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RESEARCH ARTICLE

Options pricing with Markov regime switching Heston volatility Hull–White interest rates and stochastic intensity

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Abstract
This paper proposes an options pricing model that incorporates stochastic volatility, stochastic interest rates, and stochastic jump intensity. Market shocks are modeled using a jump process, with each jump governed by an asymmetric double-exponential distribution. The model also integrates a Markov regime-switching framework for volatility and the risk-free rate, allowing the market to alternate between a finite number of distinct economic states. A closed-form solution for European option pricing is derived. To demonstrate the significance of the proposed model, a comparison with various other models is performed, and the sensitivity of the various model parameters is illustrated.

1. Introduction

A derivative contract is a financial instrument whose value is determined by the performance of an underlying asset. Options are a specific type of derivative that grants the holder the right, but not the obligation, to purchase or sell a fixed quantity of the underlying asset. The underlying asset may be bought (a call option) or sold (a put option) at a strike price and within a specified time (expiry time).

While the strike price and expiry time are predetermined in vanilla options, exotic options may be based on multiple underlying assets, the strike price may depend on several market factors, and numerous exercise dates may be possible. The two most prominent options are European and American, each with distinct exercise time conditions. There are many more options, including Asian and Bermudan options.

European options can only be exercised when the options contract expires, which is predetermined when the contract is issued. On the other hand, American options can be exercised on any trading day before the options contract expires. Options pricing has been a significant topic of theoretical exploration and practical implementation.

Fischer Black, Myron Scholes, and Robert Merton made a significant contribution to this field. Black and Scholes [1] proposed a model to price the European options with the assumption that the underlying asset follows the geometric Brownian motion. The analytical formula is derived for European option prices under this model. Despite its widespread use, the simplified assumptions made for analytical simplicity and tractability may not fully capture real-world complexities, potentially leading to some degree of mispricing.

Numerous empirical studies have shown that volatility clustering is a distinctive feature of the equity return rates [8]. The distribution of equity return rates exhibits high peaks and fat tails [2]. A negative correlation exists between share prices and volatility [6]. Lastly, mean reversion in return rates is observed [7]. The Black–Scholes model does not account for these additional characteristics.

Two types of nonconstant volatility models are widely studied in the literature to relax the assumption of constant volatility: local volatility and stochastic volatility. In the local volatility model, volatility is considered a deterministic function of the underlying asset price and time. In the stochastic volatility models, volatility is considered a different stochastic process, allowing for dynamic changes. Numerous authors have addressed the options pricing problem under stochastic volatility models. However, finding a closed-form pricing formula for European options becomes complicated after introducing an additional stochastic process, and often numerical methods are implemented for existing models, such as Hull and White [23].

Heston [22] proposed a stochastic volatility model where the volatility process follows a square root mean-reverting process, ensuring non-negativity. The Heston model accurately captures the asset dynamics, assuming a constant interest rate and the absence of sudden market shocks. It demonstrates the implied volatility smile effect, a well-known inconsistency in the Black–Scholes model. Recently, He and Chen [15] introduced a novel stochastic volatility model by incorporating a stochastic process in the long-term mean of volatility within the Heston framework. Lin and He [26] proposed an additional factor in the stochastic volatility process, which follows regime-switching. Pasricha and Goel [34] considered the Heston model for pricing exchange options. He and Lin [18] considered the long-term mean of the volatility process to follow a regime-switching stochastic process.

The assumption of a constant interest rate is not realistic, as the real market experiences fluctuations in interest rates. Therefore, the stochastic behavior of interest rates is important in modeling asset prices. To address this, various short-rate models have been introduced in the literature to accurately capture the term structure of interest rates. Vasicek [37], Cox *et al.* [3], and Hull and White [24] are some examples.

Hamilton [13] played a pivotal role in advancing the regime-switching models by proposing a class of discrete-time, Markov regime-switching, and autoregressive time-series models. Empirically, Markov regime-switching models can describe various essential characteristics of economic and financial time series, such as the asymmetry and heavy-tailedness of asset returns, time-varying conditional volatility, volatility clustering, regime-switching, nonlinearity, and other complex features.

Markov regime-switching models offer a framework for identifying variations in economic conditions. Elliott and Nishide [4] derived a closed-form expression for bond prices by considering that the short rate follows the Cox–Ingersoll–Ross (CIR) model with a Markov regime-switching. Elliott *et al.* [5] extended their work by incorporating regime-switching into the Heston stochastic volatility model and derived the price of European options. Shen and Siu [36] derived bond options price under the Markov regime-switching Hull–White model.

To account for the fluctuations caused by external factors, Merton [32] introduced the jumps in the diffusion model, and the options price is derived. Additionally, Kou [25] studied the jump-diffusion model, particularly when the jump size follows an asymmetric double exponential distribution. For further applications of the jump process in modeling stock price process, refer, Wang *et al.* [38], Han [14], Guo and Bai [12], and Lin *et al.* [29].

Grzelak *et al.* [10] considered a hybrid stochastic volatility and stochastic interest rate model for pricing European options. Guo [11] investigated the pricing of the European options using the Heston–Vasicek model. The hybrid stochastic rate and stochastic volatility model are considered to price various financial instruments (see, e.g., Grzelak and Oosterlee [9], Wu *et al.* [39]). lyu *et al.* [30] evaluated option prices within the Markov regime-switching double Heston stochastic volatility model. The model included a stochastic interest rate and Poisson jumps. Shan *et al.* [35] determined options price when the Markov-modulated Merton jump-diffusion process describes the discrete dividends. Ma *et al.* [31] priced the European options by incorporating the Heston volatility within a Markov regime-switching framework. The model considered the jump process to follow a Poisson process with stochastic intensity. The CIR model is used to model the interest rate, and the size of the jumps follows a double exponential distribution.

Lin and He [27] considered the CIR volatility process having stochastic equilibrium levels with regime-switching. He and Lin [17] extended the stochastic volatility model [15] by considering regime-switching stochastic volatility. He *et al.* [21] considered the regime-switching volatility and stochastic

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market liquidity risk for pricing European options. Lin and He [28] considered the CIR stochastic volatility with market liquidity risks for pricing variance and volatility swaps. He and Lin [20] priced the European options considering Heston stochastic volatility and market liquidity risks. He and Lin [19] proposed a stochastic interest rate model with long-run mean as another stochastic process. He and Lin [16] considered the correlation between asset price and risk-free rate using two-factor hybrid Heston Hull–White model and derived the European options price.

While the CIR model ensures positive interest rates, it does not align well with the negative interest rates seen during the 2007 financial crisis and subsequent quantitative easing policies. For more details on the significance of negative interest rates and drawbacks of the CIR model during low interest rates, refer Orlando and Bufalo [33]. Therefore, the market requires a model that can deal with negative interest rates. Vasicek model is one such model, and the Hull–White model generalizes it, giving a more general framework for interest rate dynamics. Hence, instead of the CIR model, the Hull–White model is considered.

