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Marc Hoyois

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Abstract

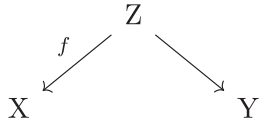
We prove the analog of the Morel–Voevodsky localization theorem for framed motivic spaces. We deduce that framed motivic spectra are equivalent to motivic spectra over arbitrary schemes, and we give a new construction of the motivic cohomology of arbitrary schemes.

In this article we show that the theory of framed motivic spaces introduced in [EHK+19] satisfies localization: if $i\colon \mathrm{Z} \hookrightarrow \mathrm{S}$ is a closed immersion of schemes, $j\colon \mathrm{U} \hookrightarrow \mathrm{S}$ is the complementary open immersion, and $\mathcal{F} \in \mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ is a framed motivic space over S, then there is a cofiber sequence

$$j_\sharp j^*\mathcal{F} \to \mathcal{F} \to i_* i^*\mathcal{F}$$

(see Theorem 10). Consequently, the theory of framed motivic spectra satisfies Ayoub's axioms [Ayo08], which implies that it admits a full-fledged formalism of six operations. Using this formalism, we show that the equivalence $\mathbf{SH}^{\mathrm{fr}}(\mathrm{S}) \simeq \mathbf{SH}(\mathrm{S})$, proved in [EHK+19] for S the spectrum of a perfect field, holds for any scheme S (see Theorem 18).

The ∞-category $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ of framed motivic spaces consists of $\mathbf{A}^1$-invariant Nisnevich sheaves on the ∞-category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_\mathrm{S})$ of smooth S-schemes and *framed correspondences*. A framed correspondence between S-schemes X and Y is a span



over S, where f is a finite syntomic morphism equipped with a trivialization of its cotangent complex in the K-theory of Z. Our result stands in contrast to the case of finite correspondences in the sense of Voevodsky, where the analog of the Morel–Voevodsky localization theorem remains unknown. The essential ingredient in our proof is the fact that the Hilbert scheme of framed points [EHK+19, Definition 5.1.7] is *smooth*.

1. Review of the Morel–Voevodsky localization theorem

We start by reviewing the localization theorem of Morel and Voevodsky [MV99, § 3, Theorem 2.21]. We refer to [Hoy14, Appendix C] for the definition of the Morel–Voevodsky ∞-category

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$\mathbf{H}(S)$ for S an arbitrary scheme. We shall denote by L_{nis}, $L_{\mathbf{A}^1}$, and L_{mot} the localization functors enforcing Nisnevich descent, $\mathbf{A}^1$-invariance, and both, respectively.

Theorem 1 (Morel–Voevodsky). *Let $i\colon Z \hookrightarrow S$ be a closed immersion of schemes, $j\colon U \hookrightarrow S$ the complementary open immersion, and $\mathcal{F} \in \mathbf{H}(S)$ a motivic space over S. Then the square*

$$\begin{array}{ccc} j_\sharp j^*\mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ j_\sharp(*) & \longrightarrow & i_*i^*\mathcal{F} \end{array}$$

is cocartesian in $\mathbf{H}(S)$.

This theorem was proved in this generality in [Hoy14, Proposition C.10], but we give here a more direct proof that was alluded to in that paper. In the sequel, we will actually use a slightly different form of this theorem; see Corollary 5 below.

Let $i\colon Z \hookrightarrow S$ be a closed immersion with open complement $j\colon U \hookrightarrow S$. For an S-scheme X and an S-morphism $t\colon Z \to X$, we define the presheaf

$$h_S(X,t)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set}$$

by the cartesian square

$$\begin{array}{ccc} h_S(X,t) & \longrightarrow & * \\ \downarrow & & \downarrow t \\ h_S(X) \sqcup_{h_S(X_U)} h_S(U) & \longrightarrow & i_*h_Z(X_Z), \end{array}$$

where $h_S\colon \mathrm{Sch}_S \to \mathrm{PSh}(\mathrm{Sch}_S)$ is the Yoneda embedding. Explicitly,

$$h_S(X,t)(Y) = \begin{cases} \mathrm{Maps}_S(Y,X) \times_{\mathrm{Maps}_Z(Y_Z,X_Z)} \{Y_Z \to Z \xrightarrow{t} X_Z\} & \text{if } Y_Z \neq \varnothing, \\ * & \text{if } Y_Z = \varnothing. \end{cases}$$

If S is a Henselian local scheme, we have the following well-known facts:

(a) if X is étale over S, then $h_S(X,t)(S)$ is contractible;
(b) if X is smooth over S, then $h_S(X,t)(S)$ is connective (i.e. not empty).

Both assertions hold by definition of $h_S(X,t)$ if $Z = \varnothing$. Otherwise, (S,Z) is an affine Henselian pair where Z has a unique closed point, so we can assume X affine. Assertion (a) is then a special case of [Gro67, Proposition 18.5.4], and assertion (b) is a special case of [Gru72, Théorème I.8]. For general S, it follows immediately that:

(a′) if X is étale over S, then $L_{nis}h_S(X,t)$ is contractible;
(b′) if X is smooth over S, then $L_{nis}h_S(X,t)$ is connective.

