# AN AXIOMATIC APPROACH TO FORCING IN A GENERAL SETTING

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**Abstract.** The technique of *forcing* is almost ubiquitous in set theory, and it seems to be based on technicalities like the concepts of genericity, forcing names and their evaluations, and on the recursively defined forcing predicates, the definition of which is particularly intricate for the basic case of atomic first order formulas. In his [3], the first author has provided an axiomatic framework for set forcing over models of ZFC that is a collection of guiding principles for extensions over which one still has *control* from the *ground model*, and has shown that these axiomatics necessarily lead to the usual concepts of genericity and of forcing extensions, and also that one can infer from them the usual recursive definition of forcing predicates. In this paper, we present a more general such approach, covering both class forcing and set forcing, over various base theories, and we provide additional details regarding the formal setting that was outlined in [3].

**§1. Introduction.** In this paper, we introduce an axiomatic framework for class forcing over models of second order set theory, that avoids the usual technicalities connected with any usual standard setup for (class) forcing, in particular the concepts of genericity, forcing names and their evaluations, and the recursively defined forcing predicates. Instead, we provide a natural collection of axioms, and show that they induce the common standard concepts: that is, they allow us to derive the usual concept of genericity, the usual recursive definitions of forcing predicates, an analogue of the structure of names for elements of generic extensions and their evaluations, thus exactly the same forcing extensions, and also the preservation of the axioms of set theory to our extensions. The aim of this paper is essentially twofold. First, it is to provide an interesting new viewpoint on what is probably the most important technical tool in modern set theory. Second, it is supposed to provide a self-contained way of introducing (class) forcing axiomatically. The only point in the paper where it is strictly necessary to refer to some sort of standard (class) forcing setup is when we briefly argue for our axioms to actually be consistent (modulo the axioms of set theory) in Section 7, after we introduce our final axiom (8). We will consider various base theories (Gödel–Bernays set theory GB, Kelley–Morse set theory KM, and some of their variants), and also deduce the somewhat simpler axioms for the special case of set forcing over ZFC and some of its weakenings in Section 8.

In this introductory section, we want to provide a rough description of our axiomatic framework, which will be followed with formal definitions in Sections 2–7.

We require forcing extensions to be based on preorders[1] in a ground model—let us fix such a transitive ground model $\mathcal{M} \in V$ for this discussion, and a class preorder $\mathbb{P}$ from $\mathcal{M}$. We require $\mathcal{M}$ to satisfy the axioms of some axiom system $T$ for set theory that is suitable for class forcing (either Gödel–Bernays set theory GB, Kelley–Morse set theory KM, or some variants of those). We think of conditions (elements) of $\mathbb{P}$ as having partial information on properties of our extensions. We require that stronger conditions have more such information, and that any particular forcing extension is based on a choice of filter on $\mathbb{P}$. We think of such a filter as a selection of conditions which have *correct* information about our extension, and we will refer to such conditions as being *correct*. The motivation for using a *filter* of conditions could be explained as follows.

- If we consider the information that a condition $q$ has to be correct, then any weaker condition $p$ has less information than $q$, and this information should therefore also be correct. This corresponds to the upwards closure property of filters.
- If $p$ and $q$ are correct conditions, we consider the information that is jointly collected by $p$ and $q$ to be correct. We require that there is a condition that collects this joint information and that we consider to be correct. This corresponds to the property of a filter that any two of its elements are compatible, as witnessed by yet another element of the filter.

We require that for any condition $p \in \mathbb{P}$, there exists a filter $G$ of correct conditions of which $p$ is an element, so that no condition is a priori incorrect. Given any particular such filter $G$, we require the *generic extension* $\mathcal{M}[G]$ to contain $\mathcal{M}$ as a subset and $G$ as an element. There are a number of natural axioms which make sure that we have *ground model control* over our generic extensions, in a sufficiently simple way. One necessary requirement for this is that elements of $\mathcal{M}[G]$ are connected to elements of the ground model so that the latter serve as a sort of *name* for the former. We require the existence of a definable relation on our ground model, which, following [3], we call the $\mathbb{P}$-membership relation. It is supposed to relate to partial knowledge about the membership relation in forcing extensions. If $a$ and $b$ are elements of $\mathcal{M}$ and $p \in \mathbb{P}$, we say that $a$ is an element of $b$ according to $p$, and write $a \in_p b$ in case the triple $\langle p, a, b \rangle$ stands in this relation.[2] We want to define a membership relation for $\mathcal{M}[G]$, letting the object denoted by $a$ be an element of the object denoted by $b$ in case $a$ is an element of $b$ according to some correct condition (that is, $\exists p \in G \ a \in_p b$). In order to be able to obtain a transitive model as our forcing extension, we thus require

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[1]A preorder is a reflexive and transitive binary relation.

[2]This relation corresponds to the relation that $\langle a, p \rangle \in b$ in a standard forcing setup, given that $a, b$ are usual forcing names.

the relation $\exists p \in \mathbb{P}\ a \in_p b$ to be well-founded. The relation $\exists p \in G\ a \in_p b$ will usually not be extensional, but we nevertheless obtain a transitive $\in$-structure (which will serve as our generic extension $\mathcal{M}[G]$) as the image of the homomorphism that is our *evaluation map* $F_G$, recursively defined by setting $F_G(b) = \{F_G(a) \mid a \in_G b\}$ for every $b \in \mathcal{M}$.

In order to be able to show that $\mathcal{M}[G]$ is well-defined and satisfies the axioms of $T$, we will need to require the following:

- A strong form of set-likeness: for any $b \in \mathcal{M}$, $\{\langle a, p \rangle \mid a \in_p b\}$ is a set in $\mathcal{M}$.
- *High degrees of freedom* for the $\mathbb{P}$-membership relation: for any relation $S$ on $\mathcal{M} \times P$ in $\mathcal{M}$, we find $b \in \mathcal{M}$ for which $\{\langle a, p \rangle \mid a \in_p b\} = S$.

