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Joining properties of automorphisms disjoint with all ergodic systems

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Abstract. We study the class $\mathrm{Erg}^{\perp}$ of automorphisms which are disjoint with all ergodic systems. We prove that the identities are the only multipliers of $\mathrm{Erg}^{\perp}$, that is, each automorphism whose every joining with an element of $\mathrm{Erg}^{\perp}$ yields a system which is again an element of $\mathrm{Erg}^{\perp}$, must be an identity. Despite this fact, we show that $\mathrm{Erg}^{\perp}$ is closed by taking Cartesian products. Finally, we prove that there are non-identity elements in $\mathrm{Erg}^{\perp}$ whose self-joinings always yield elements in $\mathrm{Erg}^{\perp}$. This shows that there are non-trivial characteristic classes included in $\mathrm{Erg}^{\perp}$.

Key words: dynamical systems, ergodic theory, non-ergodic systems, joinings of dynamicals systems
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1. *Introduction*

The study of disjointness which expresses the extreme degree of non-isomorphism (in particular, the absence of non-trivial common factors) of two measure-preserving automorphisms originated in the seminal work of Furstenberg [7] in 1967. It is an important direction of research until today and determining whether two automorphisms are disjoint still remains a challenging problem. In particular, given a class $\mathcal{A}$ of automorphisms, a full description of the class $\mathcal{A}^{\perp}$ of automorphisms disjoint with every element of $\mathcal{A}$ is often a hard task. Let us recall some classical results:

(i) the class $\mathrm{ID}^{\perp}$ of automorphisms disjoint with all identities equals the class Erg of all ergodic transformations (by ID we denote the class of all identities), see [7];
(ii) the class $\mathrm{ZE}^{\perp}$ of maps disjoint with the zero entropy automorphisms is equal to the family of K-automorphisms ([7], together with [20, 24]).

For some other classical classes, only weaker relations are known: the class Dist of distal automorphisms is disjoint from the class WM of weakly mixing automorphisms [7] or the class Rig of rigid automorphisms is disjoint from the class MM of mildly mixing automorphisms [8]. Usually, the problem of deciding about the disjointness of two automorphisms slightly simplifies when we consider them both ergodic as it is reduced to study ergodic joinings. However, if two automorphisms are disjoint, then one of them has to be ergodic (otherwise, they both have non-trivial identities as factors and these are not disjoint). In connection with item (i), it is natural to ask whether $\mathrm{Erg}^{\perp}$ goes beyond identities. It indeed does (this observation was a folklore), however, a satisfactory characterization of elements of the class $\mathrm{Erg}^{\perp}$ was only given very recently in [13], see Theorem 2 below. One of the recent reasons to study the non-ergodic case (the elements of $\mathrm{Erg}^{\perp}$, other than the one-point system, are obviously non-ergodic) came recently from Frantzikinakis and Host [6], who studied so-called (logarithmic) Furstenberg systems of the classical Liouville function and discovered that the celebrated Sarnak conjecture may fail because some (hypothetic) of its Furstenberg systems might be elements of $\mathrm{Erg}^{\perp}$. (See also the still open Problem 3.1 of a workshop [1] where Frantzikinakis' question whether an automorphism considered in Example 1 below can be realized as a Furstenberg system of the Liouville function.)

Another natural problem when studying classes of the form $\mathcal{A}^{\perp}$ is to describe the class $\mathcal{M}(\mathcal{A}^{\perp}) \subseteq \mathcal{A}^{\perp}$ of its multipliers, that is, of automorphisms $S \in \mathcal{A}^{\perp}$ such that every joining of S with every element of $\mathcal{A}^{\perp}$ also belongs to $\mathcal{A}^{\perp}$. The study of this class is, in general, very difficult and leads often to many surprising results. In 1989, Glasner and Weiss [12] pioneered the study of the class $\mathrm{WM}^{\perp}$, and they proved that $\mathrm{Dist} \subsetneq \mathrm{WM}^{\perp}$. Continuing on their result, in two papers [10, 19], it was proved that

$$\mathrm{Dist} \subsetneq \mathcal{M}(\mathrm{WM}^{\perp}) \subsetneq \mathrm{WM}^{\perp}.$$

Returning to the Erg class, we clearly have

$$\mathrm{ID} \subseteq \mathcal{M}(\mathrm{Erg}^{\perp}) \subseteq \mathrm{Erg}^{\perp}, \tag{1}$$

where it is easy to see that $\mathcal{M}(\mathrm{Erg}^{\perp}) \subsetneq \mathrm{Erg}^{\perp}$. In fact, in §4 (see Example 1), we consider a standard twist on the torus $(x, y) \mapsto (x, x + y)$ as an example of an element in $\mathrm{Erg}^{\perp}$ and directly show that it is not an element of $\mathcal{M}(\mathrm{Erg}^{\perp})$. The latter assertion follows also from our main, somewhat surprising, result. (It was formulated as a conjecture by M. Lemańczyk in 2018 (private communication).)

THEOREM 1. *We have* $\mathrm{ID} = \mathcal{M}(\mathrm{Erg}^{\perp})$.

While the class $\mathrm{Erg}^{\perp}$ is not closed under taking joinings, in §5, we prove that it is closed under Cartesian products.

To show that an ergodic automorphism T is a multiplier of a class $\mathcal{A}^{\perp}$, it is enough to show that the automorphisms determined by all self-joinings of T are disjoint from the elements of $\mathcal{A}$, see for example, [19, §5] (see also Proposition 33). We show in §6 however that this approach fails when we study the class $\mathrm{Erg}^{\perp}$ by exhibiting a non-identity automorphism T whose all self-joinings yield elements of $\mathrm{Erg}^{\perp}$ and which cannot be in $\mathcal{M}(\mathrm{Erg}^{\perp})$ by Theorem 1. Moreover, the example constructed in §6 serves to create a

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non-trivial *characteristic class* (see §2.4), which does not contain a non-trivial ergodic automorphism, yet it is not formed only of identities. This answers a question posed by Adam Kanigowski and Mariusz Lemańczyk in a private correspondence.

2. *Preliminaries*

2.1. *Measure-preserving automorphism and ergodic decomposition.* We consider invertible, bi-measurable and measure-preserving transformations T of standard Borel probability spaces $(X, \mathcal{B}, \mu)$. Recall that T is μ-preserving means that $T_*\mu = \mu$, where $T_*(\cdot)$ denotes the push-forward of a measure by the transformation T. Then, the quadruple $(X, \mathcal{B}, \mu, T)$ is called a dynamical system.

Without loss of generality, we can assume up to isomorphism that X is a compact metric space and T is a homeomorphism. Each Borel probability measure μ of a compact metric space X yields a standard Borel probability space. We denote by $\mathscr{M}(X)$ the space of such measures (it is compact in the weak*-topology). If $T : X \to X$ is a homeomorphism, then we denote by $\mathscr{M}(X, T)$ the subspace of $\mathscr{M}(X)$ consisting of T-invariant measures (it is a non-empty closed subset of $\mathscr{M}(X)$).

Two transformations T and S on a standard Borel probability space $(X, \mathcal{B}, \mu)$ are identified if $T(x) = S(x)$ for μ-almost every (a.e.) $x \in X$, and we call an *automorphism* of $(X, \mathcal{B}, \mu)$ an equivalence class of transformations modulo this identification. The group of automorphisms of $(X, \mathcal{B}, \mu)$ is denoted by $\mathrm{Aut}(X, \mathcal{B}, \mu)$. It is a classical fact that $\mathrm{Aut}(X, \mathcal{B}, \mu)$ with the so-called *weak topology* is a Polish group (see e.g. [17]).

Each automorphism T of $(X, \mathcal{B}, \mu)$ defines a unitary operator U_T on $L^2(X, \mathcal{B}, \mu)$, called a *Koopman operator*, defined by $U_T f := f \circ T$. We recall that T is said to be *ergodic* if the only U_T-invariant functions are constant. If, additionally, U_T has no other eigenfunctions, then T is said to be *weakly mixing*.

We denote by Aut the class of all automorphisms on all standard Borel probability spaces, considered up to isomorphism. By $\mathrm{Erg} \subset \mathrm{Aut}$, we denote the class of all ergodic automorphisms and by $\mathrm{WM} \subset \mathrm{Erg}$, the class of all weakly mixing automorphisms. Recall also that $\mathrm{ID} \subset \mathrm{Aut}$ stands for the class of identities on all standard Borel probability spaces. Since we allow the measures under consideration to have atoms, the class ID is uncountable.

Of course, in general, $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ need not be ergodic. Let us recall the classical concept of ergodic decomposition (see [21]). Denote by $\mathrm{Inv}(T)$ the σ-algebra of invariant sets. Let also

$$\mu = \int_{X/\mathrm{Inv}(T)} \mu_{\overline{x}} \, dP(\overline{x}) \tag{2}$$

be the disintegration of μ over $P := \mu|_{\mathrm{Inv(T)}}$. Then, there exists a measurable partition of $\overline{X} := X/\mathrm{Inv}(T)$:

$$\overline{X} = \bigcup_{n \geq 1} \overline{X}_n \cup \overline{X}_\infty \tag{3}$$

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and a standard Borel probability space $(Z, \mathcal{D}, \kappa)$, with κ non-atomic, such that the space $(X, \mathcal{B}, \mu)$ can be identified with the disjoint union of the corresponding product spaces (on $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, we consider the uniform measure)

$$\bigcup_{n\geq 1}(\overline{X}_n \times \mathbb{Z}_n) \cup (\overline{X}_\infty \times Z)$$

and the action of T in these new 'coordinates' is given by

$$\begin{aligned} T(\overline{x}, i) = (\overline{x}, T_{\overline{x}}(i)) = (\overline{x}, i+1) \quad \text{for } \overline{x} \in \overline{X}_n \text{ and} \\ T(\overline{x}, z) = (\overline{x}, T_{\overline{x}}(z)) \quad \text{for some ergodic } T_{\overline{x}} \in \mathrm{Aut}(Z, \mathcal{D}, \kappa),\ x \in \overline{X}_\infty. \end{aligned} \tag{4}$$

The map $\overline{x} \mapsto T_{\overline{x}}$ is measurable in the relevant Borel structures. The space $(\overline{X}, P)$ is called the space of ergodic components, and the representation of $(X, \mathcal{B}, \mu, T)$ given in equation (4) is called the *ergodic decomposition* of T. The ergodic decomposition is given up to a P-null subset. For example, if $X = \mathbb{T}^2$ (considered with Lebesgue measure $\mathrm{Leb}_{\mathbb{T}^2}$) and $T : (x, y) \mapsto (x, y + x)$, then it is already the ergodic decomposition of T since $\mathbb{T} \times \{0\}$ is the space of ergodic components (with $P = \mathrm{Leb}_{\mathbb{T}}$) and for P-a.e. $x \in \mathbb{T}$, $T_x(y) = x + y$ is an (ergodic) irrational rotation.

2.2. *Joinings.* Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ be two dynamical systems. We say that a probability measure λ on $(X \times Y, \mathcal{B} \otimes \mathcal{C})$ is a *joining* of T and S if:

(1) λ is $T \times S$- invariant;

(2) the marginals of λ on X- and Y-coordinates are μ and ν, respectively.