Assertion (b′) is an abstract version of Hensel's lemma in several variables. The crux of the Morel–Voevodsky localization theorem is a refinement of (b′) asserting that the motivic localization $L_{mot}h_S(X,t)$ is contractible.

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LEMMA 2. *Let $f\colon X \to Y$ be a morphism of locally finitely presented S-schemes that is étale in a neighborhood of $t(Z)$. Then the induced map $h_S(X,t) \to h_S(Y, f\circ t)$ is a Nisnevich local isomorphism.*

Proof. Since the presheaves $h_S(X,t)$ and $h_S(Y, f\circ t)$ transform cofiltered limits of quasicompact, quasiseparated (qcqs) schemes into colimits [Gro66, Théorème 8.8.2(i)], it suffices to show that the given map is an isomorphism on Henselian local schemes. Since $h_S(X,t)(T) = h_T(X_T, t_T)(T)$, we are reduced to proving that the map $h_S(X,t)(S) \to h_S(Y, f\circ t)(S)$ is an isomorphism when S is Henselian local; we will show that its fibers are contractible. Let $X' \subset X$ be an open neighborhood of $t(Z)$ where f is étale. Given a section $s\colon S \to Y$ extending $f\circ t$, we have a cartesian square

$$\begin{array}{ccc} h_S(X' \times_Y S, (t,i))(S) & \longrightarrow & * \\ \downarrow & & \downarrow{\scriptstyle s} \\ h_S(X,t)(S) & \longrightarrow & h_S(Y, f\circ t)(S). \end{array}$$

By assertion (a) above, $h_S(X' \times_Y S, (t,i))$ is contractible, as desired. □

THEOREM 3 ($\mathbf{A}^1$-Hensel lemma). *Let S be a scheme, $Z \subset S$ a closed subscheme, X an S-scheme, and $t\colon Z \to X$ an S-morphism. If X is smooth over S, then $L_{mot}h_S(X,t)$ is contractible.*

Proof. By Lemma 2, we can replace X by any open neighborhood of $t(Z)$ in X. Since the question is Nisnevich local on S, we can assume that S and X are both affine. Since $L_{nis}h_S(X,t)$ is connective, we can further assume that there exists a section $s\colon S \to X$ extending t. Then there exists an S-morphism $f\colon X \to \mathbf{V}(\mathcal{N}_s)$, étale in a neighborhood of $s(S)$, such that $f\circ s$ is the zero section of the normal bundle $\mathbf{V}(\mathcal{N}_s) \to S$. Using Lemma 2 again, we are reduced to the case where $X \to S$ is a vector bundle and $t\colon Z \to X$ is the restriction of its zero section. In this case, an obvious $\mathbf{A}^1$-homotopy shows that $L_{\mathbf{A}^1}h_S(X,t)$ is contractible. □

Remark 4. The proof of Theorem 3 actually shows that $L_{nis}L_{\mathbf{A}^1}L_{nis}h_S(X,t) \simeq *$.

COROLLARY 5. *Let $i\colon Z \hookrightarrow S$ be a closed immersion with open complement $j\colon U \hookrightarrow S$. For every $\mathcal{F} \in \mathrm{PSh}(\mathrm{Sm}_S)$, the square*

$$\begin{array}{ccc} j_\sharp j^*\mathcal{F} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{F}(\varnothing) \times h_S(U) & \longrightarrow & i_*i^*\mathcal{F} \end{array}$$

is motivically cocartesian, that is, its motivic localization is cocartesian in $\mathbf{H}(S)$.

Proof. Since this square preserves colimits in $\mathcal{F}$, we can assume that $\mathcal{F} = h_S(X)$ for some smooth S-scheme X. We must then show that the canonical map

$$h_S(X) \sqcup_{h_S(X_U)} h_S(U) \to i_*h_Z(X_Z)$$

is a motivic equivalence in $\mathrm{PSh}(\mathrm{Sm}_S)$. In fact, we will show that it is a motivic equivalence in $\mathrm{PSh}(\mathrm{Sch}_S)$. Writing the target as a colimit of representables, it suffices to show that for every morphism $f\colon T \to S$ and every map $h_S(T) \to i_*h_Z(X_Z)$, corresponding to a T-morphism

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$t\colon \mathrm{Z_T} \to \mathrm{X_T}$, the projection

$$\big(\mathrm{h_S(X)} \sqcup_{\mathrm{h_S(X_U)}} \mathrm{h_S(U)}\big) \times_{i_* \mathrm{h_Z(X_Z)}} \mathrm{h_S(T)} \to \mathrm{h_S(T)}$$

is a motivic equivalence. This map is the image by the functor $f_\sharp\colon \mathrm{PSh}(\mathrm{Sch_T}) \to \mathrm{PSh}(\mathrm{Sch_S})$ of the map

$$\mathrm{h_T}(\mathrm{X_T}, t) \to \mathrm{h_T(T)} \simeq *.$$

Indeed, this follows from the projection formula $f_\sharp(f^*(\mathrm{B}) \times_{f^*(\mathrm{A})} \mathrm{C}) \simeq \mathrm{B} \times_{\mathrm{A}} f_\sharp(\mathrm{C})$, which holds for any morphisms $\mathrm{B} \to \mathrm{A}$ in $\mathrm{PSh}(\mathrm{Sch_S})$ and $\mathrm{C} \to f^*(\mathrm{A})$ in $\mathrm{PSh}(\mathrm{Sch_T})$, and the base change equivalence $f^* i_* \simeq i_{\mathrm{T}*} f_{\mathrm{Z}}^*$. Since $f_\sharp$ preserves motivic equivalences and $\mathrm{X_T}$ is smooth over T, Theorem 3 concludes the proof. □

