Canad. Math. Bull. 2025, pp. 1–10
http://dx.doi.org/10.4153/S0008439524000894

Buried points of plane continua

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Abstract. Sets on the boundary of a complementary component of a continuum in the plane have been of interest since the early 1920s. Curry and Mayer defined the buried points of a plane continuum to be the points in the continuum which were not on the boundary of any complementary component. Motivated by their investigations of Julia sets, they asked what happens if the set of buried points of a plane continuum is totally disconnected and nonempty. Curry, Mayer, and Tymchatyn showed that in that case the continuum is Suslinian, i.e., it does not contain an uncountable collection of nondegenerate pairwise disjoint subcontinua. In an answer to a question of Curry et al., van Mill and Tuncali constructed a plane continuum whose buried point set was totally disconnected, nonempty, and one-dimensional at each point of a countably infinite set. In this paper, we show that the van Mill–Tuncali example was the best possible in the sense that whenever the buried set is totally disconnected, it is one-dimensional at each of at most countably many points. As a corollary, we find that the buried set cannot be almost zero-dimensional unless it is zero-dimensional. We also construct locally connected van Mill–Tuncali type examples.

1 Introduction

A continuum is a compact, connected metric space. A point x in a plane continuum $X \subset \mathbb{R}^2$ is **buried** if x is not in the boundary (or frontier) of any component of $\mathbb{R}^2 \backslash X$. We denote by $\mathrm{bur}(X)$ the set of all buried points of X. Note that $\mathrm{bur}(X)$ is a G_δ-set.

Motivation to study buried points comes from complex dynamics. Specifically, one of the difficult problems in complex dynamics asks whether $\mathrm{bur}(J(R))$ must be zero-dimensional in case it is punctiform, where $J(R)$ is the Julia set of a rational function on the sphere $\mathbb{C}_\infty$. The Devaney–Rocha examples of Sierpiński gasket like Julia sets have this property. Curry, Mayer, and Tymchatyn proposed to study it from a purely topological point of view. For more details, see [1, 2, 4].

In [9, Proposition 3.1], van Mill and Tuncali prove that the set of buried points of a plane continuum can be a Cantor set. They use this to provide in §4 of their paper an example of a plane Suslinian continuum whose set of buried points is totally disconnected and weakly 1-dimensional. Their example answers in the negative Question 1 of Curry, Mayer, and Tymchatyn [2], concerning whether or not a set of buried points is zero-dimensional provided it is totally disconnected. Note that if a plane continuum is regular (that is, has a basis of open sets with finite boundaries), then the set of buried points is either empty or zero-dimensional (see [2, Theorem 3]).

Received by the editors February 4, 2024; revised November 3, 2024; accepted November 22, 2024.
AMS subject classification: **37B45**, 54F15, 54F45.
Keywords: Buried points, totally disconnected, Suslinian, plane continuum.

Curry et al. showed that if X is a plane continuum with complementary components whose boundaries are locally connected and $\mathrm{bur}(X)$ is Suslinian, then in fact X has to be Suslinian [2, Theorem 6]. This is the case for the example in [9, Section 4] (the buried point set in that example is even totally disconnected). However, it is not locally connected. A plane continuum X is locally connected if and only if the boundaries of complementary components are locally connected and form a null-sequence [13, VI Theorem 4.4]. Using this fact, we construct in §5 a locally connected plane continuum whose set of buried points is totally disconnected but not zero-dimensional. The set at which the buried points are 1-dimensional is countable. This is sharp according to the following theorem, which we prove in §3.

Theorem A *Let X be a Suslinian plane continuum. If $Y \subset X$ is a totally disconnected Borel set, then Y is zero-dimensional at all but countably many points.*

Corollary B *Let X a plane continuum such that the boundary of each component of $\mathbb{R}^2 \backslash X$ is locally connected (e.g. X is locally connected). If $\mathrm{bur}(X)$ is totally disconnected, then $\mathrm{bur}(X)$ is zero-dimensional at all but countably many points.*

Countable sets are zero-dimensional, so under the assumptions of Corollary B the buried set is either zero-dimensional or weakly 1-dimensional. There is no almost zero-dimensional, weakly 1-dimensional space [7, Theorem 1]. Therefore, $\mathrm{bur}(X)$ cannot be almost zero-dimensional in the proper sense (cf. [1, Question 2.7]). To summarize:

Corollary C *Let X be a plane continuum such that each component of $\mathbb{R}^2 \backslash X$ has locally connected boundary (e.g. X is locally connected).*

(i) *If $\mathrm{bur}(X)$ is totally disconnected, then $\mathrm{bur}(X)$ is at most weakly 1-dimensional.*
(ii) *If $\mathrm{bur}(X)$ is almost zero-dimensional, then $\mathrm{bur}(X)$ is zero-dimensional.*

It is still unknown whether the Julia set of a rational function may have a buried set which is totally disconnected but not zero-dimensional (cf. [2, Question 3]).

2 Definitions

All spaces under consideration are assumed to be separable and metrizable.

A space X is **Suslinian** if each collection of pairwise disjoint nondegenerate continua in X is countable.

A space X is **totally disconnected** if every two points of X are contained in disjoint clopen sets, and **zero-dimensional** if X has a basis of clopen sets. Respectively, X is **zero-dimensional at** $x \in X$ if the point x has a neighborhood basis of clopen sets (written $\mathrm{ind}_x X = 0$, cf. [3, Problem 1.1.B]).

A space X is **almost zero-dimensional** if X has a basis of open sets whose closures are intersections of clopen sets [7, 12]. Observe that every almost zero-dimensional space is totally disconnected.

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The **dimensional kernel** of a 1-dimensional space X is defined to be the set of points at which X is not zero-dimensional; $\Lambda(X) = \{x \in X : \text{ind}_x X = 1\}$. A 1-dimensional space X is **weakly 1-dimensional** if $\Lambda(X)$ is zero-dimensional.

A **domain** is a connected open subset of the plane or topological 2-sphere. If X is a continuum on the 2-sphere, then each complementary component of X is a simply connected domain. **Riemann's mapping theorem** implies that every simply connected domain $U \subset \mathbb{C}$ is homeomorphic to the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. **Carathéodory's theorem** implies moreover that if ∂U is a simple closed curve then $\overline{U}$ is homeomorphic to the closed unit disc $\{z \in \mathbb{C} : |z| \leq 1\}$.

3 Main result

Suppose that X is a continuum and $Y \subset X$. For each $y \in Y$ let $\mathcal{U}_y$ be the collection of all open subsets U of X such that $y \in U$ and $\partial U \subset X\backslash Y$. For each $y \in Y$ put

$$F(y) = \bigcap_{U \in \mathcal{U}_y} \overline{U}.$$

Lemma 1 *If Y is totally disconnected and $y \in \Lambda(Y)$, then $F(y)$ is a nondegenerate subcontinuum of X and $F(y) \cap Y = \{y\}$.*

Proof Suppose $F(y) = \{y\}$ is degenerate. We show that Y is zero-dimensional at y. Let V be an arbitrary relatively open subset of Y that contains y. Pick an open subset W of X such that $W \cap Y = V$. Since $F(y) = \{y\} \subset V$, there is by compactness an element $U \in \mathcal{U}_y$ such that $\overline{U} \subset W$. But then $y \in U \cap Y \subset V$, and $U \cap Y$ is clopen in Y.

Now suppose that $y \in Y$ and $z \in Y\backslash\{y\}$. Pick by total disconnectedness of Y clopen subsets C_0 and $C_1 = Y\backslash C_0$ of Y such that $y \in C_0$ and $z \in C_1$. There are disjoint open subsets V_0 and V_1 of X such that $V_0 \cap Y = C_0$ and $V_1 \cap Y = C_1$. Observe that the boundary of V_0 is contained in $X\backslash Y$. Since $V_1 \cap V_0 = \varnothing$, this shows that $z \notin F(y)$. Hence, $F(y) \cap Y = \{y\}$.

It remains to prove $F(y)$ is connected. Just suppose $F(y) = A \cup B$ where A and B are disjoint closed sets with $y \in A$. Let U and V be disjoint open neighborhoods of A and B in X, respectively. By compactness, there exists $W \in \mathcal{U}_y$ such that $\overline{W} \subset U \cup V$. Then $W \cap U \in \mathcal{U}_y$, so $B = \varnothing$. ■

Remark 1 If X is locally connected and $U \in \mathcal{U}_y$, then the connected component of y in U is path-connected and belongs to $\mathcal{U}_y$.