We note that $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \lambda, T \times S)$ is a dynamical system (sometimes, we denote such a system simply by $T \vee S$). We denote the set of all joinings of T and S by $J(T, S)$. Note that $\mu \otimes \nu$ is always an element of $J(T, S)$. Following [7], we say that T and S are *disjoint* if $J(T, S) = \{\mu \otimes \nu\}$ and write $T \perp S$. Note that if T and S are ergodic and $\lambda \in J(T, S)$, then the ergodic components of λ are also elements of $J(T, S)$. In particular, we have

$$T, S \in \mathrm{Erg} \quad \text{and} \quad T \perp S \ \Rightarrow\ (T \times S, \mu \otimes \nu) \in \mathrm{Erg}. \tag{5}$$

In the case where $T = S$, we set $J_2(T) := J(T, T)$ and refer to the elements of $J_2(T)$ as *2-self-joinings*. Unless T is the 1-point dynamical system, it is never disjoint with itself. Indeed, the diagonal measure is a 2-self-joining: $\mu_{\mathrm{Id}} := (\mathrm{Id}, \mathrm{Id})_*\mu$ and $\mu_{\mathrm{Id}} = \mu \otimes \mu$ if and only if T is a 1-point dynamical system. (Here and thereafter, we denote $(f, g)(x) = (f(x), g(x))$.) More generally, if $R \in C(T)$ is an element of the centralizer of T (that is, $R \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and $R \circ T = T \circ R$), then the graph measure $\mu_R := (\mathrm{Id}, R)_*\mu$ is a member of $J_2(T)$. In particular, the *off-diagonal* self-joinings μ_{T^n}, $n \in \mathbb{Z}$, belong to $J_2(T)$. In addition, note that $C(T)$ is a closed subgroup of $\mathrm{Aut}(X, \mathcal{B}, \mu)$.

It is a classical fact that in the weak-$*$ topology, $J_2(T)$ is a compact set, see [11]. Moreover, if T is additionally ergodic, then $J_2(T)$ is a simplex and the set of extremal points of $J_2(T)$ consists of ergodic 2-self-joinings. In particular, the set of ergodic self-joinings is non-empty and we denote it by $J_2^e(T)$. Note that the graph joinings μ_R are always ergodic as long as T is ergodic. In fact, the corresponding automorphisms are isomorphic to T, where an isomorphism is given by the map $x \mapsto (x, Rx)$.

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Given $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i \geq 1$, we also consider infinite joinings $\lambda \in J(T_1, T_2, \ldots)$ (invariant measures on $X_1 \times X_2 \times \cdots$ whose marginals are μ_i, $i \geq 1$). If T_i are ergodic, then $J^e(T_1, T_2, \ldots) \neq \emptyset$. Note that if $A \subset \mathbb{N}$, then we can speak about $J(T_{i_1}, T_{i_2}, \ldots)$, where $A = \{i_1, i_2, \ldots\}$ (A can be finite here). Whenever $T_1 = T_2 = \cdots = T$, we speak about $J_\infty(T)$ the set of (infinite) self-joinings. Now, if $A \subset \mathbb{N}$ and $R_{i_k} \in C(T)$, $k \geq 2$, then we can consider the corresponding graph self-joining $\mu_{R_{i_2}, R_{i_3}, \ldots} := (\mathrm{Id}, R_{i_2}, R_{i_3}, \ldots)_* \mu$. If R_{i_k} are powers of T, then we speak about off-diagonal self-joinings. If each $\lambda \in J^e_\infty(T)$ is a product of off-diagonal self-joinings, then we say that the automorphism T has the *minimal self-joining property* (or *MSJ*), see [22].

Following [15], we say that $(X, \mathcal{B}, \mu, T)$ has the *PID property* if for every $\lambda \in J_\infty(X, \mathcal{B}, \mu, T)$, if λ projects on every pair of coordinates as $\mu \otimes \mu$, then λ is the product measure. Note that each MSJ automorphism has the PID property.

Let $(X_i, \mathcal{B}_i, \mu_i, T_i)$ for $i = 1, 2$ be a pair of dynamical systems. Let also $\mathcal{A}_i \subset \mathcal{B}_i$ be factors of those systems, that is, invariant sub-σ-algebras. Let $\mu_i = \int_{X_i/\mathcal{A}_i} \mu_{i,\bar{x}_i}\, d\mu_i|_{\mathcal{A}_i}$ be the disintegration of μ_i over the respective factor. Assume that $\lambda \in J(T_1|_{\mathcal{A}_1}, T_2|_{\mathcal{A}_2})$. Then, the formula

$$\hat{\lambda} := \int_{X_1/\mathcal{A}_1 \times X_2/\mathcal{A}_2} \mu_{1,\bar{x}_1} \otimes \mu_{2,\bar{x}_2}\, d\lambda(\bar{x}_1, \bar{x}_2)$$

defines an element in $J(T_1, T_2)$ called *the relatively independent extension of* λ. We proceed similarly for finite and countable families of automorphisms (cf. proof of Lemma 8). If additionally $T_1|_{\mathcal{A}_1}$ and $T_2|_{\mathcal{A}_2}$ are both isomorphic to some $(Y, \mathcal{C}, \nu, S)$, then we may consider the *relatively independent extension over* $\mathcal{C}$ given by

$$\mu_1 \otimes_{\mathcal{C}} \mu_2 := \int_Y \mu_{1,y} \otimes \mu_{2,y}\, d\nu(y).$$

This corresponds to taking a diagonal joining as λ in the definition of the relatively independent extension.

We are interested in properties of the class of dynamical systems disjoint with all ergodic systems, that is, with

$$\mathrm{Erg}^\perp := \{T \in \mathrm{Aut} : \ T \perp S \text{ for every } S \in \mathrm{Erg}\}.$$

In the following subsection, we give some facts describing the structure of $\mathrm{Erg}^\perp$ as well as some basic examples of elements from this class.

2.3. *Elements of* $\mathrm{Erg}^\perp$.

Let us first recall some well-known examples of automorphisms from $\mathrm{Erg}^\perp$. It is a classical fact that $\mathrm{Id} \in \mathrm{Erg}^\perp$. Also, every system of the form $(x, y) \mapsto (x, x + y)$ on $\mathbb{T}^2$ with invariant measure $\mu \otimes \mathrm{Leb}_{\mathbb{T}}$ is an element of $\mathrm{Erg}^\perp$, as long as μ is a continuous measure on $\mathbb{T}$ (see e.g. Theorem 2). In fact, a recent result from [13] gives the full characterization of elements of $\mathrm{Erg}^\perp$ in the form of the following result.

THEOREM 2. [13, Theorem 3.1] *An automorphism T belongs to $\mathrm{Erg}^\perp$ if and only if (see (2)) for $P \otimes P$-almost every $(\bar{x}, \bar{y}) \in \overline{X}^2$, the automorphisms $T_{\bar{x}}$ and $T_{\bar{y}}$ are disjoint.*

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The only ergodic element of $\mathrm{Erg}^{\perp}$ is the 1-point system. Indeed, a system (T, μ) is disjoint from itself if and only if the diagonal self-joining is the product measure and this holds if and only if μ is a Dirac measure. Hence, the elements of the class $\mathrm{Erg}^{\perp}$ in $\mathrm{Aut}(X, \mathcal{B}, \mu)$ form a meagre set (as the ergodic systems form a generic subset of $\mathrm{Aut}(X, \mathcal{B}, \mu)$ [15]). Now, in view of Theorem 2 and the disjointness of time automorphisms result in [3], the automorphism on $\mathbb{T} \times X$ given by $(t, x) \mapsto (t, T_t(x))$, where $(T_t)_{t\in\mathbb{R}}$ is a generic flow, is an element of $\mathrm{Erg}^{\perp}$.

The following fact is also useful when trying to describe the class $\mathrm{Erg}^{\perp}$.

PROPOSITION 3. *Let $T \in \mathrm{Erg}^{\perp}$ and let R be an ergodic automorphism. Then, R is disjoint from P-a.e. fibre automorphism $T_{\bar{x}}$.*

Indeed, in view of Theorem 2, the measure P is continuous if T is non-trivial. Then, Proposition 3 follows from the following result (note that the measure $P|_{\overline{X}_\infty}$ has no atoms, since otherwise, $P \otimes P$ would have an atom in some point of the form $(\bar{x}, \bar{x})$, with $T_{\bar{x}}$ not being a one-point system, which contradicts Theorem 2).

PROPOSITION 4. [13, Lemma 2.10] *Let R be an ergodic automorphism. Then, for every set $B \subset \mathrm{Aut}(X, \mathcal{B}, \mu)$ of pairwise disjoint automorphisms, R is disjoint with all but at most countably many elements of B.*

In Lemma 12, we prove that $\alpha \in \mathbb{S}^1$ is an eigenvalue of T if and only if α is an eigenvalue for a positive P-measure set of fibre automorphisms $T_{\bar{x}}$. In particular, if there exists $\alpha \in \mathbb{S}^1 \setminus \{1\}$ such that α is an eigenvalue for a set of positive P-measure fibre automorphisms, then

$$\begin{gathered} T \text{ has the (ergodic) rotation by } \alpha \text{ as a factor} \\ \text{and } T \text{ is not an element of } \mathrm{Erg}^{\perp}. \end{gathered} \tag{6}$$

2.4. *Characteristic classes.* Recall that a class $\mathcal{F} \subset \mathrm{Aut}$ of measure-preserving dynamical systems is *characteristic* if it is closed under countable joinings and taking factors. In [16], the authors give a list of many examples of such classes. In particular, they proved that ID—the class consisting of all identities of standard probability Borel spaces—is a characteristic class and is contained in every non-trivial characteristic class. A natural question arises, whether ID is the only characteristic class that does not contain a non-trivial ergodic automorphism. One of our goals is to answer positively to this question, by constructing such a class inside $\mathrm{Erg}^{\perp}$.

First, we show that for every class $\mathcal{A} \subset \mathrm{Aut}$, the set of all multipliers $\mathcal{M}(\mathcal{A}^{\perp})$ is a characteristic class. We prove the following general result.

LEMMA 5. *Let $\mathcal{A} \subset \mathrm{Aut}$. Then, $(X, \mathcal{B}, \mu, T)$ is a multiplier of $\mathcal{A}^{\perp}$ if and only if $(X^{\times\infty}, \mathcal{B}^{\otimes\infty}, \eta, T^{\times\infty})$ is a multiplier of $\mathcal{A}^{\perp}$ for every $\eta \in J_\infty(T)$.*

Proof. $\Longleftarrow$: It is enough to notice that T is isomorphic to the diagonal joining $(\mathrm{Id} \times \mathrm{Id} \times \cdots)_*\mu$.