In this article, the pricing of European options within the framework of a Heston stochastic volatility integrated with a Hull–White stochastic interest rate model is discussed. The long-run mean of the volatility process, the mean reversion level, and the volatility of the risk-free rate alternate between a finite number of regimes. These regimes correspond to the finite state space of a continuous-time Markov chain. The jump occurs based on a Poisson process with stochastic intensity, and the jump sizes follow an asymmetric double exponential distribution. To demonstrate the accuracy of the derived formula, the prices are computed numerically and compared with Monte Carlo simulation (MCS) results. The comparison of this model to the previous models and the sensitivity of price to various parameters are also demonstrated.

The paper is structured as follows: Section 2 explains the proposed model. Section 3 presents the derivation of the characteristic function of this model without considering regime-switching and then considering regime-switching. Further, the derivation of the pricing formula for the European options is presented. Section 4 consists of numerical illustrations. The final section makes concluding remarks with possible extensions of the model.

2. Model description

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\in[0,T]}, P)$ which models the uncertainty in the economy. In the absence of arbitrage, an equivalent martingale or risk-neutral measure $\mathcal{Q}$ exists. In this model, at time t, S_t represents the price of the underlying asset, v_t is the volatility of the price, r_t is the risk-free rate, N_t is the number of jumps up to and including time t, and J_i is the size of the ith jump. Let κ denotes the speed of mean reversion of volatility. The long-run mean of volatility is denoted by θ_t, and the volatility of the volatility is represented by η_v. Assume that α represents the rate of mean reversion, σ_t symbolizes the volatility, and β_t denotes the long-run mean of r_t . The jump arrival process $\{N_t, t \in [0,T]\}$ is a Poisson process with intensity process $\{\lambda_t, t \in [0,T]\}$. Then, under the risk-neutral measure $\mathcal{Q}$, the asset price, the volatility, the risk-free rate, and the intensity processes are governed by the following stochastic differential equation:

$$dS_t = (r_t - \lambda_t m)\, S_t dt + \sqrt{v_t} S_t dB_{1,t} + \left(e^J - 1\right) S_{t-} dN_t, \tag{1}$$

$$dv_t = \kappa\,(\theta_t - v_t)\, dt + \eta_v \sqrt{v_t} dB_{2,t}, \tag{2}$$

$$dr_t = \alpha\,(\beta_t - r_t)\, dt + \sigma_t dB_{3,t}, \tag{3}$$

$$d\lambda_t = \kappa_\lambda\,(\theta_\lambda - \lambda_t)\, dt + \eta_\lambda \sqrt{\lambda_t} dB_{4,t}, \tag{4}$$

where κ_λ represents the rate of mean reversion, θ_λ is the long-run mean, and η_λ is the volatility of the intensity. Here $\{B_{1,t}, t \in [0,T]\}$, $\{B_{2,t}, t \in [0,T]\}$, $\{B_{3,t}, t \in [0,T]\}$, and $\{B_{4,t}, t \in [0,T]\}$ are

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standard Weiner processes. Furthermore, $dB_{1,t}dB_{2,t} = \rho dt$ and all other processes are independent. The probability density function of the jump size process, $J = \{J_i, i = 1, 2, \ldots, N_t\}$ is defined as:

$$g(x) = \frac{pe^{\frac{-x}{\eta_1}}}{\eta_1} 1_{\{x \geq 0\}} + \frac{qe^{\frac{x}{\eta_2}}}{\eta_2} 1_{\{x<0\}}, \tag{5}$$

where $0 < \eta_1 < 1, \eta_2 > 0$ and $p + q = 1$. Let $m = \mathbb{E}_{\mathcal{Q}}[e^{J_1} - 1]$ then

$$m = \frac{p}{1-\eta_1} + \frac{q}{1+\eta_2} - 1.$$

Let θ_t, β_t, and σ_t take values $\{\theta_1, \theta_2, \ldots, \theta_n\}$, $\{\beta_1, \beta_2, \ldots, \beta_n\}$, and $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, respectively, according to a continuous time Markov chain $y = \{y_t : t \geq 0\}$ with n states whose unique stationary distribution exists. The states of y can be interpreted as different states of the economy or different phases of a business cycle. The business cycle alternates between phases of relatively fast growth in economic activity and phases of decrease in output. Without loss of generality, following [36], assume that the state space of y is identified with a set of unit vectors $\{e_1, e_2, \ldots, e_n\}$ where e_i is a unit vector in $\mathbb{R}^n$ with jth component as Kronecker delta δ_{ij}. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^n$ where $\langle a, b \rangle = a^T b$. For $\hat{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)^T$, $\hat{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)^T$, and $\hat{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)^T$, then

$$\theta_t = \langle \hat{\theta}, y_t \rangle, \quad \beta_t = \langle \hat{\beta}, y_t \rangle, \text{ and } \sigma_t = \langle \hat{\sigma}, y_t \rangle.$$

Let $\mathbb{Q} = [q_{ij}]$ be the transition rate matrix of Markov chain y where q_{ij} is the transitional intensity of the chain from regime i to regime j or from state e_i to e_j. Then,

$$dy = \mathbb{Q}^T y dt + dM, \tag{6}$$

where M is a vector martingale process. Let y_t be independent of $B_{1,t}, B_{2,t}, B_{3,t}, B_{4,t}$, and N_t. Assume the initial values $S_0 > 0$, $v_0 > 0$, $r_0 > 0$, and $\lambda_0 > 0$ are given.

Under the risk-neutral measure, the price of the European options is given by:

$$C(0) := \mathbb{E}_{\mathcal{Q}} \left[e^{-\int_0^T r(s)ds} (S_T - K)^+ \middle| \mathcal{F}_0 \right].$$

To solve this expression, a new measure is introduced. Define the forward measure $\tilde{\mathcal{Q}}$ by

$$\frac{d\tilde{\mathcal{Q}}}{d\mathcal{Q}} = \frac{e^{-\int_0^T r_u du}}{\mathbb{E}_{\mathcal{Q}} \left[e^{-\int_0^T r_u du} \right]}. \tag{7}$$

Let $B(t, T)$ be the price of a bond at time t maturing at time T. Then, $B(t, T) = \mathbb{E}_{\mathcal{Q}} \left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right]$. According to Section 3 of [36], the value $B(0, T)$ is given by

$$B(0, T) = R(T, y_0) e^{-P(T) r_0}, \tag{8}$$

where $P(T)$ and $R(T, y_0)$ are given by

$$P(T) = \frac{1}{\alpha} [1 - e^{-\alpha T}], \tag{9}$$

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$$R(T, y_0) = \mathbb{E}_{\mathcal{Q}}\left[\exp\left\{\int_0^T \left(\alpha\beta_s P(s) - \frac{1}{2}\sigma_s^2 P^2(s)\right) ds\right\}\middle| y_0\right]. \tag{10}$$