Lemma 2 *Let S be a circle in the plane, and a_1, a_2, a_3, a_4 four points of S in cyclic order. Let A_i, $i \in \{1, 2, 3, 4\}$, be pairwise disjoint arcs intersecting S only at an endpoint a_i, with opposite endpoints b_i in the bounded component of $\mathbb{R}^2\backslash S$, denoted* int(S).

If $K \subset \mathbb{R}^2\backslash(A_2 \cup S \cup A_4)$ is a continuum containing b_1 and b_3, then b_2 and b_4 are in different components of $\mathbb{R}^2\backslash(A_1 \cup S \cup A_3 \cup K)$.

Proof We may assume that S is the unit circle in the complex plane $\mathbb{C}$, and

$$a_1 = 1, a_2 = \mathrm{i}, a_3 = -1, a_4 = -\mathrm{i}.$$

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Let $\gamma \subset \mathbb{C}\backslash(A_1 \cup S \cup A_3)$ be any arc from b_2 to b_4. We will show that γ intersects K. This will imply that b_2 and b_4 are in different components of $\mathbb{C}\backslash(A_1 \cup S \cup A_3 \cup K)$, as all such components are path-connected.

In $\gamma \cup A_2 \cup A_4$, there is an arc $\hat{\gamma}$ intersecting S only at endpoints $\pm$i. Let α and β be the left and right halves of the circle S. By the θ-curve theorem [10, Lemma 64.1], $\text{int}(S)\backslash\hat{\gamma}$ is the union of two disjoint domains U and V whose boundaries are $\alpha \cup \hat{\gamma}$ and $\hat{\gamma} \cup \beta$, respectively. Let W be an open ball centered at -1 that misses $\hat{\gamma}$. Then $W \cap \text{int}(S) \subset U \cup V$. Since $W \cap \text{int}(S)$ is connected and $-1 \in \partial U$, we have $W \cap \text{int}(S) \subset U$. Thus, $U \cap A_3 \neq \varnothing$. It follows that $A_3\backslash\{-1\} \subset U$. Thus, $b_3 \in U$. Likewise, $b_1 \in V$. Now K is a connected subset of $\text{int}(S)$ meeting both U and V. So $K \cap \hat{\gamma} \neq \varnothing$. Therefore, $K \cap \gamma \neq \varnothing$. ∎

We can now prove Theorem A for locally connected X.

Theorem 3 *Let X be a locally connected Suslinian plane continuum. If $Y \subset X$ is a totally disconnected Borel set, then $\Lambda(Y)$ is countable.*

Proof For a contradiction suppose that $Y \subset X$ is totally disconnected and Borel, and $Z = \Lambda(Y)$ is uncountable. By Lemma 1 there exists $\varepsilon > 0$ such that for uncountably many z's, $\text{diam}(F(z)) \geq 5\varepsilon$. The set of all z's with this property is closed in Y, thus it is Borel and contains a Cantor set. Let us just assume that Z is a Cantor set in Y with the property $\text{diam}(F(z)) \geq 5\varepsilon$ for each $z \in Z$. We may further assume that Z lies inside of an open ball of radius ε centered at one of its points. Let S and S' be the circles of radii ε and 2ε, respectively, centered at the point. Observe that every $F(z)$ crosses S'.

In the claims below, all boundaries are respective to X (not $\mathbb{R}^2$).

Claim 4 *Let U be an open subset of X intersecting Z with $\partial U \subset (X\backslash Y) \cup S$. Then for every point $p \in \mathbb{R}^2$ there exists a connected open set $W \subset U$ intersecting Z such that $p \notin \overline{W}$ and $\partial W \subset (X\backslash Y) \cup S$.*

Proof of Claim 4 For each z in the uncountable set $Z \cap U\backslash\{p\}$ there is a non-degenerate continuum $K(z) \subset F(z)$ containing z and missing $S \cup \{p\}$. Just take any closed neighborhood N of z missing $S \cup \{p\}$, and let $K(z)$ be the component of z in $F(z) \cap N$; by the boundary bumping theorem [11, Theorem 5.4] $K(z)$ is nondegenerate. Note that $K(z) \cap Y = \{z\}$ by Lemma 1.

