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On one-sided topological conjugacy of topological Markov shifts and gauge actions on Cuntz–Krieger algebras

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Abstract. We characterize topological conjugacy classes of one-sided topological Markov shifts in terms of the associated Cuntz–Krieger algebras and their gauge actions with potentials.

Key words: Cuntz–Krieger algebra, gauge action, topological conjugacy, topological Markov shift.
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For an irreducible non-permutation matrix $A = [A(i, j)]_{i,j=1}^N$ with entries in $\{0, 1\}$, let us denote by (X_A, σ_A) the associated one-sided topological Markov shift. It consists of the compact Hausdorff space X_A of right one-sided sequences $(x_n)_{n\in\mathbb{N}}$ of $x_n \in \{1, 2, \ldots, N\}$ satisfying $A(x_n, x_{n+1}) = 1, n \in \mathbb{N}$, and the continuous surjective map of the right one-sided shift $\sigma_A : X_A \longrightarrow X_A$ defined by $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. The topology of X_A is defined by the relative topology of the infinite product topology of $\{1, 2, \ldots, N\}^{\mathbb{N}}$. A two-sided topological Markov shift $(\bar{X}_A, \bar{\sigma}_A)$ is similarly defined by replacing right one-sided sequences $(x_n)_{n\in\mathbb{N}}$ with two-sided sequences $(x_n)_{n\in\mathbb{Z}}$. See the textbooks [7, 8] for general theory of symbolic dynamical systems. By the monumental introduction of Cuntz–Krieger algebras $\mathcal{O}_A$ by Cuntz and Krieger in [6], lots of important and interesting interplays between topological Markov shifts and the Cuntz–Krieger algebras have been studied and clarified. The Cuntz–Krieger algebra $\mathcal{O}_A$ for the matrix A is defined by the universal unital C^*-algebra generated by N partial isometries $S_1, \ldots, S_N$ satisfying the relations $1 = \sum_{j=1}^N S_j S_j^*$ and $S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, i = 1, 2, \ldots, N$. As in the paper [6], the original space X_A appears in the algebra $\mathcal{O}_A$ as a maximal commutative C^*-subalgebra, written $\mathcal{D}_A$, generated by projections of the form $S_{\mu_1} \cdots S_{\mu_m} S^*_{\mu_m} \cdots S^*_{\mu_1}$ for $\mu_1, \ldots, \mu_m \in \{1, 2, \ldots, N\}$ that is canonically isomorphic to the commutative C^*-algebra $C(X_A)$ of complex-valued continuous functions on X_A, through the identification between the projection $S_{\mu_1} \cdots S_{\mu_m} S^*_{\mu_m} \cdots S^*_{\mu_1}$ and the characteristic function $\chi_{U_{\mu_1 \cdots \mu_m}}$ on X_A for the cylinder set $U_{\mu_1 \cdots \mu_m} = \{(x_n)_{n \in \mathbb{N}} \in X_A \mid x_1 = \mu_1, \ldots, x_m = \mu_m\}$. The gauge action ρ^A on $\mathcal{O}_A$ is defined by the automorphisms ρ^A_t, $t \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ satisfying $\rho^A_t(S_j) = \exp(2\pi\sqrt{-1}t) \cdot S_j$, $j = 1, 2, \ldots, N$. Cuntz and Krieger themselves proved in [6] the following fundamental results (A), (B) and (C) that show close relationships between topological dynamical systems and C^*-algebras. Let us denote by $\mathcal{K}$ and $\mathcal{C}$ the C^*-algebra of compact operators on the separable infinite-dimensional Hilbert space $\ell^2(\mathbb{N})$ and its maximal commutative C^*-subalgebra consisting of diagonal operators, respectively. Let A, B be two irreducible non-permutation matrices with entries in $\{0, 1\}$.

