $W$ are vectors of $\mathbb{R}^{n+p+2}$ and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on $\Sigma$.*

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*Proof.* By definition of $Z_X$ it is enough to compute $\nabla^2(\langle W,X\rangle V^\perp)$. Notice that

$$\nabla^2(\langle W,X\rangle V^\perp) = \Delta\langle W,X\rangle V^\perp + 2\nabla^\perp_{\nabla\langle W,X\rangle} V^\perp + \langle W,X\rangle \nabla^2 V^\perp.$$

Using lemma 3.4, lemma 3.1 items (a) and (e) and $\tilde{R}(V^\perp) = nV^\perp$, we get

$$\begin{aligned} L(\langle W,X\rangle V^\perp) &= -\nabla^2(\langle W,X\rangle V^\perp) - \langle W,X\rangle\tilde{R}(V^\perp) - \langle W,X\rangle\tilde{B}(V^\perp) \\ &= -2(n-1)\langle W,X\rangle V^\perp + \langle W,\Delta X\rangle V^\perp - 2\mathrm{div}(X)\langle W,\eta\rangle V^\perp \\ &\quad + 2\sum_{i=1}^n \langle B(e_i,W^\top),B(e_i,X)\rangle V^\perp - 2\sum_{i=1}^n\langle A_{W^\perp}e_i,\nabla_{e_i}X\rangle V^\perp \\ &\quad + 2B(\nabla\langle W,X\rangle, V^\top). \end{aligned}$$

In a similar fashion we obtain

$$\begin{aligned} -L(\langle V,X\rangle W^\perp) &= 2(n-1)\langle V,X\rangle W^\perp - \langle V,\Delta X\rangle W^\perp + 2\mathrm{div}(X)\langle V,\eta\rangle W^\perp \\ &\quad - 2\sum_{i=1}^n \langle B(e_i,V^\top),B(e_i,X)\rangle W^\perp + 2\sum_{i=1}^n\langle A_{V^\perp}e_i,\nabla_{e_i}X\rangle W^\perp \\ &\quad - 2B(\nabla\langle V,X\rangle, W^\top). \end{aligned}$$

Thus, summing these two formulas and using the notation introduced before we conclude the desired formula. □

The last result in this section provide us a useful and direct formula for the $L^2$-inner product of elements of the form $Z_{(\cdot)}$. In this result we denote by $\mu$ the measure $C(n,p)^{-1}\mathcal{H}^{n+p+1}$.

The computations are:

Lemma 3.7. *For any $X$ and $Y$ in $\Gamma(T\mathbb{R}^{n+p+2})$ the following formulas hold:*

*(a)* $\displaystyle\int_U |V^\perp|^2 \,\mathrm{d}\mu(V) = p+1;$

*(b)* $\displaystyle\int_{U\times U}\langle Z_X,Z_Y\rangle\,\mathrm{d}\mu(V)\mathrm{d}\mu(W) = 2p\langle X,Y\rangle + 2\langle X^\top,Y^\top\rangle + 2\langle X,\eta\rangle\langle Y,\eta\rangle,$

*and here $U = \mathbb{S}^{n+p+1}$ and $(\cdot)^\top$ is the projection on $T\Sigma$.*

*Proof.* The first item follows directly from lemma 3.5, because $|V^\perp|^2 = \sum_i\langle V,\eta_i\rangle^2$, where $\{\eta_1,\cdots,\eta_{p+1}\}$ is an orthonormal frame of $T\Sigma^\perp$.

For the last item, note that

$$\begin{aligned} \langle Z_X,Z_Y\rangle &= \langle W,Y\rangle\langle W,X\rangle|V^\perp|^2 - \langle W,Y\rangle\langle V,X\rangle\langle V^\perp,W\rangle \\ &\quad - \langle V,Y\rangle\langle W,X\rangle\langle W^\perp,V\rangle + \langle V,Y\rangle\langle V,X\rangle|W^\perp|^2, \end{aligned}$$

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and so, using lemma 3.5 and item (1) in this lemma, we obtain:

$$\begin{aligned}\int_{U\times U}\langle Z_X, Z_Y\rangle\,\mathrm{d}\mu(V)\mathrm{d}\mu(W) &= 2(p+1)\langle X,Y\rangle - 2\langle X, Y^{\perp}\rangle\\ &= 2p\langle X,Y\rangle + 2\langle X^{\top}, Y^{\top}\rangle + 2\langle X,\eta\rangle\langle Y,\eta\rangle,\end{aligned}$$

and in the last equality we used that $T\mathbb{R}^{n+p+2} = T\Sigma \oplus T\Sigma^{\perp} \oplus \langle \eta\rangle$. □

## 4. Statements and proof of the main results

We restate our main results, and provide a proof for each one of them. The first result reads as follows:

THEOREM 4.1. *Let $\Sigma^n$ be a closed minimal submanifold of $\mathbb{S}^{n+p+1}$, $p \geqslant 0$, and let $L$ be the stability operator acting on the space of normal sections. Then,*

$$\lambda_\alpha(L) \leqslant \lambda_{m(\alpha)}(\Delta_1) - \frac{2(n-1)}{p+1} - \frac{2p}{p+1}C,$$

