Bull. Aust. Math. Soc. **92** (2015), 353–364
doi:10.1017/S0004972715000647

ARITHMETIC PROPERTIES OF PARTITION QUADRUPLES WITH ODD PARTS DISTINCT

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(Received 24 February 2015; accepted 30 April 2015; first published online 8 July 2015)

Abstract

Let $\text{pod}_{-4}(n)$ denote the number of partition quadruples of n where the odd parts in each partition are distinct. We find many arithmetic properties of $\text{pod}_{-4}(n)$ including the following infinite family of congruences: for any integers $\alpha \geq 1$ and $n \geq 0$,

$$\text{pod}_{-4}\Big(3^{\alpha+1}n + \frac{5 \cdot 3^{\alpha} + 1}{2}\Big) \equiv 0 \pmod 9.$$

We also establish some internal congruences and some Ramanujan-type congruences modulo 2, 5 and 8 satisfied by $\text{pod}_{-4}(n)$.

2010 *Mathematics subject classification*: primary 05A17; secondary 11P83.

Keywords and phrases: congruences, partition quadruples, distinct odd parts, theta functions.

1. Introduction

Let $\text{pod}_{-k}(n)$ denote the number of partition k-tuples of n where in each partition the odd parts are distinct. For $k = 1$, $\text{pod}_{-1}(n)$ is often denoted as $\text{pod}(n)$. As usual,

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}$$

denotes one of Ramanujan's theta functions, where $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$ is the standard q-series notation. Moreover, we introduce the notation

$$(a_1, a_2, \ldots, a_n; q)_{\infty} = (a_1; q)_{\infty}(a_2; q)_{\infty} \cdots (a_n; q)_{\infty}.$$

It is not difficult to see that the generating function of $\text{pod}_{-k}(n)$ is

$$\sum_{n=0}^{\infty} \text{pod}_{-k}(n)q^n = \frac{(-q; q^2)_{\infty}^k}{(q^2; q^2)_{\infty}^k} = \frac{1}{\psi(-q)^k}. \tag{1.1}$$

In recent years, the arithmetic properties of $\text{pod}_{-k}(n)$ have drawn much attention. In 2010, Hirschhorn and Sellers [3] studied the congruence properties of $\text{pod}(n)$. They found some infinite families of Ramanujan-type congruences, including, for integers $\alpha \geq 0$ and $n \geq 0$,

$$\mathrm{pod}\Big(3^{2\alpha+3}n + \frac{23 \times 3^{2\alpha+2} + 1}{8}\Big) \equiv 0 \pmod 3.$$

They also found some internal congruences such as

$$\mathrm{pod}(81n + 17) \equiv 5\,\mathrm{pod}(9n + 2) \pmod{27}.$$

In 2011, Radu and Sellers [4] obtained further congruences for pod(n) modulo 5 and 7 by using modular forms. In 2014, the author [7] discovered many new congruences for pod(n). For example, for any integers $n \geq 0$ and $\alpha \geq 1$,

$$\mathrm{pod}\Big(5^{2\alpha+2}n + \frac{11 \cdot 5^{2\alpha+1} + 1}{8}\Big) \equiv 0 \pmod 5,$$

and

$$\mathrm{pod}\Big(5^{2\alpha+2}n + \frac{19 \cdot 5^{2\alpha+1} + 1}{8}\Big) \equiv 0 \pmod 5.$$

In 2011, Chen and Lin [2] investigated the arithmetic properties of $\mathrm{pod}_{-2}(n)$. They found two infinite families of congruences modulo 3 and 5, respectively.

Recently, the author [6] established many congruences modulo 7, 9 and 11 satisfied by $\mathrm{pod}_{-3}(n)$. For example, for any integers $\alpha \geq 1$ and $n \geq 0$,

$$\mathrm{pod}_{-3}\Big(3^{2\alpha+2}n + \frac{23 \times 3^{2\alpha+1} + 3}{8}\Big) \equiv 0 \pmod 9.$$

It is natural to ask whether we can find similar properties for $\mathrm{pod}_{-k}(n)$ for $k \geq 4$. It turns out that the case $k = 4$ is similar to those described above, but as k increases the problem becomes more difficult. In this paper, we present results about $\mathrm{pod}_{-4}(n)$, the number of partition quadruples of n where the odd parts in each partition are distinct.