Furthermore, we also require the existence of forcing predicates in $\mathcal{M}$, individually for each first order formula, and also for each second order formula in the case of KM. We do not require any particular defining instances for these predicates, we only require them to be connected to truth in generic extensions by the following two axioms (these requirements correspond to what is usually known as the *forcing theorem* in a standard class forcing setup):

- Whatever holds in $\mathcal{M}[G]$ is forced by some condition in $G$.
- Whatever is forced by some condition $p \in G$ holds true in $\mathcal{M}[G]$.

Finally, we will have to assume that our class forcing notion $\mathbb{P}$ is pretame or tame, which are technical conditions on $\mathbb{P}$ that are equivalent to the preservation of the axioms from $T$ in a standard class forcing setup.

**§2. The basic setup.** We want to verify our results for models of the base theory that is Gödel–Bernays set theory GB, and also for some of its variants. In Section 9, we will also consider the stronger theory KM, and we will consider yet another strengthening of it in Section 10. These theories are usually presented as theories in a two-sorted language, with variables for sets and for classes, and their models will be of the form $\mathcal{M} = \langle M, \mathcal{C} \rangle$, where $M$ denotes the domain of sets, and $\mathcal{C}$ denotes the domain of classes of $\mathcal{M}$.

Let $\mathcal{L}(\in)$ denote the collection of first order formulas in the language with the $\in$-predicate in which we additionally allow for second order variables, and atomic formulas of the form $x \in X$, where $x$ is a first order variable and $X$ is a second order variable. We consider equality between first or second order elements to abbreviate the statement that they have the same elements. The axioms of GB are given by the axioms of ZF for sets, allowing class parameters in the axiom schemes of separation and replacement (that is, allowing for formulas from $\mathcal{L}(\in)$ in which second order variables are replaced by second order parameters from $\mathcal{C}$), together with the class axiom of first order class comprehension, that is comprehension for classes using $\mathcal{L}(\in)$-formulas with second order parameters from $\mathcal{C}$. If $\mathcal{M} \models \mathbf{GB}$ (or any of its variants), we usually use lowercase letters to denote first order elements of $\mathcal{M}$, that is elements of $M$, and uppercase letters to denote second order

elements of $\mathcal{M}$, that is elements of $\mathcal{C}$. Note that by the separation axiom, we have $M \subseteq \mathcal{C}$.

We will also consider the strengthenings of GB that are obtained by adding the axiom of choice (GBc) in the form of the statement that every set can be well-ordered, or (GBC) the axiom that there is a well-order of all sets in order-type Ord (or equivalently, a set-like global well-order), as well as the axiom systems $\text{GB}^-$, $\text{GBc}^-$, and $\text{GBC}^-$, which are obtained from GB, GBc, or GBC respectively by removing the powerset axiom and using the axiom scheme of collection rather than replacement. We fix a base theory $T$ to be one of the above theories.

We next provide the definition of a *class forcing generic framework*, which will be the basic formal concept in our approach. As in [7], we will use the more general notion of preorders rather than (the perhaps more common restriction to) partial orders, dropping the requirement of antisymmetry. If $\vec{A}$ is a finite sequence, we will use the statement $\vec{A} \in \mathcal{C}$ to abbreviate the property that all sequents of $\vec{A}$ are elements of $\mathcal{C}$.

DEFINITION 2.1. A *class forcing generic framework* is a tuple of the form

$$\left\langle \mathcal{M}, \mathbb{P}, R, \left( \Vdash_{\varphi}^{\vec{A}} \right)_{\varphi \in \mathcal{L}(\in), \vec{A} \in \mathcal{C}}, \mathfrak{G} \right\rangle \text{ with the following properties.}$$

- $\mathcal{M}$ is a transitive set-size model of $T$ : $M$ is transitive, $\bigcup \mathcal{C} \subseteq M$, and $\mathcal{M}$ is a set such that $\mathcal{M} \models T$.
- $\mathbb{P} = \langle P, \leq \rangle$ is a preorder with weakest element $1_P$ (that is $p \leq 1_P$ for every $p \in P$) such that both $P$ and $\leq$ are in $\mathcal{C}$.
- The $\mathbb{P}$-*membership relation $R$* is a relation on $P \times M \times \mathcal{C}$ that is definable over $\mathcal{M}$ by an $\mathcal{L}(\in)$-formula $\varphi(p, a, B)$ with first order variables $p$ and $a$, and a second order variable $B$, so that for $p \in P$, $a \in M$, and $B \in \mathcal{C}$ we have $R(p, a, B)$ if and only if $\varphi(p, a, B)$.[3] We denote the property $R(p, a, B)$ as $a \in_p B$.[4]
- $\mathfrak{G}$ is a second order unary predicate on $P$, i.e., a unary predicate on $\mathcal{P}(P)$, and we require that $\mathfrak{G}(G)$ implies that $G \subseteq P$ is a filter. If $\mathfrak{G}(G)$ holds, we say that $G$ is a *generic filter*, or a $\mathbb{P}$-*generic filter on $M$*. Whenever we quantify over $G$ in the following, we assume that we quantify over $G$'s such that $\mathfrak{G}(G)$ holds.
- For every $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$ for which the number of second order variables corresponds to the length of $\vec{A}$, $\Vdash_{\varphi}^{\vec{A}} \in \mathcal{C}$ is a predicate (which we also call a *forcing relation for $\varphi$*) on $P \times M^m$, where $m$ denotes the number of free first order variables of $\varphi$. If $\langle q, a_0, \ldots, a_{m-1} \rangle \in \Vdash_{\varphi}^{\vec{A}}$, we also write $q \Vdash \varphi(a_0, \ldots, a_{m-1}, \vec{A})$.