Proof of Theorem 1. The functors $j_\sharp$, j^*, and i^* between ∞-categories of presheaves preserve motivic equivalences, as does the functor $i_*\colon \mathrm{PSh}_\Sigma(\mathrm{Sm_Z}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm_S})$ by [BH20, Proposition 2.11]. Thus, for $\mathcal{F} \in \mathbf{H}(\mathrm{S})$, the given square is the motivic localization of the square of Corollary 5. □

Remark 6. Arguing as in the proof of Corollary 11, we can deduce from Theorem 1 that

$$\mathbf{H}(\mathrm{U}) \xrightarrow{j_\sharp} \mathbf{H}(\mathrm{S}) \xrightarrow{i^*} \mathbf{H}(\mathrm{Z})$$

is a cofiber sequence of presentable ∞-categories (in fact, it is also a fiber sequence).

2. The localization theorem for framed motivic spaces

We now turn to the proof of localization for framed motivic spaces. We use the notation from [EHK+19].

Lemma 7. *The forgetful functor* $\gamma_*\colon \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm_S})) \to \mathrm{PSh}_\Sigma(\mathrm{Sm_S})$ *detects Nisnevich and motivic equivalences.*

Proof. This follows from [EHK+19, Proposition 3.2.14]. □

Proposition 8. *Let* $f\colon \mathrm{T} \to \mathrm{S}$ *be an integral morphism. Then the functor*

$$f_*\colon \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm_T})) \to \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm_S}))$$

preserves Nisnevich and motivic equivalences.

Proof. By Lemma 7, this follows from the fact that the functor $f_*\colon \mathrm{PSh}_\Sigma(\mathrm{Sm_T}) \to \mathrm{PSh}_\Sigma(\mathrm{Sm_S})$ preserves Nisnevich and motivic equivalences [BH20, Proposition 2.11]. □

Corollary 9. *Let* $f\colon \mathrm{T} \to \mathrm{S}$ *be an integral morphism. Then the functor*

$$f_*\colon \mathbf{H}^{\mathrm{fr}}(\mathrm{T}) \to \mathbf{H}^{\mathrm{fr}}(\mathrm{S})$$

preserves colimits.

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Proof. It follows from Proposition 8 that f_* preserves sifted colimits. It also preserves limits, hence finite sums since $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ is semiadditive [EHK+19, Proposition 3.2.10(iii)]. □

If $i\colon \mathrm{Z} \hookrightarrow \mathrm{S}$ is a closed immersion, it follows from Corollary 9 that we have an adjunction

$$i_* : \mathbf{H}^{\mathrm{fr}}(\mathrm{Z}) \rightleftarrows \mathbf{H}^{\mathrm{fr}}(\mathrm{S}) : i^!.$$

THEOREM 10 (Framed localization). *Let $i\colon \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with open complement $j\colon \mathrm{U} \hookrightarrow \mathrm{S}$. Then the null sequence*

$$j_\sharp j^* \to \mathrm{id} \to i_* i^*$$

of endofunctors of $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ is a cofiber sequence. Dually, the null sequence

$$i_* i^! \to \mathrm{id} \to j_* j^*$$

of endofunctors of $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ is a fiber sequence.

Proof. It suffices to prove the first statement. Since all functors involved preserve colimits by Corollary 9, it suffices to check that the sequence is a cofiber sequence when evaluated on $\gamma^*(\mathrm{X}_+)$ where X is smooth over S and affine [EHK+19, Proposition 3.2.10(i)]. By Proposition 8 and Lemma 7, it suffices to show that the map

$$\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X})/\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X}_{\mathrm{U}}) \to i_* \mathrm{h}^{\mathrm{fr}}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})$$

in $\mathrm{PSh}(\mathrm{Sm}_{\mathrm{S}})$ is a motivic equivalence, where $\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X})/\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X}_{\mathrm{U}})$ denotes the quotient in commutative monoids. Note that if $\mathrm{Y} \in \mathrm{Sch}_{\mathrm{S}}$ is *connected* then

$$\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X}_{\mathrm{U}})(\mathrm{Y}) = \begin{cases} * & \text{if } \mathrm{Y}_{\mathrm{Z}} \neq \varnothing, \\ \mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X})(\mathrm{Y}) & \text{if } \mathrm{Y}_{\mathrm{Z}} = \varnothing. \end{cases}$$