*where $m(\alpha) = \binom{n+p+2}{2}(\alpha-1)+1$ and $C$ is a lower bound for the Ricci curvature of $\Sigma$.*

*Proof.* Let $\{N_1, N_2, \dots, N_k, \dots\}$ be an orthonormal basis of the space of normal sections $\Gamma(T\Sigma^{\perp})$ formed by eigensections of $L$, where $N_i$ is an eigensection associated to $\lambda_i(L)$. Given a positive integer $\alpha$, if $\mathcal{L}_m$ denotes the direct sum of the $m$ first eigenspaces of $\Delta$, we want to find a non-zero $X \in \Gamma(T\Sigma)$ such that

$$\int_\Sigma \langle Z_X, N_1\rangle\,\mathrm{d}v_g = \int_\Sigma \langle Z_X, N_2\rangle\,\mathrm{d}v_g = \cdots = \int_\Sigma \langle Z_X, N_{\alpha-1}\rangle\,\mathrm{d}v_g = 0, \tag{4.1}$$

for all pairs of parallel vector fields $V$ and $W$. As $Z$ is a skew symmetric bilinear form on the variables $V$ and $W$ and the space of parallel vector fields of $\mathbb{R}^{n+p+2}$ has dimension equal to $n+p+2$, the problem of finding X is equivalent to the problem of finding a non-zero solution to a homogeneous system with $\binom{n+p+2}{2}(\alpha-1)$ equations and $m$ unknown variables. So, $X$ satisfying 4.1 exists if

$$m \geqslant m(\alpha) = \binom{n+p+2}{2}(\alpha-1)+1.$$

Thus, under the previous condition, the min-max principle for quadratic forms gives us

$$\begin{aligned}\lambda_\alpha(L)\int_\Sigma |Z_X|^2 &\leqslant \int_\Sigma \langle L(Z_X), Z_X\rangle\\ &= -2(n-1)\int_\Sigma |Z_X|^2 + \int_\Sigma \langle Z_{\Delta X}, Z_X\rangle + 2\int_\Sigma \langle N, Z_X\rangle,\end{aligned} \tag{4.2}$$

and here and from now on we omit the Riemannian volume element $\mathrm{d}v_g$ on $\Sigma$.

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Using lemma 3.7 item (b) we get, for $U = \mathbb{S}^{n+p+1}$:

$$\int_{U\times U} |Z_X|^2 \,\mathrm{d}\mu\mathrm{d}\mu = 2(p+1)|X|^2,$$

$$\int_{U\times U} \langle Z_{\Delta X}, Z_X\rangle \,\mathrm{d}\mu\mathrm{d}\mu = 2(p+1)\langle X, \Delta X\rangle,$$

$$\int_{U\times U} \mathrm{div}(X)\langle Z_\eta, Z_X\rangle \,\mathrm{d}\mu\mathrm{d}\mu = \mathrm{div}(X)\cdot 0 = 0,$$

$$\int_{U\times U} \langle Z_{\sum_i A_{B(X,e_i)}e_i}, Z_X\rangle \,\mathrm{d}\mu\mathrm{d}\mu = 2(p+1)\sum_j |B(X,e_j)|^2,$$

$$\int_{U\times U} \langle Z_{\sum_i B(e_i,\nabla_{e_i}X)}, Z_X\rangle \,\mathrm{d}\mu\mathrm{d}\mu = 0,$$

where $\{e_1,\cdots,e_n\}$ is an orthonormal frame on $\Sigma$. For the last term, we consider $\{\eta_1,\cdots,\eta_{p+1}\}$ a local orthonormal frame for $T\Sigma^\perp$. Denoting

$$\langle B(\nabla\langle W,X\rangle, V^\top) - B(\nabla\langle V,X\rangle, W^\top), Z_X\rangle$$

by $\Omega$, and after a direct computation we have:

$$\begin{aligned}
\Omega = {} & \langle W,\eta\rangle\langle W,X\rangle\langle B(X,V^\top),V\rangle + \langle W,X\rangle\langle V,e_i\rangle\langle B(A_{W^\perp}X,e_i),V\rangle \\
& + \langle W,\nabla_{e_j}X\rangle\langle W,X\rangle\langle V,e_i\rangle\langle B(e_j,e_i),V\rangle - \langle W,\eta\rangle\langle V,X\rangle\langle B(X,V^\top),W\rangle \\
& - \langle V,X\rangle\langle V,e_i\rangle\langle W,\eta_k\rangle\langle B(A_{\eta_k}X,e_i),W\rangle - \langle W,\nabla_{e_j}X\rangle\langle V,X\rangle\langle B(e_j,V^\top),W\rangle \\
& - \langle V,\eta\rangle\langle W,X\rangle\langle B(X,W^\top),V\rangle - \langle W,X\rangle\langle W,e_i\rangle\langle V,\eta_k\rangle\langle B(A_{\eta_k}X,e_i),V\rangle \\
& - \langle V,\nabla_{e_j}X\rangle\langle W,X\rangle\langle B(e_j,W^\top),V\rangle + \langle V,\eta\rangle\langle V,X\rangle\langle B(X,W^\top),W\rangle \\
& + \langle V,X\rangle\langle W,e_i\rangle\langle B(A_{V^\perp}X,e_i),W\rangle + \langle V,\nabla_{e_j}X\rangle\langle V,X\rangle\langle W,e_i\rangle\langle B(e_j,e_i),W\rangle,
\end{aligned}$$

and so

$$\int_{U\times U} \langle B(\nabla\langle V,X\rangle, W^\top) - B(\nabla\langle W,X\rangle, V^\top), Z_X\rangle \,\mathrm{d}\mu\mathrm{d}\mu = -2\sum_{k=1}^{p+1} |A_{\eta_k}X|^2.$$

