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# BERGMAN METRICS AND GEODESICS IN THE SPACE OF KÄHLER METRICS ON PRINCIPALLY POLARIZED ABELIAN VARIETIES

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*Abstract* It is well known in Kähler geometry that the infinite-dimensional symmetric space $\mathcal{H}$ of smooth Kähler metrics in a fixed Kähler class on a polarized Kähler manifold is well approximated by finite-dimensional submanifolds $\mathcal{B}_k \subset \mathcal{H}$ of Bergman metrics of height $k$. Then it is natural to ask whether geodesics in $\mathcal{H}$ can be approximated by Bergman geodesics in $\mathcal{B}_k$. For any polarized Kähler manifold, the approximation is in the $C^0$ topology. For some special varieties, one expects better convergence: Song and Zelditch proved the $C^2$ convergence for the torus-invariant metrics over toric varieties. In this article, we show that some $C^\infty$ approximation exists as well as a complete asymptotic expansion for principally polarized abelian varieties.

*Keywords:* geodesics; Bergman kernel; Kähler space of Kähler metrics

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## 1. Introduction

Let $(M, \omega)$ be an $m$-dimensional polarized Kähler manifold. Then the space $\mathcal{H}$ of smooth Kähler metrics in a fixed Kähler class will be an infinite-dimensional Riemannian manifold under the natural $L^2$ metric. At the level of individual metrics $\omega \in \mathcal{H}$, there exists a well-developed approximation theory [**24, 25**]: given $\omega$, one can define a canonical sequence of Bergman metrics $\omega_k \in \mathcal{B}_k$ which approximates $\omega$ in the $C^\infty$ topology. The approximation theory is based on microlocal analysis in the complex domain, specifically Bergman kernel asymptotics on and off the diagonal. Our principal aim is to study the approximation from pointwise or infinitesimal objects to more global aspects of the geometry, such as the approximation of the harmonic maps or geodesics in $\mathcal{H}$ by the corresponding objects in $\mathcal{B}_k$.

The geodesic equation for the Kähler potentials $\phi_t$ of $\omega_t$ is a complex homogeneous Monge–Ampère equation [**5**,**19**]. About the solution of this Dirichlet problem, we have the following regularity theorem: $\phi_t \in C^{1,\alpha}([0, T] \times M)$ for all $\alpha < 1$ if the endpoint metrics are smooth [**4**]. It is therefore natural to study the approximation of Monge–Ampère $\mathcal{H}$ geodesics $\phi_t$ by the much simpler $\mathcal{B}_k$ geodesics $\phi_k(t, z)$, which are defined by one parameter subgroups of $\mathrm{GL}(d_k + 1)$. The problem of approximating $\mathcal{H}$ geodesic segments between two smooth endpoints by $\mathcal{B}_k$ geodesic segments was raised by Donaldson [**5**] and Phong and Sturm [**16**]. Phong and Sturm proved that $\phi_k(t,z) \to \phi_t$ in a weak $C^0$ sense on $[0,1] \times M$; a $C^0$ result with a remainder estimate was later proved by Berndtsson [**3**].

To understand the approximation of $\mathcal{H}$ geodesics by $\mathcal{B}_k$ geodesics better, e.g. the rate of the approximation, we can test some special varieties and expect a better result. For example, in the toric varieties case, when one restricts to the torus-invariant metrics, the geodesic equation becomes the real homogeneous Monge–Ampère equation and thus can be linearized by the Legendre transform [**19**]. Thus the geodesic will be smooth if the endpoints are two smooth metrics. To such geodesics, Song and Zelditch proved the profound $C^2$ convergence in space–time derivatives with the remainder estimates. In the subsequent paper by Rubinstein and Zelditch [**18**], they proved that the harmonic map equation can be linearized and thus can be solved. Moreover, harmonic maps into $\mathcal{H}$ is also the $C^2$ limit of the corresponding ones into $\mathcal{B}_k$.

Our motivation in this article is to test the convergence of geodesics over the principally polarized abelian varieties by applying the method developed in [**23**]. Our main result is that $\phi_k(t,z) \to \phi_t(z)$ in the $C^\infty$ topology in this abelian case. Moreover, $\phi_k(t,z)$ has a complete asymptotic expansion in $k$ with the leading term $\phi_t(z)$ and the second term $\log(k^m R_\infty)$, where $R_\infty$ is the ratio of the norming constants (4.6). We also test the convergence of the harmonic maps into $\mathcal{H}_0^\Gamma$ of $(S^1)^m$-invariant metrics by the corresponding ones into $\mathcal{B}_k$ and the convergence is still in the $C^\infty$ topology.

### 1.1. Background

Let $M$ be an $m$-dimensional compact Kähler manifold, $L \to M$ an ample holomorphic line bundle. Let $h$ be a smooth hermitian metric on $L$, then $h^k$ will be the induced metric on $L^k$. The curvature of $h$ is the $(1,1)$-form on $M$ defined locally by the formula $R(h) = -\frac{1}{2}\mathrm{i}\partial\bar{\partial}\log|s(z)|_h^2$, where $s(z)$ is a local, nowhere-vanishing holomorphic section [**8**]. If we fix a hermitian metric $h_0$ and let $\omega_0 = R(h_0)$, then we define $\mathcal{H}$ as the space of Kähler metrics in the fixed class of $[\omega_0]$:

$$\mathcal{H} = \{\phi \in C^\infty(M) \colon \omega_\phi = \omega_0 + \tfrac{1}{2}\mathrm{i}\partial\bar{\partial}\phi > 0\}, \tag{1.1}$$

where $\phi$ is identified with $h = h_0 \mathrm{e}^{-\phi}$ so that $R(h) = \omega_\phi$. If we define the metric $g_{\mathcal{H}}$ on $\mathcal{H}$ as

$$\|\psi\|^2_{g_{\mathcal{H}},\phi} = \int_M |\psi|^2 \omega_\phi^m, \quad \text{where } \phi \in \mathcal{H} \text{ and } \psi \in T_\phi\mathcal{H} \simeq C^\infty(M). \tag{1.2}$$

Then formally $(\mathcal{H}, g_{\mathcal{H}})$ is an infinite-dimensional non-positively curved symmetric Riemannian manifold [**5**,**14**,**19**]. Furthermore, the geodesics of $\mathcal{H}$ in this metric are the paths $\phi_t$ which satisfy the partial differential equation:

$$\ddot{\phi} - |\partial\dot{\phi}|^2_{\omega_\phi} = 0. \tag{1.3}$$

The space $\mathcal{H}$ contains a family of finite-dimensional non-positively curved symmetric spaces $\mathcal{B}_k$ which are defined as follows. Let $H^0(M, L^k)$ be the space of holomorphic

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sections of $L^k \to M$ and let $d_k + 1 = \dim H^0(M, L^k)$. For large $k$ and for $\underline{s} = (s_0, \dots, s_{d_k})$ an ordered basis of $H^0(M, L^k)$, let

$$\iota_{\underline{s}}\colon M \to \mathbb{CP}^{d_k}, \quad z \to [s_0(z), \dots, s_{d_k}(z)] \tag{1.4}$$

be the Kodaira embedding. Then we have a canonical isomorphism $L^k = \iota_{\underline{s}}^* O(1)$. We then define a Bergman metric of height $k$ to be a metric of the form:

$$\mathrm{FS}_k(\underline{s}) := (\iota_{\underline{s}}^* h_{\mathrm{FS}})^{1/k} = \frac{h_0}{(\sum_{j=0}^{d_k} |s_j(z)|^2_{h_0^k})^{1/k}}, \tag{1.5}$$

where $h_{\mathrm{FS}}$ is the Fubini–Study Hermitian metric on $\mathcal{O}(1) \to \mathbb{CP}^{d_k}$. Note that the right-hand side of (1.5) is independent of the choice of $h_0$. We define the space of Bergman metrics as

$$\mathcal{B}_k = \{\mathrm{FS}_k(\underline{s})\colon \underline{s} \text{ a basis of } H^0(M, L^k)\}. \tag{1.6}$$

Then $\mathcal{B}_k = \mathrm{GL}(d_k + 1)/\,\mathrm{U}(d_k + 1)$ is a finite-dimensional negatively curved symmetric space. It is proved in [**24**, **25**] that the union $\mathcal{B} = \bigcup_{k=1}^{\infty} \mathcal{B}_k$ is dense in $\mathcal{H}$ in the $C^\infty$ topology. If $h \in \mathcal{H}$, then there exists $h(k) \in \mathcal{B}_k$ such that $h(k) \to h$ in $C^\infty$ topology.

In fact, there is a canonical choice of the approximating sequence $h(k)$ [**24**] which is used throughout the article. The hermitian metric $h$ on $L$ induces a natural inner product $\mathrm{Hilb}_k(h)$ on $H^0(M, L^k)$ defined by

$$\langle s_1, s_2 \rangle_{h^k} = \int_M (s_1(z), s_2(z))_{h^k} \frac{\omega_h^m}{m!}, \quad \text{where } \omega_h = R(h), \tag{1.7}$$

for any $s_1, s_2 \in H^0(M, L^k)$. In particular, the norm square of the holomorphic section is

$$\|s\|^2_{h^k} = \int_M |s|^2_{h^k} \frac{\omega_h^m}{m!}. \tag{1.8}$$

Now choose $\underline{s}(k)$ as an orthonormal basis of $H^0(M, L^k)$ with respect to the inner product $\mathrm{Hilb}_k(h)$, then we have the following $C^\infty$ asymptotics for the Bergman kernel as $k \to \infty$ [**25**] (see also [**2**]):

$$\sum_{j=0}^{d_k} |s_j(z)|^2_{h^k} = k^m + a_1(z) k^{m-1} + \cdots, \tag{1.9}$$

where $a_1(z)$ is the scalar curvature of $h$. Now let $\hat{\underline{s}}(k) = k^{-m/2} \underline{s}(k)$. Then the Bergman metric $h(k) = \mathrm{FS}_k \circ \mathrm{Hilb}_k(h) := \mathrm{FS}_k(\hat{\underline{s}}(k))$ will be an approximating sequence of $h$; to be more precise, (1.5) and (1.9) imply that for each $r > 0$,

$$\left\| \frac{h(k)}{h} - 1 \right\| = O\left(\frac{1}{k^2}\right), \qquad \|\omega(k) - \omega\| = O\left(\frac{1}{k^2}\right), \qquad \|\phi(k) - \phi\| = O\left(\frac{1}{k^2}\right), \tag{1.10}$$

where the norms are taken with respect to $C^r(\omega_0)$. Here, as before, $\omega = R(h)$, $\omega(k) = R(k)$, $h = h_0 \mathrm{e}^{-\phi}$, $h(k) = h_0 \mathrm{e}^{-\phi(k)}$.