It follows that the canonical map

$$\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X}) \sqcup_{j_\sharp \mathrm{h}^{\mathrm{fr}}_{\mathrm{U}}(\mathrm{X}_{\mathrm{U}})} \mathrm{h}_{\mathrm{S}}(\mathrm{U}) \to \mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X})/\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X}_{\mathrm{U}})$$

is an equivalence on connected essentially smooth S-schemes, hence it is a Zariski-local equivalence in $\mathrm{PSh}(\mathrm{Sm}_{\mathrm{S}})$.[1] We are thus reduced to showing that the map

$$\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X}) \sqcup_{j_\sharp \mathrm{h}^{\mathrm{fr}}_{\mathrm{U}}(\mathrm{X}_{\mathrm{U}})} \mathrm{h}_{\mathrm{S}}(\mathrm{U}) \to i_* \mathrm{h}^{\mathrm{fr}}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})$$

is a motivic equivalence. By [EHK+19, Corollary 2.3.27] and the non-framed version of Proposition 8, we can replace h^{fr} by $\mathrm{h}^{\mathrm{nfr}}$: it suffices to show that the map

$$\mathrm{h}^{\mathrm{nfr}}_{\mathrm{S}}(\mathrm{X}) \sqcup_{j_\sharp \mathrm{h}^{\mathrm{nfr}}_{\mathrm{U}}(\mathrm{X}_{\mathrm{U}})} \mathrm{h}_{\mathrm{S}}(\mathrm{U}) \to i_* \mathrm{h}^{\mathrm{nfr}}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})$$

is a motivic equivalence in $\mathrm{PSh}(\mathrm{Sm}_{\mathrm{S}})$. By [EHK+19, Theorem 5.1.5], the presheaf $\mathrm{h}^{\mathrm{nfr}}_{\mathrm{S}}(\mathrm{X})$ on *all* S-schemes is ind-representable by smooth S-schemes and compatible with any base change $\mathrm{S}' \to \mathrm{S}$. Considering $\mathrm{h}^{\mathrm{nfr}}_{\mathrm{S}}(\mathrm{X})$ as a presheaf on smooth S-schemes, this implies that $i^*(\mathrm{h}^{\mathrm{nfr}}_{\mathrm{S}}(\mathrm{X})) \simeq \mathrm{h}^{\mathrm{nfr}}_{\mathrm{Z}}(\mathrm{X}_{\mathrm{Z}})$ and $j^*(\mathrm{h}^{\mathrm{nfr}}_{\mathrm{S}}(\mathrm{X})) \simeq \mathrm{h}^{\mathrm{nfr}}_{\mathrm{U}}(\mathrm{X}_{\mathrm{U}})$. Thus, the result follows from Corollary 5. □

[1] Here, we use the fact that $\mathrm{h}^{\mathrm{fr}}_{\mathrm{S}}(\mathrm{X})$ transforms cofiltered limits of qcqs schemes into colimits (since X is locally finitely presented over S), as well as the hypercompleteness of the clopen topology on schemes.

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Corollary 11. *Let $i\colon \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with open complement $j\colon \mathrm{U} \hookrightarrow \mathrm{S}$. Then*

$$\mathbf{H}^{\mathrm{fr}}(\mathrm{U}) \xrightarrow{j_\sharp} \mathbf{H}^{\mathrm{fr}}(\mathrm{S}) \xrightarrow{i^*} \mathbf{H}^{\mathrm{fr}}(\mathrm{Z})$$

is a cofiber sequence of presentable ∞-categories, that is, the functor $i_*\colon \mathbf{H}^{\mathrm{fr}}(\mathrm{Z}) \to \mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ is fully faithful with image $(j^*)^{-1}(0)$.

Proof. Theorem 10 implies that $j^*(\mathrm{A}) \simeq 0$ if and only if $\mathrm{A} \simeq i_*i^*(\mathrm{A})$. It also implies that the unit map $i_* \to i_*i^*i_*$ is an equivalence, hence also the counit map $i_*i^*i_* \to i_*$ by the triangle identities. It remains to show that i_* is conservative. This follows immediately from the fact that every smooth Z-scheme admits an open covering by pullbacks of smooth S-schemes [Gro67, Proposition 18.1.1]. □

Remark 12. Similarly, the localization theorem holds for motivic spaces with finite étale transfers or with finite syntomic transfers, because the corresponding Hilbert schemes of points in $\mathbf{A}^n$ are smooth.

The localization theorem implies as usual the closed base change property and the closed projection formula, which states that $i_*\colon \mathbf{H}^{\mathrm{fr}}(\mathrm{Z}) \to \mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ is an $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$-module functor, as well as S^1-stable and $\mathbf{T}$-stable versions.

In the $\mathbf{T}$-stable case, using the work of Ayoub [Ayo08] and Cisinski and Déglise [CD19], we obtain for every separated morphism of finite type $f\colon \mathrm{X} \to \mathrm{Y}$ an exceptional adjunction

$$f_! : \mathbf{SH}^{\mathrm{fr}}(\mathrm{X}) \rightleftarrows \mathbf{SH}^{\mathrm{fr}}(\mathrm{Y}) : f^!$$

satisfying the usual properties. In particular, framed motivic spectra satisfy proper base change and the proper projection formula.