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[3]Note that by our choice of the domain of $R$, only *sets* from $M$ can stand in $\mathbb{P}$-membership relation to sets or classes from $\mathcal{C}$.

[4]In a standard forcing setup, this would correspond to the property that $\langle a, p \rangle \in B$.

**§3. The basic axioms.** In this section, we present our basic axioms for class forcing generic frameworks.

(1) **Existence of generic filters:** $\forall p \in P \, \exists G \; p \in G$.[5]

(2) **Well-Foundedness:** The binary relation $\exists p \in P \; a \in_p b$ on $M$ is well-founded.

(3) **Growth of Information:** For all $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$, for all $\vec{a} \in M$, and $p, q \in P$, if $p \Vdash \varphi(\vec{a}, \vec{A})$ and $q \leq p$, then $q \Vdash \varphi(\vec{a}, \vec{A})$.

Assume that $G$ is such that $\mathfrak{G}(G)$ holds. Define a relation $\in_G$ on $\mathcal{C}$ by letting $a \in_G B$ if $\exists p \in G \; a \in_p B$. Using axiom (2), this relation on $\mathcal{C}$ is well-founded, and since $\mathcal{M} \in V$, it is clearly set-like in $V$. We may thus recursively define our *evaluation function* $F_G$ along the relation $\in_G$, letting $F_G(B) = \{F_G(a) \mid a \in_G B\}$ for each $B \in \mathcal{C}$.[6] Let $\mathcal{M}[G]$ denote the $\in$-structure on the transitive set $F_G[\mathcal{C}]$:[7] That is, let $\mathcal{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$, where $M[G] = F_G[M]$, and $\mathcal{C}[G] = F_G[\mathcal{C}]$.

The next two axioms state that a natural form of the forcing theorem holds, that is based on our forcing relations. Given a finite tuple $\vec{A} = \langle A_i \mid i < n\rangle \in \mathcal{C}$, let $F_G(\vec{A}) = \langle F_G(A_i) \mid i < n\rangle$.

(4) **Truth Lemma:** For all $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$ for which the number of second order variables corresponds to the length of $\vec{A}$, all $\vec{a} \in M$, and all $G$,

$$\mathcal{M}[G] \models \varphi(F_G(\vec{a}), F_G(\vec{A})) \text{ iff } \exists p \in G \; p \Vdash \varphi(\vec{a}, \vec{A}).$$

(5) **Definability Lemma:** For all $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$ for which the number of second order variables corresponds to the length of $\vec{A}$, all $\vec{a} \in M$, and $p \in P$,

$$p \Vdash \varphi(\vec{a}, \vec{A}) \text{ iff } \forall G \ni p \; \mathcal{M}[G] \models \varphi(F_G(\vec{a}), F_G(\vec{A})).[8]$$

Our next axiom $(\bar{6})$ states that within $M$, a weak form of set-likeness holds for the $\mathbb{P}$-membership relation. We will later replace it by the stronger axiom (6).[9]

$(\bar{6})$ **Weak Set-Likeness:** If $b \in M$, then $\{a \mid \exists p \in P \; a \in_p b\} \in M$.

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[5]By our above convention, we tacitly require here that $\mathfrak{G}(G)$ holds, i.e., that $G$ is generic.

[6]It may seem like we are taking some sort of transitive collapse of the structure $\langle \mathcal{M}[G], \in_G\rangle$; however note that there is no reason to assume that $\in_G$ is extensional, or that $\in_G$ can be factorized in order to obtain an extensional relation.

[7]For the moment, this notation is somewhat ambiguous, for $\mathcal{M}[G]$ may not only depend on $\mathcal{M}$ and on $G$, but also on the $\mathbb{P}$-membership relation. In Theorem 5.1, we will however show that under additional assumptions, $\mathcal{M}[G]$ is uniquely determined.

[8]Note that we already required the forcing relations to be predicates of our model in our basic setup; however this axiom connects them with their intended meaning, and it thus seems justified to consider it to be our version of the *definability lemma*.

[9]We provide this weaker form here in order to be able to show that this form suffices for the results of Section 5.

We will also introduce two additional axioms, axioms (7) and (8), later on in our paper. For the moment, we introduce two other additional axioms, stating that all elements of $M$ have a name in $M$, and that there is a (class) name for our generic filters. They will later be replaced by the stronger axiom (7) which will imply both these axioms. Since those are only *temporary* axioms, we will not provide them with a number.

($*$)  **Names for ground model objects:**
- $\forall a \in M \; \exists \check{a} \in M \; \forall G \; F_G(\check{a}) = a$, and
- $\forall A \in \mathcal{C} \; \exists \check{A} \in \mathcal{C} \; \forall G \; F_G(\check{A}) = A$.

($**$)  **Name for generic filters:** $\exists \dot{G} \in \mathcal{C} \; \forall G \; F_G(\dot{G}) = G$.

Given $a \in M$ and $A \in \mathcal{C}$, we will use $\check{a}$ and $\check{A}$ to denote names for $a$ and $A$ as provided by axiom ($*$) above, and we will use $\dot{G}$ to denote a (class) name for $G$ as provided by axiom ($**$) above.

**§4. Forcing predicates and density.** Our axioms (1)–(5) together with the axioms ($*$) and ($**$) suffice to verify some of the basic properties of forcing, and in particular to verify that the forcing predicates satisfy their usual defining clauses, by arguments that are similar to the arguments of [3, Section 4]. For the sake of completeness, and for the benefit of our readers, we would nevertheless like to present some of these arguments here. The first step will be to verify an auxiliary result on the forcing of negated statements.

LEMMA 4.1. *For all $\varphi \in \mathcal{L}(\in)$, $p \in P$, $\vec{a} \in M$, and $\vec{A} \in \mathcal{C}$, we have that*

$$p \Vdash \neg\varphi(\vec{a}, \vec{A}) \text{ iff } \forall q \leq p \; q \not\Vdash \varphi(\vec{a}, \vec{A}).$$

PROOF. Let us assume that

(i) $p \Vdash \neg\varphi(\vec{a}, \vec{A})$.