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Now we can compare geodesics in $\mathcal{H}$ and Bergman geodesics in $\mathcal{B}_k$. Let $h_0, h_1 \in \mathcal{H}$. Then there will be a unique $C^{1,\alpha}$ Monge–Ampère geodesic $h_t = h_0 \mathrm{e}^{-\phi_t(z)}\colon [0,1] \to \mathcal{H}$ joining $h_0$ to $h_1$ for all $\alpha \in (0,1)$ [**4**]. Assume $h_0(k) = \mathrm{FS}_k(\underline{\hat{s}}^{(0)}(k))$ and $h_1(k) = \mathrm{FS}_k(\underline{\hat{s}}^{(1)}(k))$ are two sequence in $\mathcal{B}_k$ obtained by the canonical construction approximating $h_0$ and $h_1$. Then the geodesic joining $h_0(k)$ and $h_1(k)$ in the space $\mathcal{B}_k = \mathrm{GL}(d_k+1)/\,\mathrm{U}(d_k+1)$ is constructed in [**16**] as follows. Let $\sigma_k \in \mathrm{GL}(d_k+1)$ be the change of basis matrix defined by $\sigma_k \cdot \underline{\hat{s}}^{(0)}(k) = \underline{\hat{s}}^{(1)}(k)$. Without loss of generality, we may assume that $\sigma_k$ is diagonal with entries $\mathrm{e}^{\lambda_0}, \dots, \mathrm{e}^{\lambda_{d_k}}$ for some $\lambda_j \in \mathbb{R}$. Let $\underline{\hat{s}}^{(t)}(k) = \sigma_k^t \cdot \underline{\hat{s}}^{(0)}(k)$, where $\sigma_k^t$ is diagonal with entries $\mathrm{e}^{\lambda_j t}$. Define

$$h_k(t,z) = \mathrm{FS}_k(\underline{\hat{s}}^{(t)}(k)) = h_0 \mathrm{e}^{-\phi_k(t,z)}. \tag{1.11}$$

Then $h_t(k,z)$ is the smooth geodesic in $\mathrm{GL}(d_k+1)/\,\mathrm{U}(d_k+1)$ joining $h_0(k)$ to $h_1(k)$. Explicitly, use identity (1.5) again, we have

$$\phi_k(t,z) = \frac{1}{k} \log\left(\sum_{j=0}^{d_k} \mathrm{e}^{2\lambda_j t} |\hat{s}_j^{(0)}(k)|^2_{h_0^k}\right). \tag{1.12}$$

Then the main result of Phong and Sturm [**16**] is that the Monge–Ampère geodesic $\phi_t(z)$ is approximated by Bergman geodesic $\phi_k(t,z)$ in a weak $C^0$ sense on $[0,1] \times M$; a $C^0$ result with a remainder estimate was later proved by Berndtsson [**3**].

For special varieties, one expects better result. The first evidence is in [**23**]: Song and Zelditch proved the convergence of $\phi_k(t,z) \to \phi_t(z)$ is much stronger for toric hermitian metrics on the torus-invariant line bundle over the smooth toric Kähler manifold. To be more precise, they defined the space of toric Hermitian metrics:

$$\mathcal{H}(\mathbb{T}^m) = \{\phi \in \mathcal{H}\colon (\mathrm{e}^{\mathrm{i}\theta})^* \phi = \phi \text{ for all } \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}^m\}. \tag{1.13}$$

Then for the smooth geodesic in $\mathcal{H}(\mathbb{T}^m)$ with the endpoints $h_0$ and $h_1 \in \mathcal{H}(\mathbb{T}^m)$, they proved

$$\lim_{k\to\infty} \phi_k(t,z) = \phi(t,z) \quad \text{in } C^2([0,1] \times M). \tag{1.14}$$

And they also obtained the rate of the convergence and the remainder estimates. In fact, their method can be applied to the principally polarized abelian varieties. In our article, we consider the abelian case and prove the existence of $C^\infty$ convergence, moreover, we can expand $\phi_k(t,z)$ in $k$ completely with the leading term $\phi_t$.

### 1.2. Main results

Throughout the article, we will use the following notation: denote $\Gamma = (S^1)^m \cong (\mathbb{R}/\mathbb{Z})^m$, the isomorphism is given by $\mathrm{e}^{2\pi\mathrm{i}\theta} \to \theta \bmod \mathbb{Z}^m$; thus we can identify the periodic function on $\mathbb{R}^m$ with period 1 in each variable with the function defined on $\Gamma$; denote $y^2 = y_1^2 + \dots + y_m^2$ and $x \cdot y = x_1 y_1 + \dots + x_m y_m$ for $x, y \in \mathbb{R}^m$.

By performing affine transformation, it suffices to consider the principally polarized abelian variety $M = \mathbb{C}^m/\Lambda$, where $\Lambda = \mathbb{Z}^m + \mathrm{i}\mathbb{Z}^m$. We will prove our main result for this

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model case first and in § 7, we will sketch how to extend our argument to the general lattice.

Now for $M = \mathbb{C}^m/\Lambda$, where $\Lambda = \mathbb{Z}^m + \mathrm{i}\mathbb{Z}^m$, we can write each point in $M$ as $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}^m$ and they can be considered as the period coordinate in $M$. There is a natural action on $M$: the group $\Gamma = (S^1)^m$ acts on $M$ via translations in the lagrangian subspace $\mathbb{R}^m \subset \mathbb{C}^m$, i.e. the translation of $x$ in the universal covering space.

Let $L \to M$ be a principal polarization of $M$; then there is a classical hermitian metric defined on $L$ [**8**]:

$$h = \mathrm{e}^{-2\pi y^2}.$$

The curvature of $h$ is given by $R(h) = \pi \sum_{\alpha=1}^{m} \mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\alpha$ which is in the class $[\pi c_1(L)]$. Now fix $\omega_0 = R(h)$ a flat metric on $M$ with associated Kähler potential $2\pi y^2$, denote $\mathcal{H}_0^\Gamma$ as the space of $\Gamma$-invariant Kähler metrics in the fixed class $[\omega_0]$, then

$$\mathcal{H}_0^\Gamma = \{\psi \in C_\Gamma^\infty(M) \colon \omega_\psi = \omega_0 + \tfrac{1}{2}\mathrm{i}\partial\bar{\partial}\psi > 0\}.$$

Note that a smooth function $\psi(x, y)$ defined on $M$ invariant under the $\Gamma$ actions should be independent of $x$ variable; thus in fact they are smooth functions on $M/\Gamma$, i.e. $\psi$ is a smooth and periodic function on the universal covering space $y \in \mathbb{R}^m$.

All hermitian metrics $h$ on $L$ such that $R(h) = \omega_\psi \in \mathcal{H}_0^\Gamma$ will be in the form

$$h = \mathrm{e}^{-2\pi y^2 - 4\pi\psi(y)}. \tag{1.15}$$

In § 2, we will see such $h$ is a well-defined hermitian metric on $L$. And the corresponding Kähler potential is

$$\varphi(y) = 2\pi y^2 + 4\pi\psi(y), \tag{1.16}$$

where $\psi(y)$ is smooth and periodic function with period 1 and we have to assume $\nabla^2\varphi > 0$ because of the positivity of Kähler form.

The following fact about the space $\mathcal{H}_0^\Gamma$ is crucial [**5, 19**]: given any $\varphi_0$ and $\varphi_1 \in \mathcal{H}_0^\Gamma$, we can join them by a smooth geodesic $\varphi_t \in \mathcal{H}_0^\Gamma$. Thus throughout the article, we will consider the geodesic in the form $\varphi_t(y) = 2\pi y^2 + 4\pi\psi_t(y)$.

In § 7, we can extend all these settings to the case of general lattice immediately. Our main result for any principally polarized abelian variety is the following.

**Theorem 1.1.** *Let $M$ be a principally polarized abelian variety and let $L \to M$ be a principal polarization of $M$. Given $h_0$ and $h_1$ in $\mathcal{H}_0^\Gamma$ of the space of $\Gamma$-invariant Kähler metrics, let $h_t \in \mathcal{H}_0^\Gamma$ be the smooth geodesic between them. Let $h_k(t)$ be the Bergman geodesic between $h_0(k)$ and $h_1(k)$ in $\mathcal{B}_k$. Let $h_k(t) = \mathrm{e}^{-\phi_k(t,z)}h_0$ and $h_t = \mathrm{e}^{-\phi_t(z)}h_0$, then*

$$\lim_{k\to\infty} \phi_k(t, z) = \phi_t(z)$$

*in the $C^\infty([0, 1] \times M)$ topology. Moreover, we have the following $C^\infty$ complete asymptotics:*

$$\phi_k(t, z) = \phi_t(z) + mk^{-1}\log k + k^{-1}a_1(t, \mu_t) + k^{-2}a_2(t, \mu_t) + \cdots \tag{1.17}$$

*for $k$ large enough, where $\mu_t(y) = \nabla\varphi_t(y)$ where $y$ is defined in (2.1) and each $a_n$ is a smooth function of $\mu_t$ and $t$. In particular, $a_1 = \log R_\infty$ where $R_\infty$ is defined by (4.6).*

1.2.1. *Outline of the proof*

We now sketch the proof of our main result for the model case: define the inner product on $H^0(M, L^k)$ induced by $h_t^k$ in the sense of (1.7), then in Proposition 2.1 we first prove that, for any fixed $t$, the following theta functions of level $k$

$$\theta_j(z) = \sum_{n\in\mathbb{Z}^m} \exp\left(-\pi\frac{(j+kn)^2}{k} + 2\pi\mathrm{i}(j+kn)\cdot z\right), \quad j\in(\mathbb{Z}/k\mathbb{Z})^m,$$

form an orthogonal basis with respect to this inner product; in particular,

$$\dim H^0(M, L^k) = k^m.$$

Therefore, in the canonical construction, we can choose the orthonormal basis $\underline{s}^{(t)}(k)$ as $\theta_j$ normalized by $\|\theta_j\|_{h_t^k}$. Hence, if $\sigma_k \in \mathrm{GL}(k^m)$ such that $\sigma_k\cdot\hat{\underline{s}}^{(0)}(k) = \hat{\underline{s}}^{(1)}(k)$, then $\sigma_k$ can be chosen to be diagonal with entries

$$\mathrm{e}^{\lambda_j} = \frac{\|\theta_j\|_{h_0^k}}{\|\theta_j\|_{h_1^k}}.$$

Hence, the equation (1.12) of the Bergman geodesic becomes

$$\phi_k(t,z) = \frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}\left(\frac{\|\theta_j\|^2_{h_0^k}}{\|\theta_j\|^2_{h_1^k}}\right)^t \frac{|\theta_j|^2_{h_0^k}}{\|\theta_j\|^2_{h_0^k}}. \tag{1.18}$$

Our main theorem is to prove this term converges to $\phi_t(z)$ in the $C^\infty([0,1]\times M)$ topology. But

$$\phi_k(t,z) - \phi_t(z) = \frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}\left(\frac{\|\theta_j\|^2_{h_0^k}}{\|\theta_j\|^2_{h_1^k}}\right)^t \frac{|\theta_j|^2_{h_0^k}\mathrm{e}^{-k\phi_t}}{\|\theta_j\|^2_{h_0^k}},$$

denote $\rho_k(j,t) = \|\theta_j\|^2_{h_t^k}$ as the norming constant and denote

$$R_k(j,t) = \frac{\rho_k(j,t)}{(\rho_k(j,0))^{1-t}(\rho_k(j,1))^t},$$

and as usual $h_t = \mathrm{e}^{-\phi_t}h_0$, then we can rewrite

$$\phi_k(t,z) - \phi_t(z) = \frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} R_k(j,t)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}}. \tag{1.19}$$

Thus our goal is equivalent to prove this term goes to 0 in the $C^\infty$ topology as $k\to\infty$. In fact we can prove the following result which implies Theorem 1.1 immediately.