(A) If one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are topologically conjugate, then there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras such that

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho^A_t = \rho^B_t \circ \Phi, \quad t \in \mathbb{T}. \tag{1}$$

(B) If two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate, then there exists an isomorphism $\bar{\Phi} : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that

$$\bar{\Phi}(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C} \quad \text{and} \quad \bar{\Phi} \circ (\rho^A_t \otimes \mathrm{id}) = (\rho^B_t \otimes \mathrm{id}) \circ \bar{\Phi}, \quad t \in \mathbb{T}. \tag{2}$$

(C) If two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent, then there exists an isomorphism $\bar{\Phi} : \mathcal{O}_A \otimes \mathcal{K} \longrightarrow \mathcal{O}_B \otimes \mathcal{K}$ of C^*-algebras such that

$$\bar{\Phi}(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}. \tag{3}$$

The converse implications of the above three implications for each have been longstanding open problems. Matui and the author in [18] showed that the converse implication of (C) holds (cf. [2, 3, 5, 14, 19], etc). Carlsen and Rout in [4] showed that the converse implication of (B) holds by a groupoid technique (cf. [3, 5, 15], etc). Concerning the implication (A), the author in [12] showed that the condition that there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras satisfying (1) is equivalent to the condition that (X_A, σ_A) and (X_B, σ_B) are eventually conjugate, where one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *eventually conjugate* if there exist a homeomorphism $h : X_A \longrightarrow X_B$ and a non-negative integer K such that

$$\begin{cases} \sigma_B^K(h(\sigma_A(x))) = \sigma_B^{K+1}(h(x)), & x \in X_A, \\ \sigma_A^K(h^{-1}(\sigma_B(y))) = \sigma_A^{K+1}(h^{-1}(y)), & y \in X_B. \end{cases} \tag{4}$$

If one may take the integer K as zero, then the relations (4) reduce to the definition of topological conjugacy $h : X_A \longrightarrow X_B$. In [1], Brix and Carlsen found an example of irreducible topological Markov shifts (X_A, σ_A) and (X_B, σ_B) that are eventually conjugate, but not topologically conjugate. In the paper, they characterized topological

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conjugacy of one-sided topological Markov shifts not only in terms of their associated étale groupoids [1, Corollary 3.5(ii)], but also in terms of their Cuntz–Krieger algebras [1, Corollary 3.5(iii)] in the following way. Following [1], let $\tau_A : \mathcal{O}_A \longrightarrow \mathcal{O}_A$ be a completely positive map defined by $\tau_A(Y) = \sum_{i,j=1}^N S_i Y S_j^*$, $Y \in \mathcal{O}_A$. Brix and Carlsen proved that (X_A, σ_A) and (X_B, σ_B) are topologically conjugate if and only if there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \tau_A = \tau_B \circ \Phi$ [1, Corollary 3.5]. This gives rise to a characterization of one-sided topological conjugacy of one-sided topological Markov shifts in terms of C^*-algebras. We note that the gauge action also appears in their other characterization of one-sided topological conjugacy as in [1, Theorem 3.3(iv)].

In this short paper, we will attempt to characterize one-sided topological conjugacy of one-sided topological Markov shifts in terms of Cuntz–Krieger algebras and their gauge actions with potentials to compare with the characterization of eventual conjugacy as in (1). For an integer-valued continuous function $g \in C(X_A, \mathbb{Z})$ on X_A, the action $\rho^{A,g}$ is defined by the automorphisms $\rho_t^{A,g}$, $t \in \mathbb{T}$ on $\mathcal{O}_A$ satisfying $\rho_t^{A,g}(S_j) = \exp(2\pi\sqrt{-1}tg) \cdot S_j$, $j = 1, 2, \ldots, N$. The action $\rho^{A,g}$ was called a generalized gauge action in [11, 13]. In this paper, we call it the gauge action with potential g. We will prove the following theorem.

THEOREM 1. *Let A, B be two irreducible non-permutation matrices with entries in* $\{0, 1\}$. *The following assertions are equivalent.*

(i) *The one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are topologically conjugate.*

(ii) *There exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras such that $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and*

$$\Phi \circ \rho_t^{A, f\circ h} = \rho_t^{B,f} \circ \Phi \quad \text{for all } f \in C(X_B, \mathbb{Z}),\ t \in \mathbb{T}, \tag{5}$$

where $h : X_A \longrightarrow X_B$ is a homeomorphism induced by $\Phi : \mathcal{D}_A \longrightarrow \mathcal{D}_B$ satisfying $\Phi(a) = a \circ h^{-1}$ for $a \in \mathcal{D}_A$ under the canonical identification between $\mathcal{D}_A$ and $C(X_A)$.