The paper is organised as follows. In Section 2, we will present an infinite family of Ramanujan-type congruences: for any integers $\alpha \geq 1$ and $n \geq 0$,

$$\mathrm{pod}_{-4}\Big(3^{\alpha+1}n + \frac{5 \cdot 3^{\alpha} + 1}{2}\Big) \equiv 0 \pmod 9.$$

In Section 3, we prove various internal congruences such as

$$\mathrm{pod}_{-4}(27n + 14) \equiv -\mathrm{pod}_{-4}(9n + 5) \pmod{81}.$$

In Section 4, we establish some congruences for $\mathrm{pod}_{-4}(n)$ modulo 2, 5 and 8. For example, if $\sigma(m)$ denotes the sum of all positive divisors of m, then

$$\mathrm{pod}_{-4}(5n + 3) \equiv (-1)^{n+1}\sigma(2n + 1) \pmod 5.$$

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2. An infinite family of congruences modulo 9

We need some facts about $\psi(q)$ before presenting our results. The first is the 3-dissection of $\psi(q)$ (see [2, 3]):

$$\psi(q) = 1 + q + q^3 + q^6 + q^{10} + q^{15} + \cdots = A(q^3) + q\psi(q^9)$$

where

$$A(q) = \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q;q)_\infty (q^6;q^6)_\infty}.$$

The following two lemmas are also important in our discussion.

LEMMA 2.1. *Let p be a prime and α be a positive integer. Then*

$$(q;q)_\infty^{p^\alpha} \equiv (q^p;q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha},$$
$$\psi(q)^p \equiv \psi(q^p) \pmod{p}.$$

PROOF. The first congruence is [4, Lemma 1.2]. The second follows from the first and the product representation of $\psi(q)$. □

LEMMA 2.2 (See [3, Lemma 2.1]). *We have*

$$A(q^3)^3 + q^3\psi(q^9)^3 = \frac{\psi(q^3)^4}{\psi(q^9)}$$

and

$$\frac{1}{\psi(q)} = \frac{\psi(q^9)}{\psi(q^3)^4}(A(q^3)^2 - qA(q^3)\psi(q^9) + q^2\psi(q^9)^2).$$

For convenience, in this section set $s = A(q^3)$ and $t = \psi(q^9)$. We can rewrite Lemma 2.2 as

$$s^3 + q^3t^3 = \frac{\psi(q^3)^4}{\psi(q^9)},$$
$$\frac{1}{\psi(q)} = \frac{\psi(q^9)}{\psi(q^3)^4}(s^2 - qst + q^2t^2). \tag{2.1}$$

THEOREM 2.3. *For any integer $\alpha \geq 1$,*

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}\Big(3^\alpha n + \frac{3^\alpha + 1}{2}\Big) q^n \equiv (-1)^{\alpha-1} \frac{\psi(q^3)^4}{\psi(q)^8} \pmod 9.$$

PROOF. We proceed by induction on α. From (1.1) and (2.1),

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(n) q^n = \frac{1}{\psi(q)^4} = \frac{\psi(q^9)^4}{\psi(q^3)^{16}}(s^2 - qst + q^2t^2)^4. \tag{2.2}$$

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If we extract all the terms of the form q^{3k+2} in the expansion of $(s^2 - qst + q^2t^2)^4$,

$$q^2(10s^6t^2 - 16q^3s^3t^5 + q^6t^8) \equiv q^2t^2(s^3 + q^3t^3)^2 \pmod 9. \tag{2.3}$$

Hence,

$$\sum_{n=0}^{\infty}(-1)^{3n+2}\mathrm{pod}_{-4}(3n+2)q^{3n+2} \equiv \frac{\psi(q^9)^4}{\psi(q^3)^{16}}q^2t^2(s^3+q^3t^3)^2 = q^2\frac{\psi(q^9)^4}{\psi(q^3)^8} \pmod 9.$$

Dividing both sides by q^2, then replacing q^3 by q,

$$\sum_{n=0}^{\infty}(-1)^n\mathrm{pod}_{-4}(3n+2)q^n \equiv \frac{\psi(q^3)^4}{\psi(q)^8} \pmod 9.$$

Hence the result holds when $\alpha = 1$.