By axiom (5), equivalently

(ii) $\forall G \ni p \; \mathcal{M}[G] \models \neg\varphi(F_G(\vec{a}), F_G(\vec{A}))$.

By axiom (4), this is equivalent to

(iii) $\forall G \ni p \; \forall q \in G \; q \not\Vdash \varphi(\vec{a}, \vec{A})$.

We want to argue that this in turn is equivalent to our desired statement that

(iv) $\forall q \leq p \; q \not\Vdash \varphi(\vec{a}, \vec{A})$.

Thus, assume first that (iii) holds, and let $q \leq p$. By axiom (1), we may pick a generic filter $G \ni q$, which will thus also contain $p$ as an element. By (iii), we thus have that $q \not\Vdash \varphi(\vec{a}, \vec{A})$, as desired.

Conversely, assume that (iv) holds. Let $G$ be a generic filter that contains $p$ as an element, and assume for a contradiction that there is $r \in G$ such that $r \Vdash \varphi(\vec{a}, \vec{A})$. Since $G$ is a filter, we may pick $q$ below both $p$ and $r$. By axiom (3), it follows that $q \Vdash \varphi(\vec{a}, \vec{A})$, contradicting (iv). ⊣

We are now ready to show that our axioms imply generic filters to intersect all dense classes in $\mathcal{C}$.

LEMMA 4.2. *Let $D \in \mathcal{C}$ be such that $D$ is dense in $\mathbb{P}$. If $G$ is a generic filter, then $G$ intersects $D$.*

PROOF. Let $G$ be a generic filter and assume for a contradiction that $G \cap D = \emptyset$. Making use of axioms $(*)$ and $(**)$, it follows that

$$\mathcal{M}[G] \models \neg \exists x\ x \in F_G(\check{D}) \cap F_G(\dot{G}).$$

By axiom (4), we may thus find $p \in G$ such that

$$p \Vdash \neg \exists x\ x \in \check{D} \cap \dot{G}.$$

By Lemma 4.1, equivalently

$$\forall q \leq p\ q \not\Vdash \exists x\ x \in \check{D} \cap \dot{G}.$$

Since $D$ is dense, we may fix $q \leq p$ in $D$. But then,

$$q \Vdash \check{q} \in \check{D} \cap \dot{G},$$

contradicting the above. ⊣

We next need another auxiliary result on open dense sets (which could easily be extended to arbitrary dense sets, but the current version is sufficient for our purposes). We say that a subset $A$ of a preorder $P$ is *open* if it is downward closed, that is if $p \in A$ and $q \leq p$, then also $q \in A$.

LEMMA 4.3. *If $D \subseteq P$ is open, $D \in \mathcal{C}$, then $D$ is dense below $p$ if and only if*

$$(\dagger)\ \forall G \ni p\ D \cap G \neq \emptyset.$$

PROOF. Assume first that $(\dagger)$ holds. Let $r \leq p$, and using axiom (1), let $G$ be a generic filter with $r \in G$. It follows that also $p \in G$, and thus using $(\dagger)$, we obtain $s \in D \cap G$. Since $D$ is open and $G$ is a filter, we obtain $q$ below both $r$ and $s$ that is an element of $D \cap G$, showing that $D$ is dense below $p$.

On the other hand, assume that $D$ is dense below $p$, and let $G$ be a generic filter containing $p$ as an element. Let $E$ be the dense set of conditions which are either below $p$ and in $D$, or incompatible to $p$. By Lemma 4.2, it follows that $G \cap E \neq \emptyset$. Since $p \in G$ and $G$ is a filter, it thus follows that $G \cap D \neq \emptyset$, as desired. ⊣

It is now possible to show, as in [3], that the usual defining clauses for the forcing relation can be recovered from our basic axioms. The only additional clause that we need is the one for the elementhood relation with respect to classes. Its proof is very similar to the one for the elementhood relation with respect to sets; however we would like to provide one sample argument here in this paper for the benefit of our readers, and we will refer them to [3] for the other clauses below (note that since $M \subseteq \mathcal{C}$, the below lemma also

covers the case of the elementhood relation with respect to sets). Let $a \overline{\in}_p B$ if and only if $\exists q \geq p \; a \in_q B$.[10]

LEMMA 4.4. $p \Vdash a \in A$ iff $\forall r \leq p \; \exists s \leq r \; \exists x \; [x \overline{\in}_s A \wedge s \Vdash a = x]$.

PROOF. Let us assume that

(i) $p \Vdash a \in A$.

By axiom $(5)$, this is equivalent to

(ii) $\forall G \ni p \; F_G(a) \in F_G(A)$.

By the definition of $F_G$ and of $\in_G$, this in turn is equivalent to

(iii) $\forall G \ni p \; \exists x \in M \; [F_G(a) = F_G(x) \wedge \exists q \in G \; x \in_q A]$.

Using axiom $(4)$, we obtain the following equivalence:

(iv) $\forall G \ni p \; \exists x \in M \; [\exists r \in G \; r \Vdash a = x \wedge \exists q \in G \; x \in_q A]$.

Now we make use of axiom $(3)$, equivalently obtaining that

(v) $\forall G \ni p \; \exists s \in G \; \exists x \in M \; [s \Vdash a = x \wedge x \overline{\in}_s A]$.