Let $\tilde{R}(t) = \exp\left\{\int_0^t \langle\alpha\hat{\beta}P(u) - \frac{1}{2}\hat{\sigma}^2 P^2 P(u), y_u\rangle du\right\}$ and $\tilde{\mathbf{R}}(t) = \tilde{R}(t)y_t$. Then,

$$\begin{aligned}
d\tilde{\mathbf{R}}(t) &= \tilde{R}(t)dy_t + d\tilde{R}(t)y_t,\\
&= \tilde{R}(t)\left(\mathbb{Q}^T y_t dt + dM_t\right) + \left(\langle\alpha\hat{\beta}P(t) - \frac{1}{2}\hat{\sigma}^2 P^2(t), y_t\rangle\right)\tilde{R}(t)y_t dt,\\
&= \left(\mathbb{Q}^T + \langle\alpha\hat{\beta}P(t) - \frac{1}{2}\hat{\sigma}^2 P^2(t), y_t\rangle\right)\tilde{R}(t)y_t dt + \tilde{R}(t)dM_t,\\
&= \left(\mathbb{Q}^T + \text{diag}\left[\alpha\hat{\beta}P(t) - \frac{1}{2}\hat{\sigma}^2 P^2(t)\right]\right)\tilde{R}(t)y_t dt + \tilde{R}(t)dM_t.
\end{aligned}$$

Let $\hat{\mathbf{R}}(t) = \mathbb{E}_{\mathcal{Q}}[\tilde{\mathbf{R}}(t)|y_0]$ then $\hat{\mathbf{R}}(t)$ will satisfy,

$$\frac{d\hat{\mathbf{R}}(t)}{dt} = \left(\mathbb{Q}^T + \text{ diag}\left[\alpha\hat{\beta}P(s) - \frac{1}{2}\hat{\sigma}^2 P^2(s)\right]\right)\hat{\mathbf{R}}(t). \tag{11}$$

Following the similar method mentioned in [4] gives

$$\mathbb{E}_{\mathcal{Q}}[\tilde{R}(t)|y_0] = \langle\hat{\mathbf{R}}(t), 1_n\rangle = \langle e^U y_0, 1_n\rangle,$$

where $1_n = \{1, 1, \ldots, 1\}^T$ and

$$U = \int_0^t \left[\mathbb{Q}^T + \text{ diag}\left(\alpha\hat{\beta}P(s) - \frac{1}{2}\hat{\sigma}^2 P^2(s)\right)\right] ds. \tag{12}$$

The price of the call options under the forward measure $\tilde{\mathcal{Q}}$ will be

$$C(0) = B(0, T)\left(\mathbb{E}_{\tilde{\mathcal{Q}}}\left[(S_T - K)^+ \middle| \mathcal{F}_0\right]\right). \tag{13}$$

The dynamics of X_t, v_t, and λ_t remain same under the forward measure $\tilde{\mathcal{Q}}$ as in the risk neutral measure $\mathcal{Q}$ whereas the dynamics of r_t under $\tilde{\mathcal{Q}}$ will be given by

$$\begin{aligned}
dr_t &= \left(\alpha(\beta_t - r_t) - \sigma_t^2 P(t, T)\right) dt + \sigma_t d\tilde{B}_{3,t},\\
d\tilde{B}_{3,t} &= dB_{3,t} + \sigma_t P(t, T)dt,
\end{aligned}$$

where $d\tilde{B}_{3,t}$ is Brownian motion under measure $\tilde{\mathcal{Q}}$. Additionally, assume that the rate matrix of y under measure $\tilde{\mathcal{Q}}$ is given by $\tilde{\mathbb{Q}} = [\tilde{q_{ij}}]$ then

$$\tilde{q_{ij}} = \begin{cases} q_{ij}\frac{\exp(R(0,T,e_j))}{\exp(R(0,T,e_i))} & i \neq j \\ -\sum_{k\neq i} q_{ki}\frac{\exp(R(0,T,e_k))}{\exp(R(0,T,e_i))} & i = j \end{cases}.$$

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3. Options pricing

This section develops a pricing formula for the European options within the framework outlined in Section 2. Proposition 3.1 provides the characteristic function of the log process of asset price without considering the regime-switching of the volatility process. Let $X_t = \log S_t$, then

$$dX_t = \left(r_t - \lambda_t m - \frac{1}{2}v_t\right) dt + \sqrt{v_t} dB_{1,t} + d\sum_{k=1}^{N_t} J_k. \tag{14}$$

Proposition 3.1. *The conditional characteristic function $\varphi(\delta, \tau, x, v, r, \lambda)$ of X_T under the forward measure $\tilde{Q}$ is given by*

$$\varphi(\delta, \tau, x, v, r, \lambda) = \exp\{A(\delta, \tau) + B(\delta, \tau)v + C(\delta, \tau)\lambda + D(\delta, \tau)r + i\delta x\}, \tag{15}$$

where

$$\begin{aligned}
D(\delta, \tau) &= \frac{i\delta}{\alpha}(1 - e^{-\alpha\tau}),\\
B(\delta, \tau) &= \frac{2}{\eta_v^2}\left[b(\delta) + d(\delta)\left(\frac{d(\delta)\sinh(d(\delta)\tau) - b(\delta)\cosh(d(\delta)\tau)}{d(\delta)\cosh(d(\delta)\tau) - b(\delta)\sinh(d(\delta)\tau)}\right)\right],\\
C(\delta, \tau) &= \frac{2}{\eta_\lambda^2}\left[-\kappa_\lambda + d'(\delta)\left(\frac{d'(\delta)\sinh(d'(\delta)\tau) - \kappa_\lambda\cosh(d'(\delta)\tau)}{d'(\delta)\cosh(d'(\delta)\tau) - \kappa_\lambda\sinh(d'(\delta)\tau)}\right)\right],\\
A(\delta, \tau) &= \int_0^\tau \beta_u D(\delta, u)du + \kappa\int_0^\tau \theta_u B(\delta, u)du - \int_0^\tau \sigma_u^2 P(\delta, u)D(\delta, u)du\\
&\quad + \frac{1}{2}\int_0^\tau \sigma_u^2 D^2(\delta, u)du + \frac{2\kappa_\lambda\theta_\lambda}{\eta_\lambda^2}\left[-\kappa_\lambda\tau + \ln\left(d'(\delta)\cosh(d'(\delta)\tau) - \kappa_\lambda\sinh(d'(\delta)\tau)\right)\right],
\end{aligned}$$

and

$$b(\delta) = \frac{(i\rho\eta_v\delta - \kappa)}{2}, d(\delta) = \frac{\sqrt{b(\delta)^2 + \eta_v^2\delta(i + \delta)}}{2}, d'(\delta) = \frac{\sqrt{\kappa_\lambda^2 + 2\eta_\lambda^2(mi\delta - \Lambda(\delta))}}{2}.$$

