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Classifying spaces for étale algebras with generators

Abhishek Kumar Shukla and Ben Williams

Abstract. We construct a scheme $B(r;\mathbb{A}^n)$ such that a map $X \to B(r;\mathbb{A}^n)$ corresponds to a degree-n étale algebra on X equipped with r generating global sections. We then show that when $n = 2$, *i.e.*, in the quadratic étale case, the singular cohomology of $B(r;\mathbb{A}^n)(\mathbb{R})$ can be used to reconstruct a famous example of S. Chase and to extend its application to showing that there is a smooth affine $r-1$-dimensional $\mathbb{R}$-variety on which there are étale algebras $\mathcal{A}_n$ of arbitrary degrees n that cannot be generated by fewer than r elements. This shows that in the étale algebra case, a bound established by U. First and Z. Reichstein in [6] is sharp.

1 Introduction

Given a topological group G, one may form the *classifying space*, well-defined up to homotopy equivalence, as the base space of any numerable principal G-bundle $EG \to BG$ where the total space is contractible, [3, Theorem 7.5]. The space BG is a universal space for G-bundles, in that the set of homotopy classes of maps $[X, BG]$ is in natural bijection with the set of numerable principal G-bundles on X.

If G is a finite nontrivial group, then BG is necessarily infinite dimensional, [20], and so there is no hope of producing BG as a variety even over $\mathbb{C}$. Nonetheless, as in [22], one can approximate BG by taking a large representation V of G on which G acts freely outside of a high-codimension closed set Z, and such that $(V - Z)/G$ is defined as a quasiprojective scheme. The higher the codimension of Z in V, the better an approximation $(V - Z)/G$ is to the notional BG.

In this paper, we consider the case of $G = S_n$, the symmetric group on n letters. The representations we consider as our Vs are the most obvious ones, r copies of the permutation representation of S_n on $\mathbb{A}^n$. The closed loci we consider are minimal: the loci where the action is not free. We use the language of étale algebras to give an interpretation of the resulting spaces. Our main result, Theorem 3.13, says that the scheme $B(r;\mathbb{A}^n) := (V - Z)/S_n$ produced by this machine represents "étale algebras equipped with r generating global sections" up to isomorphism of these data. The schemes $B(r;\mathbb{A}^n)$ are therefore in the same relation to the group S_n as the projective spaces $\mathbb{P}^r$ are to the group scheme $\mathbb{G}_m$.

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Section 2 is concerned with preliminary results on generation of étale algebras. The main construction of the paper, that of $B(r;\mathbb{A}^n)$, is made in Section 3, and the functor it represents is described. Since we are working with schemes, and not in a homotopy category, the space $B(r;\mathbb{A}^n)$ does not classify bundles; rather it represents a functor of "bundles along with chosen generators", which we now explain.

A choice of r global sections generating an étale algebra $\mathcal{A}$ of degree n on a scheme X corresponds to a map $\phi : X \to B(r;\mathbb{A}^n)$. While the map ϕ is dependent on the chosen generating sections, we show in Section 4 that if one is prepared to pass to a limit, in a sense made precise there, the $\mathbb{A}^1$-homotopy class of a composite $\tilde{\phi} : X \to B(r;\mathbb{A}^n) \to B(\infty;\mathbb{A}^n)$ depends only on the isomorphism class of $\mathcal{A}$ and not the generators. As a practical matter, this means that for a wide range of cohomology theories, E^*, the map $E^*(\tilde{\phi})$ depends only on $\mathcal{A}$ and not on the generators used to define it.

In Section 5, working over a field, we observe that the motivic cohomology, and therefore the Chow groups, of the varieties $B(r;\mathbb{A}^2)$ has already been calculated in [4].

A degree-2 or *quadratic* étale algebra $\mathcal{A}$ over a ring R carries an involution σ and a trace map $Tr : \mathcal{A} \to R$. There is a close connection between $\mathcal{A}$ and the rank-1 projective module $\mathcal{L} = \ker(Tr)$. In Section 6, we show that the algebra $\mathcal{A}$ can be generated by r elements if and only if the projective module $\mathcal{L}$ can be generated by r elements.

A famous counterexample of S. Chase, appearing in [21], shows that there is a smooth affine $r-1$-dimensional $\mathbb{R}$-variety $\operatorname{Spec} R$ and a line bundle $\mathcal{L}$ on $\operatorname{Spec} R$ requiring r global sections to generate. This shows a that a bound of O. Forster [8] on the minimal number of sections required to generate a line bundle on $\operatorname{Spec} R$, namely $\dim R + 1$, is sharp. In light of Section 6, the same smooth affine $\mathbb{R}$-variety of dimension $r-1$ can be used to produce étale algebras $\mathcal{A}$, of arbitrary degree n, requiring r global sections to generate. This fact was observed independently by M. Ojanguren. It shows that a bound established by U. First and Z. Reichstein in [6] is sharp in the case of étale algebras: they can always be generated by $\dim R + 1$ global sections and this cannot be improved in general. The details are worked out in Section 7, and we incidentally show that the example of S. Chase follows easily from our construction of $B(r;\mathbb{A}^2)$ and some elementary calculations in the singular cohomology of $B(r;\mathbb{A}^2)(\mathbb{R})$.

Finally, we offer some thoughts about determining whether the bound of First and Reichstein is sharp if one restricts to varieties over algebraically closed fields.

1.1 Notation and other preliminaries

- All rings in this paper are assumed to be unital, associative, and commutative.
- k denotes a base ring.
- A *variety* X is a geometrically reduced, separated scheme of finite type over a field. We do not require the base field to be algebraically closed, nor do we require varieties to be irreducible.
- C_2 denotes the cyclic group of order 2.

We use the functor-of-points formalism ([5, Part IV]) heavily throughout, which is to say we view a scheme X as the sheaf of sets it represents on the big Zariski site of all schemes

$$X(U) = \mathrm{Mor}_{\mathbf{Sch}}(U, X).$$

2 Étale algebras

Let R be a ring and S an R-algebra. Then there is a morphism of rings $\mu : S \otimes_R S^{\mathrm{op}} \to S$ sending $a \otimes b$ to ab. We obtain an exact sequence

$$0 \to \ker(\mu) \to S \otimes_R S^{\mathrm{op}} \xrightarrow{\mu} S \to 0$$

We recall ([7, Chapter 4]) that an R-algebra S is called *separable* if S is a projective $S \otimes_R S^{\mathrm{op}}$-module.

Definition 2.1 Let R be a ring. A commutative R-algebra S is called *étale* if S is a flat, separable, finitely presented R-algebra.

Proposition 2.2 *Let R be a commutative ring, and S a commutative R-algebra. Then the following are equivalent:*

1. *S is an étale R-algebra.*
2. *S is a finitely presented R-algebra and* $\operatorname{Spec} S \to \operatorname{Spec} R$ *is formally étale in the sense of [9, Section 17.1].*

Proof By [7, Corollary 4.7.3], we see that 1 implies 2. Conversely, a finitely generated and formally étale map is flat and unramified [9, Corollaire 17.6.2], and a finitely generated commutative unramified R-algebra S is separable [7, Theorem 8.3.6]. ∎

Definition 2.3 An R-algebra S is called *finite étale* if S is an étale R-algebra and a finitely generated R-module.

Remark 2.4 If S is a finitely presented R-algebra that is finitely generated as an R-module then it is also finitely presented as an R-module ([11, 1.4.7]). Moreover, finitely presented and flat modules are projective ([2, tag 058Q]), so a finite étale algebra S over R is, in particular, a projective R-module of finite rank.

Definition 2.5 We say that an étale algebra is of *degree n* if the rank of S as a projective R-module is n. A degree-n étale algebra is necessarily finite étale.

Over a ring R, and for any integer $n > 0$, there exists the trivial rank n étale algebra R^n with componentwise addition and multiplication. The next lemma states that all étale algebras are étale-locally isomorphic to the trivial one.

Lemma 2.6 *Let R be a ring and S an R-algebra. The following statements are equivalent:*

- *S is an étale algebra of degree n.*
- *There is a finite étale R-algebra T such that $S \otimes_R T \cong T^n$ as T-algebras.*

A proof may be found in [7, Corollary 1.1.16, Corollary 4.4.6, Proposition 4.6.11].
We may extend this definition to schemes. Fix a ground ring k throughout.

Definition 2.7 Let X be a k-scheme. Let $\mathcal{A}$ be a locally free sheaf of $\mathcal{O}_X$-algebras of constant rank n. We say that $\mathcal{A}$ is an *étale X-algebra* or étale algebra over X if for every open affine subset $U \subset X$ the $\mathcal{O}_X(U)$-algebra $\mathcal{A}(U)$ is an étale algebra. If the algebras $\mathcal{A}(U)$ are étale of rank n, we say $\mathcal{A}$ is a *degree-n étale algebra*.

By Remark 2.4 it is clear that a sheaf of degree-n étale algebras $\mathcal{A}$ over X is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

If X is a k-scheme and n a positive integer, then there exists a trivial rank-n étale algebra $\mathcal{O}_X^n$ with componentwise addition and multiplication.