Suppose

$$\sum_{n=0}^{\infty}(-1)^n\mathrm{pod}_{-4}\Big(3^\alpha n + \frac{3^\alpha+1}{2}\Big)q^n \equiv (-1)^{\alpha-1}\frac{\psi(q^3)^4}{\psi(q)^8} \pmod 9.$$

Applying (2.1) again,

$$\frac{\psi(q^3)^4}{\psi(q)^8} = \frac{\psi(q^9)^8}{\psi(q^3)^{28}}(s^2 - qst + q^2t^2)^8. \tag{2.4}$$

If we extract all the terms of the form q^{3k+1} in the expansion of $(s^2 - qst + q^2t^2)^8$,

$$\begin{aligned}
&q(-8s^{15}t + 266q^3s^{12}t^4 - 1016q^6s^9t^7 + 784q^9s^6t^{10} - 112q^{12}s^3t^{13} + q^{15}t^{16})\\
&\quad\equiv qt(s^{15} + 5q^3s^{12}t^3 + 10q^6s^9t^6 + 10q^9s^6t^9 + 5q^{12}s^3t^{12} + q^{15}t^{15})\\
&\quad\equiv qt(s^3+q^3t^3)^5 \pmod 9.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^{3n+1}\mathrm{pod}_{-4}\Big(3^\alpha(3n+1) + \frac{3^\alpha+1}{2}\Big)q^{3n+1} &\equiv (-1)^{\alpha-1}qt(s^3+q^3t^3)^5\frac{\psi(q^9)^8}{\psi(q^3)^{28}}\\
&= (-1)^{\alpha-1}q\frac{\psi(q^9)^4}{\psi(q^3)^8} \pmod 9.
\end{aligned}$$

Dividing both sides by $-q$ and then replacing q^3 by q,

$$\sum_{n=0}^{\infty}(-1)^n\mathrm{pod}_{-4}\Big(3^{\alpha+1}n + \frac{3^{\alpha+1}+1}{2}\Big)q^n \equiv (-1)^{(\alpha+1)-1}\frac{\psi(q^3)^4}{\psi(q)^8} \pmod 9.$$

This implies that the result holds for $\alpha + 1$ and, by induction on α, completes the proof. □

For any positive integer n and prime p, we denote by $v_p(n)$ the power of p in the unique prime factorisation of n. Let $\sigma(n)$ denote the sum of the positive divisors of n.

Corollary 2.4. *For any integer $n \geq 0$,*

$$\mathrm{pod}_{-4}(3n+2) \equiv (-1)^n \sigma(2n+1) \pmod 3.$$

Moreover, $\mathrm{pod}_{-4}(3n+2) \equiv 0 \pmod 3$ *if and only if one of the following statements is true.*

(1) *There exists a prime* $p \equiv 1 \pmod 3$ *such that* $v_p(2n+1) \equiv 2 \pmod 3$*.*
(2) *There exists a prime* $p \equiv 2 \pmod 3$ *such that* $v_p(2n+1) \equiv 1 \pmod 2$*.*

Proof. Let $\alpha = 1$ in Theorem 2.3. By Lemma 2.1,

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(3n+2) q^n \equiv \frac{\psi(q^3)^4}{\psi(q)^8} \equiv \psi(q)^4 = \sum_{n=0}^{\infty} t_4(n) q^n \pmod 3,$$

where $t_4(n)$ denotes the number of representations of n as a sum of four triangular numbers. Hence,

$$\mathrm{pod}_{-4}(3n+2) \equiv (-1)^n t_4(n) \pmod 3.$$

From [1, Theorem 3.6.3], we know $t_4(n) = \sigma(2n+1)$. This proves the first assertion.

Write the prime factorisation of $2n+1$ as $2n+1 = \prod_{p|2n+1} p^{v_p(2n+1)}$. Because

$$\sigma(2n+1) = \prod_{p|2n+1} (1 + p + \cdots + p^{v_p(2n+1)}),$$

we see that $3 \mid \sigma(2n+1)$ if and only if there exists a prime $p \mid 2n+1$ such that $3 \mid (1 + p + \cdots + p^{v_p(2n+1)})$.