Note that the cofiber sequence of Corollary 11 is not part of a recollement in the sense of [Lur17, Definition A.8.1], because i^* is not left exact and the pair (i^*, j^*) is not conservative. These properties are, however, automatic in a stable setting.

Corollary 13. *Let $i\colon \mathrm{Z} \hookrightarrow \mathrm{S}$ be a closed immersion with open complement $j\colon \mathrm{U} \hookrightarrow \mathrm{S}$. Then the following pairs of fully faithful functors are recollements:*

(i) $\mathbf{SH}^{\mathrm{S}^1,\mathrm{fr}}(\mathrm{Z}) \xrightarrow{i_*} \mathbf{SH}^{\mathrm{S}^1,\mathrm{fr}}(\mathrm{S}) \xleftarrow{j_*} \mathbf{SH}^{\mathrm{S}^1,\mathrm{fr}}(\mathrm{U})$;
(ii) $\mathbf{SH}^{\mathrm{fr}}(\mathrm{Z}) \xrightarrow{i_*} \mathbf{SH}^{\mathrm{fr}}(\mathrm{S}) \xleftarrow{j_*} \mathbf{SH}^{\mathrm{fr}}(\mathrm{U})$.

Corollary 14. *Let S be a scheme locally of finite Krull dimension. Then the following pullback functors are conservative:*

(i) $\mathbf{SH}^{\mathrm{S}^1,\mathrm{fr}}(\mathrm{S}) \to \prod_{s\in \mathrm{S}} \mathbf{SH}^{\mathrm{S}^1,\mathrm{fr}}(s)$;
(ii) $\mathbf{SH}^{\mathrm{fr}}(\mathrm{S}) \to \prod_{s\in \mathrm{S}} \mathbf{SH}^{\mathrm{fr}}(s)$.

Proof. We can assume S qcqs and we prove the claim by induction on the dimension of S. For $s \in \mathrm{S}$, let $\iota_s\colon \operatorname{Spec} \mathcal{O}_{\mathrm{S},s} \to \mathrm{S}$ be the canonical map. Since ι_s is pro-smooth, the pullback functor

$$\iota_s^*\colon \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_{\mathrm{S}})) \to \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_{\mathcal{O}_{\mathrm{S},s}}))$$

preserves $\mathbf{A}^1$-invariant Nisnevich sheaves and commutes with the internal Hom from compact objects (in particular, with Ω and $\Omega_{\mathbf{T}}$). Hence, for a framed motivic S^1-spectrum or $\mathbf{T}$-spectrum

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$\mathrm{E} = (\mathrm{E}_n)_{n\geqslant 0}$ over S and a qcqs smooth S-scheme X, the Zariski stalk of $\mathrm{E}_n^{\mathrm{X}}$ at s may be computed as $\iota_s^*(\mathrm{E})_n(\mathrm{X}\times_{\mathrm{S}}\operatorname{Spec}\mathcal{O}_{\mathrm{S},s})$. By the hypercompleteness of the Zariski ∞-topos of S [CM19, § 3], equivalences between Zariski sheaves on S are detected on stalks. Since the family of functors $\mathrm{E}\mapsto \mathrm{E}_n^{\mathrm{X}}(\mathrm{S})$ is conservative, so is the family ι_s^* for $s\in \mathrm{S}$. We can therefore assume S local. Then the result follows from Corollary 13 and the induction hypothesis. □

Remark 15. Corollary 14 is also true if S is locally Noetherian of arbitrary dimension; see the proof of [Ayo14, Proposition 3.24].

3. The reconstruction theorem over a general base scheme

Next, we extend the reconstruction theorem [EHK+19, Theorem 3.5.12] to more general base schemes.

Lemma 16. *Let $f\colon \mathrm{T}\to \mathrm{S}$ be a morphism of schemes. Then the canonical transformation*

$$f^*\gamma_*\to \gamma_* f^*\colon \mathbf{H}^{\mathrm{fr}}(\mathrm{S})\to \mathbf{H}(\mathrm{T})$$

is an equivalence, and similarly for $\mathbf{SH}^{\mathrm{S}^1}$ *and* $\mathbf{SH}$.

Proof. The stable statements follow from the unstable one, using the fact that the functors γ_* and f^* can be computed levelwise on prespectra. Since f^* and γ_* preserve sifted colimits and commute with $\mathrm{L}_{\mathrm{mot}}$ [EHK+19, Propositions 3.2.14 and 3.2.15], it suffices to check that the canonical map

$$f^*\mathrm{h}_{\mathrm{S}}^{\mathrm{fr}}(\mathrm{X})\to \mathrm{h}_{\mathrm{T}}^{\mathrm{fr}}(\mathrm{X}\times_{\mathrm{S}}\mathrm{T})$$