Lemma 2.8 *Let X be a k-scheme and $\mathcal{A}$ be a finitely presented, quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Then the following are equivalent:*

- *$\mathcal{A}$ is an étale X-algebra of degree n.*
- *There is an affine étale cover $\{U_i \xrightarrow{f_i} X\}$ such that $f_i^* \mathcal{A} \cong \mathcal{O}_{U_i}^n$ as $\mathcal{O}_{U_i}$-algebras.*

Proof This is immediate from Lemma 2.6. ■

Definition 2.9 If A is an algebra over a ring R, then a subset $\Lambda \subset A$ is said to *generate* A over R if no strict R-subalgebra of A contains Λ.

If $\Lambda = \{a_1, \ldots, a_r\} \subset A$ is a finite subset, then the smallest subalgebra of A containing Λ agrees with the image of the evaluation map $k[x_1, \ldots, x_r] \xrightarrow{(a_1,\ldots,a_r)} A$. Therefore, saying that Λ generates A is equivalent to saying this map is surjective.

Proposition 2.10 *Let $\Lambda = \{a_1, \ldots, a_r\}$ be a finite set of elements of A, an algebra over a ring R. The following are equivalent:*

1. *Λ generates A as an R-algebra.*
2. *There exists a set of elements $\{f_1, \ldots, f_n\} \subset R$ that generate the unit ideal and such that, for each $i \in \{1, \ldots, n\}$, the image of Λ in A_{f_i} generates A_{f_i} as an R_{f_i}-algebra.*
3. *For each $\mathfrak{m} \in \operatorname{MaxSpec} R$, the image of Λ in $A_\mathfrak{m}$ generates $A_\mathfrak{m}$ as an $R_\mathfrak{m}$-algebra.*
4. *Let $k(\mathfrak{m})$ denote the residue field of the local ring $R_\mathfrak{m}$. For each $\mathfrak{m} \in \operatorname{MaxSpec} R$, the image of Λ in $A \otimes_R k(\mathfrak{m})$ generates $A \otimes_R k(\mathfrak{m})$ as a $k(\mathfrak{m})$-algebra.*

Proof In the case of a finite subset, $\Lambda = \{a_1, \ldots, a_r\}$, the condition that Λ generates A is equivalent to the surjectivity of the evaluation map $R[x_1, \ldots, x_r] \to A$.

The question of generation is therefore a question of whether a certain map is an epimorphism in the category of R-modules, and conditions (2)–(4) are well-known equivalent conditions saying that this map is an epimorphism. ■

Using Proposition 2.10, we extend the definition of "generation of an algebra" from the case where the base is affine to the case of a general scheme.

Definition 2.11 Let $\mathcal{A}$ be an algebra over a scheme X. For $\Lambda \subset \Gamma(X, \mathcal{A})$ we say that Λ *generates* $\mathcal{A}$ if, for each open affine $U \subset X$ the $\mathcal{O}_X(U)$-algebra $\mathcal{A}(U)$ is generated by restriction of sections in Λ to U.

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2.1 Generation of trivial algebras

Let $n \geq 2$ and $r \geq 1$. Consider the trivial étale algebra $\mathcal{O}_X^n$ over a scheme X. A global section of this algebra is equivalent to a morphism $X \to \mathbb{A}^n$, and an r-tuple Λ of sections is a morphism $X \to (\mathbb{A}^n)^r$. One might hope that the subfunctor $\mathcal{F} \subseteq (\mathbb{A}^n)^r$ of r-tuples of sections generating $\mathcal{O}_X^n$ as an étale algebra is representable, and this turns out to be the case.

In order to define subschemes of $(\mathbb{A}^n)^r$, it will be necessary to name coordinates:

$$(x_{11}, x_{12}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{r1}, \dots, x_{rn}).$$

It will also be useful to retain the grouping into n-tuples, so we define $\vec{x}_l = (x_{l1}, x_{l2}, \dots, x_{ln})$.

Notation 2.12 Fix n and r as above. For $(i, j) \in \{1, \dots, n\}^2$ with $i < j$, let $Z_{ij} \subset (\mathbb{A}^n)^r$ denote the closed subscheme given by the sum of the ideals $(x_{ki} - x_{kj})$ where k varies from 1 to n.

Write $U(r; \mathbb{A}^n)$, or $U(r)$ when n is clear from the context, for the open subscheme of $(\mathbb{A}^n)^r$ given by

$$U(r; \mathbb{A}^n) = (\mathbb{A}^n)^r - \bigcup_{i<j} Z_{ij}$$

Proposition 2.13 *Let $n \geq 2$ and $r \geq 1$. The open subscheme $U(r; \mathbb{A}^n) \subset (\mathbb{A}^n)^r$ represents the functor sending a scheme X to r-tuples $(a_1, \dots, a_r)$ of global sections of $\mathcal{O}_X^n$ that generate it as an $\mathcal{O}_X$-algebra.*

Proof Temporarily, let $\mathcal{F}$ denote the subfunctor of $(\mathbb{A}^n)^r$ defined by

$$\mathcal{F}(X) = \{\Lambda \subseteq (\Gamma(X, O_X^n)^r \mid \Lambda \text{ generates } \mathcal{O}_X^n\}.$$

It follows from Proposition 2.10 and Definition 2.11 that $\mathcal{F}$ is actually a sheaf on the big Zariski site.

Both $U(r; \mathbb{A}^n)$ and $\mathcal{F}$ are subsheaves of the sheaf represented by $(\mathbb{A}^n)^r$, and therefore in order to show they agree, it suffices to show $U(r; \mathbb{A}^n)(R) = \mathcal{F}(R)$ when R is a local ring.

Let R be a local ring. The set $U(r; \mathbb{A}^n)(R)$ consists of certain r-tuples $(\vec{a}_1, \dots, \vec{a}_r)$ of elements of R^n. Letting a_{ki} denote the ith element of $\vec{a}_k$, then the r-tuples are those with the property that for each $i \neq j$, there exists some k such that $a_{ki} - a_{kj} \in R^\times$. The proposition now follows from Lemma 2.14 below. ∎

Lemma 2.14 *Let R be a local ring, with maximal ideal $\mathfrak{m}$. Let $(\vec{a}_1, \dots, \vec{a}_r)$ denote an r-tuple of elements in R^n, and let a_{ki} denote the ith element of $\vec{a}_k$. The following are equivalent:*

- *The set $\{\vec{a}_1, \dots, \vec{a}_r\}$ generates the (trivial) étale R-algebra R^n.*
- *For each pair (i, j) satisfying $1 \leq i < j \leq n$, there is some $k \in \{1, \dots, r\}$ such that the element $a_{ki} - a_{kj}$ is a unit in $R^\times$.*

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Proof Each condition is equivalent to the same condition over $R/\mathfrak{m}$: the first by virtue of 2.10, and the second by elementary algebra. Therefore, it suffices to prove this when R is a field.

Suppose $\{\vec{a}_1, \dots, \vec{a}_r\}$ generates R^n as an algebra. Then, for any pair of indices (i, j) with $1 \le i < j \le n$, it is possible to find a polynomial $p \in R[X_1, \dots, X_r]$ such that $p(a_{1i}, a_{2i}, \dots, a_{ri}) = 1$ and $p(a_{1j}, a_{2j}, \dots, a_{rj}) = 0$. In particular, there exists some l such that $a_{li} \ne a_{lj}$.

Conversely, suppose that for each pair $i < j$, we can find some l such that $a_{li} \ne a_{lj}$. For each pair $i \ne j$, we can find a polynomial $p_{i,j} \in R[x_1, \dots, x_r]$ with the property that $p_{i,j}(a_{1i}, \dots, a_{ri}) = 1$ and $p_{i,j}(a_{1j}, \dots, a_{rj}) = 0$ by taking

$$p_{i,j} = (a_{ki} - a_{kj})^{-1}(x_k - a_{kj})$$

for instance. Consequently, we may produce a polynomial $p_i \in R[x_1, \dots, x_r]$ with the property that $p_i(a_{1j}, \dots, a_{rj}) = \delta_{i,j}$ (Kronecker delta). It follows that $\{\vec{a}_1, \dots, \vec{a}_r\}$ generates the trivial algebra. ∎

3 Classifying spaces

Fix $n \ge 2$ and $r \ge 1$.

Notation 3.1 For a given k-scheme X, *a degree-n étale algebra $\mathcal{A}$ with r generating sections* denotes the data of a degree-n étale algebra $\mathcal{A}$ over X, and an r-tuple of sections $(a_1, \dots, a_r) \in \Gamma(X, \mathcal{A})$ that generate $\mathcal{A}$. These data will be briefly denoted $(\mathcal{A}, a_1, \dots, a_r)$. A morphism $\psi : (\mathcal{A}, a_1, \dots, a_r) \to (\mathcal{A}', a_1', \dots, a_r')$ of such data consists of a map $\psi : \mathcal{A} \to \mathcal{A}'$ of étale algebras over X such that $\psi(a_i) = a_i'$ for all $i \in \{1, \dots, r\}$. It is immediate that all morphisms are isomorphisms, and between any two objects, there is at most one isomorphism. The isomorphism class of $(\mathcal{A}, a_1, \dots, a_r)$ will be denoted $[\mathcal{A}, a_1, \dots, a_r]$.

