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POSITIVE CURVATURE, PARTIAL VANISHING THEOREMS AND COARSE INDICES

JOHN ROE
Department of Mathematics, Penn State University, University Park, PA 16802, USA (john.roe@psu.edu)

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Abstract We generalize the relative index theorem of Gromov and Lawson, using coarse geometry.

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1. Introduction

Let M be a complete Riemannian manifold and let D be a generalized Dirac operator acting on sections of a Clifford bundle S over M. It is well known (see, for example, [**1**]) that there is a Weitzenbock formula

$$D^2 = \nabla^*\nabla + R,$$

where R is a certain self-adjoint endomorphism of S constructed out of the curvature. (For example, in the classical case of the Dirac operator associated with a spin-structure, R is pointwise multiplication by $\frac{1}{4}$ times the scalar curvature [**4**].)

The author's coarse index theory associates with D an index that lies in the K-theory of the 'translation C^*-algebra' $C^*(M)$. As in the classical case, the index vanishes if the curvature operator is uniformly bounded below by a positive constant. In [**7**, Proposition 3.11] this statement is generalized as follows. Suppose that there is a subset $Z \subseteq M$ such that for some constant $a > 0$ one has $R_x \geqslant a^2 I$ (as self-adjoint endomorphisms of S_x) for all $x \notin Z$; we will then say that the operator R is *uniformly positive outside* Z. The index of D then lies in the image of the map

$$K_*(C^*(Z)) \to K_*(C^*(X)),$$

where Z is considered as a metric subspace of X. In particular, if the curvature is uniformly positive outside a compact set Z (so that $C^*(Z)$ is the compact operators), one recovers the result of Gromov and Lawson [**1**, Chapter 3] that D has an index in the ordinary Fredholm sense.

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Only the briefest sketch of a proof of this proposition was included in [**7**]. This note is a response to several requests for more detail and also mentions a couple of applications of the idea.

2. The main result

Let Z be a subset of a proper metric space X and let H be an ample X-module (i.e. a Hilbert space that is a 'sufficiently large' module over $C_0(X)$, assumed fixed). (The reader is referred to [**2**] for terminology; the module action is denoted by $\rho\colon C_0(X) \to \mathfrak{B}(H)$.) The C^*-algebra $C^*(X)$ (or $C^*(X; H)$ if it is important to keep track of the particular Hilbert space) is then defined to be the norm closure of the controlled locally compact operators on H, where we recall that a *controlled* (also known as *finite propagation*) operator T has the property that there is a constant r for which

$$d(\operatorname{Supp}\varphi, \operatorname{Supp}\psi) > r \quad \Longrightarrow \quad \rho(\varphi)T\rho(\psi) = 0$$

for all $\varphi, \psi \in C_c(X)$. A controlled operator T is *supported near* Z if there is another constant r' for which

$$d(\operatorname{Supp}\varphi, Z) > r' \quad \Longrightarrow \quad \rho(\varphi)T = 0 = T\rho(\varphi).$$

The norm closure of the set of controlled locally compact operators supported near Z is an ideal in $C^*(X)$, which we denote* by $C^*(Z \subseteq X)$. It is easy to see [**3**] that the K-theory of $C^*(Z \subseteq X)$ is the same as that of $C^*(Z)$, if Z is considered as a metric space in its own right.

Now we recall the relation of these concepts to index theory. Suppose that X is actually a complete Riemannian manifold and that S is a Clifford bundle and let $H = L^2(X; S)$ in forming the algebras above. The algebra $D^*(X)$ is defined to be the norm closure of the controlled *pseudolocal* operators on H: it is a unital C^*-algebra and $C^*(X)$ is an ideal in it. The following key analytic lemma [**2**, Chapter 10] can be proved by the finite propagation speed method.

