J. Appl. Probab. 1–22 (2025)
*doi:*10.1017/jpr.2024.104

LAST EXIT TIME UNTIL FIRST EXIT TIME FOR SPECTRALLY NEGATIVE LÉVY PROCESSES

XIAOFENG YANG,* *Central South University*
JOSÉ M. PEDRAZA,** *The University of Manchester*
BIN LI,*** *University of Waterloo*

Abstract

We study the last exit time that a spectrally negative Lévy process is below zero until it reaches a positive level b, denoted by $g_{\tau_b^+}$. We generalize the results of the infinite-horizon last exit time explored by Chiu and Yin (2005) by incorporating a random horizon τ_b^+, which represents the first passage time above b. We derive an explicit expression for the joint Laplace transform of $g_{\tau_b^+}$ and τ_b^+ by utilizing a hybrid observation scheme approach proposed by Li, Willmot, and Wong (2018). We further study the optimal prediction of $g_{\tau_b^+}$ in the L_1 sense, and find that the optimal stopping time is the first passage time above a level y_b^*, with an explicit characterization of the stopping boundary y_b^*. As examples, Brownian motion with drift and the Cramér–Lundberg model with exponential jumps are considered.

Keywords: Last exit time; first exit time; spectrally negative Lévy processes; distribution; optimal prediction; optimal stopping

2010 Mathematics Subject Classification: Primary 60G51; 60G40
Secondary 91B30; 62M20

1. Introduction

Last exit times have received increasing attention in theoretical and applied probability. For example, [14] studied the joint distribution of the last exit time of a standard Markov process from a transient set and its location at that time, and [22] studied the last passage times of linear diffusions and proposed the h-transform method. Recently, this work was extended in [13] to a general setting and an application in credit risk management considered. In line with the recent trend of studies on quantitative risk management, it is natural to study the last exit time for a stochastic process X below level zero, i.e.

$$g = \sup\{s \geq 0 \colon X_s \leq 0\}, \tag{1.1}$$

where X can be interpreted as a firm value process and the distributional study of g can provide valuable information on the duration of financial distress (i.e. the period with negative firm

Received 23 June 2023; accepted 29 October 2024.

* Postal address: School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China. Email: yang17xiaofeng@163.com
** Postal address: Department of Mathematics, The University of Manchester, Manchester, M13 9PL, UK. Email: jose.pedrazaramirez@manchester.ac.uk
*** Postal address: Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada. Email: bin.li@uwaterloo.ca

value). We follow the convention that $\sup \emptyset = 0$. The distribution of g for a spectrally negative Lévy process (SNLP) was first solved in [10].

A more general form is given by $g_t = \sup\{0 \le s \le t \colon X_s \le 0\}$, which denotes the last time below zero until time t. When X is a Brownian motion, it is known from [11] that g_t follows the arcsine distribution. Apart from Brownian motions, there are few explicit results on the distribution of g_t for more general processes, and the main challenge is the lack of mathematical tractability in this finite-horizon setting. A notable breakthrough in [2] is to replace the finite horizon t by a random horizon e_q, independent and exponentially distributed, and consider $g_{e_q} = \sup\{0 \le s \le e_q \colon X_s \le 0\}$. The Laplace transform of g_{e_q} was solved in [2] for SNLPs. Later, [8, 20] studied the occupation times until g_{e_q} for SNLPs. It is worth noting that this technique of *horizon randomization* is commonly adopted in the finance literature; see the seminal work [9] on the application to option pricing.

The last exit times are not stopping times, meaning that the past history of the process is insufficient to determine whether or not last exit times have been realized. As such, for the purpose of decision making, another line of study is to *optimally predict* last exit times by stopping times. The optimal prediction of the last zero of Brownian motion with drift was studied in [12], while [15] generalizes the prediction of last passage times to a transient diffusion. Under the framework of SNLPs, [3] studied the optimal prediction of g, defined in (1.1), in L_1-distance. It was extended by [5] to the much more challenging L_p-distance. Also, the prediction of g_{e_q} was recently considered in [4] in L_1-distance.

One shortcoming of the last exit time g is that it is an infinite-horizon measure in the sense that its distributional study and applications are based on the assumption that the model for the underlying process X is unchanged and valid *forever*. This is a rather strong assumption from the *model risk* perspective. In this sense, g_t and g_{e_q} weaken the assumption as the model validation is only required to hold up to a finite or random horizon.

Our motivation for this paper is along the same lines but with the major difference that we intend to reduce the model validation assumption at the *state level* (instead of the time level as for g_t and g_{e_q}). More specifically, we consider the last exit time that X is below zero until it reaches a level $b > 0$, i.e. $g_{\tau_b^+} = \sup\{0 \le s \le \tau_b^+ \colon X_s \le 0\}$, where $\tau_b^+ = \inf\{t > 0 \colon X_t > b\}$, in which we follow the convention that $\inf \emptyset = \infty$. Within the realm of corporate risk management, this extension is a logical progression since a firm's capital structure and profitability are prone to shift as the firm expands. In this context, the parameter b can be interpreted as a critical threshold of the firm's value. Once this threshold is surpassed, the company may proceed to hire additional employees, distribute dividends to shareholders, and venture into new business areas, rendering the previous model obsolete.

The contribution of this paper is twofold. In the first part, we derive the joint Laplace transform of $g_{\tau_b^+}$ and τ_b^+ for SNLPs. It should be noted that the method employed in [2, 10] to study g and g_{e_q} cannot be directly applied to analyze $g_{\tau_b^+}$. To overcome this challenge, we adopt a hybrid observation scheme approach proposed in [19]. The merit of this approach is to unify the cases with bounded or unbounded variation sample paths. Furthermore, by letting $b \to \infty$, we can recover the distribution of g as obtained in [10], but with the added benefit of offering an alternative proof that is significantly simpler.

In the second part, we study the optimal prediction problem for $g_{\tau_b^+}$ in L_1-distance, that is, $\inf_{\tau \in \mathcal{T}} \mathbb{E}_x[|g_{\tau_b^+} - \tau|]$, where $\mathcal{T}$ is the set of all stopping times of the process X. We find that the optimal stopping time is the first time that the process X hits a fixed level, denoted by $y_b^* \in [0, b]$. The optimal stopping boundary y_b^* is explicitly derived and it is closely related to the value of b and the cumulative distribution function of $\underline{X}_\infty := \inf_{t\geq 0} X_t$. At the end, examples of Brownian motion with drift and the Cramér–Lundberg model with exponential jumps are considered to demonstrate the optimal boundary.

Our motivation to examine the optimal prediction problem for $g_{\tau_b^+}$ also comes from a theoretical point of view. It is seen in [3] that the optimal prediction of g boils down to an optimal stopping problem with an infinite time horizon, and hence the solution is a fixed boundary. However, surprisingly, in [4] it is shown that the optimal prediction of g_{e_q} is equivalent to a finite-horizon optimal stopping problem, and the optimal solution depends on a non-negative curve which is killed at the moment the mean of the exponential time is reached. Hence, unlike the Canadisation of American-type options (see [9]), optimal prediction problems with exponential time horizon do not necessarily result in infinite-horizon optimal stopping problems. With this in mind, we see that predicting $g_{\tau_b^+}$, i.e. the first exit time type of random horizon, effectively maintains the problem's infinite-horizon setting, resulting in a fixed boundary solution.

The rest of the paper is organized as follows. Section 2 provides some preliminary results concerning SNLPs. Section 3 derives an explicit expression for the joint distribution of $g_{\tau_b^+}$ and τ_b^+. Section 4 formulates the optimal prediction problem and provides the solution, while the proofs are deferred to Section 5.

2. Preliminaries

In this section we provide some preliminaries on spectrally negative Lévy processes, including scale functions and some fluctuation identities. More information on Lévy processes can be found in [6, 17, 23] for interested readers.

Let X be a spectrally negative Lévy process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by X which is naturally enlarged (see [7, Definition 1.3.38]), and we exclude the case where X has monotone paths. We assume that X is given on the canonical space Ω, which consists of all functions $\omega\colon [0, \infty) \mapsto \mathbb{R}$ that are right-continuous and have left limits, and that $X_t(\omega) = \omega_t$. We then introduce the shift operator θ_t, acting on the elements of Ω, defined by $X_s \circ \theta_t(\omega) = X_s(\theta_t(\omega)) = X_{t+s}(\omega)$, for any $s, t \geq 0$.

Throughout this paper, for all $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the law of X when started at the point $x \in \mathbb{R}$, and the associated expectation by $\mathbb{E}_x$. For simplicity, we write $\mathbb{P} = \mathbb{P}_0$ and $\mathbb{E} = \mathbb{E}_0$.

The distribution of X is characterized by its Lévy triplet (μ, σ, Π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$, and Π is the Lévy measure concentrated on $(-\infty, 0)$ with the property $\int_{(-\infty,0)} (1 \wedge y^2)\, \Pi(\mathrm{d}y) < \infty$. The Laplace exponent of X is defined by $\psi(\theta) = \log(\mathbb{E}[\mathrm{e}^{\theta X_1}])$. It is a strictly convex and infinitely differentiable function on $\mathbb{R}_+$ with $\psi(0) = 0$ and $\psi(\infty) = \infty$. We know from the Lévy–Khintchine formula that ψ is of the form

$$\psi(\theta) = -\mu\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(-\infty,0)} \big(\mathrm{e}^{\theta y} - 1 - \theta y \mathbf{1}_{\{y>-1\}}\big)\, \Pi(\mathrm{d}y)$$

for all $\theta \geq 0$. For any $q \geq 0$, the right inverse of ψ is defined as $\Phi_q = \sup\{\theta \geq 0\colon \psi(\theta) = q\}$. It is known that the behaviour of X at infinity is determined by the sign of $\psi'(0+)$. Indeed, if $\psi'(0+) > 0$ we have that X drifts to infinity, that is, $\lim_{t\to\infty} X_t = \infty$ almost surely; in this case, $\Phi_q = 0$ if and only if $q = 0$. If $\psi'(0+) < 0$, the process X drifts to minus infinity, that is, $\lim_{t\to\infty} X_t = -\infty$ almost surely, and the process oscillates, i.e. $\limsup_{t\to\infty} X_t = \infty = -\liminf_{t\to\infty} X_t$ whenever $\psi'(0+) = 0$.

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

The sample paths of X are of bounded variation if and only if $\sigma = 0$ and

$$\int_{(-1,0)} (-y)\,\Pi(\mathrm{d}y) < \infty.$$

For this particular case, we write $\psi(\theta) = \delta\theta - \int_{(-\infty,0)} (1 - \mathrm{e}^{\theta y})\,\Pi(\mathrm{d}y)$, where $\delta = -\mu - \int_{(-1,0)} y\,\Pi(\mathrm{d}y)$. Note that monotone processes are excluded from the definition of spectrally negative Lévy processes, so we assume that $\delta > 0$ when X is of bounded variation.