Proof. Let $\tau = T - t$. The conditional characteristic function of X_T by definition is $\varphi(\delta, \tau, x, v, r, \lambda) = \mathbb{E}_{\tilde{Q}}[e^{i\delta X_T}|v_t = v, r_t = r, X_t = x, \lambda_t = \lambda, y_T]$. Using Feynman–Kac theorem, $\varphi(\delta, \tau, x, v, r, \lambda)$ will satisfy the following partial differential equation (PDE),

$$\begin{aligned}
\frac{\partial\varphi}{\partial\tau} &= \left(r - \frac{1}{2}v - \lambda m\right)\frac{\partial\varphi}{\partial x} + \frac{1}{2}v_t\frac{\partial^2\varphi}{\partial x^2} + \left(\alpha(\beta - r) - \sigma^2 P(t, T)\right)\frac{\partial\varphi}{\partial r}\\
&\quad + \frac{1}{2}\sigma^2\frac{\partial^2\varphi}{\partial r^2} + \kappa(\theta_t - v)\frac{\partial\varphi}{\partial v} + \frac{1}{2}\eta_v^2 v\frac{\partial^2\varphi}{\partial v^2} + \rho\eta_v v\frac{\partial^2\varphi}{\partial X\partial v} + \kappa_\lambda(\theta_\lambda - \lambda)\frac{\partial\varphi}{\partial\lambda}\\
&\quad + \frac{1}{2}\eta_\lambda^2\lambda\frac{\partial^2\varphi}{\partial\lambda^2} + \lambda\int_{-\infty}^{+\infty}\left[\varphi(\delta, \tau, x + j, v, r, \lambda) - \varphi(\delta, \tau, x, v, r, \lambda)\right]g(j)dj.
\end{aligned}$$

Let $\Lambda(\delta) = \frac{p}{1 - i\delta\eta_1} + \frac{q}{1 + i\delta\eta_2} - 1$, then

$$\lambda\int_{-\infty}^{\infty}\left[\varphi(\delta, \tau, x + j, v, r, \lambda) - \varphi(\delta, \tau, x, v, r, \lambda)\right]g(j)dj = \Lambda(\delta)\varphi(\delta; \tau, x, v, r, \lambda). \tag{16}$$

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Assume that the characteristic function $\varphi(\delta, \tau, x, \nu, r, \lambda)$ has the following affine form:

$$\varphi(\delta, \tau, x, \nu, r, \lambda) = e^{A(\delta,\tau)+B(\delta,\tau)\nu+C(\delta,\tau)\lambda+D(\delta,\tau)r+i\delta x}. \quad (17)$$

Substituting $\varphi(\cdot)$ in the above PDE, the following set of ordinary differential equations is obtained with the boundary conditions $A(\delta, 0) = B(\delta, 0) = C(\delta, 0) = D(\delta, 0) = 0$

$$\begin{aligned}
\frac{dB}{d\tau} &= \frac{1}{2}\eta_\nu^2 B^2 + (i\rho\eta_\nu\delta - \kappa) B - \frac{i}{2}\delta - \frac{1}{2}\delta^2,\\
\frac{dC}{d\tau} &= \frac{1}{2}\eta_\lambda^2 C^2 - \kappa_\lambda C - mi\delta + \Lambda(\delta),\\
\frac{dD}{d\tau} &= -\alpha D + i\delta,\\
\frac{dA}{d\tau} &= \beta\alpha D - \sigma^2 PD + \frac{1}{2}\sigma^2 D^2 + \kappa\theta B + \kappa_\lambda\theta_\lambda C.
\end{aligned}$$

Solving the above ordinary differential equations (ODEs) will give the required values of $A(\delta, \tau)$, $B(\delta, \tau)$, $C(\delta, \tau)$, and $D(\delta, \tau)$. □

Now, to calculate the characteristic function of X_T with the regime-switching of the long-run mean of the volatility, mean reversion level of risk-free rate and volatility of risk-free rate, a method similar to solve Eq. (10) is employed.

Let $\psi(\delta, u) = (\psi_1(\delta, u), \psi_2(\delta, u), \ldots, \psi_n(\delta, u))^T$ with

$$\psi_k(\delta, u) = \left(\kappa\theta_k B(\delta, u) + \beta_k\alpha D(\delta, u) - \sigma_k^2 P(u, T) D(\delta, u) + \frac{1}{2}\sigma_k^2 D^2(\delta, u)\right).$$

The characteristic function of X_T considering regime-switching is

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathcal{Q}}}\left[e^{i\delta X_T}\middle|\mathcal{F}_t\right] &= \mathbb{E}_{\tilde{\mathcal{Q}}}\left[\mathbb{E}_{\tilde{\mathcal{Q}}}\left[e^{i\delta X_T}\middle|v_t = \nu, r_t = r, X_t = x, \lambda_t = \lambda, y_T\right]\middle|y_t\right]\\
&= \exp\{A_1(\delta, \tau) + B(\delta, \tau)\nu + C(\delta, \tau)\lambda + D(\delta, \tau)r + i\delta x\}\\
&\quad \times \mathbb{E}_{\tilde{\mathcal{Q}}}\left[\exp\{\int_0^\tau \langle\psi(\delta, u), y_u\rangle du\}\middle|y_t\right],
\end{aligned}$$

where $A(\delta, \tau) = A_1(\delta, \tau) + \int_0^\tau \langle\psi(\delta, u), y_u\rangle du$. Then,

$$\mathbb{E}_{\tilde{\mathcal{Q}}}\left[\exp\left\{\int_0^\tau \langle\psi(\delta, u), y_u\rangle du\right\}\middle|y_t\right] = \langle\Psi(\delta, 0, \tau)y_t, 1_n\rangle,$$

where $1_n = \{1, 1, \ldots, 1\}^T$ and Ψ is a $n \times n$ matrix that satisfies the following differential equation

$$\frac{d\Psi(\delta, u, \tau)}{d\tau} = \left(\tilde{\mathbb{Q}}^T + \text{diag}(\psi(\delta, \tau))\right)\Psi(\delta, u, \tau),$$

with $\Psi(\delta, u, u) = I$ for $\tau \geq u^3$. Hence, the characteristic function of X_T considering regime-switching is

$$\varphi(\delta, \tau) = \exp\{A_1(\delta, \tau) + B(\delta, \tau)\nu + C(\delta, \tau)\lambda + D(\delta, \tau)r + i\delta x\}\langle\Psi(\delta, t, \tau)y_t, 1_n\rangle. \quad (18)$$

Table 1. *The value of parameters.*

Parameters	Values	Parameters	Values	Parameters	Values
S_0	100	r_0	0.02	ν_0	0.02
ρ	–0.5	α	0.1	κ	5
λ_0	0.3	σ_1	0.04	η_ν	0.1
κ_λ	6.5	σ_2	0.06	θ_1	0.015
θ_λ	0.9	β_1	0.025	θ_2	0.07
η_λ	0.65	β_2	0.005	p	0.5
η_1	0.012	q_{01}	0.5	q	0.5
η_2	0.1	q_{10}	0.45		

The price of the call options will be

$$\begin{aligned} C(0) = {} & B(0,T)\left(\mathbb{E}_{\tilde{Q}}\left[(S_T-K)^+\Big|\mathcal{F}_0\right]\right) \\ & = B(0,T)\left(\mathbb{E}_{\tilde{Q}}\left[S_T 1_{\{X_T\geq k\}}\right] - e^k\mathbb{E}_{\tilde{Q}}\left[1_{\{X_T\geq k\}}\right]\right) \\ & = B(0,T)\left(\mathbb{E}_{\tilde{Q}}\left[S_T 1_{\{X_T\geq k\}}\right] - e^k\tilde{P}\left(X_T\geq k\right)\right). \end{aligned}$$