Lemma 2.1. *Let X be a complete Riemannian manifold, as above, and let S be a Clifford bundle over it. Let D denote the Dirac operator of S, considered as an unbounded self-adjoint operator on $H = L^2(X; S)$. If f is a bounded continuous function on $\mathbb{R}$ that has finite limits at $\pm\infty$, then $f(D) \in D^*(X; H)$. If f tends to zero at $\pm\infty$, then $f(D) \in C^*(X; H)$.*

A *normalizing function* $\chi\colon \mathbb{R} \to [-1, 1]$ is, by definition, a continuous odd function that tends to ± 1 at $\pm\infty$. Given such a function χ, it follows from the preceding lemma that $\chi(D) \in D^*(X)$ and $\chi(D)^2 - 1 \in C^*(X)$. Moreover, if χ_1 and χ_2 are two normalizing functions, then it similarly follows that $\chi_1(D) - \chi_2(D) \in C^*(X)$. Thus, the equivalence class of $\chi(D)$ gives a well-defined self-adjoint involution in $D^*(X)/C^*(X)$, defining an element $[\chi(D)] \in K_{j+1}(D^*(X)/C^*(X))$ (j is determined by the grading of the operator: it is equal to the parity of $\dim X$). We now have the following definition.

* It is denoted $C^*_X(Z)$ in [**7**], but the other notation now seems better.

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Definition 2.2. With the notation of Lemma 2.1, the *coarse index* of D is

$$\mathrm{Index}(D) = \partial[\chi(D)] \in K_j(C^*(X)),$$

where $\partial\colon K_{j+1}(D^*(X)/C^*(X)) \to K_j(C^*(X))$ is the boundary map in the long exact sequence of C^*-algebra K-theory.

Now let $Z \subseteq X$ as above. The algebra $C^*(Z \subseteq X)$ is an ideal in $D^*(X)$ (not just in $C^*(X)$). To prove our result we will need to sharpen Lemma 2.1 as follows.

Lemma 2.3. *Let the notation be as in Lemma 2.1. Suppose that the curvature operator $R = R_D$ that appears in the Weitzenbock formula for D,*

$$D^2 = \nabla^*\nabla + R_D,$$

is uniformly positive outside Z, say $R_x \geqslant a^2 I$ for $x \notin Z$. Then, for any $f \in C_c(-a, a)$ we have $f(D) \in C^*(Z \subseteq X)$.

Suppose that this lemma has been proved. Then choose a normalizing function χ such that $\chi^2 - 1$ is supported in $(-a, a)$. According to Lemma 2.3, the equivalence class of $\chi(D)$ is a (well-defined) self-adjoint involution in $D^*(X)/C^*(Z \subseteq X)$. Following the construction above, we obtain a *localized index*

$$\mathrm{Index}_Z(D) \in K_j(C^*(Z \subseteq X))$$

that maps to the previously defined $\mathrm{Index}(D)$ under the K-theory map induced by the inclusion $C^*(Z \subseteq X) \to C^*(X)$. The existence of this localized index is the precise content of [**7**, Proposition 3.11]; it implies the version of the result stated in the introduction. To state it precisely, we have the following theorem.

Theorem 2.4. *Let M be a complete Riemannian manifold and let D be a Dirac-type operator whose associated curvature endomorphism R_D is uniformly positive outside a subset Z of M. Then the construction above defines a localized coarse index*

$$\mathrm{Index}_Z(D) \in K_j(C^*(Z \subseteq X))$$

that maps to the coarse index $\mathrm{Index}(D) \in K_j(C^*(M))$ under the K-theory map induced by the inclusion $C^*(Z \subseteq X) \to C^*(X)$. (Here j is the parity of $\dim M$.)

The rest of this section will give the proof of Lemma 2.3. In order to use the finite propagation speed method, we first consider the properties of functions f that have compactly supported Fourier transforms.