$\Longrightarrow$: Assume now that $(X, \mathcal{B}, \mu, T) \in \mathcal{A}^{\perp}$ is a multiplier of $\mathcal{A}^{\perp}$. Notice first that for every $n \geq 2$ and every $\tilde{\lambda} \in J_n(T)$, the automorphism $(X^{\times n}, \mathcal{B}^{\otimes n}, \tilde{\lambda}, T^{\times n})$ is an element

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of $\mathcal{A}^\perp$ and is a multiplier of this class. Indeed, since T is a multiplier of $\mathcal{A}^\perp$, then for $n = 1$ and $(Y, \mathcal{C}, \nu, S) \in \mathcal{A}^\perp$, we have that $T \vee S \in \mathcal{A}^\perp$. Again, using the fact that T is a multiplier of $\mathcal{A}^\perp$, $T \vee (T \vee S) \in \mathcal{A}^\perp$, which settles the case $n = 2$. The argument follows by induction.

Let now $(Y, \mathcal{C}, \nu, S) \in \mathcal{A}^\perp$ and let $(Z, \mathcal{D}, \rho, R) \in \mathcal{A}$. Consider also $\eta \in J(T^{\times\infty})$ and $\zeta \in J(T^{\times\infty} \vee S, R)$. Let $\zeta_n := \pi_*^{n,Y,Z}\zeta$ be the projection on the first n X-coordinates, Y-coordinate and Z-coordinate of ζ. In particular, $\zeta_n \in J(T^{\times n} \vee S, R)$. By the finite case, $\zeta_n = \zeta_n|_{X^n \times Y} \otimes \rho$. Since finite cylinders generate the whole product σ-algebra, by passing with n to infinity, we obtain that $\zeta = \eta \otimes \rho$. □

COROLLARY 6. *Let* $\mathcal{A} \subseteq \mathrm{Aut}$. *Then,* $\mathcal{M}(\mathcal{A}^\perp)$ *is a characteristic class.*

Proof. In view of Lemma 5, the class $\mathcal{M}(\mathcal{A}^\perp)$ is closed under taking countable joinings. It remains to see that it is closed under taking factors. Let $(X, \mathcal{B}, \mu, T) \in \mathcal{M}(\mathcal{A}^\perp)$ and let $(Y, \mathcal{C}, \nu, S)$ be a factor of T. If S has a non-trivial joining with $R \in \mathcal{A}$, then so does T, via the relatively independent extension. Hence, $\mathcal{M}(\mathcal{A}^\perp)$ is also closed under taking factors. □

In view of the above corollary, a natural candidate for an example of a class that does not contain a non-trivial ergodic element and is larger than ID would be the set $\mathcal{M}(\mathrm{Erg}^\perp)$. However, in §4, we show that this class contains only identities. To show that nonetheless such a class exists, we present the following construction.

Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. Let $\mathcal{F}(T)$ be the class of measure-preserving dynamical systems, which consists of all countable self-joinings of T, as well as all factors of those joinings.

LEMMA 7. *The class* $\mathcal{F}(T)$ *is characteristic.*

This result follows from the fact that a factor of a factor of a fixed automorphism R is still a factor of R and from the following classical lemma, whose proof we provide below for the sake of completeness.

LEMMA 8. *Let* $\{(X_i, \mathcal{B}_i, \mu_i, T_i)\}_{i=1}^\infty$ *be a family of measure-preserving automorphisms. For every* $i \in \mathbb{N}$, *let* $(Y_i, \mathcal{C}_i, \nu_i, S_i)$ *be a factor of* T_i *and let* $F_i : X_i \to Y_i$ *be the factorizing map. Then, for any* $\lambda \in J(S_1, S_2, \ldots)$, *there exists a joining* $\eta \in J(T_1, T_2, \ldots)$ *such that* $(S_1 \times S_2 \times \cdots, \lambda)$ *is a factor of* $(T_1 \times T_2 \times \cdots, \eta)$.

Proof. For every $i \in \mathbb{N}$, consider the disintegration of μ_i with respect to the factor S_i:

$$\mu_i = \int_{Y_i} \mu_{i,y_i}\, d\nu_i(y_i).$$

Then, consider the measure

$$\eta := \int_{Y_1 \times Y_2 \times \cdots} (\mu_{1,y_1} \otimes \mu_{2,y_2} \otimes \cdots)\, d\lambda(y_1, y_2, \ldots)$$

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on $X_1 \times X_2 \times \cdots$. It is a non-trivial element of $J(T_1, T_2, \ldots)$. Then, $(S_1 \times S_2 \times \cdots, \lambda)$ is a factor of $(T_1 \times T_2 \times \cdots, \eta)$ through the following factorizing map:

$$(x_1, x_2, \ldots) \mapsto (F_1(x_1), F_2(x_2), \ldots).$$

□

In §6, we provide an example of a non-identity automorphism $T \in \mathrm{Erg}^{\perp}$ such that $\mathcal{F}(T)$ does not contain a non-trivial ergodic element, see Corollary 30.

2.5. *Elements of spectral theory.* Let us recall here some notions and facts concerning the spectral theory of dynamical systems. Let $(X, \mathcal{B}, \mu, T)$ be a dynamical system. For any $f \in L^2(X, \mathcal{B}, \mu)$, we define the *spectral measure* $\sigma_{f,\mu}$ (a finite positive measure defined on $\mathbb{S}^1$) which, via the Herglotz theorem, is given by its Fourier transform:

$$\hat{\sigma}_{f,\mu}(n) := \int_{\mathbb{S}^1} z^n \, d\sigma_{f,\mu} = \langle f \circ T^{-n}, f \rangle_{L^2(\mu)} := \int_X f \circ T^{-n} \cdot \overline{f} \, d\mu \tag{7}$$

for $n \in \mathbb{Z}$. Among spectral measures of functions in

$$L^2_0(X, \mathcal{B}, \mu) = \left\{ g \in L^2(X, \mathcal{B}, \mu) : \int g \, d\mu = 0 \right\},$$

there are dominating ones (in the absolute continuity sense). All such must be equivalent, and their equivalence class is called the *maximal spectral type* of U_T. If μ is understood, we write $\hat{\sigma}_f$ instead of $\hat{\sigma}_{f,\mu}$. Note that $\sigma_f(\mathbb{S}^1) = \|f\|^2_{L^2(\mu)}$. Moreover, f of norm 1 is an eigenfunction of U_T if and only if $\sigma_{f,\mu}$ is the Dirac measure at the corresponding eigenvalue.

We now show some measurability results concerning spectral measures.

LEMMA 9. *Let X be a compact metric space and $T : X \to X$ be a homeomorphism. Let $f \in C(X)$ be a complex-valued continuous function such that $|f| = 1$. Then, the map $F : \mathscr{M}(X, T) \mapsto \mathscr{M}(\mathbb{S}^1)$ defined as*

$$F(\eta) := \sigma_{f,\eta}$$

is continuous.

Proof. Recall that by the definition of weak*-convergence in $\mathscr{M}(X)$, for every $g \in C(X)$, the map $\eta \mapsto \int_X g \, d\eta$ is continuous. Thus, the function

$$\mathscr{M}(X, T) \ni \eta \overset{F_1}{\longmapsto} (\hat{\sigma}_{f,\eta}(m))_{m\in\mathbb{N}} = \left(\int_X f \circ T^{-m} \cdot \overline{f} \, d\eta \right)_{m\in\mathbb{Z}} \tag{8}$$

is continuous. It follows that the map $F_1 : \mathscr{M}(X, T) \to \mathbb{D}^{\mathbb{Z}}$ (recall that $\mathbb{D} = \{z \in \mathbb{C} : |z| \le 1\}$ and on $\mathbb{D}^{\mathbb{Z}}$, we consider the usual product metric d) given by

$$F_1(\eta) := \left(\int_X f \circ T^{-m} \cdot \overline{f} \, d\eta \right)_{m\in\mathbb{Z}}$$

is continuous. Since $\mathscr{M}(X, T)$ is compact, the image $\Upsilon := F_1(\mathscr{M}(X, T))$ is also compact in $\mathbb{D}^{\mathbb{Z}}$.

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Now, we define a function $F_2 : \Upsilon \to \mathscr{M}(\mathbb{S}^1)$ in the following way:

$$F_2((a_m)_{m\in\mathbb{Z}}) = \sigma, \quad \text{where } \hat{\sigma}(m) = a_m.$$

It is well defined via the Herglotz theorem. Note that since the family $\{z^m\}_{m\in\mathbb{Z}}$ is linearly dense in $C(\mathbb{S}^1)$, the map F_2 is also continuous. Indeed, let $(a_m^n)_{m\in\mathbb{Z}}, (b_m)_{m\in\mathbb{Z}} \in \Upsilon$ and $\sigma_n := F_2((a_m^n)_{m\in\mathbb{Z}})$, $\sigma := F_2((b_m)_{m\in\mathbb{Z}})$ and assume that $d((a_m^n)_{m\in\mathbb{Z}}, (b_m)_{m\in\mathbb{Z}}) \to 0$ as $n \to \infty$ in Υ. Then, for every $m \in \mathbb{Z}$, we have $\int_{\mathbb{T}} z^m \, d\sigma_n \to \int_{\mathbb{T}} z^m \, d\sigma$. Since the functions z^m are linearly dense, this yields the continuity of F_2. Since $F = F_2 \circ F_1$, this finishes the proof. □

LEMMA 10. *Let X be a compact metric space. Let $T : X \to X$ be a homeomorphism and let $f \in C(X)$, $|f| = 1$. Then, for every $\alpha \in \mathbb{S}^1$, the map $G : \mathscr{M}(X, T) \mapsto [0, 1]$ defined as*

$$G(\eta) := \sigma_{f,\eta}(\{\alpha\})$$

is measurable.

Proof. Fix $\alpha \in \mathbb{S}^1$. In view of Lemma 9, the map $F : \mathscr{M}(X, T) \mapsto \mathscr{M}(\mathbb{S}^1)$ given by the formula $F(\eta) = \sigma_{f,\eta}$ is continuous. In particular, the set $\Omega = F(\mathscr{M}(X, T))$ is compact in the weak*-topology. It is thus enough to show that the map $G' : \mathscr{M}(\mathbb{S}^1) \mapsto [0, 1]$ defined as $G'(\sigma) := \sigma(\{\alpha\})$ is measurable.

Let $(g_n)_{n\in\mathbb{N}} \subset C(\mathbb{S}^1)$ be a sequence of (bounded by 1) real continuous functions converging pointwise to the indicator function $\chi_{\{\alpha\}}$. Define, for every $n \in \mathbb{N}$, the map $G'_n : \mathscr{M}(\mathbb{S}^1) \to \mathbb{R}$ as $G'_n(\sigma) = \int_{\mathbb{T}} g_n \, d\sigma$ which is continuous. Then, $\lim_{n\to\infty} G'_n = G'$ pointwise. As a point limit of continuous functions, G' is measurable. Since $G = G' \circ F$, this finishes the proof. □

Remark 11. Notice that if F and G are defined as $F(\eta) := \sigma_{f-\int f\,d\eta,\eta}$ and $G(\eta) := \sigma_{f-\int f\,d\eta,\eta}(\{\alpha\})$, then, by repeating proofs of Lemmas 9 and 10, we get that F is continuous and G is measurable.

We will also make use of the following fact based on spectral theory that provides a tool to detect eigenvalues of non-ergodic dynamical systems.