Definition 3.2 For a given X, there is a set, rather than a proper class, of isomorphism classes of degree-n étale algebras over X, and so there is a set of isomorphism classes of degree-n étale algebras with r generating sections. Since generation is a local condition by Proposition 2.10, it follows that there is a functor

$$\begin{aligned} \mathcal{F}(r; \mathbb{A}^n) &: k\text{-}\mathbf{Sch} \to \mathbf{Set}, \\ \mathcal{F}(r; \mathbb{A}^n)(X) &= \{[\mathcal{A}, a_1, \dots, a_r] \mid (\mathcal{A}, a_1, \dots, a_r) \text{ is a degree-}n \\ &\qquad \text{étale algebra over } X \text{ and } r \text{ generating sections}\} \end{aligned}$$

The purpose of this section is to produce a variety $B(r; \mathbb{A}^n)$ representing the functor $\mathcal{F}(r; \mathbb{A}^n)$ on the category of k-schemes.

3.1 Descent for $\mathcal{F}(r, \mathbb{A}^n)$

Proposition 3.3 *The functor $\mathcal{F}(r; \mathbb{A}^n)$ is a sheaf on the big étale site of* Spec k.

In fact, it is a sheaf on the big fpqc site, but we will require only the étale descent condition.

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Proof Suppose X is a k-scheme and $\{ f_i : Y_i \to X\}_{i\in I}$ is an étale covering. We must identify $\mathcal{F}(r;\mathbb{A}^n)(X)$ with the equalizer in

$$E \to \prod_{i\in I} \mathcal{F}(r,\mathbb{A}^n)(Y_i) \rightrightarrows \prod_{i,j\in I^2} \mathcal{F}(r;\mathbb{A}^n)(Y_i \times_X Y_j).$$

There is clearly a map $\mathcal{F}(r;\mathbb{A}^n)(X) \to E$.

Suppose we have an i-tuple of elements $([\mathcal{A}_i, \vec{a}(i)])_{i\in I}$ in this equalizer. Choosing representatives in each case, we have degree-n étale algebras $\mathcal{A}_i$ on each Y_i, along with chosen generating global sections. The equalizer condition is that there is an isomorphism over $Y_i \times_X Y_j$ of the form $\phi_{ij} : \mathrm{pr}_1^*(\mathcal{A}_i, \vec{a}(i)) \xrightarrow{\cong} \mathrm{pr}_2^*(\mathcal{A}_j, \vec{a}(j))$. The fact that there is at most one isomorphism between étale algebras with r generating sections implies that we have a descent datum $(\mathcal{A}_i, \phi_{ij})$, and it is well known, [2, Tag 023S], that quasi-coherent sheaves satisfy étale descent. We therefore obtain a quasi-coherent sheaf of algebras $\mathcal{A}$ on X, and since $\mathcal{A}$ is an étale sheaf, the generating sections of each $\mathcal{A}_i$ glue to give generating sections of $\mathcal{A}$. This implies that $\mathcal{F}(r;\mathbb{A}^n)(X) \to E$ is surjective.

To see it is injective, suppose $(\mathcal{A}, \vec{a})$ and $(\mathcal{A}', \vec{a}')$ become isomorphic when restricted to each Y_i. Then, since there can be at most a unique isomorphism between two étale algebras with generating sections, the local isomorphisms between $(\mathcal{A}, \vec{a})$ and $(\mathcal{A}', \vec{a}')$ assemble to give an isomorphism of descent data. Since there is an equivalence of categories between descent data and quasi-coherent sheaves, [2, Tag 023S], it follows that there is an isomorphism $\phi : \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$. This isomorphism takes $\vec{a}$ to $\vec{a}'$, as required. ■

3.2 Construction of $B(r;\mathbb{A}^n)$

Proposition 3.4 *Let R be a nonzero connected ring. Then the automorphism group of the trivial étale R-algebra R^n is the symmetric group S_n, acting on the terms.*

Proof Since the equation $x^2 - x = 0$ has only the two solutions $1, 0$ in R, the condition $a^2 = a$ for $a \in R^n$ implies that each component of a is either 0 or 1.

Consider the elements

$$e_i = (0,\dots,0,1,0,\dots,0) \in R^n.$$

The set of these elements is determined by the conditions: $e_i^2 = e_i$, $e_i \neq 0$, $e_i e_j = 0$ for $i \neq j$ and $\sum_{i=1}^n e_i = 1 \in R^n$.

Therefore any automorphism of R^n as an R-algebra permutes the e_i and is determined by this permutation. ■

There is an action of the symmetric group S_n on $\mathbb{A}^n$, given by permuting the coordinates, and from there, there is a diagonal action of S_n on $(\mathbb{A}^n)^r$, and the action restricts to the open subscheme $U(r;\mathbb{A}^n)$.

Proposition 3.5 *The action of S_n on $U(r;\mathbb{A}^n)$ is scheme-theoretically free.*

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Proof It suffices to verify that the action is free on the sets $U(r;\mathbb{A}^n)(K)$ where K is a separably closed field over k. Here one is considering the diagonal S_n action on r-tuples $(\vec{a}_1,\ldots,\vec{a}_r)$ where each $\vec{a}_l \in K^n$ is a vector and such that for all indices $i \neq j$, there exists some $\vec{a}_l$ such that the ith and jth entries of $\vec{a}_l$ are different. The result follows. ∎

Construction 3.6 There is a free diagonal action of S_n on $U(r;\mathbb{A}^n)\times\mathbb{A}^n$, such that the projection $p : U(r;\mathbb{A}^n)\times\mathbb{A}^n \to U(r;\mathbb{A}^n)$ is equivariant. The quotient schemes for these actions exist by reference to [10, Exposé V, Proposition 1.8] and [15, Proposition 3.3.36]. Write $q : E(r;\mathbb{A}^n) \to B(r;\mathbb{A}^n)$ for the induced map of quotient schemes. There is a commutative square

$$\begin{array}{ccc} U(r;\mathbb{A}^n)\times\mathbb{A}^n & \xrightarrow{\pi'} & E(r;\mathbb{A}^n) \\ \downarrow p & & \downarrow q \\ U(r;\mathbb{A}^n) & \xrightarrow{\pi} & B(r;\mathbb{A}^n) \end{array}$$

Proposition 3.7 *In the notation above, the maps π and π' are finite.*

Proof We concentrate on the case of π; that of π' is similar. The map π is formed as follows (see [10, Exposé V, §1]): it is possible to cover $U(r;\mathbb{A}^n)$ by S_n-invariant open affine subschemes $\operatorname{Spec} R \subseteq \mathbb{A}^{nr}$. Then $\pi|_{\operatorname{Spec} R} : \operatorname{Spec} R \to \operatorname{Spec} R^{S_n}$, induced by the inclusion of R^{S_n} in R. The map $R^{S_n} \to R$ is of finite type, since R is of finite type over k. By [1, Exercise 5.12, p68], the extension $R^{S_n} \to R$ is integral, and being of finite type, it is finite. ∎

Corollary 3.8 *The maps $\pi : U(r;\mathbb{A}^n) \to B(r;\mathbb{A}^n)$ and $\pi' : U(r;\mathbb{A}^n)\times\mathbb{A}^n \to E(r;\mathbb{A}^n)$ are S_n-torsors.*

That is, each satisfies the conditions of [10, Exposé V, Proposition 2.6].

Remark 3.9 The sheaf of sections of the map $p : U(r;\mathbb{A}^n)\times\mathbb{A}^n \to U(r;\mathbb{A}^n)$ is the trivial degree-n étale algebra $\mathcal{O}^n_{U(r;\mathbb{A}^n)}$ on $U(r;\mathbb{A}^n)$. The action of S_n on these sections is by algebra automorphisms, and so the sheaf of sections of the quotient map $q : E(r;\mathbb{A}^n) \to B(r;\mathbb{A}^n)$ is endowed with the structure of a degree-n étale algebra $\mathcal{E}(r;\mathbb{A}^n)$ on $B(r;\mathbb{A}^n)$. We will often confuse the scheme $E(r;\mathbb{A}^n)$ over $B(r;\mathbb{A}^n)$ with the étale algebra of sections $\mathcal{E}(r;\mathbb{A}^n)$.

The map p has r canonical sections $\{s_j\}_{j=1}^r$ given as follows:

$$s_j(\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_r) = ((\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_r,),\vec{x}_j).$$

These sections are S_n-equivariant, and so induce sections $\{t_i : B(r;\mathbb{A}^n) \to E(r;\mathbb{A}^n)\}_{i=1}^r$ of the map q.

Remark 3.10 By reference to [2, Lemma 05B5], the quotient k-scheme $B(r;\mathbb{A}^n)$ is smooth over k since $U(r;\mathbb{A}^n)$ is and π is faithfully flat ([10, Exposé V, Proposition 2.6]) and locally finitely presented. Since π is finite it is a proper map. When the base k is a field, the variety $B(r;\mathbb{A}^n)$ is a quasiprojective variety but not projective. Indeed, if

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$B(r;\mathbb{A}^n) \to \mathrm{Spec}(k)$ were proper then $U(r;\mathbb{A}^n) \to \mathrm{Spec}(k)$ would be proper too, but $U(r;\mathbb{A}^n)$ is a nonempty open subvariety of affine space.