Let $f(\cdot)$ be probability density function of X_T under measure $\tilde{Q}$, then

$$\begin{aligned} \tilde{P}(X_T\geq k) = \int_k^\infty f(y)dy & = \int_k^\infty\left(\int_{-\infty}^\infty \frac{1}{2\pi}e^{-i\delta y}\varphi(\delta,\tau)d\delta\right)dy \\ & = \frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-i\delta k}\varphi(\delta,\tau)}{i\delta}d\delta \\ & = \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\delta k}\varphi(\delta,\tau)}{i\delta}\right)d\delta, \end{aligned}$$

where $i=\sqrt{-1}$. Consider another function $f_1(y)$ defined by $f_1(y)=\frac{e^y f(y)}{\varphi(-i,\tau)}$, where $\varphi(-i,\tau)=\int_{-\infty}^\infty e^y f(y)dy$. It can be verified that it is a probability density function for some random variable. The characteristic function corresponding to this random variable will be $\varphi_1(u,\tau)=\frac{\varphi(u-i,\tau)}{\varphi(-i,\tau)}$. Now consider,

$$\begin{aligned} \mathbb{E}_{\tilde{Q}}\left[S_T 1_{\{X_T\geq k\}}\right] = \int_k^\infty e^y f(y)dy & = \varphi(-i,\tau)\left(\int_k^\infty f_1(y)dy\right) \\ & = \varphi(-i,\tau)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\delta k}\varphi_1(\delta,\tau)}{i\delta}\right)d\delta\right) \\ & = \varphi(-i,\tau)\left(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\delta k}\varphi(\delta-i,\tau)}{i\delta\varphi(-i,\tau)}\right)d\delta\right) \\ & = \frac{\varphi(-i,\tau)}{2}+\left(\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\delta k}\varphi(\delta-i,\tau)}{i\delta}\right)d\delta\right). \end{aligned}$$

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Table 2. *European call options price calculated through proposed model and Monte Carlo simulation.*

K	MCS	MP	RE (%)	K	MCS	MP	RE (%)
85	19.6387	19.7158	0.391	101	10.0636	10.0843	0.206
86	18.9224	18.9952	0.383	102	9.6030	9.6212	0.189
87	18.2207	18.2894	0.376	103	9.1586	9.1743	0.17
88	17.5334	17.5987	0.371	104	8.7303	8.7433	0.148
89	16.8614	16.9235	0.367	105	8.3174	8.3280	0.127
90	16.2055	16.2639	0.359	106	7.9196	7.9282	0.108
91	15.5660	15.6202	0.347	107	7.5372	7.5437	0.086
92	14.9423	14.9926	0.335	108	7.1700	7.1742	0.058
93	14.3346	14.3812	0.324	109	6.8177	6.8193	0.023
94	13.7434	13.7862	0.311	110	6.4794	6.4789	0.009
95	13.1684	13.2077	0.298	111	6.1548	6.1524	0.039
96	12.6097	12.6458	0.285	112	5.8435	5.8398	0.064
97	12.0675	12.1004	0.272	113	5.5453	5.5404	0.088
98	11.5419	11.5716	0.257	114	5.2598	5.2541	0.108
99	11.0329	11.0594	0.24	115	4.9868	4.9804	0.128
100	10.5402	10.5637	0.222				

Abbreviations: MCS, Monte Carlo simulation; MP, model price; RE, relative error.

4. Numerical and sensitivity analysis

This section compares the price of options obtained using the derived formula with those generated through MCSs. Additionally, to highlight the importance of incorporating regime-switching and stochastic intensity in options pricing, a comparison is presented. The sensitivity of various parameters to options price is also exhibited. Unless otherwise specified, the values of the selected parameters are provided in Table 1. It is assumed that there are two regimes: one representing the state of economic boom (Regime 1) and the other economic crisis (Regime 2). In times of economic growth, the financial markets are driven by high interest rates and low volatility. Conversely, economic recessions are marked by low interest rates and high volatility. This section is divided into three subsections: Subsection 4.1 presents the numerical calculations, Subsection 4.2 provides the comparisons with other models, and Subsection 4.3 examines the sensitivity of option prices to various parameters across different strike prices.

4.1. Numerical calculation

To validate the accuracy of the derived formula, a comparison with MCSs is presented in Table 2 for option prices at various strike prices K, assuming that the economy is initially in regime 2. To calculate the options price from the MCS, $100,000$ sample paths are generated, each consisting of 500 time steps. It can be observed that the prices obtained from the derived formula (MP) are point-wise close to the corresponding prices from the MCS. The relative error between the two prices is less than 0.4% which verifies the accuracy of the formula. Figure 1 illustrates the option price when the economy is in regime 1 (economic growth) and regime 2 (economic recession). It is observed that the price of the call options increases as the strike price decreases. Additionally, Figure 1 shows that option prices are higher during an economic recession than during economic growth. This can be explained by the fact that option prices tend to increase when the underlying asset's volatility rises and the asset's returns are more volatile during a recession than economic growth.

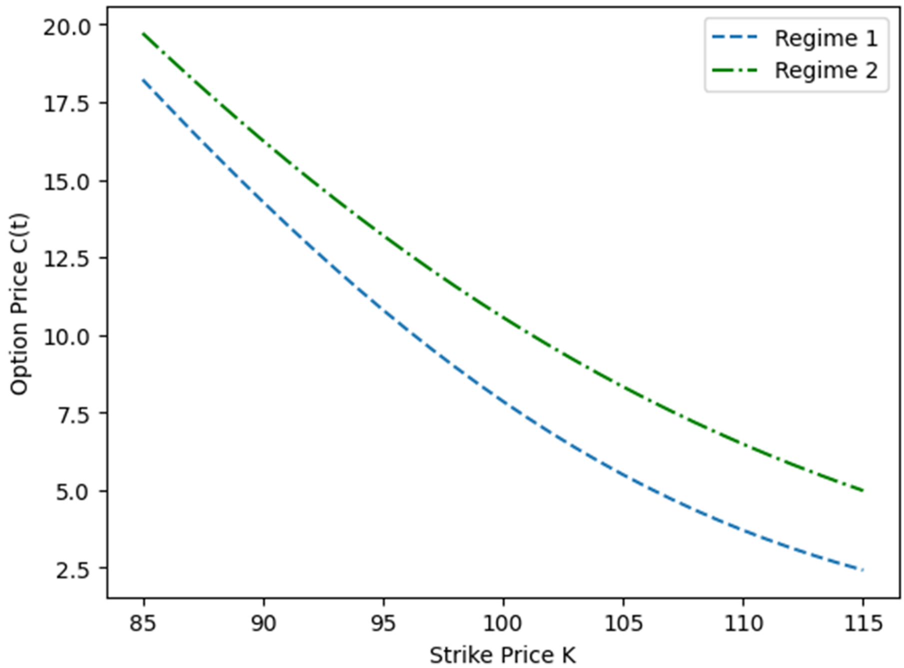

Figure 1. *Comparison of option prices against strike prices in regime 1 and regime 2.*

4.2. Model comparison

In this section, a comparative analysis of the proposed models with several models is conducted. The primary objective of this comparison is to demonstrate the significance of the proposed model.