On the other hand, we have

$$\sum_j |B(X,e_j))|^2 = \sum_j\sum_k \langle B(X,e_j),\eta_k\rangle^2 = \sum_j\sum_k \langle A_{\eta_k}X, e_j\rangle^2 = \sum_k |A_{\eta_k}X|^2,$$

and by the Gauss equation and minimality of $\Sigma$ we deduce

$$\sum_j |B(X,e_j)|^2 = (n-1)|X|^2 - \mathrm{Ric}_\Sigma(X,X).$$

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Integrating inequality (4.2) with respect to $(V, W) \in U \times U$ and using Fubini theorem we deduce, from the previous equalities, that

$$2(p+1)\lambda_\alpha(L)\int_\Sigma |X|^2 \leqslant -4(n-1)\int_\Sigma |X|^2 + 2(p+1)\int_\Sigma \langle X, \Delta X\rangle - 4p\int_\Sigma \mathrm{Ric}_\Sigma(X,X).$$

Since $X$ is a non-zero vector field in $\mathcal{L}_{m(\alpha)}$, it verifies the inequality

$$\int_\Sigma \langle \Delta X, X\rangle \leqslant \lambda_{m(\alpha)}(\Delta_1)\int_\Sigma |X|^2,$$

and hence we get

$$\lambda_\alpha(L) \leqslant \lambda_{m(\alpha)}(\Delta_1) - \frac{2(n-1)}{p+1} - \frac{2p}{p+1}C.$$

□

REMARK 4.2. As we pointed out before, in codimension one case ($p = 0$) we do not have geometric constraints on the submanifold, i.e., the lower bound on the Ricci curvature of the submanifold is not necessary.

Following the same arguments used by Savo in [**18**] and under a suitable hypothesis we are able to prove the following lower bound to the Morse index:

THEOREM 4.3. *Let $\Sigma^n$ be a closed minimal submanifold of $\mathbb{S}^{n+p+1}$, $p \geqslant 0$, and let $L$ be the stability operator acting on the space of normal sections. If $p \cdot \mathrm{Ric}_\Sigma > -(n-1)$, then*

$$\mathrm{Index}(\Sigma) \geqslant \frac{b_1(\Sigma)}{\binom{n+p+2}{2}}.$$

*Proof.* Taking

$$\alpha = \left\lfloor \frac{b_1(\Sigma) + \binom{n+p+2}{2} - 1}{\binom{n+p+2}{2}} \right\rfloor,$$

where $\lfloor x \rfloor$ is the denoting the largest integer which is $\leqslant x$. By definition, we have $m(\alpha) \leqslant b_1(\Sigma)$ and thus, using theorem 4.1, we obtain

$$\begin{aligned}\lambda_\alpha(L) &\leqslant \lambda_{b_1(\Sigma)}(\Delta_1) - \frac{2(n-1)}{p+1} - \frac{2p}{p+1}\inf_{|v|=1}(\mathrm{Ric}_\Sigma(v,v)) \\ &= -\frac{2(n-1)}{p+1} - \frac{2p}{p+1}\inf_{|v|=1}(\mathrm{Ric}_\Sigma(v,v)),\end{aligned}$$

and by our hypothesis on the Ricci curvature of $\Sigma$, we get

$$\lambda_\alpha(L) < 0.$$

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Since $\alpha \geqslant \frac{b_1(\Sigma)}{\binom{n+p+2}{2}}$, follows from the inequality above that

$$\text{Index}(\Sigma) \geqslant \frac{b_1(\Sigma)}{\binom{n+p+2}{2}}.$$

□

So, a direct consequence is the following:

COROLLARY 4.4. *Let $\Sigma^n$ be a closed minimal submanifold of $\mathbb{S}^{n+2}$, and let $L$ be the stability operator acting on the space of normal sections. If* $\text{Ric}_\Sigma > -(n-1)$, *then*

$$\text{Index}(\Sigma) \geqslant \frac{b_1(\Sigma)}{\binom{n+3}{2}}.$$

REMARK 4.5. It would be interesting to know whether there is a sequence of $n$-dimensional closed minimal submanifolds in the unit sphere with codimension $p$, and $p \cdot \text{Ric}_\Sigma > -(n-1)$ and also unbounded first Betti number. What we already know is that the first Betti's number, under the lower bound of the Ricci curvature, has an upper bound depending on the codimension and the diameter, see [**8**, theorem 5.21], which can be large for big values of the diameter.

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