Lemma 2.5. *With notation as in Lemma 2.3, suppose that $f \in \mathcal{S}(\mathbb{R})$ is an even function and has Fourier transform $\hat{f}$ supported in $(-r, r)$. Let $\varphi \in C_0(X)$ have support disjoint from a $2r$-neighbourhood of Z. Then*

$$\|f(D)\rho(\varphi)\| \leqslant \|\varphi\| \sup\{|f(\lambda)|\colon |\lambda| \geqslant a\}$$

and the same estimate applies to $\rho(\varphi)f(D)$.

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Proof. We use the Fourier cosine formula

$$f(D) = \frac{1}{\pi} \int_0^r \hat{f}(t) \cos(tD) \, \mathrm{d}t,$$

remembering that $\hat{f}(t)$ vanishes for $t > r$. Now let $U_n = \{x \in X\colon d(x, Z) > nr\}$, for $n = 1, 2$, and consider the unbounded symmetric operator that is equal to D^2 with domain $C_c^\infty(U_1)$. This operator is bounded below by $a^2 I$ and therefore it has a Friedrichs extension on the Hilbert space $L^2(U_1; S)$, which is also bounded below (with the same bound) and which we shall denote by E.

A standard finite propagation speed argument shows that if s is smooth and compactly supported in U_2, then

$$\cos(tD)s = \cos(t\sqrt{E})s \quad \text{for } 0 \leqslant t \leqslant r.$$

In particular, $\cos(tD)\rho(\varphi) = \cos(t\sqrt{E})\rho(\varphi)$ for these values of t. Via the Fourier integral above, this implies that $f(D)M_\varphi = f(\sqrt{E})M_\varphi$. But since the spectrum of $\sqrt{E}$ is bounded below by a,

$$|f(\sqrt{E})| \leqslant \sup\{|f(\lambda)|\colon |\lambda| \geqslant a\},$$

and this gives the desired estimate. □

There is a version of Lemma 2.5 without the evenness hypothesis.

Lemma 2.6. *With notation as above, suppose that $f \in \mathcal{S}(\mathbb{R})$ has Fourier transform $\hat{f}$ supported in $(-r, r)$. Let $\varphi \in C_0(X)$ have support disjoint from a $4r$-neighbourhood of Z. Then,*

$$\|f(D)\rho(\varphi)\| \leqslant 2\|\varphi\| \sup\{|f(\lambda)|\colon |\lambda| \geqslant a\}$$

and the same estimate applies to $\rho(\varphi)f(D)$.

Proof. If f is even, this is a consequence of Lemma 2.5. If f is odd, use the C^*-identity to write

$$\|f(D)\rho(\varphi)\|^2 \leqslant \|\rho(\bar{\varphi})\| \, \||f|^2(D)\rho(\varphi)\|.$$

The function $g = |f|^2$ is even, belongs to $\mathcal{S}(R)$ and has Fourier transform supported in $(-2r, 2r)$. Thus, applying Lemma 2.5 to the function g,

$$\||f|^2(D)\rho(\varphi)\| \leqslant \|\varphi\| \sup\{|f(\lambda)|^2\colon |\lambda| \geqslant a\},$$

and so we obtain (on taking the square root)

$$\|f(D)\rho(\varphi)\| \leqslant \|\varphi\| \sup\{|f(\lambda)|\colon |\lambda| \geqslant a\},$$

which gives the desired result for odd f. The general result is obtained by writing f as a sum of even and odd components (this decomposition accounts for the extra factor of 2 in the statement of Lemma 2.6). □

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Using this, let us complete the proof of Lemma 2.3. Let f be as in that lemma and let $\varepsilon > 0$ be given. There exists a smooth function g with compactly supported Fourier transform such that $\sup\{|g(\lambda) - f(\lambda)| \colon \lambda \in \mathbb{R}\} < \varepsilon$. In particular, $|g(\lambda)| < \varepsilon$ for $|\lambda| > a$. Let r be such that $\mathrm{Supp}(\hat{g}) \subseteq (-r, r)$ and let $\psi \colon X \to [0, 1]$ be a continuous function equal to 1 on a $4r$-neighbourhood of Z and vanishing off a $5r$-neighbourhood of Z. Write

$$f(D) = \rho(\psi)g(D)\rho(\psi) + \rho(1-\psi)g(D)\rho(\psi) + g(D)\rho(1-\psi) + (f(D) - g(D)).$$