4.2.1. Comparison with the Heston-CIR model with stochastic intensity

Figure 2 illustrates the comparison between the options price derived from the proposed model and those derived from the model proposed by [31]. It is assumed that initially the economy is in regime 1. The graphical comparison demonstrates the differences and similarities in the options price for a range of strike prices K. Both models exhibit a decrease in options price as the strike price increases; however, the option prices in the model proposed in this article are higher.

4.2.2. Comparison with hybrid Heston–Hull–White model with stochastic intensity

The comparison of the European call options price from the proposed model against those derived from the hybrid Heston–Hull–White model with stochastic intensity and without regime-switching is illustrated in Figure 3. If the economy is in regime 1, the markets are driven by high interest rates and low volatility, which results in lower risk, and hence option prices are less. In regime 2, the option prices are higher as compared to regime 1 as in regime 2 market has lower interest rates and higher volatility. Figure 3 illustrates that for regime 1, the option prices calculated by the regime-switching model are higher compared to the model without regime-switching, whereas for regime 2, the option prices are lower for the regime-switching model than without regime-switching model. This is because, in the regime-switching model, there is a positive probability that the economy will shift into another regime leading to difference in prices.

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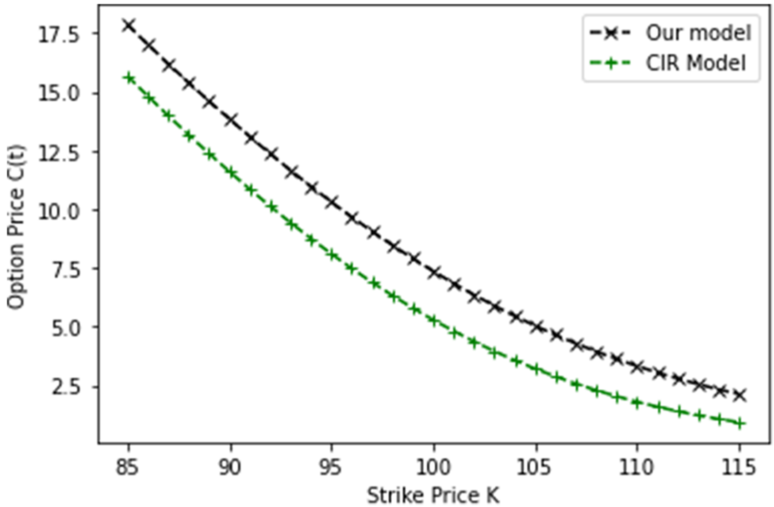

Figure 2. *Comparison of option prices between the proposed model and the CIR model.*

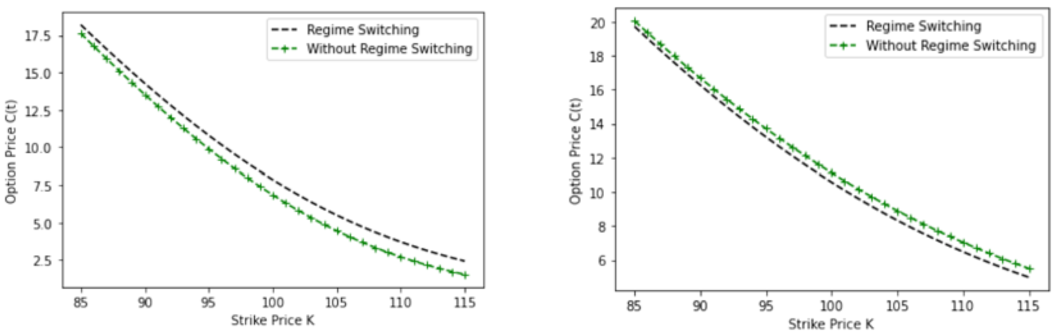

Figure 3. *Comparison of option prices against strike prices for the proposed model and the hybrid Heston–Hull–White model with stochastic intensity while considering the initial regime as Regime 1 (on the left) and Regime 2 (on the right).*

4.2.3. Comparison with regime-switching hybrid Heston–Hull–White model without stochastic intensity

To illustrate the impact of stochastic intensity, the comparison of the proposed model with the regime-switching hybrid Heston–Hull–White model having constant intensity is depicted in Figure 4. It is observed that, regardless of the initial state of the economy, the options price is higher in stochastic intensity than in constant intensity.

4.3. Sensitivity analysis

This subsection depicts the impact of the sensitivity of various parameters of the risk-free rate, volatility, and intensity processes on option prices for different strike prices. For all of these figures, it is assumed that initially, the market is in regime 2.

Initial volatility (v_0): Figure 5 exhibits a plot of the option prices and strike prices for various values of initial volatility. It is observed that as volatility increases, the options price at all strike prices tends to rise. However, at lower strike prices, the difference in options price becomes less significant.

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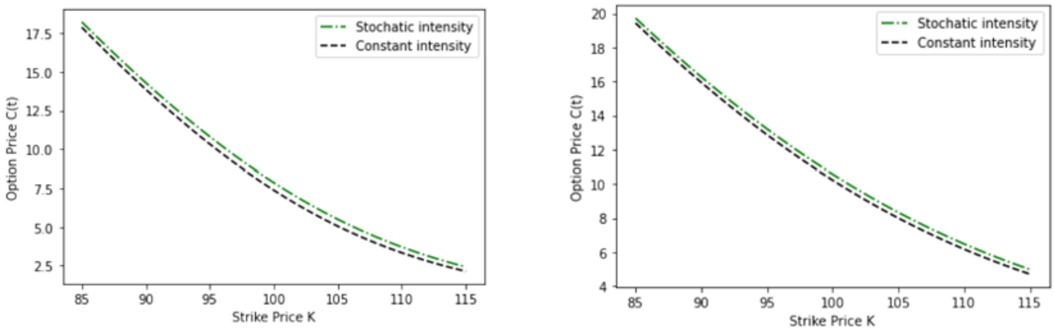

Figure 4. *Comparison of option prices against strike prices for the proposed model and the regime-switching hybrid Heston-Hull-White model without stochastic intensity while considering the initial regime as Regime 1 (on the left) and Regime 2 (on the right).*

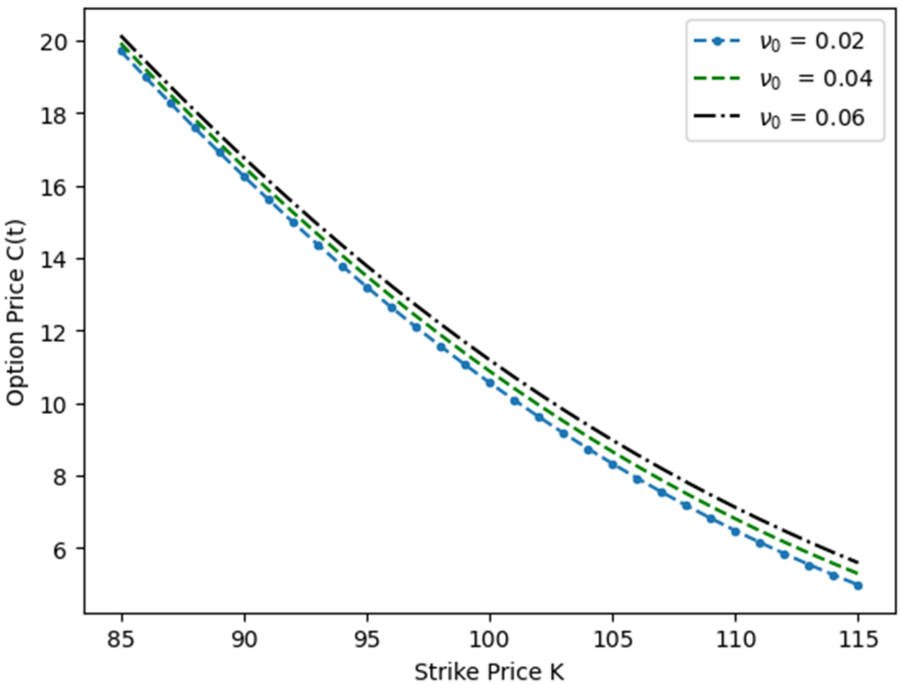

Figure 5. *Option prices against strike prices for different initial volatility levels.*

One possible explanation is that the option price depends on the intrinsic values of the option. The option's intrinsic value is higher at a lower strike price. Hence, the effect of volatility is not very significant.