Proof. (i) $\Longrightarrow$ (ii): Suppose that there exists a topological conjugacy $h : X_A \longrightarrow X_B$ between (X_A, σ_A) and (X_B, σ_B). It satisfies $h \circ \sigma_A = \sigma_B \circ h$. As $h : X_A \longrightarrow X_B$ gives rise to a continuous orbit equivalence between them in the sense of [10], a homomorphism $\Psi_h : C(X_B, \mathbb{Z}) \longrightarrow C(X_A, \mathbb{Z})$ of abelian groups is defined by setting

$$\Psi_h(f)(x) = \sum_{i=0}^{l_1(x)} f(\sigma_B^i(h(x))) - \sum_{j=0}^{k_1(x)} f(\sigma_B^j(h(\sigma_A(x)))), \quad f \in C(X_B, \mathbb{Z}),\ x \in X_A, \tag{6}$$

where $k_1(x)$, $l_1(x)$ are non-negative integers satisfying the equation

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)) \quad \text{for } x \in X_A. \tag{7}$$

By [13, Theorem 3.2], there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras such that

$$\Phi(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi \circ \rho_t^{A,\Psi_h(f)} = \rho_t^{B,f} \circ \Phi \quad \text{for all } f \in C(X_B, \mathbb{Z}),\ t \in \mathbb{T}. \tag{8}$$

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Now $h : X_A \longrightarrow X_B$ is a topological conjugacy, so that one may take the integers such as $k_1(x) = 0, l_1(x) = 1$ for all $x \in X_A$. Hence, we know that $\Psi_h(f) = f \circ h$, proving the assertion (ii).

(ii) $\Longrightarrow$ (i): Assume that there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and the equalities (5). Since the isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfies $\Phi(\mathcal{D}_A) = \mathcal{D}_B$, the homeomorphism $h : X_A \longrightarrow X_B$ satisfying $\Phi(a) = a \circ h^{-1}$ under the canonical identification between $\mathcal{D}_A$ and $C(X_A)$ gives rise to a continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) [10, Propositions 5.3 and 5.5]. Hence, as in [13, Theorem 3.2], the homeomorphism $h : X_A \longrightarrow X_B$ extends to the whole C^*-algebra $\mathcal{O}_A$, so that there exists an isomorphism $\Phi_1 : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras such that

$$\Phi_1(\mathcal{D}_A) = \mathcal{D}_B \quad \text{and} \quad \Phi_1 \circ \rho_t^{A,\Psi_h(f)} = \rho_t^{B,f} \circ \Phi_1 \quad \text{for all } f \in C(X_B, \mathbb{Z}),\ t \in \mathbb{T}, \tag{9}$$

and $\Phi_1(a) = a \circ h^{-1}$ for $a \in \mathcal{D}_A$ under the canonical identification between $\mathcal{D}_A$ and $C(X_A)$. The condition $\Phi_1(a) = a \circ h^{-1}$ for $a \in \mathcal{D}_A$ follows from the construction of $\Phi_1 : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ in [13]. Since the original isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfies the condition $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi(a) = a \circ h^{-1}, a \in \mathcal{D}_A$, the restriction of the automorphism $\Phi_1^{-1} \circ \Phi$ on $\mathcal{D}_A$ is the identity. By [9, Lemma 4.6], one may find a unitary $U_1 \in \mathcal{D}_B$ such that $\Phi_1(S_i) = U_1\Phi(S_i), i = 1, 2, \ldots, N$, where $S_i, i = 1, 2, \ldots, N$ are the canonical generating partial isometries of $\mathcal{O}_A$. By (9), we have

$$\Phi_1 \circ \rho_t^{A,\Psi_h(f)}(S_i) = \rho_t^{B,f} \circ \Phi_1(S_i) \quad \text{for } f \in C(X_B, \mathbb{Z}),\ t \in \mathbb{T}.$$

Since $\rho_t^{A,\Psi_h(f)}(S_i) = \exp(2\pi\sqrt{-1}t\Psi_h(f)) \cdot S_i$, we have

$$\Phi_1(\exp(2\pi\sqrt{-1}t\Psi_h(f))) \cdot \Phi_1(S_i) = \rho_t^{B,f}(U_1\Phi(S_i)).$$