is a motivic equivalence for every $\mathrm{X}\in \mathrm{Sm}_{\mathrm{S}}$ affine, where we regard $\mathrm{h}_{\mathrm{S}}^{\mathrm{fr}}(\mathrm{X})$ as a presheaf on Sm_{S}. By [EHK+19, Corollary 2.3.27], we can replace h^{fr} by $\mathrm{h}^{\mathrm{nfr}}$. But the map

$$f^*\mathrm{h}_{\mathrm{S}}^{\mathrm{nfr}}(\mathrm{X})\to \mathrm{h}_{\mathrm{T}}^{\mathrm{nfr}}(\mathrm{X}\times_{\mathrm{S}}\mathrm{T})$$

is an isomorphism because $\mathrm{h}_{\mathrm{S}}^{\mathrm{nfr}}(\mathrm{X})$ is a smooth ind-S-scheme that is stable under base change [EHK+19, Theorem 5.1.5]. □

Lemma 17. *Let $p\colon \mathrm{T}\to \mathrm{S}$ be a proper morphism of schemes. Then the canonical transformation*

$$\gamma^* p_*\to p_*\gamma^*\colon \mathbf{SH}(\mathrm{T})\to \mathbf{SH}^{\mathrm{fr}}(\mathrm{S})$$

is an equivalence.

Proof. If p is a closed immersion, this follows from Theorem 10 and its non-framed version. If p is smooth and proper, it follows from the ambidexterity equivalences $p_*\simeq p_\sharp\Sigma^{-\Omega_p}$. Together with Zariski descent, this implies the result for p locally projective. The general case (which we will not use) follows by a standard use of Chow's lemma; see [CD19, Proposition 2.3.11(2)] and [Hoy14, Proposition C.13] for details. □

Theorem 18 (Reconstruction theorem). *Let* S *be a scheme. Then the functor*

$$\gamma^*\colon \mathbf{SH}(\mathrm{S})\to \mathbf{SH}^{\mathrm{fr}}(\mathrm{S})$$

is an equivalence of symmetric monoidal ∞-categories

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Proof. Since the right adjoint γ_* is conservative [EHK+19, Proposition 3.5.2], it suffices to show that γ^* is fully faithful, that is, that the unit transformation $\mathrm{id} \to \gamma_*\gamma^*$ is an equivalence. By Zariski descent, we may assume S qcqs. In this case, the ∞-category $\mathbf{SH}(\mathrm{S})$ is generated under colimits by the objects $\Sigma^n_{\mathbf{T}} p_* \mathbf{1}_{\mathrm{X}}$ for $n \in \mathbf{Z}$ and $p\colon \mathrm{X} \to \mathrm{S}$ a projective morphism [Ayo08, Lemme 2.2.23]. By Lemma 17, we are thus reduced to proving that $\mathbf{1}_{\mathrm{S}} \to \gamma_*\gamma^*\mathbf{1}_{\mathrm{S}}$ is an equivalence. By Lemma 16, we can now assume that $\mathrm{S} = \operatorname{Spec} \mathbf{Z}$. By the non-framed version of Corollary 14 and again Lemma 16, the result follows from the cases $\mathrm{S} = \operatorname{Spec} \mathbf{Q}$ and $\mathrm{S} = \operatorname{Spec} \mathbf{F}_p$ for p prime, which are known by [EHK+19, Theorem 3.5.12]. □

Remark 19. The argument used in the proof of Theorem 18 can be axiomatized as follows. Let S be a qcqs scheme of finite Krull dimension, let

$$\mathbf{A}, \mathbf{B}\colon (\mathrm{Sch}^{\mathrm{qcqs}}_{\mathrm{S}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}^{\mathrm{st}}$$

be functors satisfying Ayoub's axioms [Ayo08, § 1.4.1], and let $\varphi\colon \mathbf{A} \to \mathbf{B}$ be a natural transformation that commutes with $f_\sharp$ for f smooth. Suppose that:

(i) each $\mathbf{A}(\mathrm{X})$ is cocomplete and generated under colimits by objects of the form $f_\sharp f^* p^*(\mathrm{A})$ where $f\colon \mathrm{Y} \to \mathrm{X}$ is smooth, $p\colon \mathrm{X} \to \mathrm{S}$ is the structure map, and $\mathrm{A} \in \mathbf{A}(\mathrm{S})$;
(ii) φ has a right adjoint that preserves colimits and commutes with f^* for any f;
(iii) $\varphi_s\colon \mathbf{A}(s) \to \mathbf{B}(s)$ is fully faithful for every $s \in \mathrm{S}$.

Then $\varphi_{\mathrm{X}}\colon \mathbf{A}(\mathrm{X}) \to \mathbf{B}(\mathrm{X})$ is fully faithful for every $\mathrm{X} \in \mathrm{Sch}^{\mathrm{qcqs}}_{\mathrm{S}}$.