Initial risk-free rate (r_0): Figure 6 presents how varying the values of r_0 affect the option prices by plotting option prices for different r_0 values against various strike prices K. The risk-free rate affects option price in two different ways. First, it impacts the underlying asset's price. Second, it is used as a discounting factor to determine the present value of the options price. The options price increases with the increase in value of r_0. The impact becomes less significant as the strike prices increase.

Mean reversion level of risk-free rate (β_t): To examine the impact of mean reversion level, it is assumed that $\beta_1 = \beta_2 + 0.02$. Option prices at various strike prices K corresponding to different values of the β_t are depicted in Figure 7. As the value of β_t increases, the options price also increases across

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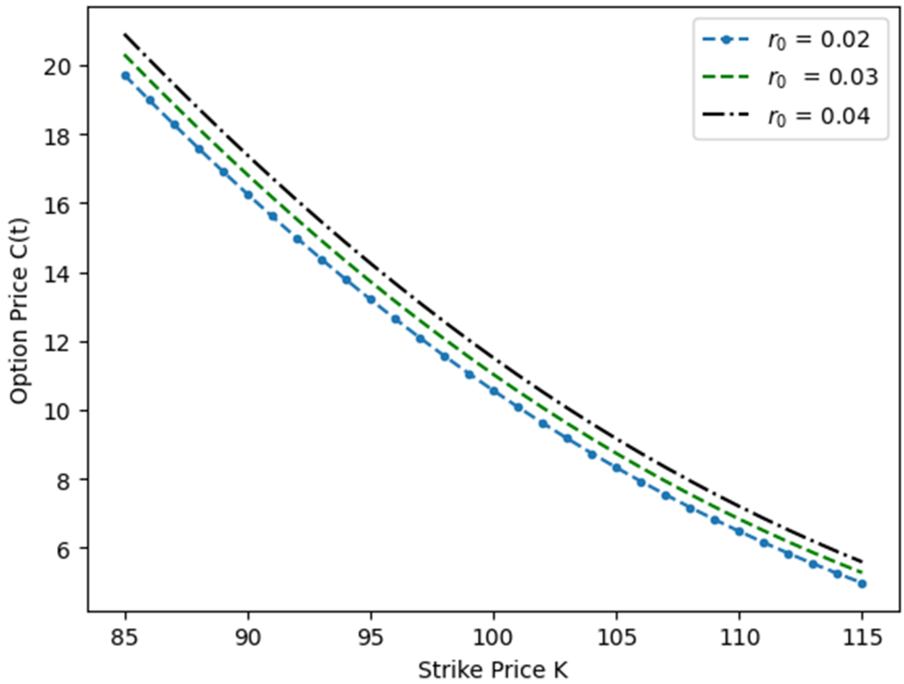

Figure 6. *Option prices against strike prices for different initial risk-free rates.*

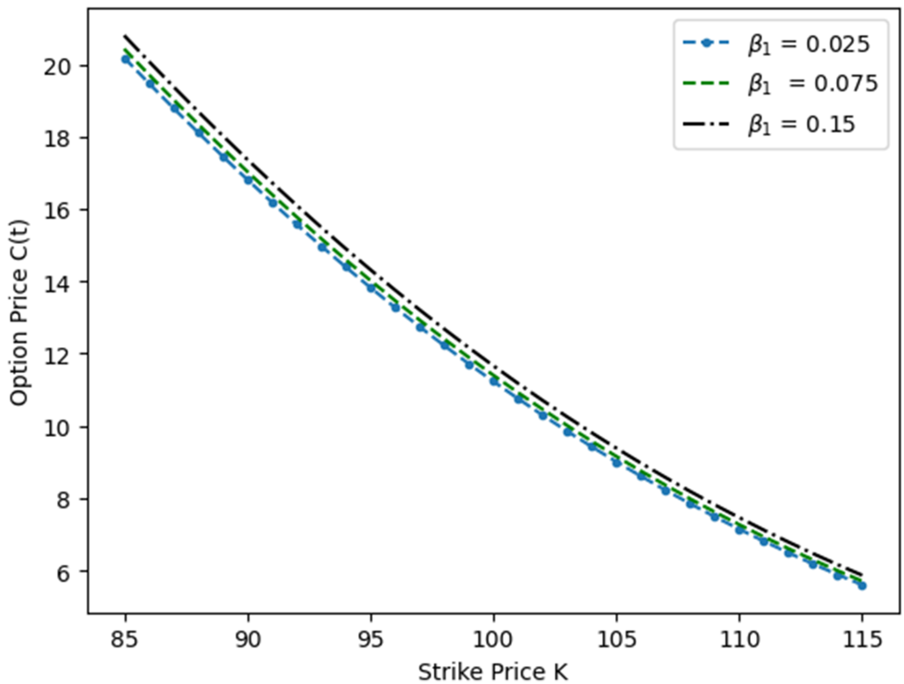

Figure 7. *Option prices against strike prices for different values of* β_{t}.

all strike prices. This is because the higher value of β_t, which is the long-run mean of the risk-free rate, implies an expectation of increasing risk-free rates.