Using Lemma 4.3 yields our final desired equivalence:

(vi) $\forall r \leq p \; \exists s \leq r \; \exists x \in M \; [x \overline{\in}_s A \wedge s \Vdash a = x]$.                    $\dashv$

In the next lemma, we list the remaining results with respect to the forcing predicates obeying their usual defining clauses, which are shown exactly as in [3, Section 5], simply carrying along additional second order predicates (in case there are any). For the detailed arguments to verify these properties, we refer the interested reader to [3]. We consider the relations $\in$ and $\neq$ as our atomic relations in the below; however $a \neq b$ could be seen as abbreviating $\neg(a = b)$, and $a \notin b$ should be seen as abbreviating $\neg(a \in b)$.

LEMMA 4.5. *Let $\varphi \in \mathcal{L}(\in)$, $p \in P$, and let $\vec{a}$ and $\vec{A}$ be finite tuples from $M$ and from $\mathcal{C}$ respectively. Then the following hold true.*

- $p \Vdash \varphi(\vec{a}, \vec{A})$ iff $\forall q \leq p \; \exists r \leq q \; r \Vdash \varphi(\vec{a}, \vec{A})$.
- $p \Vdash [\varphi \vee \psi](\vec{a}, \vec{A})$ iff $\forall r \leq p \; \exists q \leq r \; [q \Vdash \varphi(\vec{a}, \vec{A}) \vee q \Vdash \psi(\vec{a}, \vec{A})]$.
- $p \Vdash \exists x \varphi(\vec{a}, \vec{A})$ iff $\forall r \leq p \; \exists q \leq r \; \exists x \in M \; q \Vdash \varphi(x, \vec{a}, \vec{A})$.
- $p \Vdash [\varphi \wedge \psi](\vec{a}, \vec{A})$ iff $p \Vdash \varphi(\vec{a}, \vec{A}) \wedge p \Vdash \psi(\vec{a}, \vec{A})$.
- $p \Vdash a \neq b$ iff $\forall r \leq p \; \exists q \leq r \; \exists c$

$$(c \overline{\in}_q a \wedge q \Vdash c \notin b) \vee (c \overline{\in}_q b \wedge q \Vdash c \notin a).$$

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[10]For the readers who want to look up the results from [3] at this point, we should remark that in [3], one of the axioms requires that if $q \leq p$ and $a \in_p b$, then also $a \in_q b$. We did not include this axiom into our axiom system for the sake of a more elegant presentation. Note that the predicates $\overline{\in}_p$ clearly do satisfy this axiom however, and can be seen to correspond to the predicates $\in_p$ from [3].

The existence of forcing relations for atomic formulas is usually a problematic aspect in the case of class forcing (for a detailed account of this see [7]). The fundamental difference in our setup is that the definability of forcing predicates is already an integral assumption. In the above, we only show that any single step, reducing forcing statements about atomic formulas to ones for names of lower rank, proceeds as usual. We thus avoid the usual problem of having to obtain the definability of the forcing predicates for atomic formulas by recursion (which a priori is a recursion on classes, and is thus not guaranteed by the axioms of GBC, see [7], or also [6]). The existence of such recursions is an axiom of KM on the other hand, and this is the reason why every notion of class forcing satisfies the forcing theorem in KM (see [7], [6], or [1]).

**§5. An axiomatization that assumes the preservation of the axioms of set theory.** In this section, we introduce two further temporary axioms.

(∗∗∗)   **Preservation of axioms:** $\forall G \ \mathcal{M}[G] \models T$[11] and

(∗∗∗∗)  **Absoluteness of $\mathbb{P}$-membership:** The formula $\varphi$ defining the $\mathbb{P}$-membership relation $R$ is absolute for transitive models of $T$ containing $\mathcal{C}$.

Building on the terminology from [3, Section 3], let us say that a *class forcing generic extension* is a class forcing generic framework together with a particular choice of generic filter $G$. Our next theorem shows that under a suitable set of axioms, this notion corresponds to the usual notion of a class forcing extension, and in particular it shows that $\mathcal{M}[G]$ is well-defined, in the sense that it only depends on $\mathcal{M}$ and on $G$.

THEOREM 5.1. *Given a particular class forcing generic extension, assuming axioms* (2), (6̄), (∗), (∗∗), (∗∗∗), *and* (∗∗∗∗) *to hold,* $\mathcal{M}[G]$ *is the $\subseteq$-smallest model of $T$ that contains $\mathcal{C} \cup \{G\}$ as a subset of its collection of classes.*

PROOF. If $\mathcal{N} = \langle N, \mathcal{D} \rangle \models T$ is transitive such that $\mathcal{C} \cup \{G\} \subseteq \mathcal{D}$, then we can construct $M[G]$ within $\mathcal{N}$: Working in $\mathcal{N}$, define the relation $\in_G$ on $M$ by letting $a \in_G b$ if $\exists p \in G \ a \in_p b$. By axiom (∗∗∗∗), this definition of $\in_G$ is absolute between $\mathcal{N}$ and $V$. This relation on $M$ is well-founded by axiom (2), and set-like by axiom (6̄). We thus may define the restriction of $F_G$ to $M$ in $\mathcal{N}$: Let $Q(f, b)$ be the formula asserting that $f$ is a function whose domain $d$ includes $b$ as an element and is closed under $\in_G$-predecessors, and for every $a \in d$ we have $f(a) = \{f(c) \mid c \in_G a\}$. Now $F_G$ is defined by letting $F_G(b) = f(b)$ if there is an $f \in N$ such that $Q(f, b)$ holds, and is $\emptyset$ otherwise. By axioms (2) and (6̄), for any $b \in M$, such function $f \in N$ actually exists. We thus have $M[G] \subseteq N$.

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[11] Let us remark that due to axiom (5), this axiom could equivalently be replaced by a scheme of axioms, consisting of statements of the form $1_P \Vdash \varphi$ for every $\varphi \in T$. While this would increase the number of axioms, it would decrease the number of axioms that quantify over generic filters and are thus external to the model $\mathcal{M}$.