We now introduce scale functions, which play an essential role in the derivation of fluctuation identities for spectrally negative Lévy processes. For $q \geq 0$, the q-scale function is such that $W^{(q)} = 0$ for $x < 0$, and is uniquely characterised on $[0, \infty)$ as the only right-continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\psi_q(\theta)} \quad \text{for } \theta > \Phi_q, \tag{2.1}$$

where $\psi_q(\theta) = \psi(\theta) - q$. We have that $W^{(q)}$ is strictly increasing and continuous on $(0, \infty)$. For ease of notation we assume that Π has no atoms when X is of bounded variation, which guarantees that $W \in C^1(0, \infty)$. When $q = 0$, we write $W = W^{(0)}$. The value of $W^{(q)}$ at zero depends on the path variation of X. To be more precise,

$$W^{(q)}(0) = \begin{cases} 1/\delta, & \sigma = 0 \text{ and } \int_{(-1,0)} (-y)\Pi\,(\mathrm{d}y) < \infty; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Furthermore, it is established that

$$\lim_{x\to\infty} \mathrm{e}^{-\Phi_q x} W^{(q)}(x) = \Phi_q', \tag{2.3}$$

where $\Phi_q' = \partial\Phi_q/\partial q$. Moreover, from [17, (8.22)], it is known that there exists a non-increasing function g such that, for any $a > x$,

$$\log\,(W(x)) = \log\,(W(a)) - \int_x^a g(t)\,\mathrm{d}t. \tag{2.4}$$

Another class of scale function is defined by

$$Z^{(q)}(x, \theta) = \mathrm{e}^{\theta x}\left(1 - \psi_q(\theta) \int_0^x \mathrm{e}^{-\theta y} W^{(q)}(y)\,\mathrm{d}y\right), \quad x \in \mathbb{R}. \tag{2.5}$$

We note that $Z^{(q)}(x, \theta) = \mathrm{e}^{\theta x}$ for all $x \leq 0$, and $Z^{(q)}(b, \Phi_q) = \mathrm{e}^{\Phi_q b}$ for all $b \in \mathbb{R}$ and $q \geq 0$. When $\theta = 0$, $Z^{(q)}(x, \theta)$ reduces to $Z^{(q)}(x)$ as defined in [17]. By using (2.1), we can rewrite (2.5) as

$$Z^{(q)}(x, \theta) = \psi_q(\theta) \int_0^\infty \mathrm{e}^{-\theta y} W^{(q)}(x + y)\,\mathrm{d}y, \quad x \geq 0, \ \theta > \Phi_q. \tag{2.6}$$

We then introduce some results on the first passage time. For any $a \geq 0$, the Laplace transform of τ_a^+ is given by

$$\mathbb{E}\big[\mathrm{e}^{-q\tau_a^+} \mathbf{1}_{\{\tau_a^+ < \infty\}}\big] = \mathrm{e}^{-\Phi_q a}. \tag{2.7}$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Denote by τ_0^- the first time X drops below the level zero, that is, $\tau_0^- = \inf\{t > 0 \colon X_t < 0\}$. The Laplace transform of τ_a^+ on the event that X crosses above the level a before dropping below the level zero is given by

$$\mathbb{E}_x\big[\mathrm{e}^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+ < \tau_0^-\}}\big] = \frac{W^{(q)}(x)}{W^{(q)}(a)} \tag{2.8}$$

for all $q \geq 0$ and $x \leq a$. Moreover, whenever $\psi'(0+) \geq 0$, the probability of never entering the set $(-\infty, 0]$ is given by $\mathbb{P}_x(\tau_0^- = \infty) = \psi'(0+)W(x)$ for any $x \geq 0$. Hence, by writing $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$ for $t \geq 0$, we deduce that the cumulative distribution function of $-\underline{X}_\infty$ when X drifts to infinity is given by

$$F(x) := \mathbb{P}(-\underline{X}_\infty \leq x) = \mathbb{P}_x(\tau_0^- = \infty) = \psi'(0+)W(x) \tag{2.9}$$

for any $x \in \mathbb{R}$. For any $a \in \mathbb{R}$ and $q \geq 0$, the q-potential measure of X killed upon entering the set $[a, \infty)$ is absolutely continuous with respect to the Lebesgue measure, leading to

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}_x\big(X_t \in \mathrm{d}y,\, t < \tau_a^+\big)\,\mathrm{d}t = \big[\mathrm{e}^{-\Phi_q(a-x)}W^{(q)}(a-y) - W^{(q)}(x-y)\big]\,\mathrm{d}y \tag{2.10}$$

for all $x, y \leq a$.

3. Joint distribution of $g_{\tau_b^+}$ and τ_b^+

In this section we derive the joint distribution of the first passage time τ_b^+ and the last passage time $g_{\tau_b^+} = \sup\{0 \leq s \leq \tau_b^+ \colon X_s \leq 0\}$, where $b > 0$ is a fixed positive level. The results generalize [10] by letting b go to infinity. It is worth noting that the approach used in [2, 10] are not directly applicable in our case. As such, we adopt the hybrid observation scheme approach proposed in [19] to study the joint distribution of τ_b^+ and $g_{\tau_b^+}$. The merit of this approach is that the two cases of the underlying process with bounded or unbounded variation paths can be treated in a unified way, and the proof is significantly simplified.

We then introduce the hybrid observation scheme. Briefly speaking, we observe the underlying process *discretely* (with independent exponential time increments) when it is non-negative, and then switch to *continuous* observation when it becomes negative. It will be switched back to discrete observation once it recovers the level 0. Formally, we first define the following sequence of time nodes $\{\xi_n\}_{n\in\mathbb{N}}$. Let $\xi_0 = 0$, and

$$\xi_n - \xi_{n-1} = \begin{cases} e_n^\lambda & \text{if } X_{\xi_{n-1}} \geq 0, \\ \tau_0^+ \circ \theta_{\xi_{n-1}} + e_n^\lambda & \text{if } X_{\xi_{n-1}} < 0, \end{cases}$$

where $\{e_n^\lambda\}_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) exponential random variables with mean $1/\lambda > 0$, and we recall that θ is the Markov shift operator and satisfies $X_t \circ \theta_s = X_{s+t}$.

Under this hybrid observation scheme, we define the first time the process is observed below level 0 by $T_0^{\lambda,-} = \inf\{\xi_n \colon X_{\xi_n} < 0,\ n \in \mathbb{N}\}$. Note that $T_0^{\lambda,-}$ is identical to the first time below level 0 under the so-called Poissonian observation scheme; see, e.g., [1, 18]. We recall the following two formulas from the literature ([1, (15)] and [18, (3.5)]). For $q, \theta \geq 0$ and $x \leq b$,

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

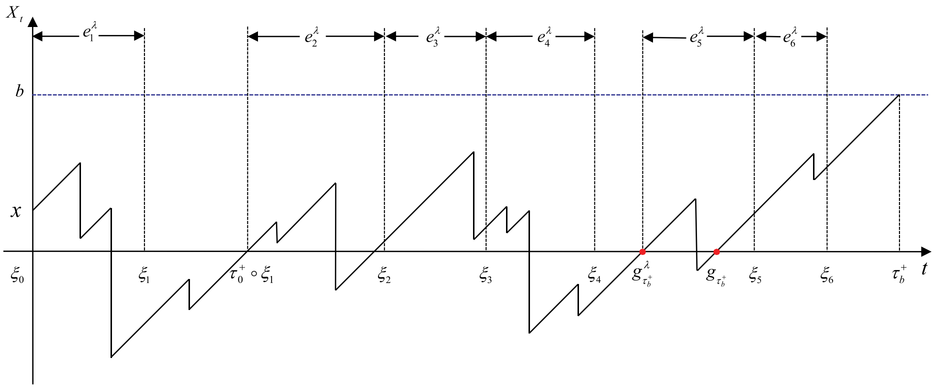


FIGURE 1. Illustration of the hybrid observation scheme, $g^\lambda_{\tau_b^+}$, and $g_{\tau_b^+}$.

$$\mathbb{E}_x\Big[\exp\big\{-qT_0^{\lambda,-}+\theta X_{T_0^{\lambda,-}}\big\}\mathbf{1}_{\{T_0^{\lambda,-}<\tau_b^+\}}\Big]=\frac{\lambda}{\lambda-\psi_q(\theta)}\bigg(Z^{(q)}(x,\theta) - Z^{(q)}(b,\theta)\frac{Z^{(q)}(x,\Phi_{\lambda+q})}{Z^{(q)}(b,\Phi_{\lambda+q})}\bigg), \tag{3.1}$$

$$\mathbb{E}_x\Big[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^{\lambda,-}\}}\Big]=\frac{Z^{(q)}(x,\Phi_{\lambda+q})}{Z^{(q)}(b,\Phi_{\lambda+q})}. \tag{3.2}$$

The last zero of X (under the hybrid observation scheme) before crossing above level b (under the continuous observation scheme) is defined by

$$g^\lambda_{\tau_b^+}:=\sup\{\xi_n+\tau_0^+\circ\theta_{\xi_n}: X_{\xi_n}<0,\ \xi_n<\tau_b^+,\ n\in\mathbb{N}\}.$$

Since the hybrid observation scheme reduces to continuous observation when the intensity λ goes to infinity, we have, for any $x\in\mathbb{R}$, $\lim_{\lambda\uparrow\infty} g^\lambda_{\tau_b^+}=g_{\tau_b^+}$ $\mathbb{P}_x$-almost surely. See Figure 1 for an illustration of the hybrid observation scheme as well as the two last passage times $g^\lambda_{\tau_b^+}$ and $g_{\tau_b^+}$.

The following theorem presents the joint distribution of $g_{\tau_b^+}$ and $\tau_b^+-g_{\tau_b^+}$.

Theorem 3.1. *For $p, q\geq 0$, $x<b$, and $b>0$,*

$$\mathbb{E}_x\Big[\exp\big\{-pg_{\tau_b^+}-q(\tau_b^+-g_{\tau_b^+})\big\}\mathbf{1}_{\{\tau_b^+<\infty\}}\Big]=\mathrm{e}^{-\Phi_p(b-x)}\frac{W^{(p)}(b)}{W^{(q)}(b)}+\frac{W^{(q)}(x)}{W^{(q)}(b)}-\frac{W^{(p)}(x)}{W^{(q)}(b)}. \tag{3.3}$$