If $p = 3$, then we have $1 + p + \cdots + p^{v_p(2n+1)} \equiv 1 \pmod 3$. If $p \equiv 1 \pmod 3$, then $1 + p + \cdots + p^{v_p(2n+1)} \equiv 1 + v_p(2n+1) \pmod 3$ and $3 \mid (1 + p + \cdots + p^{v_p(2n+1)})$ if and only if $v_p(2n+1) \equiv 2 \pmod 3$. If $p \equiv 2 \pmod 3$, then since

$$1 + p + \cdots + p^{v_p(2n+1)} = \frac{p^{v_p(2n+1)+1} - 1}{p - 1},$$

we see that $3 \mid 1 + p + \cdots + p^{v_p(2n+1)}$ if and only if $p^{v_p(2n+1)+1} \equiv 1 \pmod 3$, that is, $v_p(2n+1)$ is odd. □

Theorem 2.5. *For any integers $\alpha \geq 1$ and $n \geq 0$,*

$$\mathrm{pod}_{-4}\left(3^{\alpha+1} n + \frac{5 \cdot 3^{\alpha} + 1}{2}\right) \equiv 0 \pmod 9.$$

Proof. By [6, Lemma 2.3], the coefficient of q^{3n+2} in the series expansion of $\psi(q^3)^4/\psi(q)^8$ is divisible by 9. The result now follows from Theorem 2.3. □

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For distinct integers $\alpha \geq 1$, the arithmetic sequences $\{3^{\alpha+1}n + \frac{1}{2}(5 \cdot 3^{\alpha} + 1) : n = 0, 1, 2, \ldots\}$ are disjoint, and they account for

$$\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{\alpha+1}} + \cdots = \frac{1}{6}$$

of all nonnegative integers. The following corollary is an immediate consequence.

Corollary 2.6. *For at least* $1/6$ *of all nonnegative integers,* $\mathrm{pod}_{-4}(n)$ *is divisible by* 9.

3. Some internal congruences

If we let $\alpha = 1$ and $\alpha = 2$ in Theorem 2.3, respectively, we deduce that

$$\mathrm{pod}_{-4}(9n + 5) \equiv -\mathrm{pod}_{-4}(3n + 2) \pmod 9.$$

By more careful treatment, we can improve this congruence to the following theorem.

Theorem 3.1. *For any integer* $n \geq 0$,

$$\mathrm{pod}_{-4}(27n + 5) \equiv -\mathrm{pod}_{-4}(9n + 2) \pmod 9,$$
$$\mathrm{pod}_{-4}(27n + 14) \equiv -\mathrm{pod}_{-4}(9n + 5) \pmod{81},$$
$$\mathrm{pod}_{-4}(27n + 23) \equiv -\mathrm{pod}_{-4}(9n + 8) \pmod{27}.$$

Before we prove this theorem, we need to establish the following lemma.

Lemma 3.2. *We have*

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(3n + 2) q^n = 10 \frac{\psi(q^3)^4}{\psi(q)^8} - 36q \frac{\psi(q^3)^8}{\psi(q)^{12}} + 27q^2 \frac{\psi(q^3)^{12}}{\psi(q)^{16}},$$

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(9n + 5) q^n \equiv 35 \frac{\psi(q^3)^4}{\psi(q)^8} + 18q \frac{\psi(q^3)^8}{\psi(q)^{12}} - 27q^2 \frac{\psi(q^3)^{12}}{\psi(q)^{16}} - 27q\psi(q^3)^4 \pmod{81}.$$

Proof. Let us go back to the proof of Theorem 2.3. By (2.2) and (2.3), if we extract all the terms of the form q^{3n+2} in the expansion of $\psi(q)^{-4}$, then divide by q^2 and replace q^3 by q,

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(3n + 2) q^n = \frac{\psi(q^3)^4}{\psi(q)^{16}} (10A(q)^6 \psi(q^3)^2 - 16qA(q)^3 \psi(q^3)^5 + q^2 \psi(q^3)^8).$$

By Lemma 2.2, $A(q)^3 \psi(q^3) = \psi(q)^4 - q\psi(q^3)^4$. Substituting this formula into the identity above, after simple manipulations, we obtain the first identity.