If $A \in \mathcal{C}$, we obtain $F_G(A) = \{F_G(a) \mid a \in_G A\} \in \mathcal{D}$ by using first order class comprehension in $\mathcal{N}$, and thus we also have $\mathcal{C}[G] = F_G[\mathcal{C}] \subseteq \mathcal{D}$. It thus follows that $\mathcal{M}[G] \subseteq \mathcal{N}$, and by axioms $(*)$, $(**)$, and $(***)$, $\mathcal{M}[G]$ therefore is the $\subseteq$-smallest model of $T$ that contains all classes from $\mathcal{C}$ and the generic filter $G$ as one of its classes, if any such model exists, as desired.          ⊣

§6. **On the notion of tameness.** One of the classical results about set forcing is that it preserves the axioms of $\mathrm{ZF}^-$, $\mathrm{ZF}$, and $\mathrm{ZFC}$, and this easily extends to any of the second order theories that we consider in this paper. However the situation is completely different for class forcing, for any of the second order theories that we consider in this paper may easily be destroyed by class forcing. For example, simply consider the notion of forcing which adds a function from $\omega$ to Ord using finite conditions. This notion of class forcing will provide us with a class function mapping $\omega$ surjectively onto Ord in any of its generic extensions, clearly yielding $\mathrm{GB}^-$ to fail. A key notion in this context is that of *pretameness*, which was implicit in earlier work of A. Zarach and of M. Stanley, and which was isolated by Friedman in [5]. In the following, when we talk about sequences of classes, we will assume some suitable coding so that these sequences can themselves be viewed as classes.

DEFINITION 6.1. $\mathbb{P}$ is *pretame* (for $\mathcal{M} = \langle M, \mathcal{C} \rangle$) if for every $p \in P$ and every sequence $\langle D_i \mid i \in I \rangle \in \mathcal{C}$ of dense subclasses of $\mathbb{P}$ with $I \in M$, there is $q \leq p$ and $\langle d_i \mid i \in I \rangle \in M$ such that for every $i \in I$, $d_i \subseteq D_i$ is predense below $q$ in $\mathbb{P}$.

M. Stanley showed that pretameness is actually equivalent to the property that forcing with $\mathbb{P}$ preserves the axioms of $\mathrm{GB}^-$ in a standard class forcing setup (see [8, Theorem 3.1]), and it is shown in [8, Theorem 1.12] that pretameness is equivalent to a large number of desirable niceness features of forcing notions. It therefore seems very natural to restrict ones attention to pretame notions of forcing in the context of class forcing. Pretame notions of forcing also preserve the axiom of choice, and the existence of a global well-order in order type Ord: regarding the axiom of choice, if $x$ is a set in a generic extension $\mathcal{M}[G]$ by a pretame forcing notion $\mathbb{P}$, we pick a $\mathbb{P}$-name $\dot{x}$ for $x$, and a well-order $\prec$ of $X = \{\sigma \mid \exists p \in P \, \langle \sigma, p \rangle \in \dot{x}\}$. We then construct a well-order of $x$ as follows: given two of its elements $y$ and $z$, we let $y \lhd z$ if the $\prec$-least $\mathbb{P}$-name for $y$ in $X$ (that is, the $\prec$-least $\sigma \in X$ so that $\sigma^G = y$) is $\prec$-below any $\mathbb{P}$-name for $z$ in $X$. The argument for the preservation of the existence of a global well-order in order type Ord is essentially the same, and presented within [8, Theorem 3.1].

However, pretame notions of forcing need not preserve the powerset axiom. This can easily be observed by considering the pretame notion of forcing that adds a proper class of Cohen subsets of $\omega$. Since in any of its extensions, the powerset of $\omega$ would have to be a proper class, it cannot exist as a set. For the preservation of the powerset axiom, we need the notion of tameness. In his [4], Friedman introduces tameness of a class forcing notion $\mathbb{P}$ as the axiom requiring that $\mathbb{P}$ is pretame and that the weakest condition of $\mathbb{P}$ forces the powerset axiom to hold (in the generic extension). Since

pretameness of $\mathbb{P}$ implies the $\mathbb{P}$-forcing relation to be definable, forcing the powerset axiom to hold is a first order property of $\mathbb{P}$. Previously, in his [5], a more combinatorial definition of tameness was provided, which however seems potentially problematic to us (we do not see how a crucial step in the argument of [5, Theorem 2.21] proceeds, which is supposed to show that this combinatorial version of tameness implies the preservation of the powerset axiom), but this issue can easily be fixed by slightly strengthening the original property of tameness from [5] (we will let tameness denote this strengthened property in the below)—we will show that (our version of) tameness is equivalent to the preservation of the powerset axiom under certain additional axioms (which all hold in any standard setup for class forcing) in Section 7.[12]

DEFINITION 6.2. Let $\mathcal{M} = \langle M, \mathcal{C} \rangle \models$ GB and let $\mathbb{P} \in \mathcal{C}$ be a notion of forcing.

- If $p \in P$, a *predense* $\leq p$ *partition* (of $\mathbb{P}$) is a pair $\langle D_0, D_1 \rangle$ of classes such that $D_0 \cup D_1$ is predense below $p$ and such that $p_0 \in D_0 \wedge p_1 \in D_1$ implies that $p_0 \perp p_1$.
- Suppose that $\langle \langle D_0^i, D_1^i \rangle \mid i \in a \rangle$ and $\langle \langle E_0^i, E_1^i \rangle \mid i \in a \rangle$ are sequences of predense $\leq p$ partitions of $\mathbb{P}$. We say that they are *equivalent* $\leq p$ if for each $i \in a$,

$$\{ q \in P \mid D_0^i \text{ is predense below } q \iff E_0^i \text{ is predense below } q \}$$

  is dense below $p$ in $\mathbb{P}$.
- $\mathbb{P}$ is *tame* (for $\mathcal{M}$) if $\mathbb{P}$ is pretame (for $\mathcal{M}$) and for each $a \in M$ and $p \in P$, there is $q \leq p$ and $\alpha \in \mathrm{Ord}(M)$ s.t. whenever $r \leq q$ and

$$\vec{d} = \langle \langle d_0^i, d_1^i \rangle \mid i \in a \rangle \in M$$

  is a sequence of predense $\leq r$ partitions of $\mathbb{P}$, then there is $s \leq r$ such that $\vec{d}$ is equivalent $\leq s$ to some

$$\vec{e} = \langle \langle e_0^i, e_1^i \rangle \mid i \in a \rangle \in V_\alpha^M.\text{[13]}$$