Proof. We first derive $f(x):=\mathbb{E}_x\Big[\exp\big\{-pg^\lambda_{\tau_b^+}-q(\tau_b^+-g^\lambda_{\tau_b^+})\big\}\mathbf{1}_{\{\tau_b^+<\infty\}}\Big]$, $x<b$. For $x<0$, it follows from the spatial homogeneity and (2.7) that

$$f(x)=\mathbb{E}_x\Big[\mathrm{e}^{-p\tau_0^+}\mathbf{1}_{\{\tau_0^+<\infty\}}\Big]f(0)=\mathrm{e}^{\Phi_p x}f(0). \tag{3.4}$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Note that, conditional on the values $\{e_n^\lambda\}_{n\in\mathbb{N}}$, the times $\{\xi_n\}_{n\in\mathbb{N}}$ and $T_0^{\lambda,-}$ are stopping times of X. Hence, by conditioning on $\{e_n^\lambda\}_{n\in\mathbb{N}}$, further conditioning on $T_0^{\lambda,-}$, and applying the strong Markov property of X, we obtain that, for any $0\le x<b$,

$$\begin{aligned}f(x)&=\int_{-\infty}^{0}\mathbb{E}_x\big[\mathrm{e}^{-pT_0^{\lambda,-}}\mathbf{1}_{\{X_{T_0^{\lambda,-}}\in\mathrm{d}u,\,T_0^{\lambda,-}<\tau_b^+\}}\big]f(u)+\mathbb{E}_x\big[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^{\lambda,-}\}}\big]\\&=\int_{-\infty}^{0}\mathbb{E}_x\big[\mathrm{e}^{-pT_0^{\lambda,-}}\mathbf{1}_{\{X_{T_0^{\lambda,-}}\in\mathrm{d}u,\,T_0^{\lambda,-}<\tau_b^+\}}\big]e^{\Phi_p u}f(0)+\mathbb{E}_x\big[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^{\lambda,-}\}}\big]\\&=\mathbb{E}_x\big[\exp\big\{-pT_0^{\lambda,-}+\Phi_pX_{T_0^{\lambda,-}}\big\}\mathbf{1}_{\{T_0^{\lambda,-}<\tau_b^+\}}\big]f(0)+\mathbb{E}_x\big[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^{\lambda,-}\}}\big],\end{aligned}\tag{3.5}$$

where we used (3.4) in the second identity. By letting $x=0$ in (3.5), we can solve $f(0)$ and obtain

$$\begin{aligned}f(0)=\frac{\mathbb{E}\big[\mathrm{e}^{-q\tau_b^+}\mathbf{1}_{\{\tau_b^+<T_0^{\lambda,-}\}}\big]}{1-\mathbb{E}\big[\exp\big\{-pT_0^{\lambda,-}+\Phi_pX_{T_0^{\lambda,-}}\big\}\mathbf{1}_{\{T_0^{\lambda,-}<\tau_b^+\}}\big]}&=\frac{Z^{(q)}(0,\Phi_{\lambda+q})/Z^{(q)}(b,\Phi_{\lambda+q})}{\mathrm{e}^{\Phi_pb}Z^{(p)}(0,\Phi_{\lambda+p})/Z^{(p)}(b,\Phi_{\lambda+p})}\\&=\mathrm{e}^{-\Phi_pb}\frac{Z^{(p)}(b,\Phi_{\lambda+p})}{Z^{(q)}(b,\Phi_{\lambda+q})},\end{aligned}\tag{3.6}$$

where we used (3.1) and (3.2) in the second identity, and $Z^{(q)}(0,\theta)=1$ in the last identity. Substituting (3.6) into (3.4), for $x<0$,

$$f(x)=\mathrm{e}^{-\Phi_p(b-x)}\frac{Z^{(p)}(b,\Phi_{\lambda+p})}{Z^{(q)}(b,\Phi_{\lambda+q})}.\tag{3.7}$$

Then, substituting (3.6) into (3.5), and using (3.1) and (3.2) again, for $0\le x<b$ we have

$$f(x)=\mathrm{e}^{-\Phi_p(b-x)}\frac{Z^{(p)}(b,\Phi_{\lambda+p})}{Z^{(q)}(b,\Phi_{\lambda+q})}+\frac{Z^{(q)}(x,\Phi_{\lambda+q})}{Z^{(q)}(b,\Phi_{\lambda+q})}-\frac{Z^{(p)}(x,\Phi_{\lambda+p})}{Z^{(q)}(b,\Phi_{\lambda+q})}.\tag{3.8}$$

We furthermore unify (3.7) and (3.8) into one expression for $x<b$:

$$f(x)=\mathrm{e}^{-\Phi_p(b-x)}\frac{Z^{(p)}(b,\Phi_{\lambda+p})}{Z^{(q)}(b,\Phi_{\lambda+q})}+\left(\frac{Z^{(q)}(x,\Phi_{\lambda+q})}{Z^{(q)}(b,\Phi_{\lambda+q})}-\frac{Z^{(p)}(x,\Phi_{\lambda+p})}{Z^{(q)}(b,\Phi_{\lambda+q})}\right)\mathbf{1}_{\{0\le x<b\}}.\tag{3.9}$$

By (2.6) and the fact that $Z^{(q)}(0,\cdot)=1$, we have

$$\begin{aligned}\lim_{\lambda\to\infty}\frac{\Phi_{\lambda+q}}{\Phi_{\lambda+p}}=\lim_{\lambda\to\infty}\frac{\Phi_{\lambda+q}[Z^{(q)}(0,\Phi_{\lambda+q})]/\lambda}{\Phi_{\lambda+p}[Z^{(p)}(0,\Phi_{\lambda+p})]/\lambda}&=\lim_{\lambda\to\infty}\frac{\Phi_{\lambda+q}\int_0^\infty\mathrm{e}^{-\Phi_{\lambda+q}y}W^{(q)}(y)\,\mathrm{d}y}{\Phi_{\lambda+p}\int_0^\infty\mathrm{e}^{-\Phi_{\lambda+p}y}W^{(p)}(y)\,\mathrm{d}y}\\&=\lim_{y\to0+}\frac{W^{(q)}(y)}{W^{(p)}(y)}=1,\end{aligned}$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

where the third equality is by the initial value theorem and the last limit can be found in [1, p. 1373]. This, together with (2.6), yields

$$\lim_{\lambda\to\infty}\frac{Z^{(p)}(x,\,\Phi_{\lambda+p})}{Z^{(q)}(b,\,\Phi_{\lambda+q})}=\lim_{\lambda\to\infty}\frac{\psi_p(\Phi_{\lambda+p})}{\psi_q(\Phi_{\lambda+q})}\frac{\int_0^\infty \mathrm{e}^{-\Phi_{\lambda+p}u}W^{(p)}(x+u)\,\mathrm{d}u}{\int_0^\infty \mathrm{e}^{-\Phi_{\lambda+q}u}W^{(q)}(b+u)\,\mathrm{d}u}$$
$$=\lim_{\lambda\to\infty}\frac{\int_0^\infty \mathrm{e}^{-\Phi_{\lambda+p}u}W^{(p)}(x+u)\,\mathrm{d}u}{\int_0^\infty \mathrm{e}^{-\Phi_{\lambda+q}u}W^{(q)}(b+u)\,\mathrm{d}u}$$
$$=\lim_{\lambda\to\infty}\frac{\Phi_{\lambda+q}}{\Phi_{\lambda+p}}\frac{\Phi_{\lambda+p}\int_0^\infty \mathrm{e}^{-\Phi_{\lambda+p}u}W^{(p)}(x+u)\,\mathrm{d}u}{\Phi_{\lambda+q}\int_0^\infty \mathrm{e}^{-\Phi_{\lambda+q}u}W^{(q)}(b+u)\,\mathrm{d}u}=\frac{W^{(p)}(x)}{W^{(q)}(b)}. \tag{3.10}$$

By taking the limit $\lambda\uparrow\infty$ in (3.9) and using (3.10), we can conclude that

$$\mathbb{E}_x\big[\exp\big\{-pg_{\tau_b^+}-q(\tau_b^+-g_{\tau_b^+})\big\}\big]$$
$$=\lim_{\lambda\to\infty}\mathbb{E}_x\big[\exp\big\{-pg^\lambda_{\tau_b^+}-q(\tau_b^+-g^\lambda_{\tau_b^+})\big\}\big]$$
$$=\lim_{\lambda\to\infty}\left(\mathrm{e}^{-\Phi_p(b-x)}\frac{Z^{(p)}(b,\,\Phi_{\lambda+p})}{Z^{(q)}(b,\,\Phi_{\lambda+q})}+\left(\frac{Z^{(q)}(x,\,\Phi_{\lambda+q})}{Z^{(q)}(b,\,\Phi_{\lambda+q})}-\frac{Z^{(p)}(x,\,\Phi_{\lambda+p})}{Z^{(q)}(b,\,\Phi_{\lambda+q})}\right)\mathbf{1}_{\{0\le x<b\}}\right)$$
$$=\mathrm{e}^{-\Phi_p(b-x)}\frac{W^{(p)}(b)}{W^{(q)}(b)}+\frac{W^{(q)}(x)}{W^{(q)}(b)}-\frac{W^{(p)}(x)}{W^{(q)}(b)},$$

where in the last equality we used that $W^{(q)}$ and $W^{(p)}$ vanish on $(-\infty,0)$. □

Remark 3.1. It is worth noting that an alternative approach to proving Theorem 3.1 is to modify the roadmap proposed in [17, exercise in 8.10].

By letting $b\to\infty$ in (3.3), using (2.3) and the fact that $\lim_{b\uparrow\infty}W(b)=1/\psi'(0+)$ when X drifts to infinity (see [16, Lemma 3.3]), we obtain the Laplace transform of the last zero g, which recovers [10, Theorem 3.1] and [2, Theorem 1].

Corollary 3.1. *Suppose that* $\psi'(0+)>0$. *For* $p\ge 0$,

$$\mathbb{E}_x[\mathrm{e}^{-pg}]=\psi'(0+)\big(\Phi_p'\mathrm{e}^{\Phi_p x}+W(x)-W^{(p)}(x)\big).$$

Example 3.1. Suppose that X is a Brownian motion with drift, i.e. $X_t=\mu t+\sigma B_t$, where $\mu\in\mathbb{R}$, $\sigma>0$, and $\{B_t\}_{t\ge 0}$ is a standard Brownian motion. It follows that $\psi(\theta)=\mu\theta+\frac{1}{2}\sigma^2\theta^2$. The two roots of $\psi(\theta)-q=0$ are given by $\Phi_q=(\sqrt{\mu^2+2q\sigma^2}-\mu)/\sigma^2$ and $-\rho_q=-(\sqrt{\mu^2+2q\sigma^2}+\mu)/\sigma^2$. The q-scale function is given by

$$W^{(q)}(x)=\frac{\exp(\Phi_q x)-\exp(-\rho_q x)}{\sqrt{\mu^2+2q\sigma^2}}. \tag{3.11}$$

It follows from Theorem 3.1 that

$$\mathbb{E}_x\big[\exp\big\{-pg_{\tau_b^+}-q(\tau_b^+-g_{\tau_b^+})\big\}\big]$$
$$=\frac{\exp(\Phi_q x)-\exp(-\rho_q x)}{\exp(\Phi_q b)-\exp(-\rho_q b)}$$
$$-\frac{\sqrt{\mu^2+2q\sigma^2}}{\sqrt{\mu^2+2p\sigma^2}}\frac{\big(\exp(\Phi_p x-(2b\sqrt{\mu^2+2p\sigma^2}/\sigma^2))-\exp(-\rho_p x)\big)}{\exp(\Phi_q b)-\exp(-\rho_q b)}.$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Example 3.2. Suppose that X follows the Cramér–Lundberg model with exponential jumps, i.e. $X_t = \mu t - \sum_{i=1}^{N_t} C_i$, where $\mu \in \mathbb{R}$, $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\eta > 0$, and $\{C_i\}_{i\in\mathbb{N}}$ are i.i.d. exponential random variables with parameter $\alpha > 0$, which are independent of N_t. It is known that $\psi(\theta) = \mu\theta - \eta + (\alpha\eta/(\theta + \alpha))$. The two roots of $\psi(\theta) - q = 0$ are given by

$$\Phi_q = \frac{1}{2\mu}(\sqrt{(q+\eta-\alpha\mu)^2 + 4\mu\alpha q} + (q+\eta-\alpha\mu)),$$
$$-\rho_q = -\frac{1}{2\mu}(\sqrt{(q+\eta-\alpha\mu)^2 + 4\mu\alpha q} - (q+\eta-\alpha\mu)).$$