3.3 The functor represented by $B(r, \mathbb{A}^n)$

We now establish the identity of functors $B(r, \mathbb{A}^n)(X) = \mathcal{F}(r;\mathbb{A}^n)(X)$.

Construction 3.11 There is a canonical element $[\mathcal{E}(r;\mathbb{A}^n), t_1, \dots, t_r]$ in $\mathcal{F}(r;\mathbb{A}^n)(B(r;\mathbb{A}^n))$, see 3.9. Therefore, there exists a natural transformation of presheaves of k-schemes $B(r, \mathbb{A}^n)(\cdot) \to \mathcal{F}(r;\mathbb{A}^n)(\cdot)$ given by sending a map $\phi : X \to B(r;\mathbb{A}^n)$ to the pull-back of the canonical element.

Lemma 3.12 *If* $[A, s_1, \dots, s_r] \in \mathcal{F}(r;\mathbb{A}^n)(R)$ *where* R *is a strictly henselian local ring, then there exists a unique morphism of schemes* $\phi : \mathrm{Spec}(R) \to B(r;\mathbb{A}^n)$ *such that*

$$[A, s_1, \dots, s_r] = [\phi^*(\mathcal{E}(r;\mathbb{A}^n)), \phi^* t_1, \dots, \phi^* t_r]$$

Proof Since R is a strictly henselian local ring, there exists an R-isomorphism $A \xrightarrow{\psi} R^n$ of algebras, by virtue of [17, Proposition 1.4.4]. Let $\{\psi(s_i)\} \subset R^n$ denote the corresponding sections of R^n.

We thus obtain a map $\tilde{\phi} : \mathrm{Spec}(R) \to U(r;\mathbb{A}^n)$ defined by giving the R-point $(\psi(s_1), \dots, \psi(s_r))$. Post-composing this map with the projection $U(r;\mathbb{A}^n) \to B(r;\mathbb{A}^n)$, we obtain a morphism $\phi : \mathrm{Spec}(R) \to B(r;\mathbb{A}^n)$. It is a tautology that $\phi^*(\mathcal{E}(r;\mathbb{A}^n)) = A$ and $\phi^*(t_i) = s_i$.

It now behooves us to show that ϕ does not depend on the choices made in the construction.

Suppose $\phi' : \mathrm{Spec}\, R \to B(r;\mathbb{A}^n)$ is another morphism satisfying the conditions of the lemma. We may lift this R-point of $B(r;\mathbb{A}^n)$ to an R-point $\tilde{\phi}' : \mathrm{Spec}\, R \to U(r;\mathbb{A}^n)$, since π is an étale covering, and therefore represents an epimorphism of étale sheaves [2, Lemma 00WT]. By hypothesis we have

$$[A, s_1, \dots, s_r] = [\phi'^*(\mathcal{E}(r;\mathbb{A}^n)), \phi'^* t_1, \dots, \phi'^* t_r].$$

Thus $\tilde{\phi}$ and $\tilde{\phi}'$ differ by an automorphism of R^n, *i.e.*, by an element of S_n since local rings are connected so 3.4 applies. Therefore $\phi = \phi'$ as required. ∎

Theorem 3.13 *If* X *is a* k*-scheme, then the map*

$$B(r, \mathbb{A}^n)(X) \to \mathcal{F}(r;\mathbb{A}^n)(X)$$

is a bijection.

Proof We note that $B(r, \mathbb{A}^n)$ represents a sheaf on the big étale site of $\mathrm{Spec}\, k$, since it is a k-scheme. The presheaf $\mathcal{F}(r;\mathbb{A}^n)$ is also an étale sheaf, by virtue of Proposition 3.3. It therefore suffices to prove that $B(r, \mathbb{A}^n)(\mathrm{Spec}\, R) \to \mathcal{F}(r;\mathbb{A}^n)(\mathrm{Spec}\, R)$ when R is a strictly henselian local ring, but this is Lemma 3.12. ∎

Example 3.14 Let us consider the toy example where k is a field and $X = \mathrm{Spec}\, K$ for some field extension K/k, where $n \geq 2$, and where $r = 1$. That is, we are considering

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étale algebras A/K along with a chosen generating element $a \in A$. After base change to the separable closure, K^s, we obtain an S_n-equivariant isomorphism of K^s-algebras:

$$\psi : A_{K^s} \xrightarrow{\cong} (K^s)^{\times n}.$$

For the sake of the exposition, use ψ to identify source and target. The element $a \in A$ yields a chosen generating element $\tilde{a} \in (K^s)^n$. The element $\tilde{a}$ is a vector of n pairwise distinct elements of K^s. The element $\tilde{a}$ is a K^s-point of $U(1;\mathbb{A}^n)$. In general, this point is not defined over K, but its image in $B(1;\mathbb{A}^n)$ is.

Since $U(1;\mathbb{A}^n) \subseteq \mathbb{A}^n$, and $B(1;\mathbb{A}^n) = U(1;\mathbb{A}^n)/\Sigma_n$, the image of $\tilde{a}$ in $B(1;\mathbb{A}^n)(K^s)$ may be presented as the elementary symmetric polynomials in the a_i. To say that the image of $\tilde{a} = (a_1,\dots,a_n)$ in $B(1;\mathbb{A}^n)$ is defined over K is to say that the coefficients of the polynomial $\prod_{i=1}^n (x - a_i)$ are defined in K.

The variety $B(1;\mathbb{A}^n)$ is the k-variety parametrizing degree-n polynomials with distinct roots, *i.e.*, with invertible discriminant.

Example 3.15 To reduce the toy example even further, let us consider the case of $k = K$ a field of characteristic different from 2, and $n = 2$.

The variety $B(1;\mathbb{A}^2)$ may be presented as spectrum of the C_2-fixed subring of $k[x, y, (x-y)^{-1}]$ under the action interchanging x and y. This is $k[(x+y), (x-y)^2, (x-y)^{-2}]$, although it is more elegant to present it after the change of coordinates $c_1 = x + y$ and $c_0 = xy$:

$$B(1;\mathbb{A}^2) = \operatorname{Spec} k[c_1, c_0, (c_1^2 - 4c_0)^{-1}]$$

A quadratic étale k-algebra equipped with the generating element a corresponds to the point $(c_1, c_0) \in B(1;\mathbb{A}^2)(k)$ where a satisfies the minimal polynomial $a^2 - c_1 a + c_0 = 0$.

For instance if $k = \mathbb{R}$, the quadratic étale algebra of complex numbers $\mathbb{C}$ with generator $s + ti$ over $\mathbb{R}$ (here $t \neq 0$), corresponds to the point $(2s, s^2 + t^2) \in B(1;\mathbb{A}^2)(\mathbb{R})$, whereas $\mathbb{R} \times \mathbb{R}$, generated by $(s + t, s - t)$ over $\mathbb{R}$ (again $t \neq 0$) , corresponds to the point $(2s, s^2 - t^2)$.

4 Stabilization in cohomology

We might wish to use the schemes $B(r;\mathbb{A}^n)$ to define cohomological invariants of étale algebras. The idea is the following: suppose given such an algebra $\mathcal{A}$ on a k-scheme X, and suppose one can find generators $(a_1,\dots a_r)$ for $\mathcal{A}$. Then one has a classifying map $\phi : X \to B(r;\mathbb{A}^n)$, and one may apply a cohomology functor E^*, such as Chow groups or algebraic K-theory, to obtain "characteristic classes" for $\mathcal{A}$-along-with-$(a_1,\dots,a_r)$, in the form of $\phi^* : E^*(B(r;\mathbb{A}^n)) \to E^*(X)$. The dependence on the specific generators chosen is a nuisance, and we see in this section that this dependence goes away provided we are prepared to pass to a limit "$B(\infty)$" and assume that the theory E^* is $\mathbb{A}^1$-invariant, in that $E^*(X) \to E^*(X \times \mathbb{A}^1)$ is an isomorphism.

Definition 4.1 There are *stabilization* maps $U(r;\mathbb{A}^n) \to U(r+1;\mathbb{A}^n)$ obtained by augmenting an r-tuple of n-tuples by the n-tuple $(0, 0, \dots, 0)$. These stabilization maps are S_n-equivariant and therefore descend to maps $B(r;\mathbb{A}^n) \to B(r+1;\mathbb{A}^n)$.

The stabilization maps defined above may be composed with one another, to yield maps $B(r;\mathbb{A}^n) \to B(r';\mathbb{A}^n)$ for all $r < r'$. These maps will also be called *stabilization* maps.

Proposition 4.2 *Let X be a k-scheme. Suppose*

$$[\mathcal{A}, a_1, \dots, a_r] \in \mathcal{F}(r;\mathbb{A}^n)(X) \quad \textit{and} \quad [\mathcal{A}', a_1', \dots, a_{r'}'] \in \mathcal{F}(r';\mathbb{A}^n)(X)$$

have the property that $\mathcal{A} \cong \mathcal{A}'$ as étale algebras. Let $\phi : X \to B(r;\mathbb{A}^n)$ and $\phi' : X \to B(r';\mathbb{A}^n)$ be the corresponding classifying morphism. For $R = r + r'$, the composite maps $\tilde{\phi} : X \to B(r;\mathbb{A}^n) \to B(R;\mathbb{A}^n)$ and $\tilde{\phi}' : X \to B(r';\mathbb{A}^n) \to B(R;\mathbb{A}^n)$ given by stabilization are naively $\mathbb{A}^1$-homotopic.