LEMMA 12. *Let X be a compact metric space. Let $\mu \in \mathscr{M}(X, T)$, where T is a homeomorphism of T. Let also*

$$\mu = \int_{\overline{X}} \mu_{\bar{x}} \, dP(\bar{x})$$

be the ergodic decomposition of T and let $T_{\bar{x}}$ denote the fibre automorphism, corresponding to $\bar{x} \in \overline{X}$. Then, $\alpha \in \mathbb{S}^1$ is an eigenvalue of $(T, \mu_{\bar{x}})$ for a P-positive measure set of $\bar{x}$ if and only if α is an eigenvalue of (T, μ).

Proof. Let $\alpha \in \mathbb{S}^1$ be as in the assumption. Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions which are dense in the space of complex continuous functions on X. Note that $\sigma_{f_n,\mu} = \int_{\overline{X}} \sigma_{f_n,\mu_{\bar{x}}} \, dP(\bar{x})$. Indeed, the mth coefficient of the Fourier transform of the right-hand side measure equals

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$$
\begin{aligned}
\int_{\overline{X}} \int_{\mathbb{S}^1} z^{-m} \, d\sigma_{f_n,\mu_{\bar{x}}} \, dP(\bar{x})
&= \int_{\overline{X}} \int_X f_n \circ T^{-m} \cdot \bar{f}_n(x) \, d\mu_{\bar{x}}(x) \, dP(\bar{x}) \\
&= \int_X f_n \circ T^{-m} \cdot \bar{f}_n(x) \, d\mu(x) = \hat{\sigma}_{f_n,\mu}(m).
\end{aligned}
$$

Then, the following conditions are equivalent:

- α is an eigenvalue of (T, μ);
- there exists $n \in \mathbb{N}$ such that $\sigma_{f_n,\mu}(\{\alpha\}) > 0$;
- there exists $n \in \mathbb{N}$ and a P-positive measure set of $\bar{x}$ such that $\sigma_{f_n,\mu_{\bar{x}}}(\{\alpha\}) > 0$;
- there exists a P-positive measure set of $\bar{x}$ such that α is an eigenvalue. □

COROLLARY 13. *Let $T \in \mathrm{Erg}^{\perp}$. For any $\alpha \in \mathbb{S}^1 \setminus \{1\}$, the set of ergodic components of T which have α as an eigenvalue has 0 measure with respect to P. In particular, $P(\overline{X}_n) = 0$ for every $n \geq 2$.*

The following fact is folklore, but for the sake of completeness of the presentation, we recall its proof.

LEMMA 14. *Let X be a compact metric space. Let $\mu \in \mathscr{M}(X, T)$, where T is a homeomorphism of T. Let also*

$$
\mu = \int_{\overline{X}} \mu_{\bar{x}} \, dP(\bar{x})
$$

be the ergodic decomposition of T and let $T_{\bar{x}}$ denote the fibre automorphism, corresponding to $\bar{x} \in \overline{X}$. Then, the set

$$
\mathcal{WM} := \{\bar{x} \in \overline{X};\ T_{\bar{x}} \text{ is weakly mixing}\}
$$

is measurable.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a dense family in $C(X)$ and fix $f = f_n$. Recall that an automorphism is weakly mixing if and only if it has no non-trivial eigenvalues or, in other words, all spectral measures on L^2_0 are continuous. Let us recall that for any measure $\sigma \in \mathscr{M}(\mathbb{S}^1)$, by Wiener's lemma, we have

$$
\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} |\hat{\sigma}(i)|^2 = \sum_{\alpha \text{ is an atom of } \sigma} \sigma^2(\{\alpha\}).
$$

It is enough to show that the above equality holds for $\sigma = \sigma_{f - \int f \, d\mu, \mu}$. Note that the function $H_N : \mathscr{M}(\mathbb{S}^1) \to \mathbb{R}_{\geq 0}$ given by $H_N(\sigma) := (1/N) \sum_{i=0}^{N-1} |\hat{\sigma}(i)|^2$, by the definition of weak*-convergence, is continuous. Thus, the function $H : \mathscr{M}(\mathbb{S}^1) \to \mathbb{R}_{\geq 0}$ given by $H(\sigma) := \lim_{N\to\infty} H_N(\sigma)$, as a pointwise limit of continuous functions, is measurable.

By Remark 11, the map $F : \mathscr{M}(X, T) \mapsto \mathscr{M}(\mathbb{S}^1)$ is continuous. Moreover, by the properties of disintegration of measures, the assignment $E : \overline{X} \to \mathscr{M}(X, T)$ given by $E(\bar{x}) = \mu_{\bar{x}}$ is also measurable. Thus, the map $H \circ F \circ E$ is measurable. It remains to notice that, since f is arbitrary, $\mathcal{WM} = (H \circ F \circ E)^{-1}(\{0\})$. □

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We also recall that if $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and $f, g \in L^2_0(X, \mathcal{B}, \mu)$, then $f \perp g$ whenever $\sigma_f \perp \sigma_g$, that is, when the spectral measures are mutually singular. It follows that two automorphisms are disjoint whenever the maximal spectral types on the relevant L^2_0 spaces are mutually singular [14].

3. *Identities are multipliers of* $\mathrm{Erg}^\perp$

The following fact is classical (and follows from the spectral disjointness of ergodic automorphism with the identity maps), we recall the classical proof for completeness.

PROPOSITION 15. *Any identity map is disjoint with all ergodic systems.*

Proof. Assume that $(Z, \mathcal{D}, \rho, R)$ is ergodic and consider the identity on $(Y, \mathcal{C}, \nu)$. Let $\eta \in J(R, \mathrm{Id})$. Take $h \in L^2(Z, \rho)$ of zero mean and let $g \in L^2(Y, \nu)$. By the von Neumann theorem (and ergodicity),

$$\frac{1}{N}\sum_{n=0}^{N-1} h \circ R^n \to 0 \quad \text{in } L^2(Z, \rho),$$

so the same convergence takes place also in $L^2(Z \times Y, \eta)$. Since the strong convergence implies the weak convergence,

$$\frac{1}{N}\sum_{n=0}^{N-1} \int g \otimes h \circ R^n \, d\eta \to 0.$$

However, for each $N \geq 1$, $\int (1/N) \sum_{n=0}^{N-1} (g \otimes h \circ (\mathrm{Id} \times R)^n) \, d\eta = \int g \otimes h \, d\eta$, whence $\int g \otimes h \, d\eta = 0$. □

Now, our main goal in this section is to show that any identity is actually also a multiplier of $\mathrm{Erg}^\perp$.

PROPOSITION 16. *Let* $(Z, \mathcal{D}, \rho, R) \in \mathrm{Erg}$ *and let* $(X, \mathcal{B}, \mu, T)$ *be such that* $T \perp R$. *Consider also an identity map* $(Y, \mathcal{C}, \nu, \mathrm{Id})$ *and let* $\lambda \in J(T, \mathrm{Id})$. *Then,*

$$(X \times Y, \mathcal{B} \otimes \mathcal{C}, \lambda, T \times \mathrm{Id}) \perp R.$$

Proof. Let $\eta \in J(T, \mathrm{Id}, R)$ be such that $\eta|_{X \times Y} = \lambda$. Note that, by Proposition 15, it holds that $\eta|_{Y \times Z} = \nu \otimes \rho$ and (by assumption)

$$\eta|_{X \times Z} = \mu \otimes \rho. \tag{9}$$

Fix bounded, real functions $f \in L^2(X, \mu)$, $g \in L^2(Y, \nu)$ and $h \in L^2_0(Z, \rho)$. All we need to show is that

$$\int f \otimes g \otimes h \, d\eta = 0. \tag{10}$$

Let $f = f_1 + f_2$, where $f_1 \circ T = f_1$ and $f_2 \perp L^2(\mathrm{Inv}(T))$. Then,

$$\int f \otimes g \otimes h \, d\eta = \int f_1 \otimes g \otimes h \, d\eta + \int f_2 \otimes g \otimes h \, d\eta. \tag{11}$$

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Now, the spectral measure (all the spectral measures are computed in $L^2(X \times Y \times Z, \eta)$) of the function $f_1 \otimes g$ is equivalent to δ_1, while the spectral measure of h has no atom at 1 since R is ergodic. Hence, these spectral measures are mutually singular and, therefore, $f_1 \otimes g$ and h are orthogonal in $L^2(X \times Y \times Z, \eta)$, so the first term on the right-hand side of equation (11) disappears. In view of equation (9), we have

$$\sigma_{f_2 \otimes h} = \sigma_{f_2} * \sigma_h.$$

Suppose that this measure has an atom at 1. Then, both spectral measures σ_{f_2} and σ_h must have an atom at $c \in \mathbb{S}^1$ and $\bar{c} \in \mathbb{S}^1$, respectively, where $c \neq 1$ (as R is ergodic). However, since h is real, c is also an atom of σ_h. Then, $T \not\perp R$, which is a contradiction. Hence, the spectral measures of $f_2 \otimes h$ and g are mutually singular, and thus these functions are orthogonal in $L^2(X \times Y \times Z, \eta)$ and equation (10) holds. □

COROLLARY 17. *Let* $(X, \mathcal{B}, \mu, T) \in \mathrm{Erg}^{\perp}$ *and let* $(Y, \mathcal{C}, \nu, \mathrm{Id})$ *be an identity map. Consider* $\lambda \in J(T, \mathrm{Id})$. *Then,*

$$(X \times Y, \mathcal{B} \otimes \mathcal{C}, \lambda, T \times \mathrm{Id}) \in \mathrm{Erg}^{\perp}.$$

In other words, any identity is a multiplier of $\mathrm{Erg}^{\perp}$.

Proof. The result follows directly from Proposition 16, by applying it to T and every $R \in \mathrm{Erg}$. □

4. *Identities are the only multipliers of* $\mathrm{Erg}^{\perp}$

In this section, we prove that there is no non-identity element of $\mathrm{Erg}^{\perp}$ which is a multiplier of this class. First, we show that the twist transformation on the torus is not a multiplier of $\mathrm{Erg}^{\perp}$.

Example 1. Let $T : X \times \mathbb{T} \to X \times \mathbb{T}$ be given by $T(x, y) = (x, y + \beta(x))$, where $\beta : X \to \mathbb{T}$ is measurable. Assume that $\rho \in \mathscr{M}(X)$ is a measure on X such that for every $z \in \mathbb{T}$, we have $\rho\{x \in X,\ \beta(x) = z\} = 0$. In particular, ρ is continuous. Note that since for every $\alpha \in \mathbb{T}$, there is only countably many non-disjoint rotations, the assumptions of Theorem 2 are satisfied and, therefore, $T \in \mathrm{Erg}^{\perp}$. (Another, more elementary, proof follows by the fact that almost every ergodic component of T is disjoint (even spectrally disjoint) with a fixed ergodic automorphism R and, by the ergodicity of R, any joining between R and T is a convex combination of joinings between R and ergodic components of T.) We will show that the system $(T, \rho \otimes \mathrm{Leb}_{\mathbb{T}})$ is not a multiplier of $\mathrm{Erg}^{\perp}$.