**Lemma 1.2.** *With all assumptions and notation as above, we have*

$$\frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} R_k(j,t)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}} = mk^{-1}\log k + \log R_\infty(\mu_t,t) + k^{-2}c_1(\mu_t,t) + \cdots,$$

*where* $\mu_t(y) = \nabla\varphi_t$, $c_n(\mu_t, t) \in C^\infty(M \times [0,1])$ *and periodic in* $y$ *variables for any fixed* $t$ *and* $R_\infty$ *is defined by (4.6). Furthermore, this expansion can be differentiated any number of times on both sides with respect to* $t$ *and* $y$ *(or* $z$*). In particular, the left-hand side goes to* $0$ *in the* $C^\infty$ *topology as* $k \to \infty$*, i.e. the Bergman geodesics converge to the geodesic in the Kähler space in the* $C^\infty$ *topology.*

The proof of Lemma 1.2 is the consequence of the following two lemmas.

- The Regularity Lemma (see Lemma 4.1 below): $R_k(j,t)$ admits the complete asymptotics with the leading term given by $R_\infty(x,t)$ evaluated at the point $x_0 = -4\pi j/k$.

- The generalized Bernstein Polynomial Lemma (see Lemma 1.3 below):

$$\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(-\frac{j}{k}\right)\frac{|\theta_j|^2_{h^k}}{\|\theta_j\|^2_{h^k}}$$

  admits complete asymptotics for any periodic function $f$ defined on $\mathbb{R}^m$ with period 1.

In § 5, we will give the proof of the following generalized Bernstein Polynomial Lemma using the basic properties of theta functions and Weyl quantization.

**Lemma 1.3.** *Let* $f(x) \in C^\infty(\mathbb{R}^m)$ *and periodic in each variable with period* $1$*, let* $h \in \mathcal{H}_0^\Gamma$*, then we have the complete asymptotics*

$$\frac{1}{k^m}\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(-\frac{j}{k}\right)\frac{|\theta_j|^2_{h^k}}{\|\theta_j\|^2_{h^k}} = f(\mu) + k^{-1}b_1(\mu) + \cdots, \tag{1.20}$$

*where* $\mu(y) = y + \nabla\psi$ *and all* $b_n(\mu) \in C^\infty(\mathbb{R}^m)$*.*

The generalized Bernstein Polynomial Lemma (Lemma 1.3) has an application to Dedekind–Riemann sums for the periodic functions. Many results about the complete asymptotics of Dedekind–Riemann sums for the smooth functions with compact support over the polytope $P$ were obtained by Brion and Vergne [**1**], Guillemin and Sternberg [**10**] and many others. For purposes of comparison, Theorem 4.2 in [**10**] states that for $f \in C_0^\infty(\mathbb{R}^n)$,

$$\frac{1}{k^m}\sum_{\alpha\in\mathbb{Z}^m\cap kP} f\left(\frac{\alpha}{k}\right) = \left(\sum_F \sum_{\gamma\in\Gamma_F^\sharp} \tau_\gamma\left(\frac{1}{k}\frac{\partial}{\partial h}\right)\int_{P_h} f(x)\,\mathrm{d}x\right) \quad (h=0),$$

where $\alpha$ is the lattice point in the $k$th dilate of the polytope $kP$ and $P_h$ is a parallel dilate of $P$. We refer to [**10**] for more details.

Afterward, Zelditch related the Bernstein polynomials to the Bergman kernel for the Fubini–Study metric on $\mathbb{CP}^1$, and generalized this relation to any compact Kähler toric manifold, then he got many interesting results [**26**]. To be more precise, let $(L,h) \to$

$(M,\omega)$ be a toric Hermitian invariant line bundle over a Kähler toric manifold with associated moment polytope $P$, he proved the following complete asymptotics:

$$\sum_{\alpha\in\mathbb{Z}^m\cap kP} f\left(\frac{\alpha}{k}\right)\frac{|s_\alpha|^2_{h^k}}{\|s_\alpha\|^2_{h^k}} = f(x)+k^{-1}\mathcal{L}_1 f(x)+k^{-2}\mathcal{L}_2 f(x)+\cdots, \tag{1.21}$$

where $f\in C_0^\infty(\mathbb{R}^m)$, each $\mathcal{L}_j$ is a differential operator of order $2j$, $s_\alpha$ is the orthogonal basis of $H^0(M,L^k)$ which in fact are monomials $z^\alpha$. Then the simple integration yields

$$\frac{1}{k^m}\sum_{\alpha\in\mathbb{Z}^m\cap kP} f\left(\frac{\alpha}{k}\right) = \int_P f(x)\,\mathrm{d}x+\frac{1}{2k}\int_{\partial P} f(x)\,\mathrm{d}x+\frac{1}{k^2}\int_P \mathcal{L}_2 f(x)\,\mathrm{d}x+\cdots. \tag{1.22}$$

In [**6**], this method is then generalized to the non-compact polyhedral set.

In § 5, we will first generalize the method in [**6**, **26**] to the abelian varieties to get the Lemma 1.3. If we take the integral over $M$ on both sides of (1.20) and note

$$\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(\frac{j}{k}\right) = \sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(-\frac{j}{k}\right),$$

then we have the following Dedekind–Riemann sums for periodic functions.

**Corollary 1.4.** *Let $f(x)\in C^\infty(\mathbb{R}^m)$ and periodic in each variable with period 1, then*

$$\frac{1}{k^m}\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(\frac{j}{k}\right) = \int_{[0,1]^m} f(x)\,\mathrm{d}x+k^{-1}\int_{[0,1]^m} b_1(x)\,\mathrm{d}x+\cdots, \tag{1.23}$$

*where each $b_n(x)\in C^\infty(\mathbb{R}^m)$ and can be computed explicitly.*

### 1.3. Final remarks and further results

The $C^2$ convergence of Song and Zelditch for the toric varieties can be improved to the $C^\infty$ convergence for the abelian varieties mainly because of the Regularity Lemma (Lemma 4.1): $R_k(j,t)$ admits complete asymptotics. But for the toric case, they do not know the existence of the complete asymptotics of $R_k(\alpha,t)$, where $\alpha$ is a lattice point in $P$ which is the image of the moment map of toric varieties $\nabla_\rho\varphi\colon M\to P$. In fact, they have the following lemma:

$$\left(\frac{\partial}{\partial t}\right)^n R_k(\alpha,t) = \left(\frac{\partial}{\partial t}\right)^n R_\infty\left(\frac{\alpha}{k},t\right)+O(k^{-1/3}),\quad 0\leqslant n\leqslant 2.$$

They cannot prove the existence of complete asymptotics because they cannot get the joint asymptotics of $k$ and $\alpha$ of the norming constant $\rho_k(\alpha)=\|s_\alpha\|^2_{h^k}$, where $s_\alpha$ is the holomorphic section of the invariant line bundle. Recall that the boundary of $P$ is the image of the points with isotropy group of $\mathbb{T}^n$, $1\leqslant n\leqslant m$, under the moment map $\nabla_\rho\varphi$ and the boundary causes serious complications. To be more precise, they can rewrite $\rho_k(\alpha)$ as

$$\rho_k(\alpha) = \int_P \exp\left(-k\left(u_\varphi(x)+\left\langle\frac{\alpha}{k}-x,\nabla u_\varphi(x)\right\rangle\right)\right)\mathrm{d}x,$$

where $u_\varphi$ is the symplectic potential defined on $P$, i.e. the Legendre transform of Kähler potential $\varphi$. Note that the critical point of the phase is given by $\alpha/k$; thus they can get complete asymptotics by the stationary phase method when the point $\alpha/k$ is far away from the boundary of $P$. But they cannot get joint asymptotics by this method when the point goes to the boundary $\partial P$ as $k \to \infty$ [**23**].

But in our abelian case, we do not have such disadvantage. There is a real torus $\Gamma = (S^1)^m$ action on the abelian varieties. This action is free, i.e. there is no point with the isotropy group of $(S^1)^n$, $1 \leqslant n \leqslant m$. In § 3, we will see that the gradient of the Kähler potential induces a map $\nabla\varphi_t = 4\pi(y + \nabla\psi_t)\colon M \to M/\Gamma$, which is in fact a Lie group valued moment map for any fixed $t$. The image of $\nabla\varphi_t$ is $M/\Gamma$ which has no boundary. There is another way to look at this, in § 4, we rewrite $\rho_k(j) = \|\theta_j\|^2_{h^k}$ as an integral over the universal covering space $\mathbb{R}^m$ (4.9):

$$\rho_k(j) = \exp\left(-2\pi\frac{j^2}{k}\right)\int_{\mathbb{R}^m}\exp\left(-k\pi\left(-u(x) + \left\langle x + \frac{4\pi j}{k}, \nabla u(x)\right\rangle\right)\right)\mathrm{d}x,$$

where $u(x)$ is defined by Legendre transform of $\varphi$, thus we can apply the stationary phase method to this integral everywhere.

For example, in § 2, we can get identity (2.12) which is the exact formula for $\rho_k(j) = \|\theta_j\|^2_{h^k}$. If we assume $\psi \equiv 0$, i.e. we choose the flat metric over the abelian variety, then $\|\theta_j\|^2_{h^k}$ will be a constant independent of $j$, i.e. the joint complete asymptotics of $\rho_k(j)$ (which is in fact a constant) exist for any $j$ as $k \to \infty$. This is totally different from the toric case. For example, consider $(\mathbb{CP}^1, \omega_{\mathrm{FS}})$ with Fubini–Study metric, then

$$\|z^\alpha\|^2_{h^k_{\mathrm{FS}}} = \binom{k}{\alpha}^{-1},$$

but as proved in [**22**], for any $\alpha \in [k^{-3/4}, 1 - k^{-3/4}]$, by stationary phase method, they have

$$\binom{k}{k\alpha} \sim \frac{1}{\sqrt{2\pi k\alpha(1-\alpha)}}\mathrm{e}^{(\alpha\log\alpha + (1-\alpha)\log(1-\alpha))}.$$

Then it is easy to see that the asymptotics are highly non-uniform as $\alpha \to 0$ or $\alpha \to 1$, where 0 and 1 are two boundary points of the moment polytope $[0, 1]$ of $\mathbb{CP}^1$.

1.3.1. *Harmonic maps into* $\mathcal{H}^0_\Gamma$

A harmonic map between two Riemannian manifolds $(N_1, g_1)$ and $(N_2, g_2)$ is a critical point of the energy functional

$$E(f) = \int_{N_1} |\mathrm{d}f|^2_{g_1 \otimes f^* g_2}\,\mathrm{dVol}_{g_1}$$

on the space of smooth maps $f\colon N_1 \to N_2$. Note that this notion may also be defined when the target manifold $(N_2, g_2)$ is an infinite-dimensional weakly Riemannian manifold, e.g. $(\mathcal{H}, g_{\mathcal{H}})$. By a smooth map $f$ from $N$ to $\mathcal{H}$ we mean a function $f \in C^\infty(N \times M)$ such that $f(q, \cdot) \in \mathcal{H}$ for each $q \in N$ (see Definition 1.1 in [**18**]).