Since $\mathbf{SH}^{\mathrm{fr}}(\mathrm{S}) \simeq \mathbf{SH}(\mathrm{S}) \otimes_{\mathbf{H}(\mathrm{S})} \mathbf{H}^{\mathrm{fr}}(\mathrm{S})$, the reconstruction theorem implies that the right-lax symmetric monoidal functor $\Omega^\infty_{\mathbf{T}}\colon \mathbf{SH}(\mathrm{S}) \to \mathbf{H}(\mathrm{S})$ factors *uniquely* as

$$\begin{array}{ccc} \mathbf{SH}(\mathrm{S}) & & \\ {\scriptstyle \Omega^\infty_{\mathbf{T}}}\downarrow & \overset{!}{\searrow} & \\ \mathbf{H}(\mathrm{S}) & \underset{\gamma_*}{\longleftarrow} & \mathbf{H}^{\mathrm{fr}}(\mathrm{S}). \end{array}$$

Indeed, the ∞-groupoid of such factorizations is equivalent to that of colimit-preserving symmetric monoidal retractions of the functor $\gamma^*\colon \mathbf{SH}(\mathrm{S}) \to \mathbf{SH}^{\mathrm{fr}}(\mathrm{S})$. In particular, the underlying cohomology theory $\mathrm{Sm}^{\mathrm{op}}_{\mathrm{S}} \to \mathrm{Spc}$ of a motivic spectrum extends canonically to the ∞-category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_{\mathrm{S}})^{\mathrm{op}}$. As proved in [EHK+20, Theorem 3.3.10], this enhanced functoriality of cohomology theories can be described using the Gysin morphisms constructed using Verdier's deformation to the normal cone (see [DJK18]).

4. Application to motivic cohomology

In this final section, we obtain a simple presentation of the motivic cohomology spectrum in terms of framed correspondences. Let us denote by $\mathrm{H}\mathbf{Z}_{\mathrm{S}} \in \mathbf{SH}(\mathrm{S})$ Spitzweck's motivic cohomology spectrum over a base scheme S [Spi18]. By construction, it is stable under arbitrary base change, and when S is a Dedekind domain it represents Bloch–Levine motivic cohomology. More precisely, for such S, the presheaf $\mathrm{X} \mapsto \mathrm{Maps}_{\mathbf{SH}(\mathrm{S})}(\Sigma^\infty_{\mathbf{T}} \mathrm{X}_+, \Sigma^n_{\mathbf{T}} \mathrm{H}\mathbf{Z}_{\mathrm{S}})$ on smooth S-schemes is the Zariski sheafification of Bloch's cycle complex $\mathrm{X} \mapsto z^n(\mathrm{X}, *)$ (which is known to already be a Zariski sheaf when S is semilocal [Lev01, Theorem 1.7]). When S is the spectrum of a field, $\mathrm{H}\mathbf{Z}_{\mathrm{S}}$ is equivalent to Voevodsky's motivic cohomology spectrum.

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For any commutative monoid A, the constant sheaf A_S on Sm_S admits a canonical extension to $\mathbf{Corr}^{\mathrm{flf}}(\mathrm{Sm}_\mathrm{S})$, where 'flf' denotes the class of finite locally free morphisms: to a span

$$\begin{array}{ccccc} & & \mathrm{Z} & & \\ & \swarrow^{f} & & \searrow^{g} & \\ \mathrm{X} & & & & \mathrm{Y} \end{array}$$

with f finite locally free and a locally constant function $a\colon \mathrm{Y} \to \mathrm{A}$, we associate the locally constant function

$$X \to \mathrm{A}, \quad x \mapsto \sum_{z \in f^{-1}(x)} \deg_z(f) \cdot a(g(z))$$

(see [BH20, Lemma 13.13]). In particular, A_S can be regarded as an object of $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$ via the forgetful functor $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_\mathrm{S}) \to \mathbf{Corr}^{\mathrm{flf}}(\mathrm{Sm}_\mathrm{S})$.

If $f\colon \mathrm{T} \to \mathrm{S}$ is a morphism, there is an obvious map $\mathrm{A}_\mathrm{S} \to f_*\mathrm{A}_\mathrm{T}$ in $\mathbf{H}^{\mathrm{fr}}(\mathrm{S})$, whence by adjunction a map $f^*\mathrm{A}_\mathrm{S} \to \mathrm{A}_\mathrm{T}$ in $\mathbf{H}^{\mathrm{fr}}(\mathrm{T})$.

Lemma 20. *Let* A *be a commutative monoid and* $f\colon \mathrm{T} \to \mathrm{S}$ *a morphism of schemes. Then the canonical map* $f^*\mathrm{A}_\mathrm{S} \to \mathrm{A}_\mathrm{T}$ *in* $\mathbf{H}^{\mathrm{fr}}(\mathrm{T})$ *is an equivalence.*

Proof. We consider the following commutative triangle in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_\mathrm{T})$:

$$\begin{array}{ccc} f^*\gamma_*\mathrm{A}_\mathrm{S} & & \\ \downarrow & \searrow & \\ \gamma_* f^*\mathrm{A}_\mathrm{S} & \longrightarrow & \gamma_*\mathrm{A}_\mathrm{T}. \end{array}$$

The vertical map is a motivic equivalence by Lemma 16, and the diagonal map is trivially a Zariski equivalence. Hence, the lower horizontal map is a motivic equivalence. Since γ_* detects motivic equivalences (Lemma 7), we are done. □

Theorem 21. *Let* S *be a scheme. Then there is an equivalence of motivic* $\mathcal{E}_\infty$*-ring spectra* $\mathrm{H}\mathbf{Z}_\mathrm{S} \simeq \gamma_*\Sigma^\infty_{\mathbf{T},\mathrm{fr}}\mathbf{Z}_\mathrm{S}$.