Now we turn to the second congruence identity. If we extract all the terms of the form q^{3k+1} in the expansion of $(s^2 - qst + q^2t^2)^8$,

$$-8qs^{15}t + 266q^4s^{12}t^4 - 1016q^7s^9t^7 + 784q^{10}s^6t^{10} - 112q^{13}s^3t^{13} + q^{16}t^{16}$$
$$\equiv qt(s^3 + q^3t^3)^3(-8s^6 - 34s^3q^3t^3 + q^6t^6) \pmod{81}.$$

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By (2.4), extracting all the terms of the form q^{3k+1} in the expansion of $\psi(q^3)^4/\psi(q)^8$ and reducing modulo 81,

$$\begin{aligned}
&q\frac{\psi(q^9)^9}{\psi(q^3)^{28}}\cdot\Big(\frac{\psi(q^3)^4}{\psi(q^9)}\Big)^3(-8(s^3+q^3t^3)^2-18q^3t^3(s^3+q^3t^3)+27q^6t^6)\\
&\quad\equiv q\frac{\psi(q^9)^6}{\psi(q^3)^{16}}\Big(-8\frac{\psi(q^3)^8}{\psi(q^9)^2}-18q^3\psi(q^9)^3\frac{\psi(q^3)^4}{\psi(q^9)}+27q^6\psi(q^9)^6\Big)\\
&\quad\equiv -8q\frac{\psi(q^9)^4}{\psi(q^3)^8}-18q^4\frac{\psi(q^9)^8}{\psi(q^3)^{12}}+27q^7\frac{\psi(q^9)^{12}}{\psi(q^3)^{16}} \pmod{81}.
\end{aligned}\tag{3.1}$$

Denote by $F(q)$ the sum of all the terms of the form q^{3k+1} in the expansion of $10\psi(q^3)^4/\psi(q)^8$. Then

$$\begin{aligned}
F(q)&\equiv -80q\frac{\psi(q^9)^4}{\psi(q^3)^8}-180q^4\frac{\psi(q^9)^8}{\psi(q^3)^{12}}+270q^7\frac{\psi(q^9)^{12}}{\psi(q^3)^{16}}\\
&\equiv q\frac{\psi(q^9)^4}{\psi(q^3)^8}-18q^4\frac{\psi(q^9)^8}{\psi(q^3)^{12}}+27q^7\frac{\psi(q^9)^{12}}{\psi(q^3)^{16}} \pmod{81}.
\end{aligned}\tag{3.2}$$

Similarly, by (2.1),

$$\frac{\psi(q^3)^8}{\psi(q)^{12}}=\frac{\psi(q^9)^{12}}{\psi(q^3)^{40}}(s^2-qst+q^2t^2)^{12}.\tag{3.3}$$

If we extract all the terms of the form q^{3k} in the expansion of $(s^2-qst+q^2t^2)^{12}$,

$$\begin{aligned}
&s^{24}-352s^{21}q^3t^3+8074s^{18}q^6t^6-43252s^{15}q^9t^9+73789s^{12}q^{12}t^{12}-43252s^9q^{15}t^{15}\\
&\qquad+8074s^6q^{18}t^{18}-352s^3q^{21}t^{21}+q^{24}t^{24}\\
&\quad\equiv (s^3+q^3t^3)^8 \pmod 9.
\end{aligned}\tag{3.4}$$

Now, if we denote by $G(q)$ the sum of all the terms of the form q^{3k+1} in the expansion of $-36q\psi(q^3)^8/\psi(q)^{12}$,

$$G(q)\equiv -36q\frac{\psi(q^9)^{12}}{\psi(q^3)^{40}}(s^3+q^3t^3)^8\equiv -36q\frac{\psi(q^9)^4}{\psi(q^3)^8} \pmod{81}.\tag{3.5}$$

In the same way, since $\psi(q)^3\equiv\psi(q^3) \pmod 3$, by (2.1),

$$27q^2\frac{\psi(q^3)^{12}}{\psi(q)^{16}}\equiv 27q^2\frac{\psi(q^3)^7}{\psi(q)}=27q^2\psi(q^9)\psi(q^3)^3(s^2-qst+q^2t^2) \pmod{81}.$$

If we denote by $H(q)$ the sum of all the terms of the form q^{3k+1} in the expansion of $27q^2\psi(q^3)^{12}/\psi(q)^{16}$, then $H(q)\equiv 27q^4\psi(q^9)^3\psi(q^3)^3 \pmod{81}$. Together with (3.2),

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(3.5) and the first identity in this lemma, this gives

$$\sum_{n=0}^{\infty}(-1)^{3n+1}\mathrm{pod}_{-4}(3(3n+1)+2)q^{3n+1}$$
$$= F(q)+G(q)+H(q)$$
$$\equiv -35q\frac{\psi(q^9)^4}{\psi(q^3)^8} - 18q^4\frac{\psi(q^9)^8}{\psi(q^3)^{12}} + 27q^7\frac{\psi(q^9)^{12}}{\psi(q^3)^{16}} + 27q^4\psi(q^9)^3\psi(q^3)^3 \pmod{81}.$$