By the Suslinian property of X there exist

$$z_1, z_2, z_3, z_4 \in Z \cap U\backslash\{p\},$$

such that $K(z_i) \cap K(z_1) \neq \varnothing$ for each $i \in \{2, 3, 4\}$. Let $U_i \in \mathcal{U}_{z_i}$ be pairwise disjoint. By Remark 1, we may assume that each U_i is path-connected. Since $F(z_i)$ extends beyond S', so does U_i. Thus, there is an arc $A_i \subset U_i$ from z_i to $a_i \in S'$ that intersects S' only at a_i. By a permutation of indices $i \in \{2, 3, 4\}$, we may assume that the a_i's are in cyclic order around S'. Let $K = K(z_1) \cup K(z_3)$. Note that $p \notin K$.

Case 1: $p \in U_1 \cup S' \cup U_3$. Let W be the component of z_2 in $U_2 \cap U\backslash S$. Then W is a connected open subset of U intersecting Z, with $p \notin \overline{W}$. Note also that W is clopen in $U_2 \cap U\backslash S$, which means that $\partial W \subset \partial(U_2 \cap U\backslash S)$. The boundary of any

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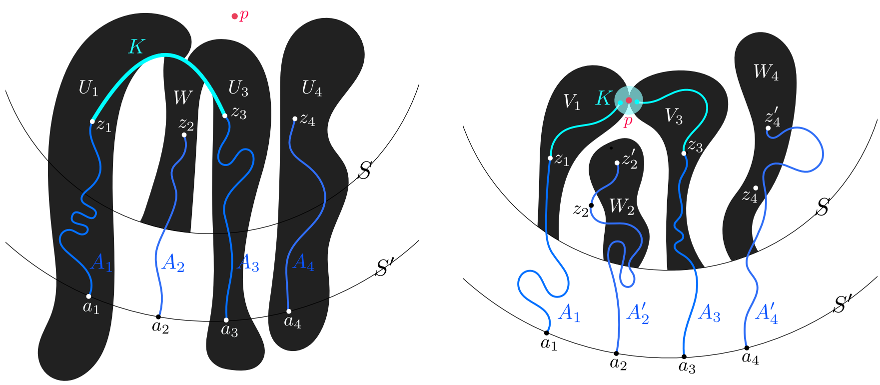

Figure 1: Proofs of Claim 4 (*left*) and Claim 5 (*right*).

finite intersection of open sets is contained in the union of the individual boundaries. Therefore, $\partial W \subset \partial U_2 \cup \partial U \cup \partial(X\backslash S) \subset (X\backslash Y) \cup S$.

Case 2: $p \notin U_1 \cup S' \cup U_3$. Then certainly $p \notin A_1 \cup S' \cup A_3$ and since $p \notin K$ we find that $p \in O = \mathbb{R}^2\backslash(A_1 \cup S' \cup A_3 \cup K)$. By Lemma 2, z_2 and z_4 are in different components of O. Of these two components, at least one does not contain p. Let's say that V is the component of z_2 in O, and $p \notin V$. The component of p in O is an open set missing V, so $p \notin \overline{V}$. Let W be the component of z_2 in $U_2 \cap U \cap V\backslash S$. Then W is a connected open subset of U intersecting Z, and $p \notin \overline{W}$. It remains to show that $\partial W \subset (X\backslash Y) \cup S$. Note the following:

- $\partial V \subset A_1 \cup S' \cup A_3 \cup K$;
- $\overline{W}$ misses $A_1 \cup A_3$ because $W \subset U_2$;
- $\overline{W}$ misses S' because it lies in the closed disc bounded by S.