As $\Phi_1(\exp(2\pi\sqrt{-1}t\Psi_h(f))) = \Phi(\exp(2\pi\sqrt{-1}t\Psi_h(f)))$, because $\exp(2\pi\sqrt{-1}t\Psi_h(f)) \in \mathcal{D}_A$, we have

$$\Phi(\exp(2\pi\sqrt{-1}t\Psi_h(f))) \cdot U_1\Phi(S_i) = U_1\rho_t^{B,f}(\Phi(S_i))$$

and hence

$$\Phi(\exp(2\pi\sqrt{-1}t\Psi_h(f))) \cdot \Phi(S_i) = \rho_t^{B,f}(\Phi(S_i)),$$

so that

$$\Phi(\rho_t^{A,\Psi_h(f)}(S_i)) = \rho_t^{B,f}(\Phi(S_i)).$$

This implies that the equality

$$\Phi \circ \rho_t^{A,\Psi_h(f)} = \rho_t^{B,f} \circ \Phi \quad \text{for all } f \in C(X_B, \mathbb{Z}) \tag{10}$$

holds. By (5) and (10), we have

$$\Psi_h(f) = f \circ h \quad \text{for all } f \in C(X_B, \mathbb{Z}). \tag{11}$$

In (5), by taking $f \equiv 1$, we have $\Phi(\mathcal{D}_A) = \mathcal{D}_B$ and $\Phi \circ \rho_t^A = \rho_t^B \circ \Phi, t \in \mathbb{T}$. Hence, (X_A, σ_A) and (X_B, σ_B) are eventually conjugate via the homeomorphism $h : X_A \longrightarrow X_B$. Hence, there exists a non-negative integer K satisfying (4). The final step to complete the proof of the implication (ii) $\Longrightarrow$ (i) is to show the following lemma. □

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LEMMA 2. *Suppose that (X_A, σ_A) and (X_B, σ_B) are eventually conjugate such that there exists a non-negative integer K satisfying (4). If the equality (11) holds, then $h : X_A \longrightarrow X_B$ gives rise to a topological conjugacy between (X_A, σ_A) and (X_B, σ_B).*

Proof. Now the non-negative integer K satisfies (4), so that we have, by (6),

$$\Psi_h(f)(x) = \sum_{i=0}^{K+1} f(\sigma_B^i(h(x))) - \sum_{j=0}^{K} f(\sigma_B^j(h(\sigma_A(x)))), \quad f \in C(X_B, \mathbb{Z}),\ x \in X_A.$$

If $K = 0$, the homeomorphism $h : X_A \longrightarrow X_B$ gives rise to a topological conjugacy between (X_A, σ_A) and (X_B, σ_B). Hence, we assume that $K \geq 1$.

By the condition (11) together with the equality $\sigma_B^{K+1}(h(x)) = \sigma_B^K(h(\sigma_A(x)))$, we see that the equality

$$\sum_{i=1}^{K} f(\sigma_B^i(h(x))) = \sum_{j=0}^{K-1} f(\sigma_B^j(h(\sigma_A(x)))), \quad f \in C(X_B, \mathbb{Z}),\ x \in X_A \tag{12}$$

holds. For a fixed $x \in X_A$, we put $y = \sigma_B(h(x)),\ w = h(\sigma_A(x))$, so that we obtain the equalities $\sigma_B^K(y) = \sigma_B^K(w)$ and

$$\sum_{i=0}^{K-1} f(\sigma_B^i(y)) = \sum_{j=0}^{K-1} f(\sigma_B^j(w)), \quad f \in C(X_B, \mathbb{Z}). \tag{13}$$

Put $y(j) = \sigma_B^j(y)$, $w(j) = \sigma_B^j(w)$, $j = 0, 1, \ldots, K-1$, and

$$Y_0 = \{y(0), y(1), \ldots, y(K-1)\}, \quad W_0 = \{w(0), w(1), \ldots, w(K-1)\}.$$

By (13), we have

$$\sum_{y(i)\in Y_0} f(y(i)) = \sum_{w(j)\in W_0} f(w(j)), \quad f \in C(X_B, \mathbb{Z}). \tag{14}$$