An "elementary $\mathbb{A}^1$-homotopy" between maps $\phi, \phi' : X \to B$ is a map $\Phi : X \times \mathbb{A}^1 \to B$ specializing to ϕ at 0 and ϕ' at 1. Two maps $\phi, \phi' : X \to B$ are "naively $\mathbb{A}^1$-homotopic" if they may be joined by a finite sequence of elementary homotopies. Two naively homotopic maps between smooth finite-type k-schemes are identified in the $\mathbb{A}^1$-homotopy theory of schemes of [18], but they do not account for all identifications in that theory.

Proof We may assume that $\mathbb{A} = \mathcal{A}'$. We may also assume that $r = r'$—if $r < r'$, then pad the vector $(a_1, \dots, a_r)$ with 0s to produce a vector $(a_1, \dots, a_r, 0, \dots, 0)$ of length r', and similarly in the other case.

Write t for the parameter of $\mathbb{A}^1$. Let $\mathcal{A}[t]$ denote the pull-back of $\mathcal{A}$ along the projection $X \times \mathbb{A}^1 \to X$.

Consider the sections $((1-t)a_1, \dots, (1-t)a_r, ta_1', \dots, ta_r')$ of $\mathcal{A}[t]$. Since either t or $(1-t)$ is a unit at all local rings of points $\mathbb{A}^1$, by appeal to Proposition 2.10 and consideration of the restrictions to $X \times (\mathbb{A}^1 - \{0\})$ and $X \times (\mathbb{A}^1 - \{1\})$, we see that $((1-t)a_1, \dots, (1-t)a_r, ta_1', \dots, ta_r')$ furnish a set of generators for $\mathcal{A}[t]$. At $t = 0$, they specialize to $(a_1, \dots, a_r, 0, \dots, 0)$, *viz.*, the generators specified by the stabilized map $\phi : X \to B(r;\mathbb{A}^n) \to B(2r;\mathbb{A}^n)$. At $t = 1$, they specialize to $(0, \dots, 0, a_1', \dots, a_r')$, which is not precisely the list of generators specified by $\phi' : X \to B(r;\mathbb{A}^n) \to B(2r;\mathbb{A}^n)$, but may be brought to this form by another elementary $\mathbb{A}^1$-homotopy. ∎

Corollary 4.3 *Let ϕ and ϕ' be as in the previous proposition. If E^* denotes any $\mathbb{A}^1$-invariant cohomology theory, then $E^*(\tilde{\phi}) = E^*(\tilde{\phi}')$.*

5 The motivic cohomology of the spaces $B(r;\mathbb{A}^2)$

For this section, let k denote a fixed field of characteristic different from 2. The motivic cohomology of the spaces $B(r;\mathbb{A}^2)$ has already been calculated in [4].

5.1 Change of coordinates

Lemma 5.1 *There is an equivariant isomorphism $U(r;\mathbb{A}^2) \cong \mathbb{A}^r\backslash\{0\} \times \mathbb{A}^r$, where C_2 acts as multiplication by -1 on the first factor $\mathbb{A}^r\backslash\{0\}$ and trivially on the second factor $\mathbb{A}^r$. Taking quotient by C_2-action yields $B(r;\mathbb{A}^2) \cong (\mathbb{A}^r\backslash\{0\})/C_2 \times \mathbb{A}^r$.*

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Proof By means of the change of coordinates

$$x_i - y_i = z_i, \quad x_i + y_i = w_i$$

we see that $U(r;\mathbb{A}^2)\cong(\mathbb{A}^r\backslash\{0\}) \times \mathbb{A}^r$. Moreover, the action of C_2 on $U(r;\mathbb{A}^2)$ is given by $z_i \mapsto -z_i$ and $w_i \mapsto w_i$. We therefore obtain an isomorphism $B(r;\mathbb{A}^2) = U(r;\mathbb{A}^2)/C_2\cong(\mathbb{A}^r\backslash\{0\})/C_2 \times \mathbb{A}^r$. Write $V(r;\mathbb{A}^2)$ for $\mathbb{A}^r\backslash\{0\}/C_2$. It is immediate that $B(r;\mathbb{A}^2)\cong V(r;\mathbb{A}^2) \times \mathbb{A}^r$, and so there is a split inclusion $V(r;\mathbb{A}^2) \to B(r;\mathbb{A}^2)$ which is moreover an $\mathbb{A}^1$-equivalence. ■

5.2 The deleted quadric presentation

Definition 5.2 Endow $\mathbb{P}^{2r-1}$ with the projective coordinates $a_1, \dots, a_r, b_1, \dots, b_r$. Let Q_{2r-2} denote the closed subvariety given by the vanishing of $\sum_{i=1}^r a_i b_i$, and let DQ_{2r-1} denote the open complement $\mathbb{P}^{2r-1}\backslash Q_{2r-2}$.

The main computation of [4] is a calculation of the modulo- 2 motivic cohomology of DQ_{2r-1}, and of a family of related spaces DQ_{2r}. Our reference for the motivic cohomology of k-varieties is [16]. For a given abelian group A, either $\mathbb{Z}$ or $\mathbb{F}_2$ in this paper, and a given variety X, the motivic cohomology $\mathrm{H}^{*,*}(X;A)$ is a bigraded algebra over the cohomology of the ground field, Spec k.

Denote the modulo- 2 motivic cohomology of Spec k by $\mathbb{M}_2$. This is a bigraded ring,

$$\mathbb{M}_2 = \bigoplus_{i,n} \mathbb{M}_2^{n,i},$$

nonzero only in degrees $0 \le n \le i$. There are two notable classes, $\rho \in \mathbb{M}_2^{1,1}$, the reduction modulo 2 of $\{-1\} \in K_1^M(k) = \mathrm{H}^{1,1}(\mathrm{Spec}\, k, \mathbb{Z})$, and $\tau \in \mathbb{M}_2^{0,1}$, corresponding to the identity $(-1)^2 = 1$. If -1 is a square in k, then $\rho = 0$, but τ is always a nonzero class.

Proposition 5.3. (Dugger–Isaksen, [4] Theorem 4.9) *There is an isomorphism of graded rings*

$$\mathrm{H}^{*,*}(DQ_{2r-1};\mathbb{F}_2)\cong\frac{\mathbb{M}_2[a,b]}{(a^2 - \rho a - \tau b, b^r)}$$

where $|a| = (1,1)$ *and* $|b| = (2,1)$.

Moreover, the inclusion $DQ_{2r-1} \to DQ_{2r+1}$ *given by* $a_{r+1} = b_{r+1} = 0$ *induces the map* $\mathrm{H}^{*,*}(DQ_{2r+1};\mathbb{F}_2) \to \mathrm{H}^{*,*}(DQ_{2r-1};\mathbb{F}_2)$ *sending a to a and b to b.*

This proposition subsumes two other notable calculations of invariants. In the first place, owing to the Beilinson–Lichtenbaum conjecture [23], it subsumes the calculation of $\mathrm{H}^*_{\text{ét}}(DQ_{2r-1};\mathbb{F}_2)$. For instance, if k is algebraically closed, then $\mathbb{M}_2 = \mathbb{F}[\tau]$, and one deduces that $\mathrm{H}^*_{\text{ét}}(DQ_{2r-1};\mathbb{F}_2)\cong\mathbb{F}_2[a,b]/(a^2 - b, b^r) = \mathbb{F}_2[a]/(a^{2r})$.

In the second, since $\mathrm{H}^{2n,n}(\cdot;\mathbb{F}_2)$ is identified with $\mathrm{CH}^n(\cdot)\otimes_{\mathbb{Z}}\mathbb{F}_2$, the calculation of the proposition subsumes that of the Chow groups modulo 2. In fact, the extension problems that prevented Dugger and Isaksen from calculating $\mathrm{H}^{*,*}(DQ_{2r-1};\mathbb{Z})$ do not arise in this range, and by reference to the appendix of [4], which in turn refers to [13], one can calculate the integral Chow rings. This is done in the first two paragraphs of the proof of [4, Theorem 4.9].

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Proposition 5.4 *One may present*

$$\mathrm{CH}^*(DQ_{2r-1}) = \frac{\mathbb{Z}[\tilde{b}]}{(2\tilde{b}, \tilde{b}^r)}, \quad |b| = 1.$$

As before, the map $DQ_{2r-1} \to DQ_{2r+1}$ given by adding 0s induces the map $b \mapsto b$ on Chow rings. Moreover $\mathrm{CH}^*(DQ_{2r-1}) \otimes_{\mathbb{Z}} \mathbb{F}_2$ can be identified with the subring of $\mathrm{H}^{*,*}(DQ_{2r-1}; \mathbb{F}_2)$ generated by b.

The reason we have explained all this is that there is a composite of maps

$$DQ_{2r-1} \to (\mathbb{A}^r \backslash \{0\})/C_2 \to B(r; \mathbb{A}^2), \tag{1}$$

both of which are $\mathbb{A}^1$-equivalences, and so Propositions 5.3 and 5.4 amount to a calculation of the motivic and étale cohomologies and Chow rings of $B(r; \mathbb{A}^2)$. Both maps in diagram (5.2) are compatible in the evident way with an increase in r, so that we may use the material of this section to compute the stable invariants of $B(r; \mathbb{A}^2)$ in the sense of Section 4.