Hence, $\overline{W} \cap \partial V \subset K\backslash\{z_1, z_3\} \subset X\backslash Y$. We conclude that

$$\begin{aligned}\partial W &\subset \overline{W} \cap \left[\partial U_2 \cup \partial U \cup \partial V \cup \partial(X\backslash S)\right] \\ &\subset \partial U_2 \cup \partial U \cup \left(\overline{W} \cap \partial V\right) \cup \partial(X\backslash S) \\ &\subset (X\backslash Y) \cup S.\end{aligned}$$

This completes the proof of Claim 4. ∎

Claim 5 *Let U be an open subset of X intersecting Z with $\partial U \subset (X\backslash Y) \cup S$. Then there are connected open subsets W_1 and W_2 of U each intersecting Z, with disjoint closures, such that $\partial W_i \subset (X\backslash Y) \cup S$.*

Proof of Claim 5 Let $z_1, z_2, z_3, z_4 \in Z \cap U$. Let $U_i \in \mathcal{U}_{z_i}$ be pairwise disjoint and path-connected. Let A_i be an arc in U_i from z_i to $a_i \in S'$, intersecting S' only at a_i. We may assume that the a_i's are in cyclic order. Let V_i be the component of z_i in

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$U_i \cap U\backslash S$. Assume that $\overline{V_1}$ and $\overline{V_3}$ have a common point p (if they are disjoint then we're done). Note that $p \notin U_2 \cup U_4$, so $p \notin A_2 \cup A_4 \cup V_2 \cup V_4$.

By Claim 4, there exist $z_2', z_4' \in Z$ and connected open subsets W_2 and W_4 of V_2 and V_4 whose closures miss p, such that $z_i' \in W_i$ and $\partial W_i \subset (X\backslash Y) \cup S$. For each $i \in \{2,4\}$, working within V_i we can modify A_i to an arc A_i' that ends at z_i' instead of z_i. The other endpoint of A_i' is still a_i, and $p \notin A_i'$. Let V be a connected neighborhood of p with closure missing $\overline{W_2} \cup \overline{W_4} \cup S' \cup A_2' \cup A_4'$. For each $i \in \{1,3\}$ let B_i be an arc in V_i from z_i into V. Let $K = B_1 \cup B_3 \cup \overline{V}$. By Lemma 2, z_2' and z_4' are in different components of $\mathbb{R}^2\backslash(A_1 \cup S' \cup A_3 \cup K)$. The continua $\overline{W_2}$ and $\overline{W_4}$ are contained in these components, hence they are disjoint. ■

By Claim 5, there are two connected open sets $W_{\langle 0\rangle}$ and $W_{\langle 1\rangle}$ in $X\backslash S$, intersecting Z, with disjoint closures and boundaries in $(X\backslash Y) \cup S$. Assuming that $\alpha \in 2^{<\mathbb{N}}$ is a finite binary sequence and W_α has been defined, apply Claim 5 to get connected open subsets $W_{\alpha^\frown 0}$ and $W_{\alpha^\frown 1}$ of W_α, each intersecting Z, with disjoint closures and boundaries in $(X\backslash Y) \cup S$. Their boundaries must meet S because each $F(z)$ meets S'. So for every infinite sequence $\alpha \in 2^{\mathbb{N}}$,

$$K_\alpha = \bigcap_{n=1}^{\infty} \overline{W_{\alpha\restriction n}},$$

is a continuum in X stretching from Z to S. Clearly if $\alpha \neq \beta$ then K_α and K_β are disjoint. Therefore, $\{K_\alpha : \alpha \in 2^{\mathbb{N}}\}$ is an uncountable collection of pairwise disjoint nondegenerate subcontinua of X, a contradiction to the Suslinian property of X. Hence, $\Lambda(Y)$ must have been countable. This completes the proof of Theorem 3. ■

Theorem A is a direct consequence of Theorem 3 and:

Theorem 6 *Every Suslinian plane continuum is contained in a locally connected Suslinian plane continuum.*