In [**18**], Rubinstein and Zelditch proved that, in the toric case, the Dirichlet problem for a harmonic map $\varphi\colon N \to \mathcal{H}(\mathbb{T}^m)$ of any compact Riemannian manifold $N$ with smooth boundary into $\mathcal{H}(\mathbb{T}^m)$ of toric invariant metrics admits a smooth solution that may be approximated in $C^2(N \times M)$ by a special sequence of harmonic maps $\varphi_k\colon N \to \mathcal{B}_k(\mathbb{T}^m) \subset \mathcal{H}(\mathbb{T}^m)$ into the subspaces $\mathcal{B}_k(\mathbb{T}^m)$ of Bergman metrics [**18**, Theorem 1.1]. This generalized the work of Song and Zelditch in the case of geodesics, i.e. where $N = [0, 1]$.

In the spirit of [**18**], we consider the harmonic maps into the space of $\mathcal{H}_0^\Gamma$ of $\Gamma$-invariant abelian metrics. Then we can prove that the approximation of the harmonic into $\mathcal{H}_0^\Gamma$ by the corresponding ones into $\mathcal{B}_k$ is still $C^\infty$.

**Theorem 1.5.** *Let $M$ be a principally polarized abelian variety and let $L \to M$ be a principal polarization. Let $(N, g)$ be a compact oriented smooth Riemannian manifold with smooth boundary $\partial N$. Let $\psi\colon \partial N \to \mathcal{H}_0^\Gamma$ denote a fixed smooth map. There exists a harmonic map $\varphi\colon N \to \mathcal{H}_0^\Gamma$ with $\varphi|_{\partial N} = \psi$ and harmonic maps $\varphi_k\colon N \to \mathcal{B}_k$ with $\varphi_k|_{\partial N} = \mathrm{FS}_k \circ \mathrm{Hilb}_k(\psi)$, then we have the following $C^\infty$ complete asymptotics,*

$$\varphi_k = \varphi + mk^{-1} \log k + k^{-1} a_1 + k^{-2} a_2 + \cdots ,$$

*where each $a_n$ is smooth and $a_1 = \log K_\infty$, where $K_\infty$ is defined by (8.3).*

The proof of Theorem 1.5 is similar to the one in [**18**]. In § 8, we will sketch the main steps of the proof for the model case.

## 2. Abelian varieties and theta functions

In this section, we will review some basic properties of principally polarized abelian varieties and theta functions. We mainly follow [**7**] and readers should refer to [**8, 15**] for more details.

Let $V$ be a $m$-dimensional complex vector space and $\Lambda \cong \mathbb{Z}^{2m}$ a maximal lattice in $V$ such that the quotient $M = V/\Lambda$ is an abelian variety, i.e. a complex torus which can be holomorphically embedded in projective space. We assume that $M$ is endowed with a principal polarization, then we can always find a basis $\lambda_1, \dots, \lambda_{2m}$ for $\Lambda$, such that $\lambda_1, \dots, \lambda_m$ is a basis of $V$ and

$$\lambda_{m+\alpha} = \sum_{\beta=1}^{m} Z_{\beta\alpha} \lambda_\beta, \quad \alpha = 1, \dots, m,$$

where $Z = (Z_{\alpha\beta})_{\alpha,\beta=1}^m$ is an $m \times m$ matrix that satisfies $Z^{\mathrm{T}} = Z$ and $\operatorname{Im} Z > 0$. Conversely, principally polarized abelian varieties are parametrized by such matrixes.

Let $x_1, \dots, x_m$, $y_1, \dots, y_m$ be the coordinates on $V$ which are dual to the generators $\lambda_1, \dots, \lambda_{2m}$ of the lattice $\Lambda$. Then $x_\alpha$ and $y_\alpha$ can also be considered as periodic coordinates in $M$, and are related to the complex ones by

$$z_\alpha = x_\alpha + \sum_{\beta=1}^{m} Z_{\alpha\beta} y_\beta, \qquad \bar{z}_\alpha = x_\alpha + \sum_{\beta=1}^{m} \bar{Z}_{\alpha\beta} y_\beta. \tag{2.1}$$

Let $L \to M$ be the holomorphic line bundle, if we further assume $L$ is a principal polarization of $M$, then the first Chern class $c_1(L)$ is given by

$$\begin{aligned}\omega_0 &= \sum_{\alpha=1}^{m} \mathrm{d}x_\alpha \wedge \mathrm{d}y_\alpha \\ &= \frac{\mathrm{i}}{2}\sum_{\alpha,\beta}(\operatorname{Im} Z)^{\alpha\beta}\, \mathrm{d}z_\alpha \wedge \mathrm{d}\bar{z}_\beta. \end{aligned} \tag{2.2}$$

The space $H^0(M, L^k)$ is naturally isomorphic with the space of holomorphic functions $\theta$ on $V$ satisfying

$$\theta(z+\lambda_\alpha) = \theta(z), \qquad \theta(z+\lambda_{m+\alpha}) = \mathrm{e}^{-2k\pi\mathrm{i}z_\alpha - k\pi\mathrm{i}Z_{\alpha\alpha}}\theta(z). \tag{2.3}$$

In fact, these theta functions are in form [**7**]

$$\theta(z) = \sum_{l\in(\mathbb{Z}/k\mathbb{Z})^m} a_l\theta_l(z, \Omega),$$

where

$$\theta_l(z,\Omega) = \sum_{n\in\mathbb{Z}^m} \exp\left(\pi\mathrm{i}(l+kn)\frac{Z}{k}(l+kn)^{\mathrm{T}}\right)\mathrm{e}^{2\pi\mathrm{i}(l+kn)\cdot z}, \quad l \in (\mathbb{Z}/k\mathbb{Z})^m. \tag{2.4}$$

In particular, $\dim H^0(M, L^k) = k^m$.

Now consider the hermitian metric $h$ defined on $L$, $h$ should be a positive $C^\infty$ function of $z$ satisfying

$$h(z)|\theta(z)|^2 = h(z+\lambda)|\theta(z+\lambda)|^2 \tag{2.5}$$

for any $\lambda \in \Lambda$; thus

$$h(z+\lambda_\alpha) = h(z), \qquad h(z+\lambda_{m+\alpha}) = |\mathrm{e}^{2\pi\mathrm{i}z_\alpha}|^2|\mathrm{e}^{\pi\mathrm{i}Z_{\alpha\alpha}}|^2 h(z). \tag{2.6}$$

Conversely, any such function $h$ defines a metric on $L$.

For simplicity, we first consider the abelian variety $M = \mathbb{C}^m/\Lambda$, where $\Lambda = \mathbb{Z}^m + \mathrm{i}\mathbb{Z}^m$, write $z = x + \mathrm{i}y$, where $x$ and $y \in \mathbb{R}^m$ and can be viewed as the periodic coordinate of $M$. Let $L \to M$ be a principal polarization of $M$, then by formula (2.4), the global holomorphic section of $H^0(M, L)$ is given by the following Riemann theta functions:

$$\theta(z) = \sum_{n\in\mathbb{Z}^m} \mathrm{e}^{-\pi n^2 + 2\pi\mathrm{i}n\cdot z}, \tag{2.7}$$

where $n^2 = n_1^2 + \cdots + n_m^2$ and $n\cdot z = n_1z_1 + \cdots + n_mz_m$. And the global holomorphic section of $H^0(M, L^k)$ is given by

$$\theta_j(z) = \sum_{n\in\mathbb{Z}^m} \exp\left(-\pi\frac{(j+kn)^2}{k} + 2\pi\mathrm{i}(j+kn)\cdot z\right) \tag{2.8}$$

with $\dim H^0(M, L^k) = k^m$. Furthermore, $\theta_j(z)$ are holomorphic functions over $\mathbb{C}^m$ and satisfy the following quasi-periodicity relations:

$$\theta_j(z_\alpha + 1) = \theta_j(z_\alpha), \qquad \theta_j(z_\alpha + i) = \mathrm{e}^{-2\pi \mathrm{i} k z_\alpha + k\pi}\theta_j(z_\alpha). \tag{2.9}$$

Now define the hermitian metric on $L$ as

$$h_t = \mathrm{e}^{-2\pi y^2 - 4\pi\psi_t(y)},$$

where $\psi_t(y)$ is a smooth and periodic function of $y \in \mathbb{R}^m$ with period 1. It is easy to check that $h_t$ satisfies conditions (2.6):

$$h_t(z_\alpha + 1) = h_t(z_\alpha), \qquad h_t(z_\alpha + \mathrm{i}) = |\mathrm{e}^{2\pi \mathrm{i} z_\alpha}|^2 \mathrm{e}^{2\pi} h_t(z_\alpha),$$

thus $h_t$ is a well-defined hermitian metric on $L$.

Now in our case, the natural Hermitian inner product (1.7) defined on the space $H^0(M, L^k)$ reads

$$\langle \theta_l, \theta_j \rangle_{h_t^k} = \int_M \theta_l(z)\overline{\theta_j(z)}\mathrm{e}^{-2k\pi y^2 - 4k\pi\psi_t(y)} \frac{\omega_{h_t}^m}{m!}, \tag{2.10}$$

where the volume form

$$\frac{\omega_{h_t}^m}{m!} = (4\pi)^m \det(I + \nabla^2\psi_t)\,\mathrm{d}x\,\mathrm{d}y.$$

**Proposition 2.1.** *$\{\theta_j,\ j \in (\mathbb{Z}/k\mathbb{Z})^m\}$ forms an orthogonal basis of $H^0(M, L^k)$ with respect to the Hermitian inner product defined by (2.10).*

**Proof.** By definition,

$$\begin{aligned}
\langle \theta_l, \theta_j \rangle_{h_t^k} &= (4\pi)^m \int_{[0,1]^m}\int_{[0,1]^m} \biggl[\sum_{n\in\mathbb{Z}^m} \exp\biggl(-\pi\frac{(l+kn)^2}{k} + 2\pi\mathrm{i}(l+kn)\cdot z\biggr)\biggr] \\
&\qquad \times \biggl[\sum_{p\in\mathbb{Z}^m} \exp\biggl(-\pi\frac{(j+kp)^2}{k} - 2\pi\mathrm{i}(j+kp)\cdot\bar{z}\biggr)\biggr] \\
&\qquad \times \mathrm{e}^{-2k\pi y^2 - 4k\pi\psi_t(y)} \det(I + \nabla^2\psi_t)\,\mathrm{d}x\,\mathrm{d}y \\
&= (4\pi)^m \biggl[\sum_{n\in\mathbb{Z}^m} \int_{[0,1]^m} \mathrm{e}^{2\pi\mathrm{i}(l+kn-j-kp)\cdot x}\,\mathrm{d}x\biggr] \\
&\qquad \times \biggl[\sum_{p\in\mathbb{Z}^m} \int_{[0,1]^m} \exp\biggl(-\pi\frac{(l+kn)^2 + (j+kp)^2}{k} \\
&\qquad\qquad - 2\pi(l+kn+j+kp)\cdot y - 2k\pi y^2 - 4k\pi\psi_t\biggr) \\
&\qquad\qquad \times \det(I + \nabla^2\psi_t)\,\mathrm{d}y\biggr].
\end{aligned} \tag{2.11}$$