By dividing both sides by $-q$, replacing q^3 by q and using $\psi(q)^3 \equiv \psi(q^3) \pmod 3$, we obtain the second congruence identity. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. The first congruence follows from Lemma 3.2 since

$$\sum_{n=0}^{\infty}(-1)^n(\mathrm{pod}_{-4}(3n+2)+\mathrm{pod}_{-4}(9n+5))q^n$$
$$\equiv 9\Big(5\frac{\psi(q^3)^4}{\psi(q)^8} - 2q\frac{\psi(q^3)^8}{\psi(q)^{12}} - 3q\psi(q^3)^4\Big) \pmod{81}. \tag{3.6}$$

Moreover, since $\psi(q^3) \equiv \psi(q)^3 \pmod 3$,

$$\sum_{n=0}^{\infty}(-1)^n(\mathrm{pod}_{-4}(3n+2)+\mathrm{pod}_{-4}(9n+5))q^n$$
$$\equiv 18\Big(\frac{\psi(q^3)^4}{\psi(q)^8} - q\frac{\psi(q^3)^8}{\psi(q)^{12}}\Big) \equiv 18(\psi(q^3)\psi(q) - q\psi(q^3)^4) \pmod{27}.$$

Since $\psi(q) = A(q^3) + q\psi(q^9)$, the terms of the form q^{3k+2} vanish on the right-hand side of this identity to yield the third congruence:

$$\mathrm{pod}_{-4}(9n+8)+\mathrm{pod}_{-4}(27n+23) \equiv 0 \pmod{27}.$$

For the second congruence, first note that by (3.4), the sum of all the terms of the form q^{3k} in the expansion of $(s^2-qst+q^2t^2)^{12}$ is congruent to $(s^3+q^3t^3)^8$ modulo 9. If we denote by $I(q)$ the sum of all the terms of the form q^{3k+1} in the expansion of $-2q\psi(q^3)^{18}/\psi(q)^{12}$, then by (3.3),

$$I(q) \equiv -2q\Big(\frac{\psi(q^3)^4}{\psi(q^9)}\Big)^8\frac{\psi(q^9)^{12}}{\psi(q^3)^{40}} \equiv -2q\frac{\psi(q^9)^4}{\psi(q^3)^8} \pmod 9.$$

By (3.1), the sum of all the terms of the form q^{3k+1} in the expansion of $5\psi(q^3)^4/\psi(q)^8$ is congruent to $-4q\psi(q^9)^4/\psi(q^3)^8$ modulo 9. Hence, if we extract all the terms of the

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form q^{3k+1} on both sides of (3.6),

$$\sum_{n=0}^{\infty}(-1)^{3n+1}(\mathrm{pod}_{-4}(9n+5)+\mathrm{pod}_{-4}(27n+14))q^{3n+1}$$
$$\equiv 27\Big(-2q\frac{\psi(q^9)^4}{\psi(q^3)^8}-q\psi(q^3)^4\Big)\equiv -81\psi(q^3)^4 \pmod{81}.$$

The second congruence follows. □

REMARK 3.3. The modulus in Theorem 3.1 cannot be replaced by a higher power of 3 because

$$\mathrm{pod}_{-4}(5)+\mathrm{pod}_{-4}(2)=3^2\times 2\times 7,$$
$$\mathrm{pod}_{-4}(14)+\mathrm{pod}_{-4}(5)=3^4\times 2^8,$$
$$\mathrm{pod}_{-4}(23)+\mathrm{pod}_{-4}(8)=3^3\times 19\times 2027.$$

4. Congruences modulo 2, 5 and 8

THEOREM 4.1. *We have*

$$\sum_{n=0}^{\infty}\mathrm{pod}_{-4}(2n)q^n=\frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^{10}(q^4;q^4)_\infty^4},$$
$$\sum_{n=0}^{\infty}\mathrm{pod}_{-4}(2n+1)q^n=4\frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^6(q^2;q^2)_\infty^2}.$$