The $\mathbb{A}^1$-equivalence $B(r; \mathbb{A}^2) \to (\mathbb{A}^r \backslash \{0\})/C_2$ was constructed above in Lemma 5.1, so it remains to prove the following.

Lemma 5.5 *Let $r \geq 1$. The variety DQ_{2r-1} is affine and has coordinate ring*

$$R = \left[\frac{k[x_1, \dots, x_r, y_1, \dots, y_r]}{(1 - \sum_{i=1}^r x_i y_i)} \right]^{C_2}$$

where the C_2 action on x_i and y_i is by $x_i \mapsto -x_i$ and $y_i \mapsto -y_i$.

Proof The variety DQ_{2r-1} is a complement of a hypersurface in $\mathbb{P}^{2r-1}$, and is therefore affine.

Let Q denote $a_1 b_1 + \cdots + a_r b_r$. The coordinate ring of DQ_{2r-1} is the ring of degree-0 terms in the graded ring $S = k[a_1, \dots, a_r, b_1, \dots, b_r, Q^{-1}]$, where $|a_i| = |b_i| = 1$ and $|Q^{-1}| = -2$. This ring is the subring of S generated by the terms $a_i a_j Q^{-1}$, $a_i b_j Q^{-1}$ and $b_i b_j Q^{-1}$.

Consider the ring

$$T = \frac{k[x_1, \dots, x_r, y_1, \dots, y_r]}{(1 - \sum_{i=1}^r x_i y_i)}.$$

One may define a map of rings $\phi : S \to T$ by sending $a_i \mapsto x_i$ and $b_i \mapsto y_i$, since $Q \mapsto 1$ under this assignment. Restricting to $\Gamma(DQ_{2r-1}, \mathcal{O}_{DQ_{2r-1}}) \subset S$, one obtains a map $\Gamma(DQ_{2r-1}, \mathcal{O}_{DQ_{2r-1}}) \to T$ for which the image is precisely the subring generated by terms $x_i x_j$, $x_i y_j$ and $y_i y_j$, *i.e.*, the fixed subring under the C_2 action given by $x_i \mapsto -x_i$ and $y_i \mapsto -y_i$.

It remains to establish this map is injective. We show that the kernel of the map $\phi : S \to T$ contains only one homogeneous element, 0, so that the restriction of this map to the subring of degree-0 terms in S is injective. The kernel of ϕ is the ideal $(Q - 1)$. Since S is an integral domain, degree considerations imply that no nonzero multiple of $(Q - 1)$ is homogeneous. ∎

Proposition 5.6 *For all r, there is an $\mathbb{A}^1$-equivalence*

$$DQ_{2r-1} \to (\mathbb{A}^r \backslash \{0\})/C_2.$$

Proof Let T be as in the proof of Lemma 5.5. It is well known that Spec T is an affine vector bundle torsor over $\mathbb{A}^r\backslash\{0\}$. In fact, for each $j \in \{1, \dots, r\}$, if we define $U_j \cong \mathbb{A}^1\backslash\{0\} \times \mathbb{A}^{r-1}$ to be the open subscheme of $\mathbb{A}^r\backslash\{0\}$ where the jth coordinate is invertible, then we arrive at a pull-back diagram

$$\begin{array}{ccc} \mathbb{A}^{r-1} \times U_j \cong \operatorname{Spec} T \times_{\mathbb{A}^r\backslash\{0\}} U_j & \longrightarrow & \operatorname{Spec} T \\ \downarrow & & \downarrow \\ U_j & \longrightarrow & \mathbb{A}^r \setminus \{0\} \end{array}$$

Since U_j inherits a free C_2-action, it follows that in the quotient we obtain a vector bundle $(\mathbb{A}^{r-1} \times U_j)/C_2 \to U_j/C_2$, and so the map $(\operatorname{Spec} T)/C_2 \to (\mathbb{A}^r\backslash\{0\})/C_2$ is an $\mathbb{A}^1$-equivalence, as claimed. ■

As a consequence of Proposition 5.6 we observe that the affine variety DQ_{2r-1} is an affine approximation of $B(r; \mathbb{A}^2)$.

6 Relation to line bundles in the quadratic case

We continue to work over a field k, and to require that the characteristic of k be different from 2.

In the case where $n = 2$, the structure group of the degree-n étale algebra is C_2, the cyclic group of order 2, which happens to be a subgroup of $\mathbb{G}_m$. More explicitly, $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R; C_2)$ is an abelian group which is isomorphic to the isomorphism classes of quadratic étale algebras on Spec R. On the other hand due to the Kummer sequence and $C_2 \subset \mathbb{G}_m$ we have

$$0 \to R^*/R^{*2} \to \mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R; C_2) \to {}_2Pic(R) \to 0$$

which means that $\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} R; C_2)$ is identified with the set of isomorphism classes of 2-torsion line bundles $\mathcal{L}$ with a choice of trivialization $\phi : \mathcal{L} \otimes \mathcal{L} \xrightarrow{\cong} \mathcal{O}_R$.

This is the basis of the following construction.

Construction 6.1 Let X be a scheme such that 2 is invertible in all residue fields, and let $\mathcal{A}$ be a quadratic étale algebra on X. There is a trace map [14, Section I.1]:

$$Tr : \mathcal{A} \to \mathcal{O}$$

and an involution $\sigma : \mathcal{A} \to \mathcal{A}$ given by $\sigma = Tr - \text{id}$. Define $\mathcal{L}$ to be the kernel of $Tr : \mathcal{A} \to \mathcal{O}$. The sequence of sheaves on X

$$0 \to \mathcal{L} \to \mathcal{A} \to \mathcal{O} \to 0$$

is split short exact, where the splitting $\mathcal{O} \to \mathcal{A}$ is given on sections by $x \mapsto \frac{1}{2}x$.

The construction of $\mathcal{L}$ from $\mathcal{A}$ gives an explicit instantiation of the map $\mathrm{H}^1_{\text{ét}}(X, C_2) \to \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m)$ on isomorphism classes. We note that $\mathcal{L}$ must necessarily be a 2-torsion line bundles, in that $\mathcal{L} \otimes \mathcal{L}$ is trivial.

It is partly possible to reverse the construction of $\mathcal{L}$ from $\mathcal{A}$.

Construction 6.2 Let X be as above, and let $\mathcal{L}$ be a line-bundle on X such that there is an isomorphism $\mathcal{L} \otimes \mathcal{L} \to \mathcal{O}$. Let $\phi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}$ be a specific choice of isomorphism. From the data $(\mathcal{L}, \phi)$, we may produce an étale algebra $\mathcal{A} = \mathcal{O} \oplus \mathcal{L}$ on which the multiplication is given, on sections, by $(r, x) \cdot (r', x') = (rr' + \phi(x \otimes x'), rx' + r'x)$.

Proposition 6.3 *Let X be a scheme such that 2 is invertible in all residue fields of points of X. Let $\mathcal{A}$ a quadratic étale algebra on X. Let $\mathcal{L}$ be the associated line bundle to $\mathcal{A}$, as in Construction 6.1. Suppose $a_1, \ldots, a_r$ are global sections of $\mathcal{A}$. Then $a_1, \ldots, a_r$ generate $\mathcal{A}$ as an algebra if and only if $a_1 - \frac{1}{2}Tr(a_1), \ldots, a_r - \frac{1}{2}Tr(a_r)$ generate $\mathcal{L}$ as a line bundle.*

Proof Write q for the map $a \mapsto a - \frac{1}{2}Tr(a)$. The questions of generation of $\mathcal{A}$ and of $\mathcal{L}$ may be reduced to residue fields at points of X, by Proposition 2.10 for the algebra and Nakayama's lemma for the line bundle.

We may therefore suppose F is a field of characteristic different from 2, and that A/F is a quadratic étale algebra. Since 2 is invertible, we may write $A = F[z]/(z^2 - c)$ for some element $c \in F^\times$. In this presentation, $\sigma(z) = -z$ and $Tr(az + b) = 2b$. The kernel of the trace map, *i.e.* $\mathcal{L}$, is therefore Fz. The map $q : A \to Fz$ is given by $q(az + b) = az$.

An r-tuple $\vec{a} = (a_1 z + b_1, \ldots, a_r z + b_r)$ of elements of A generate it as an F-algebra if and only if $q(\vec{a}) = (a_1 z, \ldots a_r z)$ do. This tuple generates A as an algebra if and only if at least one of the a_i is nonzero, which is exactly the condition for it to generate Fz as an F-vector space. ■

Remark 6.4 Let k be a field of characteristic different from 2. Let X be a k-variety. An étale algebra of degree 2 generated by r global sections corresponds to a map $X \to B(r; \mathbb{A}^2)$. A line bundle generated by r global sections corresponds to a map $X \to \mathbb{P}^{r-1}$. In light of Proposition 6.3, there must be a map of varieties $B(r; \mathbb{A}^2) \to \mathbb{P}^{r-1}$. This map is given by

$$B(r; \mathbb{A}^2) \xrightarrow{\cong} (\mathbb{A}^r \setminus \{0\})/C_2 \times \mathbb{A}^r \xrightarrow{p_1} (\mathbb{A}^r \setminus \{0\})/C_2 \to (\mathbb{A}^r \setminus \{0\})/\mathbb{G}_m \xrightarrow{\cong} \mathbb{P}^{r-1}$$

where the morphisms are, left to right, the isomorphism of Lemma 5.1, projection onto the second factor, and the map induced by the inclusion $C_2 \subset \mathbb{G}_m$.