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For the first integral, if $l_\alpha + kn_\alpha = j_\alpha + kp_\alpha$, i.e. $l_\alpha - j_\alpha = 0 \mod k$, then

$$\int_{[0,1]} e^{2\pi i(l_\alpha + kn_\alpha - j_\alpha - kp_\alpha)x_\alpha}\, dx_\alpha = 1,$$

otherwise, it is 0. Since $1 \leqslant l_\alpha, j_\alpha \leqslant k$, hence $l_\alpha + kn_\alpha = j_\alpha + kp_\alpha$ if and only if $l_\alpha = j_\alpha$ and $p_\alpha = n_\alpha$; thus the first integral is non-zero if and only if $l = j$ and $n = p$. Then (2.11) becomes

$$\langle \theta_l, \theta_j \rangle_{h_t^k} = (4\pi)^m \delta_{l,j} \sum_{n\in\mathbb{Z}^m} \int_{[0,1]^m} \exp\left(-2k\pi\left(\frac{j}{k} + n + y\right)^2\right) e^{-4k\pi\psi_t(y)} \det(I + \nabla^2\psi_t)\, dy.$$

Hence, we can see that $\{\theta_j,\ j \in (\mathbb{Z}/k\mathbb{Z})^m\}$ forms an orthogonal basis of $H^0(M, L^k)$. $\square$

Furthermore, we have:

$$\begin{aligned}\|\theta_j\|^2_{h_t^k} &= (4\pi)^m \sum_{n\in\mathbb{Z}^m} \int_{[0,1]^m} \exp\left(-2k\pi\left(\frac{j}{k} + n + y\right)^2\right) e^{-4k\pi\psi_t(y)} \det(I + \nabla^2\psi_t)\, dy \\ &= (4\pi)^m \int_{\mathbb{R}^m} \exp\left(-2k\pi\left(y + \frac{j}{k}\right)^2\right) e^{-4k\pi\psi_t(y)} \det(I + \nabla^2\psi_t)\, dy. \end{aligned} \tag{2.12}$$

In the last step, we change variable $y \to y+n$ and use the fact that $\psi_t(y)$ is a smooth and periodic function with period 1. In fact, this integral is taken over the universal covering space $\mathbb{R}^m$.

## 3. $\Gamma$-invariant metrics and geodesics

In this section, we recall some basic properties of the space $\mathcal{H}_0^\Gamma$ of $\Gamma$-invariant Kähler metric over the abelian varieties proved in [**5**].

Now consider $M = \mathbb{C}^m/\Lambda$ where $\Lambda = \mathbb{Z}^m + i\mathbb{Z}^m$, we write each point in $M$ as $z = x+iy$, where $x$ and $y \in \mathbb{R}^m$ and can be considered as periodic coordinate in $M$. Fix

$$\omega_0 = \frac{\pi i}{2} \sum_{\alpha=1}^m dz_\alpha \wedge d\bar{z}_\alpha$$

a flat metric with associated local Kähler potential $2\pi y^2$. The group $\Gamma = (S^1)^m$ acts on $M$ via translations in the lagrangian subspace $\mathbb{R}^m \subset \mathbb{C}^m$, and this induces an isometric action of $\Gamma$ on the space $\mathcal{H}$ of Kähler metrics on $M$; so $\mathcal{H}_0^\Gamma$ of $\Gamma$-invariant metrics is totally geodesic in $\mathcal{H}$. Furthermore, $\mathcal{H}_0^\Gamma$ can be viewed as the set of functions:

$$\mathcal{H}_0^\Gamma = \{\psi \in C_\Gamma^\infty(M) \colon \omega_\psi = \omega_0 + \tfrac{1}{2} i\partial\bar{\partial}\psi > 0\}. \tag{3.1}$$

In fact, functions invariant under the action of $\Gamma$ should be independent of $x$; thus they are smooth function on $M/\Gamma$, i.e. they are smooth and periodic function with period 1 defined on $y \in \mathbb{R}^m$.

The crucial point about $\mathcal{H}_0^\Gamma$ is that given any two points $\varphi_0$ and $\varphi_1$ in $\mathcal{H}_0^\Gamma$, there exists a smooth geodesics $\varphi_t(y)$ in $\mathcal{H}_0^\Gamma$ joining them. To be more precise, in the local coordinate, the geodesic is given by the path $\varphi_t(z) = 2\pi y^2 + 4\pi\psi_t(y)$ satisfying the condition

$$\ddot{\varphi} - \tfrac{1}{2}|\nabla\dot{\varphi}|^2_{\omega_\psi} = 0. \tag{3.2}$$

Moreover, $\nabla^2\varphi_t = I + \nabla^2\psi_t > 0$ because of the positivity of Kähler form; thus $\varphi_t$ is a convex function on $\mathbb{R}^m$. Then the Legendre transform of $\varphi_t(y)$

$$u_t(\mu) = \mu\cdot y - \varphi_t(y) \tag{3.3}$$

is well defined where

$$\mu = \nabla\varphi_t = 4\pi(y + \nabla\psi_t(y)). \tag{3.4}$$

For any fixed $t$, the map $\mu(y,t) = \nabla\varphi_t\colon \mathbb{R}^m \to \mathbb{R}^m$ and also induces a map $\mu\colon M \to M/\Gamma$, which is a simple example of a Lie group valued moment map. Then we have the following proposition and readers should refer to [**5**, **9**] for the proof.

**Proposition 3.1.** *$u(t,\mu)$ is linearized along the geodesic (3.2).*

According to this proposition, we can solve (3.2) in $\mathcal{H}_0^\Gamma$ as follows: given any two Kähler potential $\varphi_0$ and $\varphi_1$, make the Legendre transform $u_0 = \mathcal{L}\varphi_0$ and $u_1 = \mathcal{L}\varphi_1$, then

$$u_t = (1-t)u_0 + tu_1 \tag{3.5}$$

solves equation $\ddot{u} = 0$; thus the inverse of the Legendre transform

$$\varphi_t = \mathcal{L}^{-1}u_t \tag{3.6}$$

will solve (3.2), which is $C^\infty$.

## 4. Regularity Lemma

We first restate some results in § 3: given any two Kähler potential $\varphi_0 = 2\pi y^2 + 4\pi\psi_0$ and $\varphi_1 = 2\pi y^2 + 4\pi\psi_1$ with $\psi_0, \psi_1 \in \mathcal{H}_0^\Gamma$, we can join them by a smooth geodesic $\varphi_t = 2\pi y^2 + 4\pi\psi_t$, $\psi_t \in \mathcal{H}_0^\Gamma$. Now denote $u(t,\mu) = \mathcal{L}\varphi_t(y)$ as the Legendre transform of $\varphi_t(y)$ for any fixed $t$. By Proposition 3.1, $u(t,\mu)$ is linearized along the geodesics:

$$u(t,\mu) = (1-t)u_0(\mu) + tu_1(\mu). \tag{4.1}$$

By properties of the Legendre transform (see also § 3), we have

$$y = \nabla u, \tag{4.2}$$

$$\frac{\partial y}{\partial\mu} = (\nabla^2\varphi)^{-1} = \frac{1}{4\pi}(1 + \nabla^2\psi_t)^{-1} > 0. \tag{4.3}$$

Now denote $\rho_k(j,t) = \|\theta_j\|^2_{h_t^k}$ as norming constant, then (2.12) reads

$$\rho_k(j,t) = (4\pi)^m \int_{\mathbb{R}^m} \exp\left(-2k\pi\left(y + \frac{j}{k}\right)^2\right) \mathrm{e}^{-4k\pi\psi_t(y)} \det(I + \nabla^2\psi_t)\,\mathrm{d}y. \tag{4.4}$$

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Now denote

$$R_k(j,t) = \frac{\rho_k(j,t)}{(\rho_k(j,0))^{1-t}(\rho_k(j,1))^t}, \tag{4.5}$$

$$R_\infty(\mu,t) = \left(\frac{\det \nabla^2 u}{(\det \nabla^2 u_0)^{1-t}(\det \nabla^2 u_1)^t}\right)^{1/2}, \tag{4.6}$$

then we have following regularity lemma.

**Lemma 4.1.** *The ratio $R_\infty(\mu,t) \in C^\infty(\mathbb{R}^m \times [0,1])$ is periodic with period $4\pi$ to each $\mu$ variable for any fixed $t$, hence it is uniformly bounded. Furthermore, we have the complete asymptotics*

$$R_k(j,t) = R_\infty(\mu,t)(1 + k^{-1}a_1 + k^{-2}a_2 + \cdots + k^{-\nu}a_\nu)|_{\mu=-4\pi j/k} + O(k^{-\nu-1}),$$

*where $\nu$ is any positive integer and $O$ symbol is uniform in $t$. Moreover, each $a_\nu$ is a smooth function of $(\mu,t)$ and periodic in $\mu$ with period $4\pi$ for any fixed $t$.*

**Proof.** First, by (4.2) and (4.3), we have

$$\nabla^2 u(\mu,t) = \frac{\partial y}{\partial \mu} = \frac{1}{4\pi}(I + \nabla^2\psi_t)^{-1} > 0. \tag{4.7}$$

Then each entry of $\nabla^2 u$ will be smooth on $\mathbb{R}^m \times [0,1]$ since $\psi_t$ is smooth and so is $\det \nabla^2 u$. Moreover, by (3.4) if we change $y_\alpha \to y_\alpha + 1$, then $\mu_\alpha \to \mu_\alpha + 4\pi$, but $\nabla^2 u$ does not change by (4.7), this means $\nabla^2 u(\mu_\alpha + 4\pi) = \nabla^2 u(\mu_\alpha)$, thus $\nabla^2 u$ is a periodic matrix defined on $\mathbb{R}^m$ with period $4\pi$ for any fixed $t$, which implies that $\det \nabla^2 u$ is also a periodic function. Hence, by the definition of $R_\infty(\mu,t)$, $R_\infty(\mu,t)$ will be a positive smooth periodic function with period $4\pi$; hence it is uniformly bounded. Furthermore, there exist a constant $C > 0$ such that $R_\infty(\mu,t) > C > 0$ uniformly.