Note that if $\langle D_0, D_1 \rangle$ is a predense $\leq p$ partition of $\mathbb{P}$ for some $p \in P$, and $q \leq p$, then $\langle D_0, D_1 \rangle$ is also a predense $\leq q$ partition of $\mathbb{P}$. We will tacitly make use of this in the below.

§7. **An axiomatization that does not assume the preservation of the axioms of set theory.** Alternatively, we can replace $(\bar{6})$, $(*)$, $(**)$, $(***)$, and $(****)$ by the following axioms, and in particular we can derive the preservation of the axioms of set theory from natural axioms rather than requiring it.

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[12]It may well be the case that Friedman's original definition of tameness from [5] is equivalent to the below though, for we were not able to produce a counterexample to this being the case.

[13]The subtle difference to Friedman's original definition of tameness is that he requires $\vec{d}$ to be a sequence of predense $\leq q$ partitions of $\mathbb{P}$.

Axiom $(6)$ below is a strengthening of axiom $(\bar{6})$, which seems to be a natural requirement in the context of class forcing, and could be seen to essentially say that the objects that are our version of names for sets correspond to set-sized objects in a standard class forcing setup. Axiom $(7)$ essentially says that we can freely (from the perspective of $\mathcal{M}$) construct our version of names in $\mathcal{M}$. Axiom (8) is (pre)tameness, and is only required in the case of class forcing (for it is trivial in the case of set forcing).

(6) **Strong Set-Likeness:** If $b \in M$, then $\{\langle a, p \rangle \mid a \in_p b\} \in M$.

(7) **Universality:** If $S \in \mathcal{C}$ is a relation on $M \times P$, then there exists $B \in \mathcal{C}$ such that

$$S(a, p) \text{ if and only if } a \in_p B.$$

Moreover, if $S \in M$, then we obtain the above $B$ to be in $M$, and there is a map $\Gamma$ that is first order definable over $M$ which on input $S \in M$ yields one such witnessing $B \in M$.[14]

(8)  • **Pretameness:** $\mathbb{P}$ is pretame (for $\mathcal{M}$).
  • **Tameness:** If $T$ contains the power set axiom, then we make the stronger requirement that $\mathbb{P}$ is tame (for $\mathcal{M}$).

Let us make the important remark that our axioms (1)–(8), as well as the axioms $(*)$, $(**)$, $(***)$, and $(****)$, hold in the standard setup for class forcing, as described for example in [7]. Given a transitive model $\mathcal{M} = \langle M, \mathcal{C} \rangle$ of $T$, and a pretame, or tame (in case $T$ contains the powerset axiom) notion of class forcing $\mathbb{P} \in \mathcal{C}$ such that for every condition $p$ from $\mathbb{P}$, there is a $\mathbb{P}$-generic filter over $\mathcal{M}$ that contains $p$ as an element (this is the case in particular if $\mathcal{M}$ is countable[15]), interpreting $a \in_p B$ as $\langle a, p \rangle \in B$, we arrive at such a standard setup, and it is well-known how to verify axioms (1)–(8), $(*)$, $(**)$, $(***)$, and $(****)$ with respect to $\mathcal{M}$ and $\mathbb{P}$ in this context—for the forcing theorem, as described in our axioms (4) and (5), see [7, 8].

LEMMA 7.1. *The axioms $(*)$ and $(**)$ can be derived from axiom $(7)$.*

PROOF. Essentially, axiom (7) allows us to construct analogues of the usual canonical names for ground model objects and for the generic filter. That is, using axiom (7), by recursion on rank, for $b \in M$, we define $\check{b}$ to be such that $x \in_p \check{b}$ if and only if $p = 1$ and $x$ is of the form $\check{a}$ for some $a \in b$, and we do this in a definable way, making use of the map $\Gamma$. Note that using $\Gamma$, we in fact obtain a strong form of axiom $(*)$: we have the extra property that the map from $b$ to $\check{b}$ described above is definable over $M$. We then use this to define $\dot{G}$ to be such that $x \in_p \dot{G}$ if and only if $p \in P$ and $x = \check{p}$.  ⊣

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[14]This additional definability assumption is of course redundant when $T$ yields the existence of a global well-order. In fact, it could also be omitted if $T$ includes the powerset axiom: in that case, instead of a single object $B$ witnessing an instance of axiom (7), we would obtain a set of rank-minimal objects witnessing such an instance, and this would suffice to verify Lemma 7.1, Theorem 7.3, and Proposition 7.4.

[15]This is supposed to mean that both $M$ and $\mathcal{C}$ are countable.

Making heavy use of axioms (6) and (7), our next lemma shows that standard forcing names and their evaluations essentially coincide with elements of our ground model $\mathcal{M}$ and their images under our evaluation function $F_G$.