If $Y_0 \cap W_0 = \emptyset$, one may find $f_0 \in C(X_B, \mathbb{Z})$ such that

$$f_0(y(i)) = 1, \quad f_0(w(i)) = 0 \quad \text{for all } i = 0, 1, \ldots, K-1,$$

a contradiction to (13) unless $K = 0$. Hence, $Y_0 \cap W_0 \neq \emptyset$. Take $i_0, j_0 \in \{0, 1, \ldots, K-1\}$ such that $y(i_0) = w(j_0)$. We put

$$Y_1 = Y_0 \backslash \{y(i_0)\}, \quad W_1 = W_0 \backslash \{w(j_0)\},$$

so that we have

$$\sum_{y(i)\in Y_1} f(y(i)) = \sum_{w(j)\in W_1} f(w(j)), \quad f \in C(X_B, \mathbb{Z}). \tag{15}$$

Inductively, we finally know that $Y_0 = W_0$ unless $K = 0$. Hence, we may find $p, q \in \{0, 1, \ldots, K-1\}$ such that $y = \sigma_B^p(w)$, $w = \sigma_B^q(y)$. If $q = 0$, then we have $h(\sigma_A(x)) = \sigma_B(h(x))$. If $q \neq 0$, we have $y = \sigma_B^{p+q}(y)$ and hence y is periodic. Therefore, we conclude that the equality $h(\sigma_A(x)) = \sigma_B(h(x))$ holds for $x \in X_A$ such that $y = \sigma_B(h(x))$ is not periodic. A point $x \in X_A$ is said to be eventually periodic if $\sigma_A^L(x)$ is periodic for

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some non-negative integer L. The set of non-eventually periodic points is dense in the topological Markov shift for an irreducible non-permutation matrix. Since a continuous orbit equivalence preserves the set of eventually periodic points, we know that the equality $h(\sigma_A(x)) = \sigma_B(h(x))$ holds for all $x \in X_A$. □

Remark 3. The equality (5) is equivalent to the following equality:

$$\Phi \circ \rho_t^{A,g} = \rho_t^{B,g\circ h^{-1}} \circ \Phi \quad \text{for all } g \in C(X_A, \mathbb{Z}),\ t \in \mathbb{T}. \tag{16}$$

Let A, B be irreducible, non-permutation matrices with entries in $\{0, 1\}$. As in [10], one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are said to be *continuously orbit equivalent* if there exist non-negative integer-valued continuous functions k_1, l_1 on X_A and k_2, l_2 on X_B such that

$$\sigma_B^{k_1(x)}(h(\sigma_A(x))) = \sigma_B^{l_1(x)}(h(x)), \quad x \in X_A, \tag{17}$$

$$\sigma_A^{k_2(y)}(h^{-1}(\sigma_B(y))) = \sigma_A^{l_2(y)}(h^{-1}(y)), \quad y \in X_B. \tag{18}$$

If one may take $k_1 \equiv 0, l_1 \equiv 1, k_2 \equiv 0, l_2 \equiv 1$, then the above equalities (17) and (18) reduce to the definition that (X_A, σ_A) and (X_B, σ_B) are topologically conjugate. If one may take $k_1 \equiv K, l_1 \equiv K + 1, k_2 \equiv K, l_2 \equiv K + 1$ for some constant non-negative integer K, then the above equalities (17) and (18) reduce to the definition that (X_A, σ_A) and (X_B, σ_B) are eventually conjugate. If one may take $l_1 - k_1 = 1 + b_1 - b_1 \circ \sigma_A$ and $l_2 - k_2 = 1 + b_2 - b_2 \circ \sigma_B$ for some integer-valued continuous functions $b_1 : X_A \longrightarrow \mathbb{Z}$ and $b_2 : X_B \longrightarrow \mathbb{Z}$, respectively, then the above equalities (17) and (18) reduce to the definition that (X_A, σ_A) and (X_B, σ_B) are *strongly continuous orbit equivalent* [11]. The continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) is completely characterized by the condition that there exists an isomorphism $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$. Take a homeomorphism $h : X_A \longrightarrow X_B$ such that $\Phi(a) = a \circ h^{-1}$ for $a \in \mathcal{D}_A$. In particular, we see that $\Phi(g) = g \circ h^{-1}$ for $g \in C(X_A, \mathbb{Z})$. We finally summarize characterization of these subequivalence relations of continuous orbit equivalence in one-sided topological Markov shifts in the following way.