The q-scale function is given by

$$W^{(q)}(x) = \Phi'_q \exp(\Phi_q x) - \rho'_q \exp(-\rho_q x), \tag{3.12}$$

where

$$\Phi'_q = \frac{1}{2\mu}\left(\frac{q+\eta+\alpha\mu}{\sqrt{(q+\eta-\alpha\mu)^2 + 4\mu\alpha q}} + 1\right), \qquad \rho'_q = \frac{1}{2\mu}\left(\frac{q+\eta+\alpha\mu}{\sqrt{(q+\eta-\alpha\mu)^2 + 4c\alpha q}} - 1\right).$$

It follows from Theorem 3.1 that

$$\begin{aligned}
&\mathbb{E}_x\Big[\exp\Big\{-p g_{\tau_b^+} - q(\tau_b^+ - g_{\tau_b^+})\Big\}\Big] \\
&\quad = \frac{\Phi'_q \exp(\Phi_q x) - \rho'_q \exp(-\rho_q x)}{\Phi'_q \exp(\Phi_q b) - \rho'_q \exp(-\rho_q b)} \\
&\qquad - \frac{\rho'_p\big(\exp(\Phi_p x - (b/\mu)\sqrt{(p+\eta-\alpha\mu)^2 + 4\mu\alpha p}) - \exp(-\rho_p x)\big)}{\Phi'_q \exp(\Phi_q b) - \rho'_q \exp(-\rho_q b)}.
\end{aligned}$$

4. Optimal prediction of $g_{\tau_b^+}$

In this section, we consider the optimal prediction problem for $g_{\tau_b^+}$, that is, for fixed $b > 0$ and $x \in \mathbb{R}$,

$$\inf_{\tau\in\mathcal{T}} \mathbb{E}_x\big[\big|g_{\tau_b^+} - \tau\big|\big], \tag{4.1}$$

where $\mathcal{T}$ is the set of all stopping times. To avoid triviality (i.e. the mean of $g_{\tau_b^+}$ being infinite), we assume that X drifts to infinity. Indeed, if X does not drift to infinity, we obtain from Theorem 3.1 that $\mathbb{E}_x\big[g_{\tau_b^+}\big] = \infty$. Thus, for any stopping time τ with finite mean,

$$\mathbb{E}\big[\big|g_{\tau_b^+} - \tau\big|\big] \geq \mathbb{E}\big[g_{\tau_b^+}\big] - \mathbb{E}[\tau] = \infty.$$

Hence, in the context of a minimization problem with stopping times that have a finite mean, any stopping time can provide a solution.

The following theorem presents the solution to the optimal prediction problem (4.1). Its proof is postponed to Section 5. Recall from (2.9) that $F(x) = \psi'(0+)W(x)$ is the cumulative distribution function of $-\underline{X}_\infty$, and it is continuous and strictly increasing in $x \in (0, \infty)$. From (2.2), we know that $F(0) = 0$ if X is of unbounded variation, and $F(0) = \psi'(0+)/\delta \in (0, 1)$ when X is of bounded variation.

Theorem 4.1. *Suppose that X is a spectrally negative Lévy process that drifts to infinity. We define*

$$y_b^* = \inf\left\{y \in [0, b] \colon \int_{[0,y]} F(y-z)\, F(\mathrm{d}z) \geq \frac{F(b)}{2}\right\},$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

where we follow the convention that $\inf \emptyset = b$. *Then the stopping time* $\tau_{y_b^*} = \inf\{t \geq 0 \colon X_t \geq y_b^*\}$ *is optimal for* (4.1) *for any* $x \in \mathbb{R}$. *More specifically:*

(i) *If* $F(0) \geq \sqrt{\frac{1}{2}}$, $y_b^* = 0$ *for all* $b > 0$.

(ii) *If* $\frac{1}{2} \leq F(0) < \sqrt{\frac{1}{2}}$ *then, for* $b \leq F^{-1}(2F(0)^2)$, $y_b^* = 0$. *For* $b > F^{-1}(2F(0)^2)$, *the value* $y_b^* \in (0, b)$ *is the unique solution to*

$$\int_{[0,y]} F(y-z)\, F(\mathrm{d}z) - \frac{F(b)}{2} = 0.$$

(iii) *If* $F(0) < \frac{1}{2}$, *let* $b_0 > 0$ *be the unique solution to*

$$\int_{[0,b]} F(b-z)\, F(\mathrm{d}z) - \frac{F(b)}{2} = 0.$$

For $b \leq b_0$, $y_b^* = b$. *For* $b > b_0$, $y_b^* \in (0, b)$ *is the unique solution to*

$$\int_{[0,y]} F(y-z)\, F(\mathrm{d}z) - \frac{F(b)}{2} = 0.$$

Remark 4.1. Since F is the cumulative distribution function of $-\underline{X}_\infty$, the function $y \mapsto \int_{[0,y]} F(y-z)\, F(\mathrm{d}z)$ is actually the cumulative distribution function of $-\underline{X}_\infty - \underline{Y}_\infty$, where $\underline{Y}_\infty$ is an independent copy of $\underline{X}_\infty$. Therefore, Theorem 4.1 indicates that the optimal stopping boundary y_b^* corresponds to the minimum in between b and the $F(b)/2$-quantile of the random variable $-\underline{X}_\infty - \underline{Y}_\infty$.

In [3], the optimally predicting problem for the last zero $g = \sup\{t \geq 0 \colon X_t \leq 0\}$, i.e.

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}_x[|g - \tau|], \tag{4.2}$$

is solved when X is a spectrally negative Lévy process. Upon assuming that the Lévy measure Π satisfies $\int_{(-\infty,-1)} y^2\, \Pi(\mathrm{d}y) < \infty$, which ensures the existence of the first moment of g, the stopping time $\tau_{a^*} = \inf\{t \geq 0 \colon X_t \geq a^*\}$ is optimal for (4.2), where

$$a^* = \inf\left\{y \geq 0 \colon \int_{[0,y]} F(y-z)\, F(\mathrm{d}z) \geq \frac{1}{2}\right\}.$$

Since $\lim_{b\to\infty} F(b) = 1$, we obtain the following corollary which verifies the convergence of the optimal stopping boundary.

Corollary 4.1. $\lim_{b\to\infty} y_b^* = a^*$.

Example 4.1. Suppose that X is a Brownian motion with positive drift, i.e. $X_t = \mu t + \sigma B_t$, where $\mu > 0$, $\sigma > 0$, and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. It is known from (3.11) and $F(x) = \psi'(0+)W(x) = \mu W(x)$ that $F(x) = 1 - \exp(-2\mu x/\sigma^2)$, $x \geq 0$. That is, $-\underline{X}_\infty \sim \mathrm{Exp}(2\mu/\sigma^2)$, which implies that $\int_{[0,y]} F(y-z)\, F(\mathrm{d}z)$ corresponds to the cumulative distribution function of a Gamma$(2, 2\mu/\sigma^2)$ random variable given by

$$\int_{[0,y]} F(y-z)\, F(\mathrm{d}z) = 1 - \frac{2\mu y}{\sigma^2} \exp\left(-\frac{2\mu y}{\sigma^2}\right) - \exp\left(-\frac{2\mu y}{\sigma^2}\right), \quad y \geq 0.$$

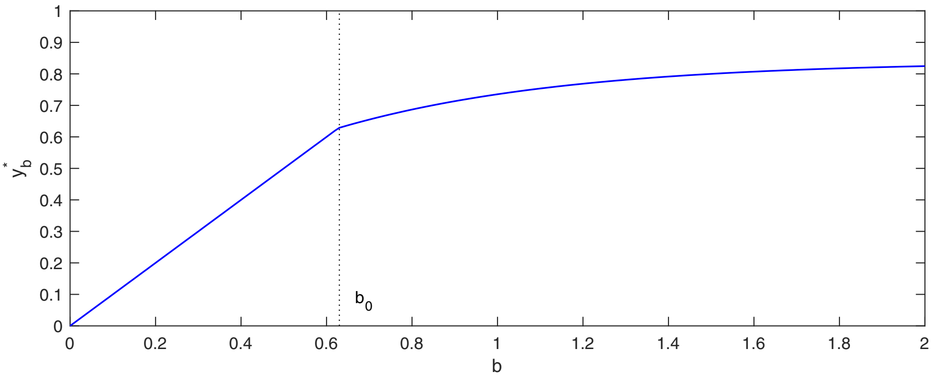


FIGURE 2. Optimal stopping boundary y_b^* for the drifted Brownian motion model with $\mu = 1 = \sigma$.

Hence, it follows from Theorem 4.1 that the optimal stopping boundary is

$$y_b^* = \inf\left\{y \in [0, b]\colon 1 - \frac{2\mu y}{\sigma^2}\exp\left(-\frac{2\mu y}{\sigma^2}\right) - \exp\left(-\frac{2\mu y}{\sigma^2}\right) \geq \frac{1-\exp(-2\mu b/\sigma^2)}{2}\right\}.$$

Figure 2 plots the mapping $b \mapsto y_b^*$ with $\mu = 1 = \sigma$. Note that X is of unbounded variation with $F(0) = 0$. Consistent with Theorem 4.1(iii) and Corollary 4.1, it can be seen that y_b^* increases linearly for $b < b_0$, and y_b^* converges to the value $a^* \approx 0.8391$ when $b \to \infty$.

Example 4.2. Suppose that X follows a Cramér–Lundberg model with exponential jumps, i.e. $X_t = \mu t - \sum_{i=1}^{N_t} C_i$, where $\mu > 0$, $\{N_t\}_{\{t\geq 0\}}$ is a Poisson process with intensity $\eta > 0$, and $\{C_i\}_{i\in\mathbb{N}}$ are i.i.d. exponential random variables with parameter $\alpha > 0$, which are independent of N_t. In this case, X is a spectrally negative Lévy process drifting to infinity by assuming $\psi'(0+) = \mu - (\eta/\alpha) > 0$. It follows from (3.12) and $F(x) = \psi'(0+)W(x) = (\mu - (\eta/\alpha))W(x)$ that

$$F(x) = 1 - \frac{\eta}{\alpha\mu}\exp\left(-\left(\alpha - \frac{\eta}{\mu}\right)x\right), \quad x \geq 0.$$