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and consider the automorphism

$$R(x, z) = (x, z + \beta(x) + \alpha)$$

on $\mathbb{T}^2$, which preserves the measure $\rho \otimes \mathrm{Leb}_{\mathbb{T}}$. One can check that R also satisfies the assumptions of Theorem 2, and hence $R \in \mathrm{Erg}^{\perp}$. Now, consider the transformation P on $X \times \mathbb{T}^2$ given by $P(x, y, z) = (x, y + \beta(x), z + \beta(x) + \alpha)$. It is easy to see that P is an

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automorphism of $(X \times \mathbb{T}^2, \rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}})$. Moreover, we can treat $\rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}}$ as a measure on $X^2 \times \mathbb{T}^2$ (up to a permutation of coordinates):

$$\rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}}(A_1 \times A_2 \times A_3 \times A_4) = \rho(A_1 \cap A_3)\mathrm{Leb}_{\mathbb{T}}(A_2)\mathrm{Leb}_{\mathbb{T}}(A_4),$$

where $A_i \in \mathcal{B}(\mathbb{T})$ for $i = 1, 2, 3, 4$. In other words, we consider the relatively independent extension over the common factor on the first coordinate, which is the identity map on X. Notice that then, we have

$$\rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}}(\{(x, y, x, z) : x \in X,\ \ y, z \in \mathbb{T}\}) = 1$$

and the measure we consider is $T \times R$-invariant, so it is easy to see that it is a joining of automorphisms T and R. Now, notice that $(T \times R, \rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}})$ has the rotation by α as a factor. Indeed, consider the map $\Pi : (X \times \mathbb{T}^2, \rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}}) \to (\mathbb{T}, \mathrm{Leb}_{\mathbb{T}})$ given by $\Pi(x, y, z) = z - y$. Then, we have

$$\Pi(P(x, y, z)) = \Pi(x, y + \beta(x), z + \beta(x) + \alpha) = z - y + \alpha$$

and

$$R_\alpha(\Pi(x, y, z)) = R_\alpha(z - y) = z - y + \alpha,$$

where R_α denotes the ergodic rotation by α on $\mathbb{T}$. In particular, in view of equation (6), the automorphism $(T \times R, \rho \otimes \mathrm{Leb}_{\mathbb{T}} \otimes \mathrm{Leb}_{\mathbb{T}})$ is not an element of $\mathrm{Erg}^{\perp}$.

As one of the crucial steps to prove the main result of this section, we show first an important special case, when all ergodic components of the considered system are weakly mixing.

For every Borel space $(X, \mathcal{B}, \mu)$, we consider the *flip* map $R = R_X : X \times X \to X \times X$ given by

$$R(x, y) := (y, x).$$

Note that R preserves $\mu \otimes \mu$ and for every $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, we have

$$R \in C(T \times T, \mu \otimes \mu). \tag{12}$$

This map will serve as a tool to produce ergodic factors of appropriately constructed self-joinings of considered systems. Hence, we first prove some auxiliary lemmas concerning the flip map.

LEMMA 18. *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing system. Then, $(X \times X, (T \times T) \circ R_X, \mu \otimes \mu)$ is also weakly mixing.*

Proof. Since (T, μ) is weakly mixing, then so is (T^2, μ) and hence $(X \times X, T^2 \times T^2, \mu \otimes \mu)$ is also weakly mixing. The result now follows from the fact that, by equation (12), we have

$$((T \times T) \circ R)^2 = T^2 \times T^2.$$

□

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LEMMA 19. *Let $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ be two weakly mixing systems such that*

$$(T \times T, \mu \otimes \mu) \perp (S \times S, \nu \otimes \nu).$$

Then,

$$((T \times T) \circ R_X, \mu \otimes \mu) \perp ((S \times S) \circ R_Y, \nu \otimes \nu).$$

Proof. Let $\rho \in J((T \times T) \circ R_X, (S \times S) \circ R_Y)$. We will show that $\rho = (\mu \otimes \mu) \otimes (\nu \otimes \nu)$.

Note that

$$\tfrac{1}{2}\rho + \tfrac{1}{2}(R_X \times R_Y)_*\rho \in J(T \times T, S \times S).$$

Thus, $\frac{1}{2}\rho + \frac{1}{2}(R_X \times R_Y)_*\rho = (\mu \otimes \mu) \otimes (\nu \otimes \nu)$, since $(T \times T)$ and $(S \times S)$ are disjoint. However, by Lemma 18, we have that

$$(((T \times T) \circ R_X) \times ((S \times S) \circ R_Y), (\mu \otimes \mu) \otimes (\nu \otimes \nu)) \in \text{Erg}.$$

Since an ergodic measure cannot be a non-trivial convex combination of other invariant measures, we obtain $\rho = (\mu \otimes \mu) \otimes (\nu \otimes \nu)$, which finishes the proof. □

We may now proceed to the proof of the special case of the main result of this section.

LEMMA 20. *Let $T \in \text{Erg}^\perp$ be such that almost every ergodic component is weakly mixing and not a one-point system. Then, T is not a multiplier of $\text{Erg}^\perp$.*

Proof. Let $T \in \text{Erg}^\perp$ be such that almost every ergodic component is weakly mixing and not a one-point system. Assume that $T \in \mathcal{M}(\text{Erg}^\perp)$. Let

$$\mu = \int_{\overline{X}} \mu_{\bar{x}} \, dP(\bar{x})$$

be the ergodic decomposition of (T, μ). Let $\tilde{\mu} \in J_2(T, \mu)$ be given by

$$\tilde{\mu} = \int_{\overline{X}} \mu_{\bar{x}} \otimes \mu_{\bar{x}} \, dP(\bar{x}). \tag{13}$$

Since P-a.e ergodic component of T is weakly mixing, then equation (13) actually gives the ergodic decomposition of $\tilde{\mu}$.

Note that, since for P-a.e. $\bar{x} \in \overline{X}$, the map $(T, \mu_{\bar{x}})$ is weakly mixing and not a one-point system, it follows that $\mu_{\bar{x}}$ is non-atomic and we have that

$$\tilde{\mu}\{(x, x) : x \in X\} = 0. \tag{14}$$

Consider $R = R_X$, the flip map on $X \times X$. Since R preserves $\mu_{\bar{x}} \otimes \mu_{\bar{x}}$ for all $\bar{x} \in \overline{X}$, we have that $\tilde{\mu}$ is $(T \times T) \circ R$-invariant. Moreover, by Lemma 18, the systems $\big((T \times T) \circ R, \mu_{\bar{x}} \otimes \mu_{\bar{x}}\big)$, $\bar{x} \in \overline{X}$, are ergodic and thus equation (13) also yields an ergodic decomposition of $\tilde{\mu}$ with respect to $(T \times T) \circ R$.

As a consequence of Lemma 5, we get that

$$(T \times T, \tilde{\mu}) \in \mathcal{M}(\text{Erg}^\perp) \subset \text{Erg}^\perp. \tag{15}$$

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In particular, by Theorem 2, for $P \otimes P$-a.e. $(\bar{x}, \bar{y}) \in \overline{X}^2$, we have

$$(T \times T, \mu_{\bar{x}} \otimes \mu_{\bar{x}}) \perp (T \times T, \mu_{\bar{y}} \otimes \mu_{\bar{y}}).$$

Hence, by Lemma 19, we obtain that for $P \otimes P$-a.e. $(\bar{x}, \bar{y}) \in \overline{X}^2$,

$$((T \times T) \circ R, \mu_{\bar{x}} \otimes \mu_{\bar{x}}) \perp ((T \times T) \circ R, \mu_{\bar{y}} \otimes \mu_{\bar{y}}).$$

This, again by Theorem 2, gives that $((T \times T) \circ R, \tilde{\mu}) \in \mathrm{Erg}^{\perp}$.

Denote by $\mathcal{R}$ the σ-algebra of R-invariant sets. Note that $\mathcal{R}$ is a factor σ-algebra for both $T \times T$ and $(T \times T) \circ R$. Hence, we may consider the associated relatively independent extension $\tilde{\mu} \otimes_{\mathcal{R}} \tilde{\mu} \in J\big((T \times T, \tilde{\mu}), ((T \times T) \circ R, \tilde{\mu})\big)$. Note that for $\tilde{\mu} \otimes_{\mathcal{R}} \tilde{\mu}$-a.e. (x_1, x_2, y_1, y_2), the following hold:

- $(x_1, x_2) = (y_1, y_2)$ or $(x_1, x_2) = (y_2, y_1)$ (because the restriction of $\tilde{\mu} \otimes_{\mathcal{R}} \tilde{\mu}$ to $\mathcal{R} \otimes \mathcal{R}$ is the diagonal joining);
- $x_1 \neq x_2$ and $y_1 \neq y_2$ by equation (14).

Therefore, we can consider the map $\varphi : (X \times X)^2 \to \mathbb{Z}_2$, defined $\tilde{\mu} \otimes_{\mathcal{R}} \tilde{\mu}$-almost everywhere, by

$$\varphi(x_1, x_2, y_1, y_2) = \begin{cases} 0 & \text{if } (x_1, x_2) = (y_1, y_2), \\ 1 & \text{if } (x_1, x_2) = (y_2, y_1). \end{cases}$$

Let A be the addition of 1 on $\mathbb{Z}_2$. Then, φ is a factor map between $((T \times T) \times (T \times T) \circ R$, $\tilde{\mu} \otimes_{\mathcal{R}} \tilde{\mu})$ and the ergodic rotation $(\mathbb{Z}_2, \frac{1}{2}(\delta_0 + \delta_1), A)$. Indeed, for $\tilde{\mu} \otimes_{\mathcal{R}} \tilde{\mu}$-a.e. (x_1, x_2, y_1, y_2), it holds that

$$\begin{aligned} \varphi \circ ((T \times T) \times (T \times T) \circ R)(x_1, x_2, y_1, y_2) \\ &= \varphi(Tx_1, Tx_2, Ty_2, Ty_1) \\ &= \begin{cases} 0 & \text{if } (Tx_1, Tx_2) = (Ty_1, Ty_2), \\ 1 & \text{if } (Tx_1, Tx_2) = (Ty_2, Ty_1), \end{cases} \\ &= \begin{cases} 0 & \text{if } (x_1, x_2) = (y_1, y_2), \\ 1 & \text{if } (x_1, x_2) = (y_2, y_1), \end{cases} \\ &= A \circ \varphi(x_1, x_2, y_1, y_2). \end{aligned}$$

Hence, we found a joining of $(T \times T, \tilde{\mu}) \in \mathcal{M}(\mathrm{Erg}^{\perp})$ (see equation (15)) and $((T \times T) \circ R, \tilde{\mu}) \in \mathrm{Erg}^{\perp}$, which has an ergodic rotation as a factor. This contradiction proves that $T \notin \mathcal{M}(\mathrm{Erg}^{\perp})$. □

Before passing to prove the main result of this section, let us deal with the measurability issues concerning the choice of a discrete factor. For this purpose, let us recall the classical Kuratowski–Ryll-Nardzewski theorem.

THEOREM 21. (Kuratowski and Ryll-Nardzewski [**18**]) *Let $(X, \mathcal{B})$ be a Polish space with the Borel σ-algebra and let $(\Omega, \mathcal{C})$ be a measurable space. Let f be a multifunction on Ω which takes values in closed subsets of X. Assume moreover that f is weakly measurable,*

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that is, for every open $A \subset X$, the set $\{\omega \in \Omega;\ f(\omega) \cap A \neq \emptyset\}$ is measurable. Then, there exists a measurable selector $F : \Omega \to X$ of f, that is, $F(\omega) \in f(\omega)$ for every $\omega \in \Omega$.