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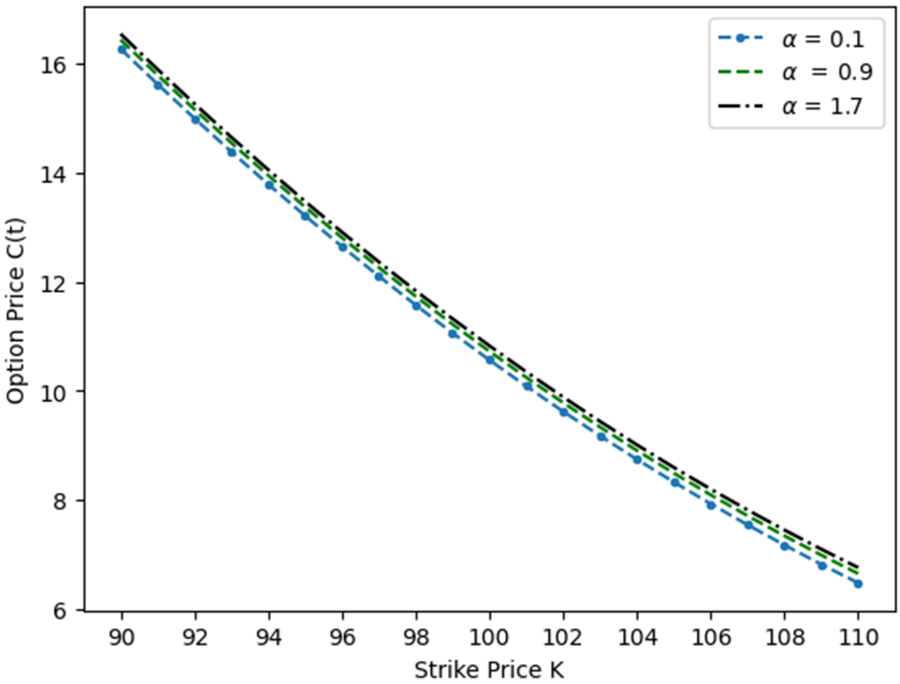

Figure 8. *Option prices against strike prices for different values of* α*.*

Mean reversion speed of risk-free rate (α): The effect of varying α on option prices is investigated in Figure 8. This figure analyzes the option prices at various strike prices corresponding to different values of mean reversion speed (α). The figure shows that the higher value of α leads to an increased options price for every strike price.

Volatility of risk-free rate (σ_t): Figure 9 displays the options price against the parameter σ_t at various strike prices K. To plot this figure, it is assumed that $\sigma_2 = \sigma_1 + 0.02$. It is observed that option prices are positively correlated with the volatility of risk-free rate. A higher σ_t increases option prices at every strike price.

Long-run mean of intensity process (θ_λ): Figure 10 illustrates the options price at different values of long-run mean of intensity process. From the figure, it can be observed that as the value of θ_λ increases, there is a corresponding increase in options price for all strike prices. This is because as the value of θ_λ increases, the expectation of jump intensity increases.

5. Conclusions and future work

This study proposes a pricing formula for European options using the regime-switching framework with stochastic intensity. The closed-form formula for options price is derived using the Feynman–Kac theorem with asset volatility and risk-free rate controlled by the economic environment. The results highlight the importance of incorporating stochastic intensity in option pricing and its significant impact. Furthermore, a comparison is made with models without regime-switching framework to depict the difference in option prices. This divergence occurs because the regime-switching model adjusts for fluctuating market conditions and volatility patterns over time, resulting in more accurate pricing that reflects economic movements. As a result, including regime-switching can result in more accurate pricing, better reflecting the complexity of real-world financial markets.

An interesting extension may involve studying the valuation of American options within this framework, considering their flexibility to be exercised at any time before or on the maturity date. Considering

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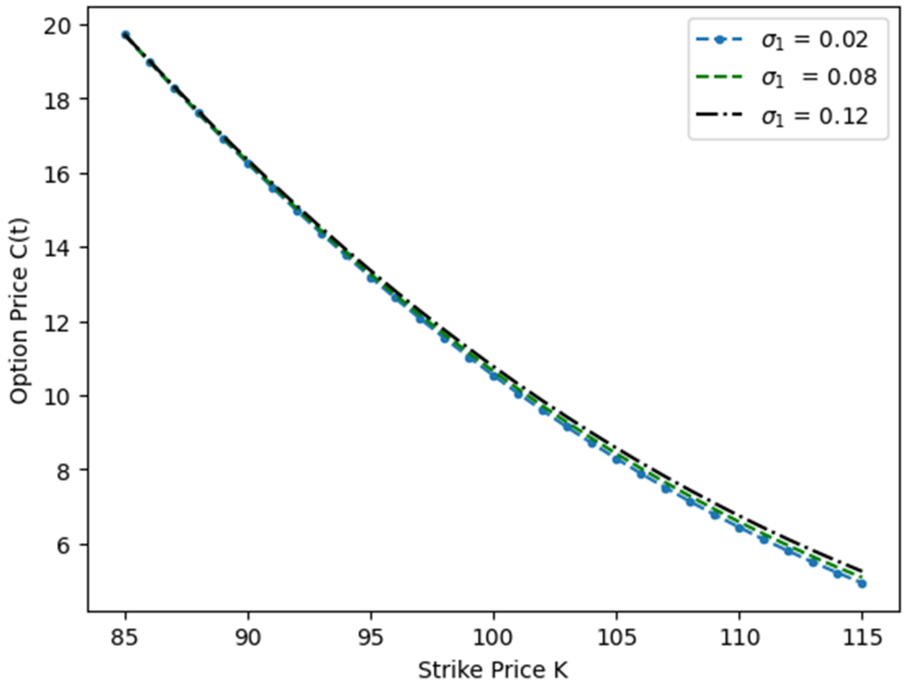

Figure 9. *Option prices against strike prices for different values of* σ_t.

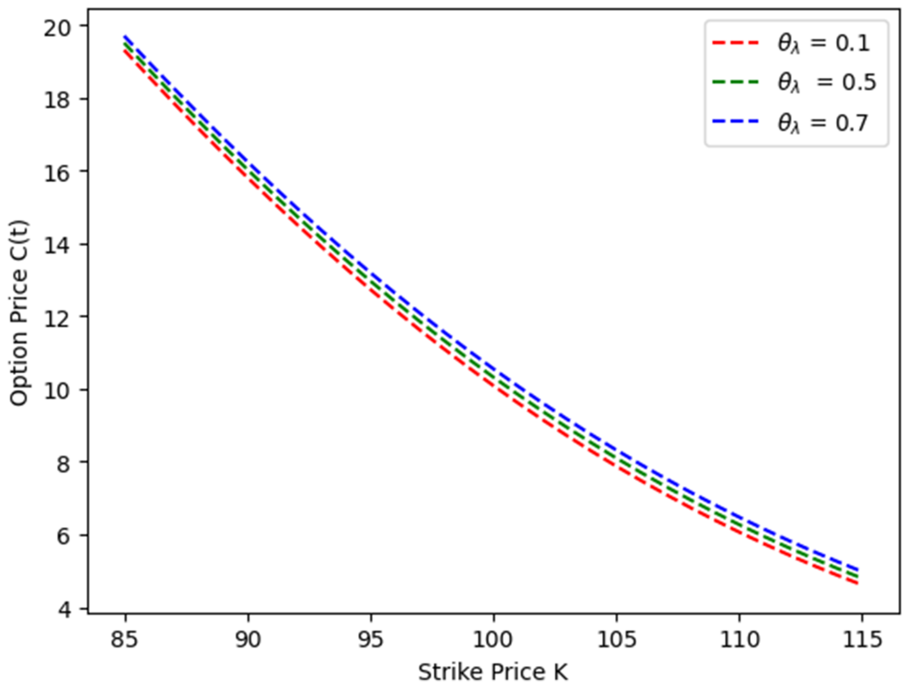

Figure 10. *Option prices against strike price for different values of* θ_λ.

the relationship between the intensity process and the jump arrival process allows for a more realistic scenario. Hence, a more general jump arrival process like the Hawkes process may be considered for asset price modeling.

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Conflict of interest. The authors declare that they have no competing interests.

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