In particular, $F(0) = 1 - (\eta/\alpha\mu) > 0$. Hence

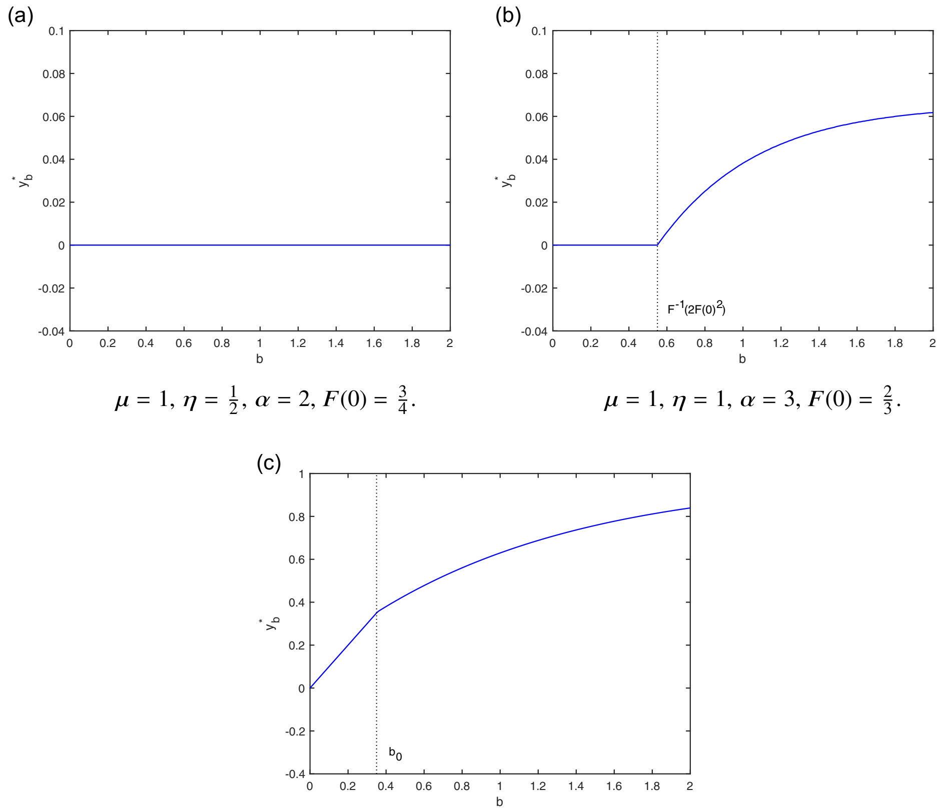
$$\int_{[0,y]} F(y-z)\,F(\mathrm{d}z) = 1 + \frac{\eta}{\alpha\mu}\left(\frac{\eta}{\alpha\mu} - 2\right)\exp\left(-\left(\alpha - \frac{\eta}{\mu}\right)y\right) - \left(\frac{\eta}{\alpha\mu}\right)^2\left(\alpha - \frac{\eta}{\mu}\right)y\exp\left(-\left(\alpha - \frac{\eta}{\mu}\right)y\right), \quad y \geq 0.$$

The optimal boundary follows from Theorem 4.1:

$$y_b^* = \inf\left\{y \in [0, b]\colon \int_{[0,y]} F(y-z)\,F(\mathrm{d}z) \geq \frac{1-(\eta/\alpha\mu)\exp(-(\alpha-(\eta/\mu))b)}{2}\right\}.$$

Figure 3 depicts the mapping $b \mapsto y_b^*$ under the Cramér–Lundberg model for the three cases in Theorem 4.1. Note that in this case X is of bounded variation with $F(0) = 1 - (\eta/\alpha\mu) > 0$. The parameters in Figure 3(a) are set to $\mu = 1$, $\eta = \frac{1}{2}$, $\alpha = 2$, which implies $F(0) = \frac{3}{4}$. From Theorem 4.1(i), $y_b^* = 0$ for any $b > 0$. In Figure 3(b), we set $\mu = 1$, $\eta = 1$, $\alpha = 3$, which implies

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press



$\mu = 1,\ \eta = \frac{1}{2},\ \alpha = 2,\ F(0) = \frac{3}{4}.$ $\mu = 1,\ \eta = 1,\ \alpha = 3,\ F(0) = \frac{2}{3}.$

$\mu = 1,\ \eta = 2,\ \alpha = 3,\ F(0) = \frac{1}{3}.$

FIGURE 3. Optimal stopping boundary y_b^* for the Cramér–Lundberg model with exponential jumps.

$F(0) = \frac{2}{3} \in \big[\frac{1}{2}, \sqrt{\frac{1}{2}}\big)$. From Theorem 4.1(ii), $y_b^* = 0$ for $b < F^{-1}(2F(0)^2)$, and then increases with b. It converges to $a^* \approx 0.0656$ as $b \to \infty$ according to Corollary 4.1. In Figure 3(c), we set $\mu = 1$, $\eta = 2$, $\alpha = 3$, which implies $F(0) = \frac{1}{3}$. From Theorem 4.1(iii) and Corollary 4.1, we see that $y_b^* = b$ for $b < b_0$, and y_b^* converges to the value $a^* \approx 0.9711$ when $b \to \infty$.

5. Proof of Theorem 4.1

Since the random time $g_{\tau_b^+}$ is only $\mathbb{F}$-measurable, we first use the following lemma to establish an equivalence between the optimal prediction problem (4.1) and an optimal stopping problem.

Lemma 5.1. *For any* $\tau \in \mathcal{T}$, $\mathbb{E}_x\big[|g_{\tau_b^+} - \tau|\big] = \mathbb{E}_x\big[g_{\tau_b^+}\big] + \mathbb{E}_x\big[\int_0^\tau G_b(s, X_s)\,\mathrm{d}s\big]$, *where* $G_b(s, x) := 2\big[\mathbf{1}_{\{\tau_b^+ \le s\}} + \mathbf{1}_{\{\tau_b^+ > s\}} W(x)/W(b)\big] - 1$. *Then the optimal prediction problem (4.1) is equivalent to the optimal stopping problem*

$$V_b(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\bigg[\int_0^\tau G_b(s, X_s)\,\mathrm{d}s\bigg]. \tag{5.1}$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Proof. For any stopping time $\tau \in \mathcal{T}$ and $x \in \mathbb{R}$, $|g_{\tau_b^+} - \tau| = \int_0^\tau (2\mathbf{1}_{\{g_{\tau_b^+} \le s\}} - 1)\,\mathrm{d}s + g_{\tau_b^+}$. From Fubini's theorem and the tower property of conditional expectations, it follows that

$$\begin{aligned}\mathbb{E}_x\big[\big|g_{\tau_b^+} - \tau\big|\big] &= \mathbb{E}_x\bigg[\int_0^\tau (2\mathbf{1}_{\{g_{\tau_b^+} \le s\}} - 1)\,\mathrm{d}s\bigg] + \mathbb{E}_x\big[g_{\tau_b^+}\big] \\ &= \mathbb{E}_x\bigg[\int_0^\infty \mathbf{1}_{\{s<\tau\}}\big(2\mathbb{E}_x\big[\mathbf{1}_{\{g_{\tau_b^+} \le s\}} \mid \mathcal{F}_s\big] - 1\big)\,\mathrm{d}s\bigg] + \mathbb{E}_x\big[g_{\tau_b^+}\big] \\ &= \mathbb{E}_x\bigg[\int_0^\tau (2\mathbb{P}_x\big(g_{\tau_b^+} \le s \mid \mathcal{F}_s\big) - 1)\,\mathrm{d}s\bigg] + \mathbb{E}_x\big[g_{\tau_b^+}\big]. \end{aligned} \tag{5.2}$$

Note that in the event of $\{\tau_b^+ \le s\}$ we have $g_{\tau_b^+} \le \tau_b^+ \le s$ and thus

$$\mathbb{P}_x\big(g_{\tau_b^+} \le s \mid \mathcal{F}_s\big) = \mathbf{1}_{\{\tau_b^+ \le s\}} + \mathbb{P}_x\big(g_{\tau_b^+} \le s,\ \tau_b^+ > s \mid \mathcal{F}_s\big). \tag{5.3}$$

Further, the event $\{g_{\tau_b^+} \le s,\ \tau_b^+ > s\}$ is equal to $\{X_u \ge 0 \text{ for all } u \in [s, \tau_b^+],\ \tau_b^+ > s\}$ (up to a $\mathbb{P}$-null set). Hence, for all $s \ge 0$,

$$\begin{aligned}\mathbb{P}_x\big(g_{\tau_b^+} \le s,\ \tau_b^+ > s \mid \mathcal{F}_s\big) &= \mathbb{P}_x(X_u \ge 0 \text{ for all } u \in [s, \tau_b^+],\ \tau_b^+ > s \mid \mathcal{F}_s) \\ &= \mathbf{1}_{\{\tau_b^+ > s\}}\mathbb{P}_x\Big(\inf_{0 \le u \le \tau_b^+ - s} X_{u+s} \ge 0 \mid \mathcal{F}_s\Big) \\ &= \mathbf{1}_{\{\tau_b^+ > s\}}\mathbb{P}_{X_s}\big(\underline{X}_{\tau_b^+} \ge 0\big) \\ &= \mathbf{1}_{\{\tau_b^+ > s\}}\mathbb{P}_{X_s}\big(\tau_b^+ < \tau_0^-\big) = \mathbf{1}_{\{\tau_b^+ > s\}}\frac{W(X_s)}{W(b)},\end{aligned}$$

where we recall that $\underline{X}_t = \inf_{0 \le s \le t} X_s$, and the last equality follows from the Markov property of X and (2.8). Substituting into (5.3) yields

$$\mathbb{P}_x\big(g_{\tau_b^+} \le s \mid \mathcal{F}_s\big) = \mathbf{1}_{\{\tau_b^+ \le s\}} + \mathbf{1}_{\{\tau_b^+ > s\}}\frac{W(X_s)}{W(b)}.$$

Further, substituting this expression into (5.2) yields, for any $x \in \mathbb{R}$,

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}_x\big[\big|g_{\tau_b^+} - \tau\big|\big] = \mathbb{E}_x\big[g_{\tau_b^+}\big] + \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\bigg[\int_0^\tau \bigg(2\bigg[\mathbf{1}_{\{\tau_b^+ \le s\}} + \mathbf{1}_{\{\tau_b^+ > s\}}\frac{W(X_s)}{W(b)}\bigg] - 1\bigg)\,\mathrm{d}s\bigg],$$

completing the assertion of the proof. □

We then focus on the optimal stopping problem (5.1). The next lemma provides some basic properties of the value function V_b.

Lemma 5.2.

(i) *$V_b(x)$ is non-decreasing in x.*

(ii) *$V_b(x) \in (-\infty, 0]$ for all $x \in \mathbb{R}$. In particular, $V_b(x) = 0$ for all $x \ge b$.*

(iii) *For any $x \in \mathbb{R}$ and $b > 0$, we can write*

$$V_b(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\bigg[\int_0^{\tau \wedge \tau_b^+} \bigg(2\frac{W(X_s)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg]. \tag{5.4}$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

(iv) *Fix $b > 0$. Let $D = \{x \in \mathbb{R}: V_b(x) = 0\}$ and $\tau_D = \inf\{t \geq 0: X_t \in D\}$. Then D is a closed set and τ_D is an optimal stopping time for (5.1).*

Proof. (i). For $x \in \mathbb{R}$ and $b > 0$, by the spatial homogeneity of X,

$$V_b(x) = \inf_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau \left(2\left[\mathbf{1}_{\{\tau_{b-x}^+ \leq s\}} + \mathbf{1}_{\{\tau_{b-x}^+ > s\}} \frac{W(X_s + x)}{W(b)} \right] - 1 \right) \mathrm{d}s \right].$$

Fix $s \geq 0$. We define the auxiliary function

$$F_s(x, b) = \mathbf{1}_{\{\tau_{b-x}^+ \leq s\}} + \mathbf{1}_{\{\tau_{b-x}^+ > s\}} \frac{W(X_s + x)}{W(b)}.$$

We then show that, for fixed $b > 0$, the mapping $x \mapsto F_s(x, b)$ is non-decreasing. Take $x_1 \leq x_2$, then $\tau_{b-x_2}^+ \leq \tau_{b-x_1}^+$. First, if $\omega \in \left\{ s < \tau_{b-x_2}^+ \right\}$, then we have $\omega \in \left\{ s < \tau_{b-x_1}^+ \right\}$ and

$$F_s(x_1, b)(\omega) = \frac{W(X_s + x_1)}{W(b)} \leq \frac{W(X_s + x_2)}{W(b)} = F_s(x_2, b)(\omega),$$

where we used that W is increasing. Second, if $\omega \in \left\{ \tau_{b-x_2}^+ \leq s < \tau_{b-x_1}^+ \right\}$, we have $X_s(\omega) \leq b - x_1$ and then

$$F_s(x_1, b)(\omega) = \frac{W(X_s + x_1)}{W(b)} \leq 1 = F_s(x_2, b)(\omega).$$

Lastly, if $\omega \in \left\{ \tau_{b-x_2}^+ < \tau_{b-x_1}^+ \leq s \right\}$, we have $F_s(x_1, b)(\omega) = 1 = F_s(x_2, b)(\omega)$. Therefore, by integrating $2F_s(x, b) - 1$ with respect to $\mathrm{d}s \times \mathbb{P}$, and taking the infimum upon all stopping times, we deduce that $x \mapsto V_b(x)$ is non-decreasing.