We use the above result to show that we can measurably assign an eigenvalue to any element of ergodic decomposition. Recall the notions of space of ergodic components $\overline{X}$ and its subsets $\overline{X}_n$ and $\overline{X}_\infty$ from equation (3).

LEMMA 22. *Let $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$ and let*

$$\mu = \int_{\overline{X}} \mu_{\bar{x}}\, dP(\bar{x})$$

be the ergodic decomposition. Assume that P-a.e. fibre automorphism is non-weakly mixing (that is, possesses a non-trivial eigenvalue). Then, there exists a measurable map $F : \overline{X}_\infty \to L^2(Z, \mathcal{D}, \kappa)$, where $(Z, \mathcal{D}, \kappa)$ is defined in equation (4), such that $F(\bar{x})$ is an eigenfunction of modulus 1 of $T_{\bar{x}}$ in $L^2_0(Z, \kappa)$ (identified with $L^2_0(X, \mu_{\bar{x}})$).

Proof. We will define F on the set $\overline{X}_\infty$ by considering a multifunction H defined below and taking F as a measurable selector given by Theorem 21. Therefore, the remainder of the proof is devoted to picking a proper multifunction and checking that the assumptions of Theorem 21 are satisfied.

Let $W \subset L^2_0(Z, \mathcal{D}, \kappa)$ be the subset of function of integral 0 and modulus 1 (note that W is a Polish space in L^2-topology). Let H be a multifunction which assigns to each element $T_{\bar{x}} := (T, \mu_{\bar{x}})$ the set $H(T_{\bar{x}}) \subset W$ of its eigenfunctions. Note that this is a closed set. Indeed, assume that $f_n \to f \in L^2_0(Z, \mathcal{D}, \kappa)$, where $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions in $H(T_{\bar{x}})$. Then, f is also of modulus 1. Moreover, by using the compactness of the circle and passing to a subsequence if necessary, we also have that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of the corresponding eigenvalues converges to some number $\lambda \in \mathbb{S}^1$. It is now easy to check that f is an eigenfunction corresponding to the eigenvalue λ.

We now prove that H is weakly measurable so that we can apply Theorem 21. Let $A \subset W$ be open. Without loss of generality, we can assume that $\overline{X}_\infty$ is a metric compact space. Consider the map $\varphi_A : \overline{X}_\infty \times \mathbb{S}^1 \times A \to L^2(Z, \mathcal{D}, \kappa)$, given by

$$\varphi_A(\bar{x}, \lambda, f) = f \circ T_{\bar{x}} - \lambda f.$$

By equation (4), $T_{\bar{x}} \in \mathrm{Aut}(Z, \mathcal{D}, \kappa)$. Hence, the right-hand side of the above formula is an element of $L^2(Z, \mathcal{D}, \kappa)$. Let

$$\pi_{\overline{X}} : \overline{X}_\infty \times \mathbb{S}^1 \times A \to \overline{X},\ \pi_{\overline{X}}(\bar{x}, \lambda, f) = \bar{x}$$

be the projection on the first coordinate. To prove the weak measurability of H, we need to show that the set

$$\{\bar{x} \in \overline{X}_\infty :\ H(T_{\bar{x}}) \cap A \neq \emptyset\} = \pi_{\overline{X}}(\varphi_A^{-1}(0))$$

is measurable. Since the map $\bar{x} \mapsto T_{\bar{x}}$ is Borel measurable, then so is φ_A. Thus, $\pi_{\overline{X}}(\varphi_A^{-1}(0))$ is analytic, and hence measurable with respect to $\mathcal{C}$ being the σ-algebra of P-measurable sets. It remains to use Theorem 21 for $\Omega = \overline{X}_\infty$ and $\mathcal{C}$. □

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Remark 23. The following result follows immediately from the fact that the class of automorphisms whose almost all ergodic components have discrete spectra is a characteristic class as proved in [**16**]. Hence, for T, there exists the largest factor of T belonging to this characteristic class, and this factor considered on each fibre $T_{\bar{x}}$ is the Kronecker factor of $T_{\bar{x}}$, see [**16**, §2.3.1]. However, we need a special form of this factor, and hence we give the complete proof below.

LEMMA 24. *Let T satisfy the assumptions of Lemma 22. Then, there exists a non-trivial factor of T whose ergodic decomposition consists of ergodic rotations.*

Proof. Let F be given by Lemma 22. Note that for every $\bar{x} \in \overline{X}_\infty$, the corresponding eigenvalue $\varphi(\bar{x})$ is equal to $F(\bar{x}) \circ T_{\bar{x}}/F(\bar{x})$, and hence it depends measurably on $\bar{x}$. Thus, in view of the fact that the map $\mathbb{S}^1 \ni \alpha \mapsto R_\alpha \in \mathrm{Aut}(\mathbb{S}^1, \mathrm{Leb}_{\mathbb{S}^1})$ is continuous, the automorphism $S \in \mathrm{Aut}(\overline{X} \times \mathbb{S}^1, P \otimes \mathrm{Leb}_{\mathbb{S}^1})$,

$$S(\bar{x}, r) := (\bar{x}, \varphi(\bar{x})r) \tag{16}$$

is the desired factor and factorizing map $J : \overline{X} \times Z \to \overline{X} \times \mathbb{S}^1$ is given by

$$J(\bar{x}, z) = (\bar{x}, F(\bar{x})(z)).$$

□

Proof of Theorem 1. Let $(X, \mathcal{B}, \mu, T) \in \mathcal{M}(\mathrm{Erg}^\perp)$ and let $\mu = \int_{\overline{X}} \mu_{\bar{x}}\, dP(\bar{x})$ be its ergodic decomposition. In view of Corollary 13, we have $P(\overline{X}_1 \cup \overline{X}_\infty) = 1$. We will show that $P(\overline{X}_1) = 1$. Assume by contradiction that $P(\overline{X}_\infty) > 0$.

Consider first the case when $P(\mathcal{WM}) > 0$, where $\mathcal{WM} := \{\bar{x} \in \overline{X}_\infty : T_{\bar{x}} \text{ is weakly mixing}\}$ (it is measurable via Lemma 14). Then, T can be decomposed into a disjoint action of two automorphisms $\tilde{T}$ and $\tilde{T}^*$. Here, $\tilde{T}$ stands for the restriction of T to the union of fibres corresponding to $\mathcal{WM}$, that is, it is measure $\tilde{\mu}$-preserving, where the ergodic decomposition of $\tilde{T}$ is of the form

$$\tilde{\mu} = \frac{1}{P(\mathcal{WM})} \int_{\mathcal{WM}} \mu_{\bar{x}}\, dP(\bar{x})$$

and the automorphism $\tilde{T}^*$ is the restriction of T to the union of the fibres corresponding to $\overline{X} \setminus \mathcal{WM}$. Then, $(\tilde{T}, \tilde{\mu})$ satisfies the assumption of Lemma 20 and thus, it is not a multiplier of $\mathrm{Erg}^\perp$, that is, there exists $S \in \mathrm{Erg}^\perp$ and a joining $\tilde{\eta} \in J(\tilde{T}, S)$ such that $(\tilde{T} \times S, \tilde{\eta}) \notin \mathrm{Erg}^\perp$. It is now enough to join $\tilde{T}^*$ and S independently to get a non-trivial joining $\eta \in J(T, S)$ such that $(T \times S, \eta) \notin \mathrm{Erg}^\perp$, which is a contradiction with the fact that $T \in \mathrm{Erg}^\perp$.

Thus, we can assume that for P-a.e. $\bar{x} \in \overline{X}_\infty$, the automorphism $T_{\bar{x}}$ has a non-trivial eigenvalue. Let $\mu_\infty = (1/P(\overline{X}_\infty)) \int_{\overline{X}_\infty} \mu_{\bar{x}}\, dP(x)$. Then, by Lemma 24, there exists a factor $\hat{T}$ of (T, μ_∞) such that all ergodic components $\hat{T}_{\bar{x}}$ of $\hat{T}$ are rotations on the circle. Note that for a.e. pair of $(\bar{x}, \bar{y}) \in \overline{X}_\infty \times \overline{X}_\infty$, the associated rotations are disjoint in view of Theorem 2. Then, $\hat{T}$ has exactly the form of Example 1, which is not in $\mathcal{M}(\mathrm{Erg}^\perp)$. Since, by Corollary 6, $\mathcal{M}(\mathrm{Erg}^\perp)$ is a characteristic class, we get that $(T, \mu_\infty) \notin \mathcal{M}(\mathrm{Erg}^\perp)$. Thus, again by joining the restriction of T to the fibres in $\overline{X}_1$ independently, we get that $(T, \mu) \notin \mathcal{M}(\mathrm{Erg}^\perp)$.

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Hence, $P(\overline{X}_1) = 1$, which means that T is an identity. This proves that $\mathcal{M}(\mathrm{Erg}^\perp) \subset \mathrm{ID}$. The opposite inclusion follows directly from Corollary 17. This finishes the proof. $\square$

5. $\mathrm{Erg}^\perp$ *is closed under taking products*

As Theorem 1 shows, the class $\mathrm{Erg}^\perp$ is not closed under taking joinings due to an interplay between fibre automorphisms over the 'common' part of ergodic components. It turns out that this phenomenon can not happen if the spaces of ergodic components are independent—in this section, we show the class $\mathrm{Erg}^\perp$ is actually closed under taking the Cartesian products.

THEOREM 25. *Assume that* $(X, \mathcal{B}, \mu, T), (Y, \mathcal{C}, \nu, S) \in \mathrm{Erg}^\perp$. *Then,*

$$(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu, T \times S) \in \mathrm{Erg}^\perp .$$

Proof. Let $(Z, \mathcal{D}, \rho, R)$ be an arbitrary ergodic system. Recall first that all systems under consideration may be viewed as measure-preserving homeomorphisms of compact metric spaces. The above observation allows one to use Lemma 10 later in the proof.

On the space $X \times Y \times Z$, for every $A \in \{X, Y, Z, X \times Y, X \times Z, Y \times Z\}$, denote by $\pi^A : X \times Y \times Z \to A$ the standard projection on the corresponding coordinates. Consider any joining $\lambda \in J((T \times S, \mu \otimes \nu), (R, \rho))$. We aim at showing that $\lambda = \mu \otimes \nu \otimes \rho$.