Proof Let X be a Suslinian plane continuum. Let $U_0, U_1, U_2, \ldots$ be the connected components of $\mathbb{R}^2\backslash X$ with U_0 unbounded. In each domain U_m, there is a sequence $S_0^m, S_1^m \ldots$ of disjoint, concentric simple closed curves limiting to the boundary of U_m. Let $A_0^0 = \varnothing$, and for each $m \geq 1$ let A_0^m be the closed topological disc in U_m that is bounded by S_0^m. For each $n \geq 1$, let A_n^m be the closed annulus in U_m that is bounded by S_{n-1}^m and S_n^m. In A_n^m there is a finite collection of arcs $\mathcal{I}_n^m$ covering ∂A_n^m such that each component of $A_n^m\backslash\bigcup\mathcal{I}_n^m$ has diameter less than $1/(m+n)$. We can easily arrange that the boundaries of these components are simple closed curves, and $X_n^m = \bigcup\mathcal{I}_n^m$ is connected. For the reader who'd like more details, one can assume that X is a subset of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, so that all complementary components of X are simply connected. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Given a component U of $\hat{\mathbb{C}}\backslash X$, apply the Riemann mapping theorem to get a continuous bijection $f : \mathbb{D} \to U$ and transfer the circles $\{z \in \mathbb{C} : |z| = 1 - 2^{-n}\}$ to U to get the curves S_n. Apply uniform continuity on each compact annulus $\{z \in \mathbb{C} : 1 - 2^{-n} \leq |z| \leq 1 - 2^{-n-1}\}$ to divide it further into a grid whose cells have images with a small enough diameter. See Figure 2.

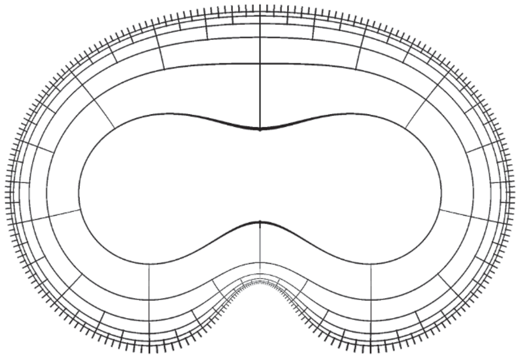

Figure 2: Dividing a domain into cells via a conformal mapping of $\mathbb{D}$.

Put

$$X' = X \cup \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} X_n^m.$$

The components of $\mathbb{R}^2 \backslash X'$ form a null sequence of domains whose boundaries are simple closed curves, so the continuum X' is locally connected [13, VI Theorem 4.4]. Moreover, X' is Suslinian because $X' \backslash X$ is a countable union of arcs. ■

4 Examples

First, we describe a locally connected extension of the continuum in [9] with the same buried set, which is the totally disconnected and weakly 1-dimensional space by Kuratowski [6], [3, Exercise 1.2.E].

Remark 2 Kuratowski's space K is the graph of a particular function $f : C \to [-1, 1]$ defined on the middle-thirds Cantor set C. It has a simple algebraic formula based on binary representations of members of C (see §4 of [9]). Clearly K is totally disconnected, and it is weakly 1-dimensional by the following. Let C_1 be the countable set consisting of 0 and all right endpoints of the intervals

$$\left(\tfrac{1}{3}, \tfrac{2}{3}\right), \left(\tfrac{1}{9}, \tfrac{2}{9}\right), \left(\tfrac{7}{9}, \tfrac{8}{9}\right), \ldots$$

removed from $[0, 1]$ to obtain C. For every $x \in C_1$, it happens that K is 1-dimensional at $\langle x, f(x)\rangle$. On the other hand, f is continuous at each point of $C_0 = C \backslash C_1$. Hence, if $x \in C_0$, then K is zero-dimensional at $\langle x, f(x)\rangle$. Therefore,

$$\Lambda(K) = \left\{\langle x, f(x)\rangle : x \in C_1\right\},$$

and K is weakly 1-dimensional.

See Figure 3 for an illustration of K and its closure in $C \times [-1, 1]$.

Example 1 There exists a locally connected plane continuum whose buried set is totally disconnected and 1-dimensional.

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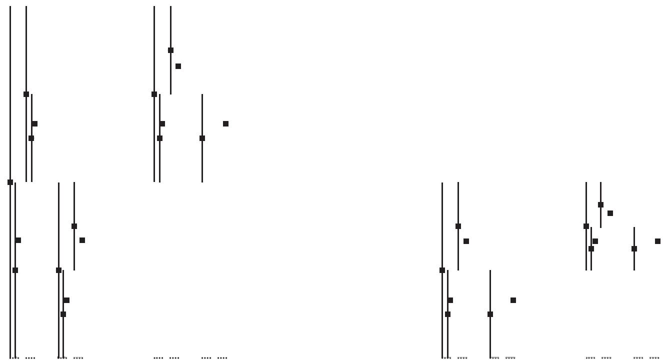

Figure 3: The closure of K consists of the points of K and vertical arcs through points of $\Lambda(K)$.