Now we turn to prove the existence of the complete asymptotic of $R_k(j,t)$. First from (4.3), we have the following relation of the volume form:

$$\mathrm{d}\mu = (4\pi)^m \det(I + \nabla^2\psi_t)\,\mathrm{d}y. \tag{4.8}$$

Now plug (3.3), (4.2) and (4.8) into (4.4), then we can rewrite the norming constant $\rho_k(j,t)$ as

$$\rho_k(j,t) = \exp\left(-2\pi\frac{j^2}{k}\right)\int_{\mathbb{R}^m} \exp\left(-k\left(\mu\cdot\frac{\partial u}{\partial\mu} - u + \frac{4\pi j}{k}\cdot\frac{\partial u}{\partial\mu}\right)\right)\mathrm{d}\mu. \tag{4.9}$$

Hence, by definition, in order to get the complete asymptotics of $R_k(t,j)$ it is sufficient to get the complete asymptotics of $\rho_k(t,j)$ for all $t \in [0,1]$, i.e. we try to get the complete asymptotics of

$$\int_{\mathbb{R}^m} \exp\left(-k\left(\mu\cdot\frac{\partial u}{\partial\mu} - u + \frac{4\pi j}{k}\cdot\frac{\partial u}{\partial\mu}\right)\right)\mathrm{d}\mu$$

by applying the stationary phase method. The formula of the stationary phase method is as follows [**11**, Theorem 7.7.5]:

$$\int u(x)\mathrm{e}^{\mathrm{i}k\Psi(x)}\,\mathrm{d}x = \frac{\mathrm{e}^{\mathrm{i}k\Psi(x)}}{\sqrt{\det(k\nabla^2\Psi(x)/2\pi\mathrm{i})}}\sum_{\lambda=0}^{\infty} k^{-\lambda}L_\lambda u|_{x=x'}, \tag{4.10}$$

where $x'$ is the critical point of $\Psi$, $\operatorname{Im}\Psi \geqslant 0$ and $L_\lambda$ is a differential operator of order $2\lambda$ defined by

$$L_\lambda u = \sum_{\alpha-\beta=\lambda}\sum_{2\alpha\geqslant 3\beta}\frac{1}{2^\alpha \mathrm{i}^\lambda \beta!\alpha!}L_\Psi^\alpha[g_{x'}^\beta u(x)]|_{x=x'}, \tag{4.11}$$

where $g_{x'}(x) = \Psi(x) - \Psi(x') - \langle\nabla^2\Psi(x')(x-x'), x-x'\rangle/2$. Furthermore, the coefficients $L_\lambda u$ are rational homogeneous functions of degree $-\lambda$ in $\nabla^2\Psi(x'), \dots, \Psi^{(2\lambda+2)}(x')$ with the denominator $(\det\nabla^2\Psi(x'))^{3\lambda}$. Note that in [**11**], $u(x)$ is assumed to has compact support, but in fact this formula is true for any $u(x) \in C^\infty(\mathbb{R}^m)$. The strategy is to choose a cut-off function $\chi$ in a neighbourhood of $x'$ and rewrite the amplitude $u$ to be $\chi u + (1-\chi)u$, then we will separate the integration into two parts correspondingly. To the integration with the amplitude $\chi u$, we use the formula of stationary phase method directly; to the second part, by Theorem 1.1.4 in [**21**], it is $O(k^{-\infty})$.

To our case, note that the hypotheses of [**11**] are satisfied since we can add some constant to ensure our phase function has positive imaginary part. Now the critical point of the phase

$$\Psi = \mu\cdot\frac{\partial u}{\partial\mu} - u + \frac{4\pi j}{k}\cdot\frac{\partial u}{\partial\mu}$$

satisfies

$$\left(\mu' + \frac{4\pi j}{k}\right)\cdot\nabla^2 u = 0.$$

As proved above, the matrix $\nabla^2 u > 0$ and each entry is smooth and periodic on $\mathbb{R}^m$, then there will be a constant $c > 0$, such that $\nabla^2 u > cI$ which is always non-degenerated; thus the critical point of the phase is given by $\mu' = -4\pi j/k$. And the Hessian of the phase at the critical point is $\nabla^2\Psi|_{\mu=\mu'} = \nabla^2 u(x', t) > cI$. Thus by the formula of the stationary phase method, we have

$$\begin{aligned}
&\int_{\mathbb{R}^m}\exp\left(-k\left(\mu\cdot\frac{\partial u}{\partial\mu} - u + \frac{4\pi j}{k}\cdot\frac{\partial u}{\partial\mu}\right)\right)\mathrm{d}\mu \\
&\quad = k^{-m/2}\left(\exp\left(-k\left(\mu\cdot\frac{\partial u}{\partial\mu} - u + \frac{4\pi j}{k}\cdot\frac{\partial u}{\partial\mu}\right)\right)\sqrt{\det\nabla^2 u}\right) \\
&\qquad\qquad \times (1 + k^{-1}L_1(t,x) + k^{-2}L_2(t,\mu) + \cdots)|_{\mu'=-4\pi j/k} \\
&\quad = k^{-m/2}(\mathrm{e}^{ku}\sqrt{\det\nabla^2 u})(1 + k^{-1}L_1(t,\mu) + k^{-2}L_2(t,\mu) + \cdots)|_{\mu'=-4\pi j/k}.
\end{aligned} \tag{4.12}$$

Now we can show that each $L_\lambda$ is a smooth function of $(\mu, t)$ and periodic of $\mu$ for any fixed $t$. In our case, by simple computations, all derivatives of the phase function $\Psi(\mu)$ at the point $\mu' = -4\pi j/k$ is $\Psi^{(n)}(\mu') = (n-1)u^{(n)}(\mu')$, e.g. easy to check $\nabla^2\Psi(\mu') = \nabla^2 u(\mu')$, $\Psi^{(3)}(\mu') = 2u^{(3)}(\mu')$; thus by formula (4.11), each coefficient $L_\lambda$ evaluated at

$\mu'$ is equal to a functional $\mathcal{F}_\lambda(\nabla^2 u(\mu), \ldots, u^{(2\lambda+1)})$ evaluated at $\mu'$, hence let $L_\lambda(\mu) = \mathcal{F}_\lambda(\nabla^2 u, \ldots, u^{(2\lambda+1)})$, then $L_\lambda(\mu)$ will be smooth and periodic since $\nabla^2 u$ and $u^{(2\lambda+1)}$ are smooth and periodic matrix.

Now we can get the following expression of $R_k(j,t)$ by expanding each term in denominator and numerator:

$$\begin{aligned} R_k(j,t) &= \mathrm{e}^{k\pi(u-(1-t)u_0-tu_1)}\left(\frac{\det \nabla^2 u}{(\det \nabla^2 u_0)^{1-t}(\det \nabla^2 u_1)^t}\right)^{1/2} \\ &\quad \times \frac{1+k^{-1}L_1(t,\mu)+\cdots}{(1+k^{-1}L_1(0,\mu)+\cdots)^{1-t}(1+k^{-1}L_1(1,\mu)+\cdots)^t}\bigg|_{\mu=-4\pi j/k} \\ &= R_\infty(\mu,t)(1+k^{-1}a_1+k^{-2}a_2+\cdots+k^{-\nu}a_\nu)|_{\mu=-4\pi j/k} + O(k^{-\nu-1}). \end{aligned} \tag{4.13}$$

In the last step, we plug in the identity (4.1):

$$u(t,\mu) = (1-t)u_0(\mu) + tu_1(\mu),$$

which implies that $\mathrm{e}^{k\pi(u-(1-t)u_0-tu_1)} = 1$. Then we apply the Taylor expansion $(1+x)^\gamma = 1+\gamma x+\cdots$ to the term

$$(1+k^{-1}L_1(t,\mu)+\cdots)(1+k^{-1}L_1(0,\mu)+\cdots)^{t-1}(1+k^{-1}L_1(1,\mu)+\cdots)^{-t},$$

just choose $\gamma$ as $t-1$ and $-t$. If we expand these three terms completely, we will get the complete asymptotics, and we can compute each term step by step. For example, the first term is just 1 and the second term is $k^{-1}(L_1(t,\mu)-(1-t)L_1(0,\mu)-tL_1(1,\mu))$. Moreover, $a_\nu$ is a polynomial of $t$ and $L_\lambda(t,\mu)$ for some $\lambda$, hence each $a_\nu$ is smooth and uniformly bounded on $[0,1]\times M$, and periodic for any fixed $t$. Furthermore, if we combine this with the fact that $R_\infty(\mu,t)$ is uniformly bounded, then the error term $R_\infty(\mu,t)a_{\lambda+1}$ is uniformly bounded, i.e. the symbol $O$ is uniformly bounded. □

## 5. The generalized Bernstein Polynomial Lemma

In this section, we will prove the Lemma 1.3. We first introduce the definition and some basic properties of the Bergman kernel (refer to [**20**, **25**, **26**] for more details).

Let $(L,h) \to (M,\omega)$ be a positive holomorphic line bundle over a compact Kähler manifold of complex dimension $m$. We assume

$$\omega = -\tfrac{1}{2}\mathrm{i}\partial\bar{\partial}\log|s(z)|_h^2,$$

where $s(z)$ is a local holomorphic frame. We now define the Bergman kernel as the orthogonal projection from the $L^2$ integral sections to the holomorphic sections:

$$\Pi_k\colon L^2(M,L^k) \to H^0(M,L^k). \tag{5.1}$$

Furthermore, if $\{s_j^k\}_{j=0}^{d_k}$ is an orthonormal basis of $H^0(M,L^k)$ with respect to the inner product defined by (1.7), then

$$\Pi_k(z,w) = \sum_{j=0}^{d_k} s_j^k(z)\otimes\overline{s_j^k(w)}, \tag{5.2}$$

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where $d_k+1=\dim H^0(M,L^k)$. Then we have the following proposition for the Bergman kernel off the diagonal to any $m$-dimensional Kähler manifold [**2**, **20**].

**Proposition 5.1.** *For any $C^\infty$ positive hermitian line bundle $(L,h)$, we have*

$$\Pi_k(z,w)=\exp(k(\phi(z,\bar w)-\frac{1}{2}(\phi(z)+\phi(w))))A_k(z,w)+O(k^{-\infty}), \tag{5.3}$$

*where $\phi$ is the smooth local Kähler potential for $h$, $\phi(z,\bar w)$ is the almost analytic extension of $\phi(z)$ and $A_k(z,w)=k^m(1+k^{-1}a_1(z,w)+\cdots)$ a semi-classical symbol of order $m$.*

Now we turn to the proof of Lemma 1.3.