PROOF. By (1.1),

$$\sum_{n=0}^{\infty}\mathrm{pod}_{-4}(n)q^n=\frac{(-q;q^2)_\infty^4}{(q^2;q^2)_\infty^4}=\frac{(q^2;q^4)_\infty^4}{(q;q)_\infty^4}=\frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^4(q^4;q^4)_\infty^4}.$$

From [5, Corollary 2.4],

$$\frac{(q^2;q^2)_\infty^{14}}{(q;q)_\infty^4(q^4;q^4)_\infty^4}=\frac{(q^4;q^4)_\infty^{10}}{(q^8;q^8)_\infty^4}+4q\frac{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4}{(q^4;q^4)_\infty^2}.$$

Hence

$$\sum_{n=0}^{\infty}\mathrm{pod}_{-4}(n)q^n=\frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^{10}(q^8;q^8)_\infty^4}+4q\frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^6(q^4;q^4)_\infty^2},$$

from which the theorem follows. □

THEOREM 4.2. *We have the following results.*

(1) *For any integer* $n\ge 0$, $\mathrm{pod}_{-4}(4n+2)\equiv 0 \pmod 2$.
(2) *If* $n=k(k+1)$ *for some integer* k, *then* $\mathrm{pod}_{-4}(2n+1)\equiv 4 \pmod 8$. *Otherwise,* $\mathrm{pod}_{-4}(2n+1)\equiv 0 \pmod 8$.

Proof. (1) By Theorem 4.1 and Lemma 2.1,

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n)q^n = \frac{1}{(q;q)_\infty^{10}} \cdot \frac{(q^2;q^2)_\infty^{10}}{(q^4;q^4)_\infty^4} \equiv \frac{1}{(q^2;q^2)_\infty^3} \pmod 2.$$

Note that terms of the form q^{2n+1} do not appear on the right-hand side of the above equation and hence $\mathrm{pod}_{-4}(4n+2) \equiv 0 \pmod 2$.

(2) Again by Theorem 4.1 and Lemma 2.1,

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1)q^n = \frac{4}{(q;q)_\infty^6} \cdot \frac{(q^4;q^4)_\infty^4}{(q^2;q^2)_\infty^2} \equiv 4(q^2;q^2)_\infty^3 \pmod 8.$$

By Jacobi's identity [1, Theorem 1.3.4],

$$(q^2;q^2)_\infty^3 = \sum_{k=0}^{\infty} (-1)^k(2k+1)q^{k(k+1)} \equiv \sum_{k=0}^{\infty} q^{k(k+1)} \pmod 2.$$

Hence

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1)q^n \equiv 4\sum_{k=0}^{\infty} q^{k(k+1)} \pmod 8.$$

Comparing the coefficients of q^n on both sides completes the proof. □

Theorem 4.3. *Let $p \geq 3$ be a prime and $m \equiv -1 \pmod p$. If r is an integer such that $8r+1$ is a quadratic nonresidue modulo p, then*

$$\mathrm{pod}_{-m}(pn+r) \equiv 0 \pmod p.$$

Proof. By Lemma 2.1,

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-m}(n)(-q)^n = \frac{1}{\psi(q)^m} \equiv \left(\frac{1}{\psi(q^p)}\right)^{(m+1)/p} \psi(q) \pmod p.$$

Note that $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$ and the congruence $r \equiv n(n+1)/2 \pmod p$ is equivalent to $8r+1 \equiv (2n+1)^2 \pmod p$. If $8r+1$ is a quadratic nonresidue modulo p, then q^{pn+r} vanishes in the expansion of the right-hand side. This completes the proof. □

The particular case $p = 5$ and $m = 4$ gives the corollary.

Corollary 4.4. *For any integer $n \geq 0$,*

$$\mathrm{pod}_{-4}(5n+2) \equiv \mathrm{pod}_{-4}(5n+4) \equiv 0 \pmod 5.$$

Lemma 4.5. *We have*

$$\psi(q) = A(q^5) + qB(q^5) + q^3\psi(q^{25}),$$

where

$$A(q) = (-q^2, -q^3, q^5; q^5)_\infty, \quad B(q) = (-q, -q^4, q^5; q^5)_\infty.$$

https://doi.org/10.1017/S0004972715000647 Published online by Cambridge University Press