First, note that $(\pi^{X\times Z})_*\lambda = \mu \otimes \rho$ and $(\pi^{Y\times Z})_*\lambda = \nu \otimes \rho$ as $T, S \in \mathrm{Erg}^\perp$. Let

$$\mu = \int_{\overline{X}} \mu_{\bar{x}} \, dP(\bar{x})$$

be the ergodic decomposition of μ (with $P = \mu|_{\mathrm{Inv(T)}}$). Since $\mathrm{Inv}(T)$ is also a factor of $(T \times S \times R, \lambda)$ and $(\pi^X)_*\lambda = \mu$,

$$\lambda = \int_{\overline{X}} \lambda_{\bar{x}} \, dP(\bar{x}) \tag{17}$$

and $(\pi^X)_*\lambda_{\bar{x}} = \mu_{\bar{x}}$ P-almost everywhere, by the uniqueness of disintegrations. Moreover, note that

$$\mu \otimes \rho = \int_{\overline{X}} \mu_{\bar{x}} \otimes \rho \, dP(\bar{x}).$$

Since $(\pi^{X\times Z})_*\lambda = \mu \otimes \rho$, again by using the uniqueness of disintegration, we get

$$(\pi^{X\times Z})_*\lambda_{\bar{x}} = \mu_{\bar{x}} \otimes \rho \quad P\text{-almost everywhere.} \tag{18}$$

Analogously, since $(\pi^{X\times Y})_*\lambda = \mu \otimes \nu$, we also have

$$(\pi^{X\times Y})_*\lambda_{\bar{x}} = \mu_{\bar{x}} \otimes \nu \quad P\text{-almost everywhere.} \tag{19}$$

In particular, $(\pi^Y)_*\lambda_{\bar{x}} = \nu$. Thus, by equation (18), we have

$$\lambda_{\bar{x}} \in J((S, \nu), (T \times R, \mu_{\bar{x}} \otimes \rho)) \quad P\text{-almost everywhere.}$$

What remains to show is that

$$P(\{\bar{x} : (T \times R, \mu_{\bar{x}} \otimes \rho) \text{ is not ergodic}\}) = 0. \tag{20}$$

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Indeed, if equation (20) holds, then since $S \in \text{Erg}^\perp$, we obtain that

$$\lambda_{\bar{x}} = \mu_{\bar{x}} \otimes \nu \otimes \rho \quad P\text{-almost everywhere},$$

which together with equation (17) yields $\lambda = \mu \otimes \nu \otimes \rho$.

To show equation (20), recall first that R can have at most countably many eigenvalues. Moreover, the Cartesian product of two ergodic systems is not ergodic if and only if they have a non-trivial common eigenvalue. It is thus enough to show that for any fixed $\alpha \in \mathbb{T} \setminus \{0\}$, we have

$$P(\{\bar{x} : \alpha \text{ is an eigenvalue of } (T, \mu_{\bar{x}})\}) = 0. \tag{21}$$

Assume that equation (21) does not hold, that is,

$$P(\{\bar{x} : \alpha \text{ is an eigenvalue of } (T, \mu_{\bar{x}})\}) > 0. \tag{22}$$

Then, by Lemma 12, α is an eigenvalue of T. This is a contradiction with equation (6). Hence, we proved equation (21) which in turn completes the proof of the theorem. □

By induction, we obtain from the above result the following corollary.

COROLLARY 26. *If $(X_i, \mathcal{B}_i, \mu_i, T_i)_{i\in\mathbb{N}}$ is a sequence of elements of* $\text{Erg}^\perp$*, then*

$$\left(\prod_{i=0}^{\infty} X_i, \bigotimes_{i=0}^{\infty} \mathcal{B}_i, \bigotimes_{i=0}^{\infty} \mu_i, \prod_{i=0}^{\infty} T_i\right) \in \text{Erg}^\perp .$$

6. *An automorphism whose self-joinings are disjoint with ergodic automorphisms*

We are going to construct an automorphism whose self-joinings are all elements of $\text{Erg}^\perp$. To do that, we will rely on the PID property. We need the following result.

THEOREM 27. (Ryzhikov [**23**]) *Assume that $(X, \mathcal{B}, \mu, T)$ has the PID property and $(Y_i, \mathcal{B}_i, \nu_i, S_i)$, $i = 1, 2$, are measure-preserving automorphisms. If $\lambda \in J(T, S_1, S_2)$ projects as product measures on each pair of coordinates, then*

$$\lambda = \mu \otimes \nu_1 \otimes \nu_2.$$

In the proof of Theorem 29, we will use several times the following corollary of the above result.

COROLLARY 28. *Let $n \geq 2$ and let $(X_i, \mathcal{B}_i, \mu_i, T_i)$, $i = 1, \ldots, n$, be measure-preserving automorphisms satisfying the PID property. Let $(Y, \mathcal{C}, \nu, S)$ be a measure-preserving automorphism. If $\lambda \in J(S, T_1, \ldots, T_n)$ projects as product measures on each pair of coordinates, then*

$$\lambda = \nu \otimes \mu_1 \otimes \cdots \otimes \mu_n.$$

Proof. We use induction. For $n = 2$, the result follows directly from Theorem 27. Assume that it is true for some $n \geq 2$, we will prove that it holds for $n + 1$. Let $\lambda \in J(S, T_1, \ldots, T_{n+1})$ and let $\tilde{\lambda}$ be the projection of λ on $Y \times X_1 \times \cdots \times X_n$ and

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let $\bar{\lambda}$ be the projection of λ on $X_1 \times \cdots \times X_{n+1}$. By the induction assumption, we get that

$$\tilde{\lambda} = \nu \otimes \mu_1 \otimes \cdots \otimes \mu_n \quad \text{and} \quad \bar{\lambda} = \mu_1 \otimes \cdots \otimes \mu_{n+1}.$$

The result now follows from Theorem 27 by taking $T := T_{n+1}$, $S_1 := S$ and $S_2 := T_1 \times \cdots \times T_n$. □

We are now ready to present and prove the main result of this section.

THEOREM 29. *Let $T : X \to X$ be a μ-preserving homeomorphism of a compact metric space X. Let*

$$\mu = \int_{\overline{X}} \mu_{\bar{x}} \, dP(\bar{x})$$

be the ergodic decomposition and let $T_{\bar{x}}$ denote the action of T on the fibre corresponding to $\bar{x} \in \overline{X}$. Assume that P is a continuous probability measure. Assume moreover that:

- *for every $\bar{x} \in \overline{X}$, there exists a set of $\bar{y} \in \overline{X}$, whose complement is countable, such that for every $\bar{y}$ from this set, we have $T_{\bar{x}} \perp T_{\bar{y}}$;*
- *for every $\bar{x}, \bar{y} \in \overline{X}$, if $T_{\bar{x}} \not\perp T_{\bar{y}}$, then $T_{\bar{x}}$ and $T_{\bar{y}}$ are isomorphic;*
- *for every $\bar{x} \in \overline{X}$, the automorphism $(T_{\bar{x}}, \mu_{\bar{x}})$ has the MSJ property (in particular, it has the PID property).*

Then, $(T^{\times\infty}, \eta) \in \mathrm{Erg}^{\perp}$ for every $\eta \in J_\infty(T, \mu)$.

Proof. It is enough to prove that for every $n \geq 1$ and every $\eta \in J_n(T, \mu)$, $(T^{\times n}, \eta) \in \mathrm{Erg}^{\perp}$. Note that since P is continuous, for $P \otimes P$-a.e. $(\bar{x}, \bar{y}) \in \overline{X} \times \overline{X}$, we have $T_{\bar{x}} \perp T_{\bar{y}}$. Then, by Theorem 2, $T \in \mathrm{Erg}^{\perp}$.

Let $2 \leq n < \infty$, fix $\eta \in J_n(T, \mu)$ and let $(R, \rho) \in \mathrm{Erg}$ be arbitrary. Let $\psi \in J((T^{\times n}, \eta), (R, \rho))$. We want to show that $\psi = \eta \otimes \rho$. To distinguish between the n copies of X and $\overline{X}$, we denote by X_k and $\overline{X}_k$ the domain and the space of ergodic components of T on the kth coordinate for every $k = 1, \ldots n$. We assign the coordinate $(n+1)$ to the automorphism R. Finally, we set $\mathbf{X} = X_1 \times \cdots \times X_n$ and by $\pi^{k_1,\ldots,k_m} : \mathbf{X} \to X_{k_1} \times \cdots \times X_{k_m}$, we denote the projection on $k_1, \ldots, k_m$ coordinates.

Consider the ergodic decomposition of $(T^{\times n}, \eta)$:

$$\eta = \int_{\overline{\mathbf{X}}} \eta_{\bar{\mathbf{x}}} \, dQ(\bar{\mathbf{x}}).$$

Thus, we also have the following disintegration of ψ:

$$\psi = \int_{\overline{\mathbf{X}}} \psi_{\bar{\mathbf{x}}} \, dQ(\bar{\mathbf{x}}).$$

By uniqueness of ergodic decomposition, we have that $(\pi^{\mathbf{X}})_* \psi_{\bar{\mathbf{x}}} = \eta_{\bar{\mathbf{x}}}$ and by ergodicity of R, we get $(\pi^{n+1})_* \psi_{\bar{\mathbf{x}}} = \rho$. Thus, $\psi_{\bar{\mathbf{x}}} \in J((T^{\times n}, \eta_{\bar{\mathbf{x}}}), (R, \rho))$ for Q-a.e. $\bar{\mathbf{x}}$. To show that $\psi = \eta \otimes \rho$, we just have to show that

$$\psi_{\bar{\mathbf{x}}} = \eta_{\bar{\mathbf{x}}} \otimes \rho \quad \text{for } Q\text{-a.e. } \bar{\mathbf{x}} \in \overline{\mathbf{X}}. \tag{23}$$

Note that by uniqueness of ergodic decomposition, for Q-a.e. $\bar{\mathbf{x}} \in \overline{\mathbf{X}}$ and for every $k = 1, \ldots, n$, we have

$$(\pi^k)_* \psi_{\bar{\mathbf{x}}} = (\pi^k)_* \eta_{\bar{\mathbf{x}}} = \mu_{\bar{x}_k} \quad \text{for some } \bar{x}_k(\bar{\mathbf{x}}) = \bar{x}_k \in \overline{X}_k.$$

Moreover, all marginals of Q are equal to P. Therefore, since P is continuous, we get that for every $\bar{x} \in \overline{X}$, we have

$$Q\{\bar{\mathbf{x}} \in \overline{\mathbf{X}} : \ \mu_{\bar{x}_k} = \mu_{\bar{x}} \text{ for some } k = 1, \ldots, n\} = 0. \tag{24}$$

In particular, since there are only up to countably many ergodic components of T isomorphic to $\mu_{\bar{x}}$, we get by equation (24) that for every $\bar{x} \in \overline{X}$,

$$Q(\{\bar{\mathbf{x}} \in \overline{\mathbf{X}} : \ (T, \mu_{\bar{x}_k}) \text{ is isomorphic to } (T, \mu_{\bar{x}}) \text{ for some } k = 1, \ldots, n\}) = 0. \tag{25}$$

Now, to show equation (23), we consider cases which depend on the form of $\eta_{\bar{\mathbf{x}}}$. More precisely, we consider the number of coordinates on which the projection yields isomorphic maps (recall that by assumption, the ergodic components are either isomorphic or disjoint).