Proof Let Z be the plane continuum from [9] which has $\operatorname{bur}(Z) = K$. By construction, $\mathbb{R}^2 \setminus Z$ is a countable union of disjoint connected open sets with simple closed curves as boundaries. One complementary component of Z, say W_0, is unbounded and the others, say $W_1, W_2, \ldots$, form a sequence of bounded domains. Let $\mathcal{I}_n$, for $n \geq 1$, be a finite set of arcs in $\overline{W_n}$ such that the collection $\mathcal{V}_n$ of components of $W_n \setminus \bigcup \mathcal{I}_n$ has mesh less than $1/n$. We can easily arrange that the boundary of every element of $\mathcal{V}_n$ is a simple closed curve, and $Z_n = \partial W_n \cup \bigcup \mathcal{I}_n$ is connected. Here one can even apply Carathéodory's theorem to get Z_n from a grid in the closed unit disc. As in Theorem 6,

$$Z' = Z \cup \bigcup_{n=1}^{\infty} Z_n,$$

is a locally connected continuum. Clearly, $\operatorname{bur}(Z) \subset \operatorname{bur}(Z')$. For the other inclusion, note that points in $Z' \setminus Z$ have finite graph neighborhoods in Z', so are not in $\operatorname{bur}(Z')$. Now suppose $z \in Z \setminus \operatorname{bur}(Z)$. Then $z \in \partial W_n$ for some n, and there exists $V \in \mathcal{V}_n$ such that $z \in \partial V$. So $z \notin \operatorname{bur}(Z')$. We conclude that $\operatorname{bur}(Z') = \operatorname{bur}(Z) = K$. ■

The next example shows that Theorem A is false outside of the plane. By a **dendroid**, we shall mean a hereditarily unicoherent, arcwise-connected continuum.

Example 2 There exists a Suslinian dendroid whose endpoint set is totally disconnected and 1-dimensional at every point.

We briefly indicate the construction of such an example below. For further details, see [8, Section 5.3].

Proof Begin with a *Lelek function* $\varphi : C \to [0,1]$ whose positive graph

$$E = \{\langle x, \varphi(x)\rangle : x \in C \text{ and } \varphi(x) > 0\},$$

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is 1-dimensional at every point [5]. Let

$$L = \bigcup_{x \in C} \{x\} \times [0, \varphi(x)] \subset C \times [0,1].$$

The quotient of L that is obtained by shrinking $C \times \{0\}$ to a point is a continuum known as the Lelek fan. We will identify arcs in $L \backslash E$ to produce a Suslinian continuum (a dendroid, in fact) that homeomorphically contains E.

For each $n = 1, 2, 3, \ldots$ let $\mathscr{C}_n$ be the natural partition of C into 2^n pairwise disjoint intervals of length 3^{-n}. For each $i = 1, \ldots, 2^n - 1$ define

$$A_{n,i} = \left\{x \in C : \varphi(x) \geq \tfrac{2i+1}{2^{n+1}}\right\}.$$

Put $\langle x, 0\rangle \sim \langle x', 0\rangle$ for all $x, x' \in C$. If $y > 0$ then define $\langle x, y\rangle \sim \langle x', y\rangle$ if there exists n, i such that $x, x' \in A_{n,i}$ belong to the same member of $\mathscr{C}_n$, and $y \leq i/2^n$. It is possible to see that $\sim$ is an equivalence relation, the equivalence classes under $\sim$ form an upper semicontinuous decomposition of L, and the quotient $D = L/\sim$ is a Suslinian dendroid with endpoint set E. ■

Remark 3 The endpoint set of a smooth dendroid is always G_δ; therefore, E is Borel and Polish.

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https://doi.org/10.4153/S0008439524000894 Published online by Cambridge University Press

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https://doi.org/10.4153/S0008439524000894 Published online by Cambridge University Press