The first term is a locally compact operator supported near Z, the second and third terms have norm bounded by 2ε by Lemma 2.6, and the fourth term has norm bounded by ε by the spectral theorem. Thus, $f(D)$ lies within 5ε of a locally compact operator supported near Z. Since ε is arbitrary, $f(D) \in C^*(Z \subseteq X)$, as was to be shown.

3. Vanishing results

As a consequence of the discussion above, if the curvature operator R is uniformly positive outside Z, and if the K-theory map $K_*(C^*(Z)) \to K_*(C^*(X))$ is zero, then the index $\mathrm{Index}(D) \in K_*(C^*(X))$ must vanish. The usual vanishing theorem establishes this result when $Z = \emptyset$ (i.e. when we have uniformly positive curvature on the whole of M), so we can regard these sorts of results as a generalization where one allows a 'small amount' of non-positive curvature.

For example, we have the following proposition.

Proposition 3.1. *Let M be a complete connected non-compact Riemannian manifold and let D be a Dirac-type operator whose associated curvature R is uniformly positive outside a compact set. Then,* $\mathrm{Index}(D) = 0$.

Proof. Let K be a compact set outside of which the curvature is uniformly positive and let Z be the union of K and a geodesic ray from one of its points to infinity. The index of D then lies in the image of $K_*(C^*(Z)) \to K_*(C^*(X))$ by the discussion above. But Z is coarsely equivalent to $\mathbb{R}^+$, so $K_*(C^*(Z)) = 0$. □

As another example, imagine that we are in the situation of the 'partitioned manifold index theorem' of [**5**]. So, let M be a non-compact manifold that is partitioned by a compact hypersurface N, which (say) is spin and of non-zero $\hat{\mathcal{A}}$-genus, into two pieces M^+ and M^-.

Proposition 3.2. *A partitioned manifold as described above admits no complete metric that has uniformly positive scalar curvature on just one of the partition components (M^+ or M^-).*

Proof. Suppose that M has such a metric. Using the distance from N, construct a proper coarse map $g \colon M \to \mathbb{R}$ that induces the given partition. By definition, the partitioned manifold index is

$$g_*(\mathrm{Index}\, D) \in K_1(C^*(|\mathbb{R}|)) = \mathbb{Z},$$

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and the index theorem of [**5**] equates this to the $\hat{\mathcal{A}}$-genus of N. Now suppose that M has positive scalar curvature over M^+. Then by our main result, the coarse index factors through $K_1(C^*(M^- \subseteq M))$. But considering the commutative diagram

$$\begin{array}{ccc} K_1(C^*(M^- \subseteq M)) & \longrightarrow & K_1(C^*(M)) \\ \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle g_*} \\ K_1(C^*(\mathbb{R}^- \subseteq \mathbb{R})) & \longrightarrow & K_1(C^*(\mathbb{R})) \end{array}$$

and noting that the bottom left-hand group is zero, we see that the coarse index vanishes. □

4. The relative index theorem

The key technical result of [**1**, Chapter 4] is a relative index theorem that may be expressed as follows.

Suppose that M_1 and M_2 are complete Riemannian manifolds equipped with generalized Dirac operators D_1 and D_2, respectively, acting on (graded) Clifford bundles S_1 and S_2. Suppose further that these items *agree near infinity*: in other words, suppose that there exist compact sets $Z_i \subseteq M_i$ and an isometry $h\colon M_1 \setminus Z_1 \to M_2 \setminus Z_2$ that is covered by a bundle isomorphism from S_1 to S_2, and that this isomorphism conjugates D_1 to D_2.