(ii) and (iii). By taking $\tau \equiv 0$ in the definition of V_b (see (5.1)), we deduce that $V_b(x) \leq 0$ for all $x \in \mathbb{R}$. Moreover, for $x \geq b$, it is obvious that $\inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left[\left| g_{\tau_b^+} - \tau \right| \right] = \inf_{\tau \in \mathcal{T}} \mathbb{E}_x[\tau] = 0$, where the infimum is attained by $\tau \equiv 0$. It follows from Lemma 5.1 that the stopping time $\tau \equiv 0$ is also optimal for (5.1), and then $V_b(x) = 0$ when $x \geq b$.

Next, we proceed to show that (5.4) holds. By (5.1),

$$\begin{aligned} V_b(x) &= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left[\int_0^\tau \left(2\left[\mathbf{1}_{\{\tau_b^+ \leq s\}} + \mathbf{1}_{\{\tau_b^+ > s\}} \frac{W(X_s)}{W(b)} \right] - 1 \right) \mathrm{d}s \right] \\ &= \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left[\int_0^\tau \left(\mathbf{1}_{\{\tau_b^+ \leq s\}} + \mathbf{1}_{\{\tau_b^+ > s\}} \left[2\frac{W(X_s)}{W(b)} - 1 \right] \right) \mathrm{d}s \right] \\ &\geq \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left[\int_0^{\tau \wedge \tau_b^+} \left(2\frac{W(X_s)}{W(b)} - 1 \right) \mathrm{d}s \right]. \end{aligned}$$

On the other hand, from the definition of $V_b(x)$ in (5.1), for an arbitrary stopping time τ and any $x \in \mathbb{R}$,

$$V_b(x) \leq \mathbb{E}_x\left[\int_0^{\tau \wedge \tau_b^+} \left(2\left[\mathbf{1}_{\{\tau_b^+ \leq s\}} + \mathbf{1}_{\{\tau_b^+ > s\}} \frac{W(X_s)}{W(b)} \right] - 1 \right) \mathrm{d}s \right] = \mathbb{E}_x\left[\int_0^{\tau \wedge \tau_b^+} \left(2\frac{W(X_s)}{W(b)} - 1 \right) \mathrm{d}s \right].$$

Thus, we deduce that, for any $x \in \mathbb{R}$,

$$V_b(x) \leq \inf_{\tau \in \mathcal{T}} \mathbb{E}_x\left[\int_0^{\tau \wedge \tau_b^+} \left(2\frac{W(X_s)}{W(b)} - 1 \right) \mathrm{d}s \right].$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

The claim then follows.

It remains to show that $V_b(x) > -\infty$ for all $x \in \mathbb{R}$. For all $s < \tau_b^+$, because W is increasing and non-negative,

$$-1 \le 2\frac{W(X_s)}{W(b)} - 1 \le 1.$$

Thus, by differentiating (2.7) and setting $q = 0$, we deduce that, for all $x \in \mathbb{R}$ and $b > 0$,

$$\mathbb{E}_x\bigg[\sup_{t\ge 0}\bigg|\int_0^{t\wedge\tau_b^+}\bigg(2\frac{W(X_s)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg|\bigg] \le \mathbb{E}_x[\tau_b^+] = \frac{b-x}{\psi'(0+)} < \infty,$$

where we used that $\psi(\Phi_q) = q$ so that $\Phi_q' = 1/\psi'(\Phi_q)$. Therefore, $V_b(x) > -\infty$ for all $x \in \mathbb{R}$.

(iv). First, we show that $x \mapsto V_b(x)$ is upper semicontinuous. Note that it suffices to take the infimum in the definition of V_b over stopping times with finite mean. Indeed, for any stopping time τ with $\mathbb{E}[\tau] = \infty$, since $\mathbb{E}[\tau_b^+] < \infty$, we have that

$$\mathbb{E}_x\bigg[\int_0^{\tau}\bigg(\mathbf{1}_{\{\tau_b^+\le s\}} + \mathbf{1}_{\{\tau_b^+ > s\}}\bigg[2\frac{W(X_s)}{W(b)} - 1\bigg]\bigg)\,\mathrm{d}s\bigg] \ge -\mathbb{E}\big[\tau\wedge\tau_b^+\big] + \mathbb{E}\big[\tau - \big(\tau\wedge\tau_b^+\big)\big] = \infty.$$

Due to the fact that the infimum of an upper semicontinuous function is upper semicontinuous, it suffices to show that, for each $\tau \in \mathcal{T}$ with finite mean, the mapping

$$x \mapsto \mathbb{E}\bigg[\int_0^{\tau}(2F_s(x, b) - 1)\,\mathrm{d}s\bigg] \tag{5.5}$$

is upper semicontinuous. Since $x \mapsto F_s(x, b)$ is non-decreasing as shown in part (i), the upper semicontinuity follows if the mapping given in (5.5) is right-continuous. We then show the right continuity of (5.5). Note that the stochastic process $\{\tau_t^+ : t \ge 0\}$ is a subordinator (see, e.g., [17, Corollary 3.14]) and thus stochastically continuous. Indeed, the stochastic continuity of $\{\tau_t^+, t \ge 0\}$ implies that, for any $h > 0$, $y > 0$, and $\varepsilon > 0$, $\lim_{h\downarrow 0}\mathbb{P}\big(\big|\tau_{y-h}^+ - \tau_y^+\big| > \varepsilon\big) = 0$. In other words, τ_{y-h}^+ converges in probability to τ_y^+ when $h \downarrow 0$. Moreover, since the mapping $h \mapsto \tau_{y-h}^+$ is decreasing, we deduce that the convergence holds almost surely. Thus, we obtain, by the right continuity of W,

$$\begin{aligned}\lim_{h\downarrow 0} F_s(x+h, b) &= \lim_{h\downarrow 0}\bigg[\mathbf{1}_{\{\tau_{b-x-h}^+\le s\}} + \mathbf{1}_{\{\tau_{b-x-h}^+ > s\}}\frac{W(X_s + x + h)}{W(b)}\bigg] \\ &= \bigg[\mathbf{1}_{\{\tau_{b-x}^+\le s\}} + \mathbf{1}_{\{\tau_{b-x}^+ > s\}}\frac{W(X_s + x)}{W(b)}\bigg] = F_s(x, b)\end{aligned}$$

almost surely. Hence, from the dominated convergence theorem we deduce that, for any stopping time τ with finite mean,

$$\lim_{h\downarrow 0}\mathbb{E}\bigg[\int_0^{\tau}(2F_s(x+h, b) - 1)\,\mathrm{d}s\bigg] = \mathbb{E}\bigg[\int_0^{\tau}(2F_s(x, b) - 1)\,\mathrm{d}s\bigg].$$

Therefore, the mapping $x \mapsto \mathbb{E}\big[\int_0^{\tau}(2F_s(x, b) - 1)\,\mathrm{d}s\big]$ is right-continuous and thus upper semicontinuous as claimed. Hence, we have that D is a closed set.

Next, we proceed to show that τ_D is optimal for V_b. For $a \in \mathbb{R}$, $y > 0$, and $x \le y$, we define the stochastic process $\{L_t^{a,y,x}, t \ge 0\}$ as follows:

$$L_t^{a,y,x} = a + \int_0^t \mathbf{1}_{\{y\vee(\overline{X}_s + x) < b\}}\bigg(2\frac{W(X_s + x)}{W(b)} - 1\bigg)\,\mathrm{d}s,$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

where $\overline{X}_t = \sup_{0\le r\le t} X_r$. For any $t \ge 0$, we simply write $L_t = L_t^{0,0,0}$. It can be seen that the family of probability measures $\mathbb{P}_{a,y,x} = \text{Law}((L^{a,y,x}, y \vee (\overline{X} + x), X + x) \mid \mathbb{P})$ is Markovian (see [21, Section III.6]). Then we have that, for any $a \in \mathbb{R}$, $y > 0$, and $x \le y$,

$$\begin{aligned}
\widetilde{V}(a, y, x) := \inf_{\tau\in\mathcal{T}} \mathbb{E}_{a,y,x}[L_\tau] = \inf_{\tau\in\mathcal{T}} \mathbb{E}\big[L_\tau^{a,y,x}\big] & \\
&= a + \inf_{\tau\in\mathcal{T}} \mathbb{E}\bigg[\int_0^\tau \mathbf{1}_{\{y\vee(\overline{X}_s+x)<b\}}\bigg(2\frac{W(X_s+x)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg] \\
&= a + \inf_{\tau\in\mathcal{T}} \mathbb{E}\bigg[\int_0^\tau \mathbf{1}_{\{y<b\}}\mathbf{1}_{\{\overline{X}_s+x<b\}}\bigg(2\frac{W(X_s+x)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg] \\
&= a + \mathbf{1}_{\{y<b\}} \inf_{\tau\in\mathcal{T}} \mathbb{E}_x\bigg[\int_0^\tau \mathbf{1}_{\{\overline{X}_s<b\}}\bigg(2\frac{W(X_s)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg] \\
&= a + \mathbf{1}_{\{y<b\}} V_b(x).
\end{aligned}$$

Since $V_b(x)$ is non-positive and upper semicontinuous, it follows that $\widetilde{V}$ is upper semicontinuous. From the general theory of optimal stopping (see, e.g., [21, Corollary 2.9]), we deduce that an optimal stopping time for $\widetilde{V}$, and hence also optimal for V_b, is given by

$$\inf\{t \ge 0\colon \widetilde{V}(L_t, \overline{X}_t, X_t) = L_t\} = \inf\{t \ge 0\colon L_t + \mathbf{1}_{\{\overline{X}_t<b\}} V_b(X_t) = L_t\} = \tau_D \wedge \tau_b^+ = \tau_D,$$

where in the last equality we used that $V_b(x) = 0$ for all $x \ge b$, so then $[b, \infty) \subset D$. The proof is now complete. □

Since V_b is a non-decreasing function and D is a closed set we deduce that $D = [y_b^*, \infty)$ for some value y_b^* to be determined. Moreover, since W vanishes on $(-\infty, 0)$, we deduce from (5.4) that, for any $x < 0$,

$$V_b(x) \le \mathbb{E}_x\bigg[\int_0^{\tau_0^+}\bigg(2\frac{W(X_s)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg] = -\mathbb{E}_x[\tau_0^+] < 0.$$

This implies that $y_b^* \ge 0$, and then $y_b^* \in [0, b]$. Hence, we deduce that $\tau_D \le \tau_b^+$ and, from (5.4),

$$V_b(x) = \inf_{y\in[0,b]} V_{b,y}(x) = V_{b,y_b^*}(x) \tag{5.6}$$

for all $x \in \mathbb{R}$ and $b > 0$, where

$$V_{b,y}(x) := \mathbb{E}_x\bigg[\int_0^{\tau_y^+}\bigg(2\frac{W(X_s)}{W(b)} - 1\bigg)\,\mathrm{d}s\bigg].$$

The following lemma ensures that the convolution of W with W (and W') is sufficiently smooth. These facts will be helpful in the upcoming lemmas.