7 The example of Chase

The following will be referred to as "the example of Chase".

Construction 7.1 Let $S = \mathbb{R}[z_1, \ldots, z_r]/\left(\sum_{i=1}^r z_i^2 - 1\right)$ and equip this with the C_2-action given by $z_i \mapsto -z_i$. Let $R = S^{C_2}$. The dimension of both R and S is $r - 1$.

The ring R carries a projective module of rank 1, *i.e.*, a line bundle, that requires r global sections in order to generate it. This example is given in [21, Theorem 4].

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Remark 7.2 In fact, the line bundle in question is of order 2 in the Picard group, so Proposition 6.3 applies and there is an associated quadratic étale algebra on Spec $R = Y(r)$ requiring r generators. The algebra is, of course, dependent on a choice of trivialization of the square of the line bundle, but one may choose the trivialization so the étale algebra in question is S itself as an R-algebra.

Remark 7.3 This construction shows that the bound of First and Reichstein, [6], on the number of generators required by an étale algebra of degree 2 is tight. This was first observed, to the best of our knowledge, by M. Ojanguren in private communication.

Even better, replacing S by $S \times R^{n-2}$ over R, one produces a degree-n étale algebra over R requiring r elements to generate, so the bound is tight in the case of étale algebras of arbitrary degrees. We owe this observation to Zinovy Reichstein.

The original method of proof that the line bundle in the example of Chase cannot be generated by fewer than r global sections uses the Borsuk–Ulam theorem. Here we show that a variation on that proof follows naturally from our general theory of classifying objects. The Borsuk–Ulam theorem is a theorem about the topology of $\mathbb{R}P^r$, so it can be no surprise that it is replaced here by facts about the singular cohomology of $\mathbb{R}P^r$.

7.1 The homotopy type of the real points of $B(r; \mathbb{A}^2)$

In addition to the general results about the motivic cohomology of $B(r; \mathbb{A}^2)$, we can give a complete description of the homotopy type of the real points $B(r; \mathbb{A}^2)(\mathbb{R})$.

If X is a nonsingular $\mathbb{R}$-variety, then it is possible to produce a complex manifold from X by first extending scalars to $\mathbb{C}$ and then employing the usual Betti realization functor to produce a manifold $X(\mathbb{C})$. Since X is defined over $\mathbb{R}$, however, the resulting manifold is equipped with an action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong C_2$. We write $X(\mathbb{R})$ for the Galois-fixed points of $X(\mathbb{C})$.

Remark 7.4 The real realization functor $X \rightsquigarrow X(\mathbb{R})$ preserves finite products, so that if $f, g : X \to Y$ are two maps of varieties and $H : X \times \mathbb{A}^1 \to X'$ is an $\mathbb{A}^1$-homotopy between them, then $f(\mathbb{R}), g(\mathbb{R})$ are homotopic maps of varieties, via the homotopy obtained by restricting $H(\mathbb{R}) : X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R}) = X(\mathbb{R}) \times \mathbb{R} \to X'(\mathbb{R})$ to the subspace $X(\mathbb{R}) \times [0, 1]$.

Using Lemma 5.1, present $U(r; \mathbb{A}^2)$ as the variety of $2r$-tuples

$$(z_1, \dots, z_r, w_1, \dots, w_r) \quad \text{such that} \quad (z_1, \dots, z_r) \neq (0, \dots, 0).$$

This variety carries an action by C_2 sending $z_i \mapsto -z_i$ and fixing the w_i. We know $U(r; \mathbb{A}^2)$ and $B(r; \mathbb{A}^2)$ are naively homotopy equivalent to $\mathbb{A}^r \setminus \{0\}$ and $\mathbb{A}^r \setminus \{0\}/C_2$ respectively.

Construction 7.5 We now consider an inclusion that is not, in general, an equivalence. Let $P(r) = \mathrm{Spec}\, S$ denote the subvariety of $\mathbb{A}^r \setminus \{0\}$ consisting of r-tuples $(z_1, \dots, z_r)$ such that $\sum_{i=1}^r z_i^2 = 1$. This is an $(r-1)$-dimensional closed affine subscheme of $\mathbb{A}^r \setminus \{0\}$, invariant under the C_2 action on $\mathbb{A}^r \setminus \{0\}$. The quotient of $P(r)$ by C_2 is $Y(r) = \mathrm{Spec}\, R$, and is equipped with an evident map $Y(r) \to (\mathbb{A}^r \backslash \{0\})/C_2 \to B(r;\mathbb{A}^2)$. Here S and R take on the same meanings as in Construction 7.1.

Proposition 7.6 *Let notation be as in Construction 7.5. The real manifold $B(r;\mathbb{A}^2)(\mathbb{R})$ has the homotopy type of*

$$B(r;\mathbb{A}^2)(\mathbb{R}) \simeq \mathbb{R}P^{r-1} \amalg \mathbb{R}P^{r-1}.$$

The closed inclusion $Y(r) \to B(r;\mathbb{A}^2)$ includes $Y(r)(\mathbb{R}) \to B(r;\mathbb{A}^2)(\mathbb{R})$ as a deformation retract of one of the connected components.

Proof By Lemma 5.1 and Remark 7.4, the manifold $B(r,\mathbb{A}^2)(\mathbb{R})$ is homotopy equivalent to $(\mathbb{A}^r \backslash \{0\}/C_2)(\mathbb{R})$. The manifold $(\mathbb{A}^r \backslash \{0\}/C_2)(\mathbb{C})$ consists of equivalence classes of r-tuples of complex numbers $(z_1, ..., z_r)$, where the z_i are not all 0, under the relation

$$(z_1, \dots, z_r) \sim (-z_1, \dots, -z_r).$$

The real points of $(\mathbb{A}^r \backslash \{0\})/C_2$ consist of Galois-invariant equivalence classes. There are two components of this manifold: either the terms in $(z_1, \dots, z_r)$ are all real or they are all imaginary. In either case, the connected component is homeomorphic to the manifold $\mathbb{R}P^{r-1}$.

We now consider the manifold $Y(r)(\mathbb{R})$. This arises as the Galois-fixed points of $Y(r)(\mathbb{C})$, which in turn is the quotient of $P(r)(\mathbb{C})$ by a sign action. That is, $P(r)(\mathbb{C})$ is the complex manifold of r-tuples $(z_1, \dots, z_r)$ satisfying $\sum_{i=1}^r z_i^2 = 1$. Again, in the $\mathbb{R}$-points, the z_i are either all real or all purely imaginary. The condition $\sum_{i=1}^r z_i^2 = 1$ is incompatible with purely imaginary z_i, so $Y(r)(\mathbb{R})$ is the manifold of r-tuples of real numbers $(z_1, \dots, z_r)$ satisfying $\sum_{i=1}^r z_i^2 = 1$, taken up to sign. In short, $Y^r(\mathbb{R}) = \mathbb{R}P^{r-1}$.

As for the inclusion $Y(r)(\mathbb{R}) \to B(r;\mathbb{A}^2)(\mathbb{R})$, it admits the following description, as can be seen by tracing through all the morphisms defined so far. Suppose given an equivalence class of real numbers $(z_1, \dots, z_r)$, satisfying $\sum_{i=1}^r z_i^2 = 1$, taken up to sign. Then embed $(z_1, \dots, z_r)$ as the point of $B(r;\mathbb{A}^2)(\mathbb{R})$ given by the class of $(z_1, z_2, \dots, z_r, 0, \dots, 0)$. That is, embed $\mathbb{R}P^{r-1}$ in $\mathbb{R}^r \times \left(\mathbb{R}^{r-1} \backslash \{0\}\right)/C_2$ by embedding $\mathbb{R}P^{r-1} \subset (\mathbb{R}^r \backslash \{0\})/C_2$ as a deformation retract, and then embedding the latter space as the zero section of the trivial bundle. It is elementary that this composite is also a deformation retract. ∎

Remark 7.7 We remark that the functor $X \rightsquigarrow X(\mathbb{R})$ does not commute with colimits. For instance $U(r;\mathbb{A}^2)(\mathbb{R})/C_2$, which is connected, is not the same as $B(r;\mathbb{A}^2)(\mathbb{R})$.

In fact, the two components of $B(r;\mathbb{A}^2)(\mathbb{R})$ as calculated above correspond to two isomorphism classes of quadratic étale $\mathbb{R}$-algebras: one component corresponds to the split algebra $\mathbb{R} \times \mathbb{R}$, and the other to the nonsplit $\mathbb{C}$.

We will need two properties of $\mathrm{H}^*(\mathbb{R}P^r;\mathbb{F}_2)$ here. Both are standard and may be found in [12].

- $\mathrm{H}^*(\mathbb{R}P^r;\mathbb{F}_2) \cong \mathbb{F}_2[\theta]/(\theta^{r+1})$ where $|\theta| = 1$.