In these circumstances one can define a *relative topological index* $\operatorname{Index}_r(D_1, D_2) \in \mathbb{Z}$. There are several ways to define this quantity. For instance, one can compactify each of the M_i identically outside Z_i (thus obtaining *compact* manifolds $\tilde{M}_i$ with elliptic operators $\tilde{D}_i$) and then take the difference of the ordinary Fredholm indices, $\operatorname{Index}(\tilde{D}_1) - \operatorname{Index}(\tilde{D}_2)$, to define the relative index. Alternatively, one can take the Chern–Weil forms $\mathfrak{a}_i$ that are the representatives of the indices of D_i according to the local index theorem, and 'integrate their difference' over $M_1 \cup M_2$: specifically, note that h^* takes $\mathfrak{a}_2$ to $\mathfrak{a}_1$, so that if we let $\mathfrak{a}$ be any smooth form on M_2, supported outside Z_2 and agreeing with $\mathfrak{a}_2$ near infinity, then the difference

$$\int_{M_1} (\mathfrak{a}_1 - h^*\mathfrak{a}) - \int_{M_2} (\mathfrak{a}_2 - \mathfrak{a})$$

is well defined (the integrands are compactly supported) and independent of the choice of $\mathfrak{a}$, and may be taken as the definition of the 'integral of the difference of Chern–Weil forms'. The equality of these two definitions of relative index is essentially Proposition 4.6 of [**1**]: it shows both that the first definition is independent of the choice of compactification and that the second definition yields an integer.

Remark 4.1. Either definition implies that the relative index $\operatorname{Index}_r(D_1, D_2)$ depends only on the geometry of M_1 and M_2 (and the associated operators) in a neighbourhood of the 'regions of disagreement' Z_1 and Z_2. This stability property of the relative index is the basis for several calculations in [**1**].

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Now suppose further that D_1 and D_2 have uniformly positive Weitzenbock curvature operators at infinity. Then D_1 and D_2, individually, are Fredholm operators, by [**1**, Theorem 3.2] (a special case of our Theorem 2.4). The relative index theorem then states the following proposition.

Proposition 4.2 (Gromov and Lawson [1, Theorem 4.18]). *In the circumstances described above one has*

$$\operatorname{Index}(D_1) - \operatorname{Index}(D_2) = \operatorname{Index}_r(D_1, D_2).$$

We are going to generalize this result by allowing the 'regions of disagreement' Z_i to be non-compact. The first thing that we need to do is to *define* the relative index in this case. The following discussion, which is based on the ideas of [**6**], leads up to the generalized definition of the relative index, Definition 4.5.

Let M_1 and M_2 be complete Riemannian manifolds (as above) and let D_1 and D_2 be generalized Dirac operators. Suppose that M_1 and M_2 are equipped with coarse maps q_1 and q_2 to a *control space* X (a proper metric space) and that Z is a subset of X. Put $Z_i = q_i^{-1}(Z) \subseteq M_i$ for $i = 1, 2$. Suppose that there is a diffeomorphism $h\colon M_1 \setminus Z_1 \to M_2 \setminus Z_2$ that is covered by an isomorphism of Clifford bundles and Dirac operators and is *compatible with the control maps* in the sense that $q_1 = q_2 \circ f$.

From these data one can define a relative index in $K_j(C^*(Z))$. Let H_i be the Hilbert space $L^2(M_i; S_i)$ and regard each H_i as an X-module via the control map q_i. In this way we obtain translation algebras $C^*(X; H_i)$, $i = 1, 2$, each of which contains an ideal $C^*(Z \subseteq X; H_i)$ corresponding to Z. The isometry h between the M_i outside Z_i passes to a unitary isomorphism V between the $L^2(M_i \setminus Z_i; S_i)$ and it is easy to see that conjugation by this unitary induces an isomorphism of quotient C^*-algebras

$$\Phi\colon C^*(M_1; H_1)/C^*(Z_1 \subseteq M_1; H_1) \to C^*(M_2; H_2)/C^*(Z_2 \subseteq M_2; H_2).$$

Lemma 4.3. *Let the notation be as above and let $f \in C_0(\mathbb{R})$. Then,*

$$\Phi[f(D_1)] = [f(D_2)]$$

in the quotient algebra $C^*(M_2; H_2)/C^*(Z_2 \subseteq M_2; H_2)$.