Lemma 5.3. *Let $f\colon \mathbb{R} \mapsto \mathbb{R}^+$ be any non-negative function such that $\int_0^y f(z)\,\mathrm{d}z < \infty$ for any $y \ge 0$. Further assume that $W \in C^1(0, \infty)$ and $f \in C(0, \infty)$. Then, the convolution function $J(y) := \int_0^y W(y-z)f(z)\,\mathrm{d}z$, $y \ge 0$, belongs to $C^1(0, \infty)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}y}J(y) = W(0)f(y) + \int_0^y W'(y-z)f(z)\,\mathrm{d}z.$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Proof. For any $u \geq 0$ and $y > 0$ we define the auxiliary function

$$T(u, y) := \int_0^y W'(y - z + u)f(z)\,\mathrm{d}z.$$

We first prove that T is continuous in each argument. That is, for fixed $u \geq 0$, the mapping $y \mapsto T(u, y)$ is continuous on $(0, \infty)$ and, for fixed $y > 0$, the mapping $u \mapsto T(u, y)$ is continuous on $[0, \infty)$. For arbitrarily fixed $y > 0$ and $u \geq 0$, let $\delta > 0$ and $\varepsilon > 0$ be such that $\delta < y$ and $u < \varepsilon$. Then, for any $z \in (0, \delta]$,

$$\begin{aligned} W'(y - z + u)f(z) &\leq f(z) \sup\nolimits_{v \in [y-\delta+u, y+u]} W'(v) \\ &\leq f(z) \sup\nolimits_{v \in [y-\delta, y+\varepsilon]} W'(v) = C_1 f(z), \end{aligned}$$

where $C_1 := \sup_{u \in [y-\delta, y+\varepsilon]} W'(u)$. It follows from the continuity of W' on $[y - \delta, y + \varepsilon]$ that $0 < C_1 < \infty$. On the other hand, for any $z \in (\delta, y]$ we see that

$$W'(y - z + u)f(z) \leq W'(y - z + u) \sup\nolimits_{v \in [\delta, y]} f(v) = C_2 W'(y - z + u),$$

where $C_2 := \sup_{v \in [\delta, y]} f(v)$ and we used that $W'(x) > 0$ for all $x > 0$. Since f is a non-negative and continuous function in $(0, \infty)$, $0 < C_2 < \infty$. Hence, since W' and f are non-negative we obtain that, for any $z \in (0, y)$,

$$0 \leq W'(y - z + u)f(z) \leq C_1 f(z) + C_2 W'(y - z + u).$$

Moreover,

$$\int_0^y [C_1 f(z) + C_2 W'(y - z + u)]\,\mathrm{d}z = C_1 \int_0^y f(z)\,\mathrm{d}z + C_2[W(y + u) - W(u)] < \infty.$$

Therefore, since W' is continuous on $(0, \infty)$, we deduce from the extended dominated convergence theorem that T is continuous in each argument.

Next, for any $y > 0$, we prove

$$\lim_{h \downarrow 0} \frac{1}{h}[J(y + h) - J(y)] = W(0)f(y) + \int_0^y W'(y - z)f(z)\,\mathrm{d}z.$$

For any $h > 0$ and $y > 0$, it follows that

$$\begin{aligned} &\frac{1}{h}[J(y + h) - J(y)] \\ &\quad = \frac{1}{h}\bigg(\int_0^{y+h} W(y + h - z)f(z)\,\mathrm{d}z - \int_0^y W(y - z)f(z)\,\mathrm{d}z \bigg) \\ &\quad = \frac{1}{h}\bigg(\int_0^y [W(y + h - z) - W(y - z)]f(z)\,\mathrm{d}z + \int_y^{y+h} W(y + h - z)f(z)\,\mathrm{d}z \bigg). \end{aligned} \tag{5.7}$$

For the first term on the right-hand side we see from Fubini's theorem that

$$\begin{aligned} \frac{1}{h}\int_0^y [W(y + h - z) - W(y - z)]f(z)\,\mathrm{d}z &= \frac{1}{h}\int_0^y \int_0^h W'(y - z + u)f(z)\,\mathrm{d}u\,\mathrm{d}z \\ &= \frac{1}{h}\int_0^h T(u, y)\,\mathrm{d}u. \end{aligned}$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Therefore, by the fundamental theorem of calculus we see that, for any $y > 0$,

$$\lim_{h\downarrow 0} \frac{1}{h} \int_0^y [W(y+h-z) - W(y-z)]f(z)\,\mathrm{d}z = T(0, y) = \int_0^y W'(y-z)f(z)\,\mathrm{d}z.$$

For the second term on the right-hand side of (5.7), since W is strictly increasing on $[0, \infty)$, we have

$$\frac{1}{h}\int_y^{y+h} W(y+h-z)f(z)\,\mathrm{d}z \le \frac{1}{h}W(h)\int_y^{y+h} f(z)\,\mathrm{d}z,$$
$$\frac{1}{h}\int_y^{y+h} W(y+h-z)f(z)\,\mathrm{d}z \ge W(0)\frac{1}{h}\int_y^{y+h} f(z)\,\mathrm{d}z.$$

Thus, we can deduce from the fact that f and W are continuous on $(0, \infty)$ and the fundamental theorem of calculus that

$$\lim_{h\downarrow 0}\frac{1}{h}\int_y^{y+h} W(y+h-z)f(z)\,\mathrm{d}z = W(0)f(y).$$

Therefore, we conclude that, for any $y > 0$,

$$\lim_{h\downarrow 0}\frac{1}{h}[J(y+h) - J(y)] = W(0)f(y) + \int_0^y W'(y-z)f(z)\,\mathrm{d}z.$$

Using similar arguments we can show that, for any $y > 0$,

$$\lim_{h\downarrow 0}\frac{1}{h}[J(y) - J(y-h)] = f(y)W(0) + \int_0^y W'(y-z)f(z)\,\mathrm{d}z.$$

Finally, the continuity of $(\mathrm{d}/\mathrm{d}y)J(y)$ on $(0, \infty)$ follows by the continuity of f and $T(0, y)$. □

The next lemma provides an analytical expression for the function $V_{b,y}$ in terms of the scale function W. We recall that $F(x) = \psi'(0+)W(x)$ is the cumulative distribution function of $-\underline{X}_\infty$.

Lemma 5.4. *For any $b > 0$, $y \ge 0$, and $x \le y$,*

$$V_{b,y}(x) = H_b(y) - H_b(x), \tag{5.8}$$

where $H_b(y) = (2/W(b))\int_0^y W(z)W(y-z)\,\mathrm{d}z - (y/\psi'(0+))$. Moreover, $V_b(x) = V_{b,y_b^*}(x)$, where y_b^* is given by

$$y_b^* = \inf\left\{y \in [0, b] : \int_{[0,y]} F(y-z)\,F(\mathrm{d}z) \ge \frac{F(b)}{2}\right\}. \tag{5.9}$$

Proof. Fix $y \ge 0$. For any $x \le y$, it follows from (2.10) that

$$\begin{aligned} V_{b,y}(x) &= \mathbb{E}_x\left[\int_0^{\tau_y^+}\left(2\frac{W(X_s)}{W(b)} - 1\right)\mathrm{d}s\right] \\ &= \int_{-\infty}^{y} 2\frac{W(z)}{W(b)}\int_0^\infty \mathbb{P}_x(X_s \in \mathrm{d}z,\, s < \tau_y^+)\,\mathrm{d}s - \mathbb{E}_x[\tau_y^+] \\ &= \frac{2}{W(b)}\int_{-\infty}^{y} W(z)[W(y-z) - W(x-z)]\,\mathrm{d}z - \frac{y-x}{\psi\prime(0+)} = H_b(y) - H_b(x), \end{aligned}$$

where in the last equality we used that W vanishes on $(-\infty, 0)$.

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Let y_b^* be such that $\inf_{y\in[0,b]} H_b(y) = H_b(y_b^*)$. We then show that y_b^* is given by (5.9). For $y \geq 0$, by Lemma 5.3, we have

$$\begin{aligned} H_b'(y) &= \frac{2}{W(b)} W(y)W(0) + \frac{2}{W(b)} \int_0^y W(z)W'(y-z)\,\mathrm{d}z - \frac{1}{\psi\prime(0+)} \\ &= \frac{2}{W(b)} W(y)W(0) + \frac{2}{W(b)} \int_0^y W(y-z)W'(z)\,\mathrm{d}z - \frac{1}{\psi\prime(0+)}. \end{aligned}$$

Since W is non-negative and strictly increasing, we deduce that $H_b'(y)$ is strictly increasing. Thus, the function $H_b(y)$ attains its infimum on $[0, b]$ at the point

$$\begin{aligned} &\inf\left\{y \in [0, b]\colon \frac{2}{W(b)} W(y)W(0) + \frac{2}{W(b)} \int_0^y W(y-z)W'(z)\,\mathrm{d}z - \frac{1}{\psi\prime(0+)} \geq 0\right\} \\ &\quad = \inf\left\{y \in [0, b]\colon \int_{[0,y]} F(y-z)\,F(\mathrm{d}z) \geq \frac{F(b)}{2}\right\} = y_b^*, \end{aligned}$$

where we used $F(z) = \psi'(0+)W(z)$ for all $z \geq 0$, and that the measure $F(\mathrm{d}z) = \psi\prime(0+)\,W(\mathrm{d}z)$ may have an atom at zero (when X is of bounded variation). Since y_b^* is defined within $[0, b]$, we follow the convention that $\inf \emptyset = b$.