COROLLARY 4. (Theorem 1 and [13, Corollary 3.5]; see also [12, Theorem 1.5] and [11, Theorem 6.7]) *Let $\Phi : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ be an isomorphism of C^*-algebras satisfying $\Phi(\mathcal{D}_A) = \mathcal{D}_B$. Let $h : X_A \longrightarrow X_B$ be the homeomorphism satisfying $\Phi(a) = a \circ h^{-1}$ for $a \in \mathcal{D}_A$.*

(i) *The homeomorphism $h : X_A \longrightarrow X_B$ gives rise to a topological conjugacy between (X_A, σ_A) and (X_B, σ_B) if and only if*

$$\Phi \circ \rho_t^{A,g} = \rho_t^{B,\Phi(g)} \circ \Phi \quad \textit{for all } g \in C(X_A, \mathbb{Z}),\ t \in \mathbb{T}. \tag{19}$$

(ii) *The homeomorphism $h : X_A \longrightarrow X_B$ gives rise to an eventual conjugacy between (X_A, σ_A) and (X_B, σ_B) if and only if*

$$\Phi \circ \rho_t^{A} = \rho_t^{B} \circ \Phi, \quad t \in \mathbb{T}. \tag{20}$$

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(iii) *The homeomorphism $h : X_A \longrightarrow X_B$ gives rise to a strongly continuous orbit equivalence between (X_A, σ_A) and (X_B, σ_B) if and only if there exists a unitary one-cocycle $v_t \in \mathcal{D}_B$ for the gauge action ρ^B such that*

$$\Phi \circ \rho_t^A = \mathrm{Ad}(v_t) \circ \rho_t^B \circ \Phi, \quad t \in \mathbb{T}. \tag{21}$$

Proof. (i) The 'if' part follows from Theorem 1 (ii) $\Longrightarrow$ (i) and its proof by noticing Remark 3. We will show the 'only if' part. Suppose that $h : X_A \longrightarrow X_B$ is a topological conjugacy between (X_A, σ_A) and (X_B, σ_B). By Theorem 1 (i) $\Longrightarrow$ (ii) and its proof, one may find an isomorphism $\Phi_1 : \mathcal{O}_A \longrightarrow \mathcal{O}_B$ of C^*-algebras such that $\Phi_1(\mathcal{D}_A) = \mathcal{D}_B$, $\Phi_1(a) = a \circ h^{-1}$ for $a \in \mathcal{D}_A$ and

$$\Phi_1 \circ \rho_t^{A,g} = \rho_t^{B,\Phi_1(g)} \circ \Phi_1 \quad \text{for all } g \in C(X_A, \mathbb{Z}),\ t \in \mathbb{T}. \tag{22}$$

Hence, Φ_1 coincides with Φ on the subalgebra $\mathcal{D}_A$. By using a similar argument to the proof of Theorem 1 (ii) $\Longrightarrow$ (i), one may find a unitary U_1 in $\mathcal{D}_B$ such that $\Phi_1(S_i) = U_1\Phi(S_i)$, $i = 1, 2, \ldots, N$, where S_i, $i = 1, 2, \ldots, N$, are the canonical generating partial isometries of $\mathcal{O}_A$, so that we have $\Phi \circ \rho_t^{A,g} = \rho_t^{B,\Phi_1(g)} \circ \Phi$ by the same argument as the one obtained from (9) to (10). As $\Phi_1(g) = \Phi(g)$, we conclude the equality (19).

(ii) and (iii) The 'if' parts of (ii) and (iii) follow from [**13**, Corollary 3.5(i)] and [**13**, Corollary 3.5(ii)] (see also [**13**, Theorem 3.3(i)] and [**13**, Theorem 3.3(ii)]) and their proofs, respectively. The 'only if' parts follow from [**13**, Theorem 3.3(i)] and [**13**, Theorem 3.3(ii)] and their proofs, respectively, by using a similar argument to the 'only if' part of the above proof (i). □

A generalization of Theorem 1 to more general subshifts treated in the paper [**16**] will be studied in a forthcoming paper [**17**].

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