Case 1: No isomorphic components. Let $\mathcal{D}_0 \subset \overline{\mathbf{X}}$ be the set of elements satisfying the following:

$$(T, \mu_{\bar{x}_1}), \ldots, (T, \mu_{\bar{x}_n}) \quad \text{are pairwise disjoint.} \tag{26}$$

Then, by Corollary 28 and mutual disjointness assumption, for every $\bar{\mathbf{x}} \in \mathcal{D}_0$, we get

$$\eta_{\bar{\mathbf{x}}} = \mu_{\bar{x}_1} \otimes \cdots \otimes \mu_{\bar{x}_n}. \tag{27}$$

Note also that for Q-a.e. $\bar{\mathbf{x}} \in \mathcal{D}_0$, the automorphism (R, ρ) is disjoint with $(T, \mu_{\bar{x}_k})$ for every $k = 1, \ldots, n$. Indeed, otherwise for some $1 \leq k_0 \leq n$, we would have that

$$\begin{aligned} &Q(\{\bar{\mathbf{x}} \in \overline{\mathbf{X}}; \ (T, \mu_{\bar{x}_{k_0}}) \text{ and } (R, \rho) \text{ are not disjoint }\}) \\ &\quad = P(\{\bar{x} \in \overline{X}; \ (T, \mu_{\bar{x}}) \text{ and } (R, \rho) \text{ are not disjoint }\}) > 0. \end{aligned}$$

(The measurability of sets considered follows from [**13**, Corollary 3.8].)

By Proposition 4 and the assumptions of the theorem, the set considered above can be at most countable. This is a contradiction with the fact that P is continuous.

Thus, for Q-a.e. $\bar{\mathbf{x}} \in \mathcal{D}_0$, the automorphism (R, ρ) is disjoint with $(T, \mu_{\bar{x}_k})$ for every $k = 1, \ldots, n$. In particular, by equation (27), for every such $\bar{\mathbf{x}}$, the measure $\psi_{\bar{\mathbf{x}}}$ projects on each pair of coordinates as a product measure. By Corollary 28, we get that

$$\psi_{\bar{\mathbf{x}}} = \eta_{\bar{\mathbf{x}}} \otimes \rho$$

for Q-a.e. $\bar{\mathbf{x}} \in \mathcal{D}_0$. This finishes the proof of Case 1.

Case 2: There exist isomorphic components. Consider now the elements $\bar{\mathbf{x}} \in \overline{\mathbf{X}}$ such that, up to a permutation of coordinates, there exist $m < n$ and indices $0 = \ell_0 < \ell_1 < \cdots < \ell_{m-1} < \ell_m = n$ such that $(T, \mu_{\bar{x}_{\ell_{k-1}+1}}), \ldots, (T, \mu_{\bar{x}_{\ell_k}})$ are isomorphic for every $k = 1, \ldots, m$ and m is minimal in this representation. For every $m = 1, \ldots, n-1$,

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denote by $\mathcal{D}_m \subset \overline{\mathbf{X}}$ the set of elements for which there are exactly m groups of indices in the above decomposition. We will show that this case reduces to Case 1.

Fix $m = 1, \ldots, n-1$ and let $\bar{\mathbf{x}} \in \mathcal{D}_m$. Let $\ell_0, \ldots, \ell_m$ be as above. Since $\eta_{\bar{\mathbf{x}}}$ is ergodic, for every $k = 1, \ldots, m$, the projection $(\pi^{\ell_{k-1}+1,\ldots,\ell_k})_*\eta_{\bar{\mathbf{x}}}$ is an ergodic measure for the map $T^{\times \ell_k - \ell_{k-1}}$ (up to an isomorphism, it is a self-joining of $(T, \mu_{\bar{x}_{\ell_k}})$). Moreover, recalling that $(T, \mu_{\bar{x}})$ has the MSJ property for every $\bar{x} \in \overline{X}$, we obtain that

$$(\pi^{\ell_{k-1}+1,\ldots,\ell_k})_*\eta_{\bar{\mathbf{x}}} \text{ is a product of } r_k \leq \ell_k - \ell_{k-1} \text{ off-diagonal self-joinings.}$$

Thus, the measure $\eta_{\bar{\mathbf{x}}}$ is actually a joining of $r_1 + \cdots + r_m$ automorphisms such that any two of them are either disjoint or isomorphic, and in the latter case, $\eta_{\bar{\mathbf{x}}}$ projects on the corresponding coordinates as the product measure. Hence, by using Corollary 28, we get that $\eta_{\bar{\mathbf{x}}}$ is a product joining of $r_1 + \cdots + r_m$ ergodic components of T. In other words, we reduced the problem, where equation (27) is satisfied for a smaller number of indices. The remainder of the proof of Case 2 is analogous to the proof of Case 1 by taking $n := r_1 + \cdots + r_m$. □

Recalling the definition of a characteristic class and $\mathcal{F}(T)$ from §2.4, we get the following result.

COROLLARY 30. *If T satisfies the assumptions of Theorem 29, then $\mathcal{F}(T)$ is a characteristic class of elements from* $\mathrm{Erg}^{\perp}$.

Proof. By Lemma 7, Theorem 29 and the fact that any factor of an element of $\mathrm{Erg}^{\perp}$ is also a member of this class. □

We now construct an example to show that the set of automorphisms satisfying the assumptions of Theorem 29 is non-empty. We build it by using the classical cutting and stacking construction of systems of rank 1. We have the following fact, which is a special case of the results by Gao and Hill [9] and Danilenko [2].

PROPOSITION 31. *Let $\bar{a} = \{a_n\}_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$ be the binary expansion of $a \in [0, 1]$. Let $T_a : [0, 1] \to [0, 1]$ be a rank 1 automorphism defined in the following way, via the cutting and stacking procedure:*

- *on each step, the cutting parameter is equal to 3;*
- *on step n, we add a spacer over the first tower if $a_n = 0$ and over the second if $a_n = 1$.*

Assume that $|a - b| \neq k/2^l$ for any $k, l \in \mathbb{N}$. Then, T_a and T_b are disjoint. Otherwise, they are isomorphic.

By [4], the systems considered in the above proposition are ergodic and satisfy the MSJ property. Note that the sequence heights of towers in the cutting and stacking construction described in the above theorem is universal for all $a \in [0, 1]$. We leave the proof of the following easy lemma to the reader.

LEMMA 32. *The map $[0, 1] \ni a \mapsto T_a \in \mathrm{Aut}(X, \nu)$ is well defined and continuous in $[0, 1] \setminus \mathbb{Q}$.*

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Let us consider $S \in \mathrm{Aut}([0,1] \times X, \mathrm{Leb}_{[0,1]} \otimes \nu)$ defined in the following way:

$$S(a, x) := (a, T_a x).$$

Then, in view of Proposition 31 and Lemma 32, by previous remarks, S satisfies the assumptions of Theorem 29. In particular, we have $S \in \mathrm{Erg}^{\perp}$.

7. *A remark on multipliers*

We will now show some connections between the multipliers of a class $\mathcal{A}^{\perp}$ and the characteristic classes included in $\mathcal{A}^{\perp}$. We recall that $\mathcal{M}(\mathcal{A}^{\perp})$ is always a characteristic class (see Corollary 6).

PROPOSITION 33. *For any T and R, we have that $T \in \mathcal{M}(\{R\}^{\perp})$ provided that for each $\lambda \in J_2(T)$, we have $(T \times T, \lambda) \perp R \times R$, where $R \times R$ is considered with product measure.*

The proof of the proposition follows by word for word repetition (ignoring the ergodicity of T) of the proof of [**19**, Proposition 5.1]. Notice moreover that this proposition applies (non- trivially) only if $R \in \mathrm{WM}$ (because we consider all self-joinings of T, and hence also non-ergodic; however, $R \times R$ which is considered with product measure is not ergodic whenever R is not weakly mixing; finally use that two non-trivial non-ergodic automorphisms are not disjoint).

COROLLARY 34. *Assume that $\mathcal{A}$ is a class closed under taking Cartesian squares. Then, $T \in \mathcal{M}(\mathcal{A}^{\perp})$ if and only if for each $\lambda \in J_2(T)$, we have $(T \times T, \lambda) \perp R$ for each $R \in \mathcal{A}$.*

Note that the necessity in the corollary is obvious and the other direction follows from Proposition 33. Similarly to Proposition 33, the above corollary only applies when $\mathcal{A} \subset \mathrm{WM}$. Corollary 34 implies also the following result.

COROLLARY 35. *Assume that $\mathcal{A}$ is a class closed under taking Cartesian squares. Then, $\mathcal{M}(\mathcal{A}^{\perp})$ is the largest characteristic class included in $\mathcal{A}^{\perp}$. In particular, the result holds when $\mathcal{A} = \mathrm{WM}$.*

Note that ID satisfies the assumptions of Corollary 35. It would be interesting to know whether the assertion of Corollary 35 holds for any class $\mathcal{A}$. Notice that for $\mathcal{A} = \mathrm{Erg}$, the assumption on the Cartesian squares is not satisfied as the Cartesian square of any (non-trivial) rotation is not ergodic. More than that, in $\mathrm{Erg}^{\perp}$, the class of multipliers is ID, and it is the smallest characteristic class included in $\mathrm{Erg}^{\perp}$ (the existence of larger than ID characteristic classes follows from Corollary 30), so the assertion of Corollary 35 fails for $\mathrm{Erg}^{\perp}$.

A general question arises, whether given a characteristic class $\mathcal{C} \subset \mathcal{A}^{\perp}$, there exists a maximal characteristic class $\mathcal{C}' \subset \mathcal{A}^{\perp}$ such that $\mathcal{C} \subset \mathcal{C}'$. To see that this holds, we apply the Kuratowski–Zorn lemma. Consider a chain $(\mathcal{C}_i)_{i \in \mathbb{I}}$ of characteristic classes such that $\mathcal{C} \subset \mathcal{C}_i \subset \mathcal{A}^{\perp}$. Let $\tilde{\mathcal{C}}$ be the smallest characteristic class containing $\bigcup_{i \in \mathbb{I}} \mathcal{C}_i$. We need to show that $\tilde{\mathcal{C}} \subset \mathcal{A}^{\perp}$. It is enough to check that if $T_n \in \bigcup_{i \in \mathbb{I}} \mathcal{C}_i$, $n = 1, 2, \ldots$, then for every $\eta \in J(T_1, T_2, \ldots)$, we have $(T_1 \times T_2 \times \cdots, \eta) \in \mathcal{A}^{\perp}$.

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Let $T_k \in \mathcal{C}_{i_k}$, $k = 1, \ldots, n$, then, since $(\mathcal{C}_i)_{i \in \mathbb{I}}$ is a chain, without loss of generality, we can assume that $T_1, T_2, \ldots, T_n \in \mathcal{C}_{i_n}$. Since $\mathcal{C}_{i_n} \subset \mathcal{A}^\perp$ is a characteristic class, we receive that all joinings of $T_1, T_2, \ldots, T_n$ are in $\mathcal{C}_{i_n} \subset \mathcal{A}^\perp$. To conclude, it remains to use the fact that every infinite joining is an inverse limit of finite joinings.

COROLLARY 36. *For every T satisfying the assumptions of Theorem 29, there exists maximal characteristic class $\mathcal{C} \ni T$ such that $\mathcal{M}(\mathrm{Erg}^\perp) \subset \mathcal{C} \subsetneq \mathrm{Erg}^\perp$.*

We have not been able to describe the maximal characteristic classes in $\mathrm{Erg}^\perp$, it is even not clear whether there is only one.

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