**Proof.** Assume $M=\mathbb{C}^m/\Lambda$ where $\Lambda=\mathbb{Z}^m+\mathrm{i}\mathbb{Z}^m$ and $L\to M$ is a principal polarization of $M$. Choose Kähler potential $\varphi(y)=2\pi y^2+4\pi\psi(y)$ as before. From Proposition 2.1, $\{\theta_j,\ j\in(\mathbb{Z}/k\mathbb{Z})^m\}$ forms an orthogonal basis of $H^0(M,L^k)$ with respect to the Hermitian inner product defined by (1.7); thus by formula (5.2), the Bergman kernel is given by

$$\Pi_k(z,w)=\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}\frac{\theta_j(z)\overline{\theta_j(w)}\exp(-\frac{1}{2}k\varphi(\operatorname{Im} z)-\frac{1}{2}k\varphi(\operatorname{Im} w))}{\|\theta_j\|^2_{h^k}}. \tag{5.4}$$

For any function $f(x)\in C^\infty(\mathbb{T}^m)$, we can define the following translation operator $T_k\colon f(x)\to f(x-(1/k))$ on the universal covering space. If we consider this operator acting on the vector space $H^0(M,L^k)$ of holomorphic theta functions, then we have the following Weyl quantization [**12**, **13**]:

$$\mathrm{Op}_k(f)=\sum_{n\in\mathbb{Z}^m}\hat f(n)T_k^n, \tag{5.5}$$

where $\hat f(n)$ is the Fourier coefficients of $f$. Now apply $T_k$ to theta functions

$$\theta_j(z)=\sum_{n\in\mathbb{Z}^m}\exp\left(-\pi\frac{(j+kn)^2}{k}+2\pi\mathrm{i}(j+kn)\cdot z\right),$$

then for any $x\in\mathbb{R}^m$, it is easy to see that

$$T_k(\theta_j(z+x))=\mathrm{e}^{-2\pi\mathrm{i}(j/k)}\theta_j(z+x), \tag{5.6}$$

where $\mathrm{e}^{-2\pi\mathrm{i}(j/k)}=\mathrm{e}^{-2\pi\mathrm{i}(j_1/k)}\cdots\mathrm{e}^{-2\pi\mathrm{i}(j_m/k)}$. Next apply $\mathrm{Op}_k(f)$ to theta functions, we have

$$\mathrm{Op}_k(f)\theta_j(z+x)=\left(\sum_{n\in\mathbb{Z}^m}\hat f(n)\mathrm{e}^{-2\pi\mathrm{i}(j/k)\cdot n}\right)\theta_j(z+x)=f\left(-\frac{j}{k}\right)\theta_j(z+x). \tag{5.7}$$

Now apply this operator to the Bergman kernel off the diagonal (5.4), we have

$$\begin{aligned}\mathrm{Op}_k&(f)\Pi_k(z+x,w)|_{x=0}\\&=\mathrm{Op}_k(f)\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}\frac{\theta_j(z+x)\overline{\theta_j(w)}\exp(-\frac{1}{2}k\varphi(\operatorname{Im} z)-\frac{1}{2}k\varphi(\operatorname{Im} w))}{\|\theta_j\|^2_{h^k}}|_{x=0}\\&=\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}f\left(-\frac{j}{k}\right)\frac{\theta_j(z)\overline{\theta_j(w)}\exp(-\frac{1}{2}k\varphi(\operatorname{Im} z)-\frac{1}{2}k\varphi(\operatorname{Im} w))}{\|\theta_j\|^2_{h^k}}.\end{aligned} \tag{5.8}$$

Here we use the fact that $\varphi(\operatorname{Im}(z+x)) = \varphi(\operatorname{Im} z) = \varphi(y)$. Now choose $z = w$, we have

$$\frac{1}{k^m} \sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(-\frac{j}{k}\right) \frac{|\theta_j|_{h^k}}{\|\theta_j\|^2_{h^k}} = \frac{1}{k^m} \mathrm{Op}_k(f) \mathit{\Pi}_k(z+x,z)|_{x=0}. \tag{5.9}$$

Now we get the complete asymptotics of $\mathit{\Pi}_k(z+x,z)$ as follows: in our abelian case, by assumption, our Kähler potential only depends on $y = \operatorname{Im} z$, i.e.

$$\varphi(z) = \varphi(y) = \varphi\left(\frac{z-\bar{z}}{2\mathrm{i}}\right),$$

thus the almost analytic extension of $\varphi$ is given by

$$\varphi(z,\bar{w}) = \varphi\left(\frac{z-\bar{w}}{2\mathrm{i}}\right). \tag{5.10}$$

Hence, formula (5.3) reads

$$\begin{aligned} \mathit{\Pi}_k(z+x,z) &= \exp(k(\varphi(z+x,\bar{z}) - \tfrac{1}{2}(\varphi(z+x)+\varphi(z))))A_k(z+x,z) \\ &= \exp(k(\varphi(z+x,\bar{z}) - \varphi(z)))A_k(z+x,z), \end{aligned} \tag{5.11}$$

where $A_k(z+x,z) = k^m(1+k^{-1}a_1(z+x,z)+\cdots)$ since $M$ is $m$ dimensional. In the last step, we use the fact that $\varphi(z+x) = \varphi(z) = \varphi(\operatorname{Im} z)$ again.

Now apply the operator $(1/k^m)\mathrm{Op}_k(f)$ on both sides of (5.11), then

$$\begin{aligned} &\frac{1}{k^m}\mathrm{Op}_k(f)\mathit{\Pi}_k(z+x,z)|_{x=0} \\ &\quad= \frac{1}{k^m}\sum_{n\in\mathbb{Z}^m} \hat{f}(n) T_k^n \mathit{\Pi}_k(z+x,z)|_{x=0} \\ &\quad= \frac{1}{k^m}\sum_{n\in\mathbb{Z}^m} \hat{f}(n) \mathit{\Pi}_k\left(z-\frac{n}{k},z\right) \\ &\quad= \frac{1}{k^m}\sum_{n\in\mathbb{Z}^m} \hat{f}(n) \exp\left(k\left(\varphi\left(z-\frac{n}{k},\bar{z}\right)-\varphi(z)\right)\right)A_k\left(z-\frac{n}{k},z\right) \\ &\quad= \frac{1}{k^m}\sum_{n\in\mathbb{Z}^m} \hat{f}(n) \exp\left(k\left(\varphi\left(y-\frac{n}{2\mathrm{i}k}\right)-\varphi(y)\right)\right)A_k\left(z-\frac{n}{k},z\right). \end{aligned} \tag{5.12}$$

In the last step, by identity (5.10), the almost analytic extension

$$\varphi\left(z-\frac{n}{k},\bar{z}\right) = \varphi\left(\frac{z-(n/k)-\bar{z}}{2\mathrm{i}}\right) = \varphi\left(y-\frac{n}{2k\mathrm{i}}\right).$$

To the last equation in (5.12), if we apply the Taylor expansion to

$$\exp\left(k\left(\varphi\left(y-\frac{n}{2\mathrm{i}k}\right)-\varphi(y)\right)\right)$$

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and use the complete asymptotic of

$$A_k\left(z-\frac{n}{k},z\right)=k^m\left(1+k^{-1}a_1\left(z-\frac{n}{k},z\right)+\cdots\right),$$

we will get the complete asymptotic of $\mathrm{Op}_k(f)\Pi(z+x,z)|_{x=0}$. For example, we can compute the leading term as follows: first,

$$\begin{aligned}\exp\left(k\left(\varphi\left(y-\frac{n}{2\mathrm{i}k}\right)-\varphi(y)\right)\right)&=\exp\left(-\nabla\varphi\cdot\frac{n}{2\mathrm{i}}+O(k^{-1})\right)\\&=\exp\left(-\nabla\varphi\cdot\frac{n}{2\mathrm{i}}\right)+O(k^{-1});\end{aligned}$$

second,

$$\frac{1}{k^m}A_k\left(z-\frac{n}{k},z\right)=1+O(k^{-1}).$$

Hence the leading term is given by

$$\sum_{n\in\mathbb{Z}^m}\hat{f}(n)\exp\left(-\nabla\varphi\cdot\frac{n}{2\mathrm{i}}\right)=f\left(\frac{\nabla\varphi}{4\pi}\right)=f(\mu),\tag{5.13}$$

where $\mu=y+\nabla\psi$. Hence, we can get the complete asymptotics step by step if we expand

$$\exp\left(k\left(\varphi\left(y-\frac{n}{2\mathrm{i}k}\right)-\varphi(y)\right)\right)$$

and $A_k$ more terms. □

As a remark, if we replace $f$ and $h$ to be a path of smooth periodic function $f_t$ and any path $h_t$ in $\mathcal{H}_0^\Gamma$, then the lemma still holds with the leading term $f_t(\mu)$. Furthermore, we can differentiate the complete asymptotics with respect to $t$ on both sides.

## 6. $C^\infty$ convergence of Bergman geodesics

In this section, we will apply the Regularity Lemma and the generalized Bernstein Polynomial Lemma to prove Lemma 1.2.

**Proof.** We first apply Lemma 4.1, and denote $A_\nu(\mu,t)\sim R_\infty(\mu,t)a_\nu(\mu)$, then $A_\nu(\mu,t)$ is periodic in $\mu$ since $R_\infty(\mu,t)$ and $a_\nu(\mu)$ are periodic, then

$$\begin{aligned}&\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}R_k(j,t)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}}\\&\quad\sim\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}R_\infty(\mu,t)(1+k^{-1}a_1+k^{-2}a_2+\cdots)|_{\mu=-4\pi j/k}\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}}\\&\quad\sim\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}R_\infty\left(-\frac{4\pi j}{k},t\right)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}}+\frac{1}{k}\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}A_1\left(-\frac{4\pi j}{k},t\right)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}}+\cdots.\end{aligned}$$

Since $R_\infty(\mu,t)$ is periodic with period $4\pi$, then $R_\infty(4\pi\mu)$ will be periodic with period 1, thus if we apply Lemma 1.3 to function $R_\infty(4\pi\mu)$, we have

$$\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} R_\infty\left(-\frac{4\pi j}{k},t\right)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}} \sim k^m(R_\infty(\mu,t)+k^{-1}b_{11}(\mu,t)+\cdots),$$

where $\mu = 4\pi(y+\nabla\psi_t)$. In fact, we can apply Lemma 1.3 to each coefficient, e.g.

$$\frac{1}{k}\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} A_1\left(-\frac{4\pi j}{k},t\right)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}} \sim k^m(k^{-1}A_1(\mu,t)+\cdots)$$

and so on, then we have the complete asymptotics:

$$\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} R_k(j,t)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}} \sim k^m(R_\infty(\mu,t)+k^{-1}(A_1+b_{11})+\cdots).$$

We can divide $R_\infty$ since in Lemma 4.1, we prove this term is strictly positive, uniformly bounded and smooth. Hence,

$$\begin{aligned}\frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} R_k(j,t)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}} &\\ &\sim k^{-1}\log\left[k^m R_\infty(\mu,t)\left(1+\frac{1}{k}\frac{A_1+b_{11}}{R_\infty}+\cdots\right)\right]\\ &\sim mk^{-1}\log k + k^{-1}\log R_\infty + k^{-1}\log\left(1+\frac{1}{k}\frac{A_1+b_{11}}{R_\infty}+\cdots\right)\\ &\sim mk^{-1}\log k + k^{-1}\log R_\infty + k^{-2}\frac{A_1+b_{11}}{R_\infty}+\cdots.\end{aligned}$$

In the last step, we use the Taylor expansion $\log(1+x)\sim x-\frac{1}{2}x^2+\cdots$. Moreover, in each step, according to Lemmas 4.1 and 1.3, we do not change the regularity of smoothness, hence,

$$\frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} R_k(j,t)\frac{|\theta_j|^2_{h_t^k}}{\|\theta_j\|^2_{h_t^k}} \to 0$$

in $C^\infty$ topology as $k\to\infty$. This implies that the Bergman geodesics converge to the geodesic in the Kähler space in $C^\infty$ topology. □

## 7. General lattice

In this section, we will sketch the proof of our main theorem for any principally polarized abelian variety.