It remains to verify that $V_b(x) = V_{b,y_b^*}(x)$. Since $V_{b,y}(x) = 0$ for $x > y$, we obtain from (5.6) and (5.8) that

$$\begin{aligned} V_b(x) = \inf_{y\in[0,b]} V_{b,y}(x) = 0 \wedge \inf_{y\in[x,b]} V_{b,y}(x) = 0 \wedge \left(\inf_{y\in[x,b]} H_b(y) - H_b(x)\right) \\ = \inf_{y\in[x,b]} H_b(y) - H_b(x). \end{aligned}$$

If $x \leq y_b^*$, it is clear that $V_b(x) = H_b(y_b^*) - H_b(x) = V_{b,y_b^*}(x)$. If $x > y_b^*$, we have $V_{b,y_b^*}(x) = 0$. Further, since $H_b(y)$ is strictly increasing for $y \in [y_b^*, b]$ and $x > y_b^*$, we have $\inf_{y\in[x,b]} H_b(y) = H_b(x)$ and thus $V_b(x) = 0$. □

Before we give a more explicit characterization of the optimal stopping boundary y_b^*, we introduce some auxiliary results. For $a \geq 0$, we define

$$K(a) = \int_{[0,a]} F(a-z)\,F(\mathrm{d}z) - \frac{F(a)}{2}.$$

Lemma 5.5. *We have the following results for the auxiliary function $K(a)$:*

(i) *If $F(0) \geq \frac{1}{2}$, $K(0) \geq 0$ and $K(a) > 0$ for all $a > 0$.*

(ii) *If $F(0) < \frac{1}{2}$, the equation $K(a) = 0$ has a unique solution $b_0 \in (0, \infty)$ with $K(a) < 0$ for all $a \in [0, b_0)$ and $K(a) > 0$ for all $a \in (b_0, \infty)$.*

Proof. First, since both $y \mapsto \int_{[0,y]} F(y-z)\,F(\mathrm{d}z)$ and F are cumulative distribution functions, it can be seen that $\lim_{a\to\infty} K(a) = \frac{1}{2}$. Hence, there exists $N > 0$ sufficiently large that $[N, \infty) \subset \{a \geq 0\colon K(a) \geq 0\}$.

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

From (2.4), we deduce that $F'(x)/F(x)$ is non-increasing. Therefore, for any $a > 0$, it follows from Lemma 5.3 that

$$\begin{aligned} K'(a) &= 2F(0)F'(a) + \int_0^a F'(a-z)F'(z)\,\mathrm{d}z - \frac{F'(a)}{2} \\ &= 2F(0)F'(a) + \int_0^a \frac{F'(a-z)}{F(a-z)} F(a-z)F'(z)\,\mathrm{d}z - \frac{F'(a)}{2} \\ &\geq 2F(0)F'(a) + \frac{F'(a)}{F(a)} \int_0^a F(a-z)F'(z)\,\mathrm{d}z - \frac{F'(a)}{2} \\ &= \frac{F'(a)}{F(a)}[F(a)F(0) + K(a)]. \end{aligned}$$

In fact, this inequality is strict when X is of unbounded variation. If not, we have

$$\int_0^a \frac{F'(a-z)}{F(a-z)} F(a-z)F'(z)\,\mathrm{d}z = \frac{F'(a)}{F(a)} \int_0^a F(a-z)F'(z)\,\mathrm{d}z,$$

for some $a \geq 0$. Since F and F' are strictly positive on $(0, \infty)$, we deduce that $F'(y)/F(y) = C$ for some $C \geq 0$ and all $y \in (0, a)$. This implies that $F(y) = B\mathrm{e}^{Cy}$ for some $B \geq 0$ and any $y \in (0, a)$. Using the fact that X is of unbounded variation, so then $F(0+) = \psi'(0+)W(0+) = 0$, we deduce that $B = 0$. This contradicts the fact that F is strictly increasing. Hence, when X is of unbounded variation, we have

$$K'(a) > \frac{F'(a)}{F(a)} K(a). \tag{5.10}$$

Since $F(0) > 0$ when X is of bounded variation, the strict inequality (5.10) holds regardless of the path variation of X.

Further, it is easy to show from (5.10), the continuity of K and K', and the behaviour of K near 0 and at infinity that there exists a value $b_0 \geq 0$ such that $\{a \geq 0 \colon K(a) \geq 0\} = [b_0, \infty)$. Moreover, $K(a) < 0$ for all $a \in [0, b_0)$, and $K(a) > 0$ for all $a \in (b_0, \infty)$. In particular, if $F(0) \geq \frac{1}{2}$, $K(0) = F(0)^2 - F(0)/2 = F(0)\big(F(0) - \frac{1}{2}\big) \geq 0$, so that $b_0 = 0$. Otherwise, if $F(0) < \frac{1}{2}$, by the continuity of K and (5.10), we deduce that $b_0 > 0$ and b_0 is the unique solution to the equation $K(a) = 0$. □

The following lemma provides a more explicit characterization of the optimal stopping boundary y_b^*.

Lemma 5.6.

(i) *If* $F(0) \geq \sqrt{\frac{1}{2}}$, $y_b^* = 0$ *for all* $b > 0$.

(ii) *If* $\frac{1}{2} \leq F(0) < \sqrt{\frac{1}{2}}$, *then for* $b \leq F^{-1}(2F(0)^2)$, $y_b^* = 0$. *For* $b > F^{-1}(2F(0)^2)$, *the value* $y_b^* \in (0, b)$ *is the unique solution to* $\int_{[0,y]} F(y-z)\,F(\mathrm{d}z) - \frac{1}{2}F(b) = 0$.

(iii) *If* $F(0) < \frac{1}{2}$, *let* $b_0 > 0$ *be the unique solution to* $\int_{[0,b]} F(b-z)\,F(\mathrm{d}z) = \frac{1}{2}F(b)$. *For* $b \leq b_0$, $y_b^* = b$. *For* $b > b_0$, $y_b^* \in (0, b)$ *is the unique solution to*

$$\int_{[0,y]} F(y-z)\,F(\mathrm{d}z) - \frac{F(b)}{2} = 0.$$

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

Proof. We define the function $P_b(y) = \int_{[0,y]} F(y-z)\, F(\mathrm{d}z) - \frac{1}{2}F(b)$ for $y \geq 0$ and $b > 0$. Hence, $P_b(b) = K(b)$ and $y_b^* = \inf\{y \in [0, b] : P_b(y) \geq 0\}$.

(i). If $F(0) \geq \sqrt{\frac{1}{2}}$, $P_b(0) = F(0)^2 - \frac{1}{2}F(b) \geq \frac{1}{2}(1 - F(b)) \geq 0$ for all $b > 0$. Since P_b is strictly increasing, and from the definition of y_b^*, we deduce that $y_b^* = 0$ for any $b > 0$.

(ii). Suppose that $\frac{1}{2} \leq F(0) < \sqrt{\frac{1}{2}}$. From Lemma 5.5 we know that $K(a) > 0$ for all $a > 0$. This implies that $y_b^* < b$ for all $b > 0$. Note that $F^{-1}(2F(0)^2)$ is well defined since $2F(0)^2 \geq F(0)$. For $b \leq F^{-1}(2F(0)^2)$, for any $y \in (0, b]$, $P_b(y) > P_b(0) = F(0)^2 - \frac{1}{2}F(b) \geq 0$. Thus, we deduce that $y_b^* = 0$. On the other hand, for $b > F^{-1}(2F(0)^2)$, $P_b(0) < 0$. This, together with $P_b(b) = K(b) > 0$ and $P_b(y)$ being strictly increasing on $(0, \infty)$, mean we can deduce that y_b^* is the unique solution to the equation $P_b(y) = 0$.

(iii). Suppose that $F(0) < \frac{1}{2}$. From Lemma 5.5, we know that the value b_0 is such that $K(a) \leq 0$ for all $a \leq b_0$ and $K(a) > 0$ for all $a > b_0$. Hence, for $b \leq b_0$, $P_b(y) \leq P_b(b) = K(b) \leq 0$ for all $y \leq b$. Thus, $y_b^* = b$. Otherwise, for $b > b_0$,

$$P_b(0) = F(0)^2 - \frac{F(b)}{2} < \frac{F(0) - F(b)}{2} \leq 0,$$

and $P_b(b) = K(b) > 0$. Thus, the value y_b^* is the unique solution to the equation $P_b(y) = 0$. □

Acknowledgements

We would like to thank two anonymous referees for their helpful comments and suggestions.

Funding information

Xiaofeng Yang is supported by the China Scholarship Council (No. CSC202206370081) and Hunan Provincial Innovation Foundation for Postgraduate (No. CX20220247). Bin Li gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada (No. 04338).

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] ALBRECHER, H., IVANOVS, J. AND ZHOU, X. (2016). Exit identities for Lévy processes observed at Poisson arrival times. *Bernoulli* **22**, 1364–1382.
[2] BAURDOUX, E. J. (2009). Last exit before an exponential time for spectrally negative Lévy processes. *J. Appl. Prob.* **46**, 542–558.
[3] BAURDOUX, E. J. AND PEDRAZA, J. M. (2020). Predicting the last zero of a spectrally negative Lévy process. In *XIII Symposium on Probability and Stochastic Processes*. Birkhäuser, Cham, pp. 77–105.
[4] BAURDOUX, E. J. AND PEDRAZA, J. M. (2023). Predicting the last zero before an exponential time of a spectrally negative Lévy process. *Adv. Appl. Prob.* **55**, 611–642.
[5] BAURDOUX, E. J. AND PEDRAZA, J. M. (2024). L_p optimal prediction of the last zero of a spectrally negative Lévy process. *Ann. Appl. Prob.* **34**, 1350–1402.
[6] BERTOIN, J. (1998). *Lévy Processes*. Cambridge University Press.
[7] BICHTELER, K. (2002). *Stochastic Integration with Jumps*. Cambridge University Press.

https://doi.org/10.1017/jpr.2024.104 Published online by Cambridge University Press

[8] CAI, C. AND LI, B. (2018). Occupation times of intervals until last passage times for spectrally negative Lévy processes. *J. Theoret. Prob.* **31**, 2194–2215.

[9] CARR, P. (1998). Randomization and the American put. *Rev. Financial Studies* **11**, 597–626.

[10] CHIU, S. N. AND YIN, C. (2005). Passage times for a spectrally negative Lévy process with applications to risk theory. *Bernoulli* **11**, 511–522.

[11] CHUNG, K. L. (1976). Excursions in Brownian motion. *Ark. Mat.* **14**, 155–177.

[12] DU TOIT, J., PESKIR, G. AND SHIRYAEV, A. (2008). Predicting the last zero of Brownian motion with drift. *Stochastics* **80**, 229–245.

[13] EGAMI, M. AND KEVKHISHVILI, R. (2020). Time reversal and last passage time of diffusions with applications to credit risk management. *Finance Stoch.* **24**, 795–825.

[14] GETOOR, R. AND SHARPE, M. (1973). Last exit times and additive functionals. *Ann. Prob.* **1**, 550–569.

[15] GLOVER, K. AND HULLEY, H. (2014). Optimal prediction of the last-passage time of a transient diffusion. *SIAM J. Control Optim.* **52**, 3833–3853.

[16] KUZNETSOV, A., KYPRIANOU, A. E. AND RIVERO, V. (2013). The theory of scale functions for spectrally negative Lévy processes. In *Lévy Matters II*, Springer, Berlin, pp. 97–186.

[17] KYPRIANOU, A. E. (2014). *Fluctuations of Lévy Processes with Applications*. Springer, Berlin.

[18] LANDRIAULT, D., LI, B., WONG, J. T. AND XU, D. (2018). Poissonian potential measures for Lévy risk models. *Insurance Math. Econom.* **82**, 152–166.

[19] LI, B., WILLMOT, G. E. AND WONG, J. T. (2018). A temporal approach to the Parisian risk model. *J. Appl. Prob.* **55**, 302–317.

[20] LI, Y., YIN, C. AND ZHOU, X. (2017). On the last exit times for spectrally negative Lévy processes. *J. Appl. Prob.* **54**, 474–489.

[21] PESKIR, G. AND SHIRYAEV, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, Basel.

[22] SALMINEN, P. (1984). One-dimensional diffusions and their exit spaces. *Math. Scand.* **54**, 209–220.

[23] SATO, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.