Proof. Since $\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$, and the residue of $n(n+1)/2$ modulo 5 can only be 0, 1 or 3, there is a dissection

$$\psi(q) = A(q^5) + qB(q^5) + q^3C(q^5).$$

Because $n(n+1)/2 \equiv 0 \pmod 5$ if and only if $n \equiv 0, 4 \pmod 5$, we have

$$A(q^5) = \sum_{n=0}^{\infty} q^{5n(5n+1)/2} + \sum_{n=0}^{\infty} q^{(5n+4)(5n+5)/2}.$$

Replacing q^5 by q, replacing the index n by $-n-1$ in the second summation, so that the summation over n runs from $-\infty$ to -1, and applying Jacobi's triple product identity,

$$A(q) = \sum_{n=-\infty}^{\infty} q^{(5/2)n^2+(1/2)n} = (-q^2, -q^3, q^5; q^5)_\infty.$$

Since $n(n+1)/2 \equiv 1 \pmod 5$ if and only if $n \equiv 1, 3 \pmod 5$, in a similar way we can show that $B(q) = (-q, -q^4, q^5; q^5)_\infty$. Finally, since $n(n+1)/2 \equiv 3 \pmod 5$ if and only if $n \equiv 2 \pmod 5$,

$$q^3C(q^5) = \sum_{n=0}^{\infty} q^{(5n+2)(5n+3)/2} = q^3\psi(q^{25}).$$

The proof of this lemma is now complete. □

Theorem 4.6. *For any integer $n \geq 0$,*

$$\mathrm{pod}_{-4}(5n+3) \equiv (-1)^{n+1}\sigma(2n+1) \pmod 5.$$

Moreover, $\mathrm{pod}_{-4}(5n+3) \equiv 0 \pmod 5$ *if and only if one of the following statements is true.*

(1) *There exists a prime* $p \equiv 1 \pmod 5$ *such that* $v_p(2n+1) \equiv 4 \pmod 5$.
(2) *There exists a prime* $p \equiv 2, 3$ *or* $4 \pmod 5$ *such that* $v_p(2n+1) \equiv 3 \pmod 4$.

Proof. By Lemma 2.1, we have $\psi(q)^5 \equiv \psi(q^5) \pmod 5$. By Lemma 4.5,

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(n) q^n = \frac{\psi(q)}{\psi(q)^5} \equiv \frac{1}{\psi(q^5)}(A(q^5) + qB(q^5) + q^3\psi(q^{25})) \pmod 5.$$

If we extract all the terms of the form q^{5n+3}, then divide by $-q^3$ and replace q^5 by q and apply $\psi(q)^5 \equiv \psi(q^5) \pmod 5$ again,

$$\sum_{n=0}^{\infty} (-1)^n \mathrm{pod}_{-4}(5n+3) q^n \equiv -\frac{\psi(q^5)}{\psi(q)} \equiv -\psi(q)^4 = -\sum_{n=0}^{\infty} t_4(n) q^n \pmod 5.$$

Since $t_4(n) = \sigma(2n+1)$, comparing the coefficients of q^n on both sides yields

$$\mathrm{pod}_{-4}(5n+3) \equiv (-1)^{n+1}\sigma(2n+1) \pmod 5.$$

Write the prime factorisation of $2n+1$ as $2n+1=\prod_{p|2n+1} p^{v_p(2n+1)}$. Then

$$\sigma(2n+1)=\prod_{p|2n+1}(1+p+\cdots+p^{v_p(2n+1)}).$$

Let p be any prime factor of $2n+1$. For $p=5$, $1+p+\cdots+p^{v_p(2n+1)}\equiv 1 \pmod 5$. If $p\equiv 1 \pmod 5$, $1+p+\cdots+p^{v_p(2n+1)}\equiv 1+v_p(2n+1) \pmod 5$ and $5 \mid 1+p+\cdots+p^{v_p(2n+1)}$ if and only if $v_p(2n+1)\equiv 4 \pmod 5$. If $p\equiv 2, 3$ or $4 \pmod 5$, then

$$1+p+\cdots+p^{v_p(2n+1)}=\frac{p^{v_p(2n+1)+1}-1}{p-1} \pmod 5$$

and $5 \mid 1+p+\cdots+p^{v_p(2n+1)}$ if and only if $p^{v_p(2n+1)+1}\equiv 1 \pmod 5$, which is also equivalent to $v_p(2n+1)\equiv 3 \pmod 4$. □

Acknowledgement

The author would like to thank the referee for his/her helpful comments.

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