There is also a 'D^*-version' of this discussion. Namely, following [**9**] we can define ideals $D^*(Z_i \subseteq M_i; H_i)$ as the closure of the finite propagation pseudolocal* operators that are supported near Z_i and are locally compact on $M_i \setminus Z_i$. Once again, conjugation by U induces an isomorphism of quotient C^*-algebras

$$\Psi\colon D^*(M_1; H_1)/D^*(Z_1 \subseteq M_1; H_1) \to D^*(M_2; H_2)/D^*(Z_2 \subseteq M_2; H_2).$$

Lemma 4.4. *Let notation be as above and let χ be a normalizing function. Then*

$$\Psi[\chi(D_1)] = [\chi(D_2)]$$

in the quotient algebra $D^*(M_2; H_2)/D^*(Z_2 \subseteq M_2; H_2)$.

* 'Finite propagation' is defined with respect to the control space X via the control maps q_i; 'pseudolocal' is defined with respect to the ambient manifold M_i.

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Proof. The proofs of Lemmas 4.3 and 4.4 both rely on the finite propagation speed method. First we give the proof for Lemma 4.3. Suppose that $f \in \mathcal{S}(\mathbb{R})$ and has Fourier transform $\hat{f}$ supported in $(-r, r)$. As usual, we write

$$f(D) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) \mathrm{e}^{\mathrm{i}tD} \, \mathrm{d}t$$

and use the fact that, for a Dirac-type operator D, $\mathrm{e}^{\mathrm{i}tD}$ has propagation $|t|$. Let $\psi_i \colon M_i \to [0, 1]$ be a smooth function equal to 1 on an r-neighbourhood of Z_i and vanishing off a $2r$-neighbourhood of Z_i, and such that $\psi_1 = \psi_2 \circ h$ on $M_1 \setminus Z_1$. Write

$$f(D_i) = f(D_i)\rho(\psi_i) + f(D_i)(1 - \rho(\psi_i)).$$

Since $f(D_i)$ has propagation r, the first term belongs to $C^*(Z_i \subseteq H_i)$. By finite propagation speed we have

$$V^* \mathrm{e}^{\mathrm{i}tD_1} \rho(1 - \psi_1) V = \mathrm{e}^{\mathrm{i}tD_2} \rho(1 - \psi_2) \quad \text{for } |t| < r.$$

Consequently,

$$V^* f(D_1)(1 - \rho(\psi_1))V = f(D_2)(1 - \rho(\psi_2))$$

and the proof is complete for f having compactly supported Fourier transform. The general result follows, since such f are norm-dense in $C_0(\mathbb{R})$.

The proof of Lemma 4.4 follows a similar pattern, where the Fourier transform $\hat{\chi}$ must now be understood as a distribution with a mild singularity at 0. The only additional argument that is needed is to show that

$$(V\chi(D_1)V^* - \chi(D_2))\rho(\varphi) \tag{4.1}$$

is compact for $\varphi \in C_0(M_2 \setminus Z_2)$. Suppose in fact that φ is compactly supported. Then there is a constant $r > 0$ such that $d(Z_2, \mathrm{Supp}(\varphi)) > r$ and if we should choose the normalizing function χ to have Fourier transform supported in $(-r, r)$, then finite propagation speed shows that the displayed quantity in 4.1 is not just compact: it is actually *zero*! The general case follows from this particular one, since any two normalizing functions differ by some $g \in C_0(\mathbb{R})$ and we already know that for such g the *individual* terms $g(D_1)$ and $g(D_2)$ are locally compact. □