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- The standard inclusion of $\mathbb{R}P^r \hookrightarrow \mathbb{R}P^{r+1}$ given by augmenting by 0 induces the evident reduction map $\theta \mapsto \theta$ on cohomology.

Proposition 7.8 *We continue to work over $k = \mathbb{R}$. Let $s_r : B(r; \mathbb{A}^2) \to B(r+1; \mathbb{A}^2)$ be the stabilization map of Definition 4.1. The induced map on cohomology groups*

$$s_r^* : \mathrm{H}^j(B(r+1; \mathbb{A}^2)(\mathbb{R}); \mathbb{F}_2) \to \mathrm{H}^j(B(r; \mathbb{A}^2)(\mathbb{R}); \mathbb{F}_2)$$

is an isomorphism when $j \leq r$ and is 0 otherwise.

Proof The map s_r^* is arrived at by considering the inclusion $U(r; \mathbb{A}^2) \to U(r+1; \mathbb{A}^2)$, which is given by augmenting an r-tuple of pairs $(a_1, b_1, \ldots, a_r, b_r)$ by $(0, 0)$, and then taking the quotient by C_2. After $\mathbb{R}$-realization, one is left with a map $B(r; \mathbb{A}^2)(\mathbb{R}) \to B(r+1; \mathbb{A}^2)(\mathbb{R})$ which on each connected component is homotopy equivalent to the standard inclusion $\mathbb{R}P^r \to \mathbb{R}P^{r+1}$. The result follows. ■

Proposition 7.9 (Ojanguren) *Let S and R be as in Construction 7.1. The quadratic étale algebra S/R cannot be generated by fewer than r elements.*

Sketch of proof Write $Y(r) = \operatorname{Spec} R$ as in Construction 7.5. The morphism $Y(r) \to B(r; \mathbb{A}^2)$ of Construction 7.5 classifies a quadratic étale algebra over $Y(r)$, and we can identify this algebra as S.

The map $\phi : Y(r) \to B(r; \mathbb{A}^2)$ induces stable maps $\tilde{\phi} : Y(r) \to B(R; \mathbb{A}^2)$. Any such stable map induces a surjective map

$$\tilde{\phi}^* : \mathrm{H}^*(B(R; \mathbb{A}^2)(\mathbb{R}); \mathbb{F}_2) \to \mathrm{H}^*(Y(r)(\mathbb{R}); \mathbb{F}_2)$$

by Proposition 7.6 and 7.8. In particular, it is a surjection when $* = r - 1$.

Suppose S can be generated by $r-1$ elements, then there is a classifying map $\phi' : Y(r) \to B(r-1; \mathbb{A}^2)$, from which one can produce a stable map

$$(\tilde{\phi}')^* : \mathrm{H}^*(B(R; \mathbb{A}^2)(\mathbb{R}) : \mathbb{F}_2) \to \mathrm{H}^*(B(r-1; \mathbb{A}^2); \mathbb{F}_2) \to \mathrm{H}^*(Y(r)(\mathbb{R}); \mathbb{F}_2).$$

By reference to Corollary 4.3, for sufficiently large values of R, the maps $\tilde{\phi}^*$ and $(\tilde{\phi}')^*$ agree. But $(\tilde{\phi}')^*$ induces the 0-map when $* = r - 1$, since $\mathrm{H}^*(B(r-1; \mathbb{A}^2)(\mathbb{R}); \mathbb{F}_2)$ is a direct sum of two copies of $\mathbb{F}_2[\theta]/(\theta^{r-1})$. This contradicts the surjectivity of $\tilde{\phi}^*$ in this degree.

7.2 Algebras over fields containing a square root of −1

Remark 7.10 When the field k contains a square root i of -1, the analogous construction to that of Chase exhibits markedly different behaviour. For simplicity, suppose r is an even integer. Consider the ring

$$S' = \frac{k[z_1, \ldots, z_r]}{\left(\sum_{i=1}^r z_i^2 - 1\right)}$$

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with the action of C_2 given by $z_i \mapsto -z_i$. Let $R' = (S')^{C_2}$. After making the change of variables $x_j = z_{2j-1} + iz_{2j}$ and $y_j = z_{2j-1} - iz_{2j}$, we see that S' is isomorphic to

$$\frac{k[x_1, \dots, x_{r/2}, y_1, \dots, y_{r/2}]}{\left(\sum_{j=1}^{r/2} x_j y_j - 1\right)}$$

and R' is isomorphic to the subring consisting of terms of even degree. The smallest R'-subalgebra of S' containing the $r/2$-terms $x_1, \dots, x_{r/2}$ contains each of the y_j because of the relation

$$y_j = \sum_{l=1}^{r/2} x_l (y_l y_j)$$

so S' may be generated over R' by $r/2$ elements. In fact, R' is the coordinate ring of DQ_{r-1}, by Lemma 5.5. In Proposition 7.13 below, we show that S' cannot be generated by fewer than $r/2$ elements over R'.

One may reasonably ask, therefore, over a field k containing a square root of -1:

Question 7.11 For a given dimension d, is there a smooth d-dimensional affine variety $\operatorname{Spec} R$ and a finite étale algebra $\mathcal{A}$ over $\operatorname{Spec} R$ such that $\mathcal{A}$ cannot be generated by fewer than $d+1$ elements?

The result of [6] implies that if $d+1$ is increased, then the answer is negative.

Remark 7.12 If $d = 1$, the answer to the question is positive. An example can be produced using any smooth affine curve Y for which ${}_2\mathrm{Pic}(Y) \neq 0$. Specifically, one may take a smooth elliptic curve and discard a point to produce such a Y. A nontrivial 2-torsion line bundle $\mathcal{L}$ on Y cannot be generated by 1 section, since it is not trivial. One may choose a trivialization $\phi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{O}$, and therefore endow $\mathcal{L} \oplus \mathcal{O}$ with the structure of a quadratic étale algebra, as in Construction 6.2, and this algebra also cannot be generated by 1 element.

Proposition 7.13 *Let k be a field containing a square root i of -1. Let T denote the ring*

$$T = \frac{k[x_1, \dots, x_r, y_1, \dots, y_r]}{(\sum_{i=1}^{r} x_i y_i - 1)}$$

endowed with the C_2 action given by $x_i \mapsto -x_i$ and $y_i \mapsto -y_i$. Let $R = T^{C_2}$. Then the quadratic étale algebra T over R can be generated by the r elements $x_1, \dots, x_r$, but cannot be generated by fewer than r elements.

Proof The ring R is the coordinate ring of the variety DQ_{2r-1} in Lemma 5.5. In particular, there is an $\mathbb{A}^1$-equivalence $\phi : DQ_{2r-1} \to B(r; \mathbb{A}^2)$, as in equation (5.2). Tracing through this composite, one sees it classifies the quadratic étale algebra generated by $x_1, \dots, x_r$, *i.e.*, T itself—the argument being as given for DQ_{r-1} in Remark 7.10.

Suppose for the sake of contradiction that T can be generated by $r-1$ elements over R. Let $\phi' : DQ_{2r-1} \to B(r-1; \mathbb{A}^2)$ be a classifying map for some such $r-1$-tuple

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of generators. Let $\tilde{\phi}$ and $\tilde{\phi}'$ denote the composite maps $DQ_{2r-1} \to B(2r-1; \mathbb{A}^2)$. By Corollary 4.3, these maps induce the same map on Chow groups. But in degree $r-1$, the map $\tilde{\phi}^* : \mathrm{CH}^{r-1}(B(2r-1;\mathbb{A}^2)) \to \mathrm{CH}^{r-1}(B(r;\mathbb{A}^2)) \to \mathrm{CH}^{r-1}(DQ_{2r-1})$ is an isomorphism of cyclic groups of order 2, by reference to Proposition 5.4, while by the same proposition, $(\tilde{\phi}')^* : \mathrm{CH}^{r-1}(B(2r-1;\mathbb{A}^2)) \to \mathrm{CH}^{r-1}(B(r-1;\mathbb{A}^2)) \to \mathrm{CH}^{r-1}(DQ_{2r-1})$ is 0. ■

The following shows that the bound of [6] is not quite sharp when applied to quadratic étale algebras over smooth $\bar{k}$-algebras where $\bar{k}$ is an algebraically closed field.

Proposition 7.14 *Let $\bar{k}$ be an algebraically closed field. Let $n \geq 2$, and* Spec *R an n-dimensional smooth affine $\bar{k}$-variety. If $\mathcal{A}$ is a quadratic étale algebra on* Spec *R, then $\mathcal{A}$ may be generated by n global sections.*

Proof Let $\mathcal{L}$ be a torsion line bundle on Spec R, or, equivalently, a rank- 1 projective module on R. A result of Murthy's, [19, Corollary 3.16], implies that $\mathcal{L}$ may be generated by n elements if and only if $c_1(\mathcal{L})^n = 0$. By another result of Murthy's, [19, Theorem 2.14], the group $\mathrm{CH}^n(R)$ is torsion free, so it follows that if $\mathcal{L}$ is a 2-torsion line bundle, then $\mathcal{L}$ can be generated by n elements. The proposition follows by Proposition 6.3. ■

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Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
e-mail: abhisheks@math.subc.ca tbjw@math.ubc.ca