Let $M = \mathbb{C}^m/\Lambda$ where $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{\lambda_1,\ldots,\lambda_{2m}\}$ is a lattice in $\mathbb{C}^m$ with its normalized period matrix given by $\Omega := [I, Z]$, where $Z^{\mathrm{T}} = Z$ and $\operatorname{Im} Z > 0$. Choose

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$\{x_1,\dots,x_m,y_1,\dots,y_m\}$ as the coordinates of the basis dual to $\{\lambda_1,\dots,\lambda_{2m}\}$ such that [**8**]

$$z_\alpha = x_\alpha + \sum_{\beta=1}^{m} Z_{\alpha\beta} y_\beta$$

and

$$\bar{z}_\alpha = x_\alpha + \sum_{\beta=1}^{m} \bar{Z}_{\alpha\beta} y_\beta.$$

Assume $L \to M$ is a principal polarization of $M$, then the holomorphic sections of $H^0(M, L^k)$ are given by the following theta functions:

$$\theta_l(z,\Omega) = \sum_{n\in\mathbb{Z}^m} \exp\left(\pi \mathrm{i}(l+kn)\cdot \frac{Z}{k}(l+kn)^{\mathrm{T}}\right) \mathrm{e}^{2\pi \mathrm{i}(l+kn)\cdot z}, \quad l \in (\mathbb{Z}/k\mathbb{Z})^m,$$

where $n\cdot z = n_1 z_1 + \cdots + n_m z_m$.

Now consider the Kähler potential in the form

$$\varphi(t,y) = 2\pi y X y^{\mathrm{T}} + 4\pi\psi(t, Xy^{\mathrm{T}}), \tag{7.1}$$

where $y=(y_1,\dots,y_m)$, $X = \operatorname{Im} Z$. We assume $\varphi$ is convex in $y$ and $\psi$ is smooth on $\mathbb{R}^m$ and periodic with period 1 in each variable $y_j$ for any fixed $t$. Then it is easy to check that such Kähler potential satisfies conditions (2.6).

By choosing such Kähler potential, Proposition 2.1 still holds depending on the following computations (see also [**7**]):

$$\begin{aligned}
&\langle \theta_{l'}(z,\Omega), \theta_l(z,\Omega)\rangle_{h_t^k} \\
&\quad = \int_{[0,1]^m\times[0,1]^m} \left(\sum_{n'\in\mathbb{Z}^m} \exp\left(-\mathrm{i}\pi(l+kn')\cdot\frac{Z}{k}(l+kn')^{\mathrm{T}}\right)\mathrm{e}^{-2\pi\mathrm{i}(l+kn')\cdot z}\right) \\
&\qquad \times \left(\sum_{n\in\mathbb{Z}^m} \exp\left(-\mathrm{i}\pi(l+kn)\cdot\frac{\bar{Z}}{k}(l+kn)\right)\mathrm{e}^{-2\pi\mathrm{i}(l+kn)^{\mathrm{T}}\cdot\bar{z}}\right) \\
&\qquad\qquad \times \mathrm{e}^{-2k\pi y X y^{\mathrm{T}} - 4k\pi\psi(t,Xy)}\cdot \det\nabla^2\varphi_t(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \delta_{l,l'} \sum_{n\in\mathbb{Z}^m} \int_{[0,1]^m} \exp\left(-2k\pi\left(y+\frac{l+kn}{k}\right)\cdot X\left(y+\frac{l+kn}{k}\right)^{\mathrm{T}}\right) \\
&\qquad\qquad \times \mathrm{e}^{-4k\pi\psi(t,Xy)}\cdot\det\nabla^2\varphi_t(y)\,\mathrm{d}y.
\end{aligned}$$

Thus $\{\theta_l(z,\Omega),\ l\in(\mathbb{Z}/k\mathbb{Z})^m\}$ forms an orthogonal basis of $H^0(M,L^k)$. Furthermore,

$$\|\theta_l(z,\Omega)\|^2_{h_t^k} = \int_{\mathbb{R}^m} \exp\left(-2k\pi\left(y+\frac{l}{k}\right)X\left(y+\frac{l}{k}\right)^{\mathrm{T}}\right)\mathrm{e}^{-4k\pi\psi(t,Xy)}\cdot\det\nabla^2\varphi_t(y)\,\mathrm{d}y. \tag{7.2}$$

Then all main steps in the model case can be extended to the general case immediately.

- Define $u(t, y)$ as the Legendre transform of $\varphi(t, y)$ with respect to $y$ variables for any fixed $t$, then we can still linearize $u(t, y)$ along the geodesics since the Proposition 3.1 is only the property of convex functions [**17**, p. 106].

- By substituting $\varphi$ by the Legendre transform $u(t, y)$, we rewrite (7.2) as

$$\exp\left(-2k\pi\left(\frac{j}{k}\right)X\left(\frac{j}{k}\right)^{\mathrm{T}}\right)\int_{\mathbb{R}^m}\exp\left(-4k\pi\left(\nabla u\cdot X\cdot\left(\frac{j}{k}\right)^{\mathrm{T}}+u-\mu\cdot\nabla u\right)\right)\mathrm{d}\mu,$$

  where $\mu = \nabla\varphi$.

  By applying the stationary phase method, we can get the complete asymptotics of this integration evaluated at

$$\mu' = -X\cdot\left(\frac{4\pi j}{k}\right)^{\mathrm{T}},$$

  which is the critical point of the phase function. Thus $R_k(j, t)$ which is the ratio of the norming constants will be asymptotic to $R_\infty(\mu, t)$ as

$$R_k(j,t)\sim R_\infty(\mu,t)(1+k^{-1}a_1(\mu,t)+\cdots)|_{\mu=-X\cdot(4\pi j/k)^{\mathrm{T}}}.$$

  If we change variable to $\mu\cdot(4\pi X)^{-1} = \nu$, then $R_\infty(\nu, t)$ and each $a_j(\nu, t)$ are smooth functions over $\mathbb{R}^m$ and periodic with period 1 in variables $\nu$ for any fixed $t$.

- In the general case, we define the operator $T_k\colon f(x) \to f(x - (1/k))$. Then for general theta functions $\theta_l(z, \Omega)$, we still have

$$T_k(\theta_l(z,\Omega)) = \mathrm{e}^{-2\pi\mathrm{i}(l/k)}\theta_l(z,\Omega),$$

  where $\mathrm{e}^{-2\pi\mathrm{i}(l/k)}$ denotes $\mathrm{e}^{-2\pi\mathrm{i}(l_1/k)}\cdots\mathrm{e}^{-2\pi\mathrm{i}(l_m/k)}$. Then by applying the Weyl quantization to the Bergman kernel and using Fourier transform and Taylor expansion, for any $f(4\pi X\cdot x^{\mathrm{T}}) \in C^\infty(\mathbb{R}^m)$ which is also periodic with period 1 in $x$ variables, following the proof in § 5, we can prove

$$\frac{1}{k^m}\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m} f\left(-X\cdot\left(\frac{4\pi j}{k}\right)^{\mathrm{T}}\right)\frac{|\theta_j(z,\Omega)|_{h^k}}{\|\theta_j(z,\Omega)\|_{h^k}}\sim f(\mu)+k^{-1}b_1(\mu)+\cdots,$$

  where $\mu = \nabla\varphi$. Our main result with the same formula as the model case holds if we apply this formula to $R_\infty(\mu, t)$ and each $a_j(\mu, t)$ and follow the steps in § 6.

Thus our main result holds for any principally polarized abelian variety.

## 8. Proof of Theorem 1.5

The proof of Theorem 1.5 is similar to the one in [**18**]. For simplicity, we just sketch the main steps for the model case $M = \mathbb{C}^m/\Lambda$, where $\Lambda = \mathbb{Z}^m + \mathrm{i}\mathbb{Z}^m$.

The crucial formula in the toric case is the identity (4.1) in [**18**], while in our abelian case, we modify it to be

$$\begin{aligned}\varphi_k(q,z)-\varphi(q,z)\\ =\frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}\exp\biggl(\int_{\partial N}\partial_{\nu(p)}G(p,q)\log\|\theta_j(z)\|^2_{h^k_\psi(p)}\,\mathrm{d}V_{\partial N}(p)\biggr)|\theta_j(z)|^2_{h^k_\varphi(q)},\end{aligned}\tag{8.1}$$

where $G(q,p)$ denotes the positive Dirichlet–Green kernel for the Laplacian $\triangle_{N,g}$, $\mathrm{d}V_{\partial N}$ is the induced measure on $\partial N$ by restricting the Riemannian volume form $\mathrm{d}V_N$ from $N$ to $\partial N$ and $\nu(q)$ is an outward unit normal to $\partial N$. Then proving Theorem 1.5 is equivalent to proving that (8.1) admits complete asymptotics. Denote

$$K_k(q,j)=\exp\biggl(-\int_{\partial N}\partial_{\nu(p)}G(p,q)\log\frac{\|\theta_j(z)\|^2_{h^k_\varphi(q)}}{\|\theta_j(z)\|^2_{h^k_\psi(p)}}\,\mathrm{d}V_{\partial N}(p)\biggr).$$

Then we can rewrite (8.1) as

$$\varphi_k(q,z)-\varphi(q,z)=\frac{1}{k}\log\sum_{j\in(\mathbb{Z}/k\mathbb{Z})^m}K_k(q,j)\frac{|\theta_j(z)|^2_{h^k_\varphi(q)}}{\|\theta_j(z)\|^2_{h^k_\varphi(q)}}.\tag{8.2}$$

Put $u_q:=u_{\varphi(q)}=u(q,\cdot)$ is the Legendre transform of $\varphi_q(y)\in\mathcal{H}_0^\Gamma$ for $q\in N$. Denote

$$K_\infty(q,x)=\exp\biggl(-\frac{1}{2}\int_{\partial N}\partial_{\nu(p)}G(p,q)\log\frac{\det\nabla^2u_q(x)}{\det\nabla^2u_p(x)}\,\mathrm{d}V_{\partial N}(p)\biggr),\tag{8.3}$$

where $x=\nabla\varphi$.

From the proof of the Regularity Lemma (Lemma 4.1), if we plug in the complete asymptotic expansion of the norming constants $\|\theta_j(z)\|^2_{h^k_\varphi(q)}$ and $\|\theta_j(z)\|^2_{h^k_\psi(p)}$, we have the following complete asymptotic expansion:

$$K_k(q,j)=K_\infty(q,x)+k^{-1}b_1(q,x)+\cdots|_{x=-4\pi j/k}.\tag{8.4}$$

If we plug (8.4) into the right-hand side of (8.2), we obtain the following expansion:

$$\frac{1}{k}\log\biggl(\sum_j K_\infty\Bigl(q,\frac{-4\pi j}{k}\Bigr)\frac{|\theta_j(z)|^2_{h^k_\varphi(q)}}{\|\theta_j(z)\|^2_{h^k_\varphi(q)}}+k^{-1}\sum_j b_1\Bigl(q,\frac{-4\pi j}{k}\Bigr)\frac{|\theta_j(z)|^2_{h^k_\varphi(q)}}{\|\theta_j(z)\|^2_{h^k_\varphi(q)}}+\cdots\biggr).$$

Hence, Theorem 1.5 yields if we apply the generalized Bernstein Lemma (Lemma 1.3) to each summation above and follow the steps in § 6.

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