Now let π_i denote the quotient map $C^*(M_i) \to C^*(M_i)/C^*(Z_i \subseteq M_i)$ or $D^*(M_i) \to D^*(M_i)/D^*(Z_i \subseteq M_i)$ as appropriate. Let us define A to be the pull-back C^*-algebra

$$A = \{(T_1, T_2) \in C^*(M_1; H_1) \oplus C^*(M_2; H_2) \colon \Phi(\pi_1(T_1)) = \pi_2(T_2)\}.$$

Similarly, define B to be the pull-back C^*-algebra

$$B = \{(T_1, T_2) \in D^*(M_1; H_1) \oplus D^*(M_2; H_2) \colon \Phi(\pi_1(T_1)) = \pi_2(T_2)\}.$$

Then A is an ideal in B. Let D denote the Dirac operator on the disjoint union $M_1 \sqcup M_2$. Lemmas 4.3 and 4.4 show that for a normalizing function χ, the operator $\chi(D)$ is an

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element of B, and that for a function $f \in C_0(\mathbb{R})$, the operator $f(D)$ is an element of the ideal A. Consequently, there is defined an index of D,

$$\operatorname{Index}_Z(D) \in K_j(A). \tag{4.2}$$

The group $K_j(A)$ can be decomposed as a direct sum. In fact, let $U\colon H_1 \to H_2$ be a *covering isometry* for the identity map [**2**, Definition 6.3.9] that agrees on $L^2(M_1 \setminus Z_1)$ with the isomorphism $L^2(M_1 \setminus Z_1) \to L^2(M_2 \setminus Z_2)$ induced by h. (The hypothesis that h boundedly commutes with the control maps assures the existence of such an isometry.) There is then a split short exact sequence

$$0 \to C^*(Z_1 \subseteq M_1) \to A \to C^*(M_2) \to 0, \tag{4.3}$$

where the first map is $a \mapsto (a, 0)$, the second is $(a_1, a_2) \mapsto a_2$, and the splitting maps a to (U^*aU, a). From this split short exact sequence we obtain a direct sum decomposition

$$K_j(A) = K_j(C^*(Z_1 \subseteq M_1)) \oplus K_j(C^*(M_2)).$$

Definition 4.5. The *relative index* of the above data is the component in $K_j(C^*(Z_1 \subseteq M_1)) = K_j(C^*(Z))$ of $\operatorname{Index}_Z(D) \in K_j(A)$. We denote it by $\operatorname{Index}_r(D_1, D_2)$.

The generalization of Gromov–Lawson's relative index theorem is then the following theorem.

Theorem 4.6. *Let (M_i, D_i, q_i) be a set of relative-index data over (X, Z) with the notation described above. Suppose that the operators D_i have uniformly positive Weitzenbock curvature operators outside Z_i. Then each D_i has a localized coarse index in $K_j(C^*(Z))$, by Theorem 2.4, and the identity*

$$\operatorname{Index}_Z(D_1) - \operatorname{Index}_Z(D_2) = \operatorname{Index}_r(D_1, D_2)$$

holds in $K_j(C^*(Z))$.

(The case considered by Gromov and Lawson can be recovered by taking $X = \mathbb{R}^+$, $Z = \{0\}$.)

Proof. Let A be the pull-back algebra that we introduced in our definition of the relative index (so that A consists of pairs (T_1, T_2), $T_i \in C^*(M_i)$, that 'agree away from Z'). Let J be the ideal in A that consists of pairs (T_1, T_2), where each T_i belongs to $C^*(Z_i \subseteq M_i)$; in fact, J is simply the direct sum $C^*(Z_1 \subseteq M_1) \oplus C^*(Z_2 \subseteq M_2)$. Let D denote the Dirac operator on $M_1 \sqcup M_2$.