Proof. By Lemmas 16 and 20, it suffices to prove this when S is a Dedekind domain. In this case, there is an isomorphism of presheaves of commutative rings

$$\Omega^\infty_\mathbf{T}\mathrm{H}\mathbf{Z}_\mathrm{S} \simeq \mathbf{Z}_\mathrm{S}.$$

We claim that this isomorphism is compatible with the framed transfers on either side, the ones on the left coming from Theorem 18. Since we are dealing with discrete constant sheaves, it suffices to compare the transfers for a framed correspondence of the form $\eta \leftarrow \mathrm{T} \to \eta$ where η is a generic point of a smooth S-scheme. Thus we may assume that S is a field, in which case we can compute the framed transfers on $\Omega^\infty_\mathbf{T}\mathrm{H}\mathbf{Z}_\mathrm{S}$ using [EHK+19, Proposition 5.3.6], verifying the claim.

By adjunction, we obtain a morphism of $\mathcal{E}_\infty$-algebras

$$\varphi_\mathrm{S}\colon \Sigma^\infty_{\mathbf{T},\mathrm{fr}}\mathbf{Z}_\mathrm{S} \to \gamma^*\mathrm{H}\mathbf{Z}_\mathrm{S}$$

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in $\mathbf{SH}^{\mathrm{fr}}(S)$. We show that φ_S is an equivalence. By construction, φ_S is functorial in S. By Corollary 14(2), we may therefore assume that S is the spectrum of a perfect field. In this case, the recognition principle [EHK+19, Theorem 3.5.14(i)] implies that φ_S exhibits $\gamma_*\Sigma^\infty_{\mathbf{T},\mathrm{fr}}\mathbf{Z}_S$ as the very effective cover of $\mathrm{H}\mathbf{Z}_S$. Since $\mathrm{H}\mathbf{Z}_S$ is already very effective [BH20, Lemma 13.7], we conclude that φ_S is an equivalence. □

If S is a Dedekind domain, the motivic spectrum $\mathrm{H}\mathbf{Z}_S \in \mathbf{SH}(S)$ lies in the heart of the effective homotopy t-structure [BH20, Lemma 13.7]. It follows that it admits a unique strictly commutative monoid structure in $\mathbf{SH}(S)$, in the sense of [Hoy18, § 7]. Hence, for any scheme S, $\mathrm{H}\mathbf{Z}_S \in \mathbf{SH}(S)$ is a module over the Eilenberg–Mac Lane spectrum $\mathbf{Z} \in \mathrm{Spt}$. In particular, for any $A \in \mathbf{Mod}_{\mathbf{Z}}(\mathrm{Spt})$, we can form the tensor product $\mathrm{HA}_S = \mathrm{H}\mathbf{Z}_S \otimes_{\mathbf{Z}} A$. This construction defines a symmetric monoidal functor

$$\mathbf{Mod}_{\mathbf{Z}}(\mathrm{Spt}) \to \mathbf{Mod}_{\mathrm{H}\mathbf{Z}_S}(\mathbf{SH}(S)), \quad A \mapsto \mathrm{HA}_S.$$

When S is the spectrum of a field and A is an abelian group, HA_S is equivalent to Voevodsky's motivic Eilenberg–Mac Lane spectrum with coefficients in A.

Corollary 22. *Let* S *be a scheme and* A *an abelian group (respectively, a ring; a commutative ring). Then there is a canonical equivalence of* $\mathrm{H}\mathbf{Z}_S$*-modules (respectively, of* $\mathcal{A}_\infty$*-*$\mathrm{H}\mathbf{Z}_S$*-algebras; of* $\mathcal{E}_\infty$*-*$\mathrm{H}\mathbf{Z}_S$*-algebras)* $\mathrm{HA}_S \simeq \gamma_*\Sigma^\infty_{\mathbf{T},\mathrm{fr}}A_S$.

Proof. By Lemmas 16 and 20, we may assume that S is a Dedekind domain. Since the equivalence of Theorem 21 takes place in the heart of the effective homotopy t-structure, it can be uniquely promoted to an equivalence of $\mathcal{E}_\infty$-rings in strictly commutative monoids. Hence, for any $A \in \mathbf{Mod}_{\mathbf{Z}}(\mathrm{Spt}_{\geqslant 0})$, we obtain an equivalence $\mathrm{HA}_S \simeq \gamma_*\Sigma^\infty_{\mathbf{T},\mathrm{fr}}(\mathbf{Z}_S \otimes_{\mathbf{Z}} A)$. To conclude, note that $\mathbf{Z}_S \otimes_{\mathbf{Z}} A \simeq A_S$ when A is discrete. □

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Marc Hoyois marc.hoyois@ur.de
Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

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