Because of the positive curvature away from Z, it follows from Lemma 2.3 that, for $f \in C_0(\mathbb{R})$, $f(D)$ belongs to the ideal J. Thus, in this case, the index $\operatorname{Index}_Z(D)$ defined in (4.2) in fact belongs to $K_j(J) = K_j(C^*(Z)) \oplus K_j(C^*(Z))$, and it is apparent from the definitions that, in terms of this direct sum decomposition,

$$\operatorname{Index}_Z(D) = (\operatorname{Index}_Z(D_1), \operatorname{Index}_Z(D_2)).$$

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The definition of the relative index tells us to take the component of $\mathrm{Index}_Z(D)$ in $K_j(C^*(Z))$ in the direct sum decomposition coming from the split short exact sequence (4.3). Restricted to J, this sequence takes the form

$$0 \to C^*(Z) \to C^*(Z) \oplus C^*(Z) \to C^*(Z) \to 0,$$

where the first map is inclusion on the first factor, the second is projection on the second factor and the splitting used is $a \mapsto (a, a)$. Using this splitting, one finds that the relevant component of $\mathrm{Index}_Z(D) = (\mathrm{Index}_Z(D_1), \mathrm{Index}_Z(D_2))$ is $\mathrm{Index}_Z(D_1) - \mathrm{Index}_D(Z_2)$, as required. □

As we observed above, it is an important feature of the Gromov–Lawson relative index that it depends only on the geometry of a neighbourhood of the 'region of disagreement'. The corresponding result is also true in our more general context and is a key to the applications of the relative index concept in [**7**].

Proposition 4.7 (Roe [7, Theorem 3.12]). *The relative index of Definition 4.5 depends only on the geometry of a metric neighbourhood of Z_1 and Z_2 and the operators thereon.*

Notice that this statement is independent of any positive-curvature hypotheses.

Proof. This follows from the results of [**8**]. In that paper, it was shown that to a set of relative index data (as described in this section), one may associate a *relative homology class* that lies in the K-homology group $K_*(Z)$. Moreover, comparison of the definitions shows that our coarse relative index is simply the image of this relative homology class under the coarse assembly map

$$A : K_*(Z) \to K_*(C^*(Z)).$$

The result is therefore a consequence of [**8**, Proposition 4.8], which states that in fact the relative *homology class* of a set of relative index data depends only on the geometry in a neighbourhood of the region of disagreement. □

References

1. M. Gromov and H. B. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, *Publ. Math. IHES* **58**(1) (1983), 83–196.
2. N. Higson and J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs (Oxford University Press, 2000).
3. N. Higson, J. Roe and G. Yu, A coarse Mayer–Vietoris principle, *Math. Proc. Camb. Phil. Soc.* **114** (1993), 85–97.
4. A. Lichnerowicz, Spineurs harmoniques, *C. R. Acad. Sci. Paris Sér. I* **257** (1963), 7–9 (in French).
5. J. Roe, Partitioning non-compact manifolds and the dual Toeplitz problem, in *Operator algebras and applications* (ed. D. Evans and M. Takesaki), pp. 187–228 (Cambridge University Press, 1989).

https://doi.org/10.1017/S0013091514000236 Published online by Cambridge University Press

6. J. Roe, A note on the relative index theorem, *Q. J. Math.* **42** (1991), 365–373.
7. J. Roe, *Index theory, coarse geometry, and the topology of manifolds*, CBMS Conference Proceedings, Volume 90 (American Mathematical Society, Providence, RI, 1996).
8. J. Roe and P. Siegel, Sheaf theory and Paschke duality, *K-Theory* **12** (2013), 213–234.
9. P. Siegel, Homological calculations with analytic structure groups, PhD thesis, Penn State (2012).

https://doi.org/10.1017/S0013091514000236 Published online by Cambridge University Press