LEMMA 7.2.     (i) *If $\sigma$ is a (class) $\mathbb{P}$-name in $M$ (or in $\mathcal{C}$) in the standard sense, then there is $a \in M$ (or $a \in C$) such that $F_G(a) = \sigma^G$, where $\sigma^G$ denotes the standard evaluation of the name $\sigma$ by the generic filter G.*

(ii) *If $a \in M$ (or $a \in \mathcal{C}$), then there is a (class) $\mathbb{P}$-name $\sigma$ in $M$ (or in $\mathcal{C}$) such that $\sigma^G = F_G(a)$.*

*Thus, in particular, $\mathcal{M}[G]$ is the standard generic extension of the model $\mathcal{M}$ by the generic filter G.*

PROOF. Let us first verify (i). Let $\sigma$ be a $\mathbb{P}$-name in $M$. Making use of the map $\Gamma$ that was introduced in axiom (7), we recursively define a *translation function h*, letting

$$h(\tau) = \Gamma(\{\langle h(v), p \rangle \mid \langle v, p \rangle \in \tau\}) \in M.$$

It follows that $F_G(h(\sigma)) = \{F_G(a) \mid \exists p \in G \; a \in_p h(\sigma)\}$

$$= \{F_G(a) \mid a = h(\tau) \text{ and } \exists p \in G \; \langle \tau, p \rangle \in \sigma\} = \{\tau^G \mid \exists p \in G \; \langle \tau, p \rangle \in \sigma\} = \sigma^G.$$

Now if $\sigma \in \mathcal{C}$, using axiom (7), we can define $h(\sigma)$ to be such that for all $\mathbb{P}$-names $\tau \in M$ and $p \in P$,

$$h(\tau) \in_p h(\sigma) \text{ if and only if } \langle \tau, p \rangle \in \sigma.$$

We thus obtain $h(\sigma) \in \mathcal{C}$, and then the same argument as above shows that $F_G(h(\sigma)) = \sigma^G$.

Now let us verify (ii), in a similar way. Let $b \in M$, and making use of axiom (6), we obtain a map $E \in \mathcal{C}$ with domain $M$ such that for every $a \in M$, $E(a) = \{\langle x, p \rangle \mid x \in_p a\} \in M$. Making use of this map, we recursively define another translation function $g$, letting $g(b) = \{\langle g(a), p \rangle \mid a \in E(b), a \in_p b\}$ for $b \in M$. Similar to before, it now follows that

$$g(b)^G = \{g(a)^G \mid \exists p \in G \; \langle g(a), p \rangle \in g(b)\} = \{F_G(a) \mid \exists p \in G \; a \in_p b\} = F_G(b).$$

If $B \in \mathcal{C}$, we can define $g(B) = \{\langle g(a), p \rangle \mid a \in_p B\}$, and then the same argument as above shows that $g(b)^G = F_G(b)$.      ⊣

Our next goal is to show that we can derive axiom (∗∗∗) as well as the consequences of Theorem 5.1 from axioms (1)–(8). There are two somewhat different ways to do this. The first possibility is to make use of Lemma 7.2, and then invoke the relevant standard results about the preservation of the axioms of set theory in standard generic extensions,[16] and also about their minimality. The second possibility is to directly verify the individual axioms of set theory in our generic extension $\mathcal{M}[G]$ from our axioms. The advantage

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[16]For the case of the powerset axiom, using our new definition of tameness, such a proof can easily be isolated from our proof of the powerset axiom in the second proof of Theorem 7.3.

of this second option is that we do not make use of any standard results in this case (but of course, the arguments to verify the individual axioms in our generic extension are essentially analogous to the corresponding classical arguments). We can then infer minimality of $\mathcal{M}[G]$ by invoking Theorem 5.1 with respect to the standard setup for class forcing.[17] We would like to present both approaches in the below.

THEOREM 7.3. *Axioms* (1)–(8) *imply that* $\mathcal{M}[G]$ *is the* $\subseteq$-*smallest model of T that contains* $\mathcal{C} \cup \{G\}$ *as a subset of its collection of classes. In particular thus, axiom* (***) *holds.*

FIRST PROOF. Lemma 7.2 shows that our extension $\mathcal{M}[G]$ is exactly the standard class forcing extension of $\mathcal{M}$ by $G$, and thus one can apply the standard arguments to show that $\mathcal{M}[G]$ is the $\subseteq$-smallest model $\mathcal{N}$ of $T$ that contains $\mathcal{C} \cup \{G\}$ as a subset of its collection of classes. ⊣

SECOND PROOF. We directly verify the individual axioms in our generic extension. This argument mostly proceeds similar to the proof of [8, Theorem 3.1(1)]. The verification of the powerset axiom in $\mathcal{M}[G]$ proceeds somewhat similar to the proof of [5, Theorem 2.21], however making use of our modified definition of tameness.

Since $\mathcal{M}[G]$ is a transitive $\in$-structure, it clearly satisfies Regularity and Extensionality. Using axiom (7), it is easy to see that $M[G]$ satisfies Pairing, and by axiom (∗), it satisfies Infinity.

Let us treat the union axiom: Let $a \in M[G]$ be given; we need to show that for some $b \in M[G]$, $\bigcup a \subseteq b$. Let $X = \{c \mid \exists p\ c \in_p a\} \in M$ by axiom ($\bar{6}$). Let $Y = \{d \mid \exists c \in X\ \exists q\ d \in_q c\} \in M$ by axiom ($\bar{6}$). Using axiom (7), let $\dot{b} \in M$ be such that $d \in_r \dot{b}$ if and only if $d \in Y$ and $r = 1$. Using the definition of $F_G$, it is straightforward to check that $b = F_G(\dot{b})$ is as desired.

Let us verify first order class comprehension in $\mathcal{M}[G]$. If $\varphi \in \mathcal{L}(\in)$ and $B \in \mathcal{C}$, using axiom (7), let $A \in \mathcal{C}$ be such that $a \in_p A$ if and only if $p \Vdash \varphi(a, B)$. It then follows that $F_G(A) = \{x \in M[G] \mid \mathcal{M}[G] \models \varphi(x, F_G(B))\}$.

We now show that $\mathcal{M}[G]$ satisfies collection. Let $a \in M$, $A \in \mathcal{C}$, and assume that $\mathcal{M}[G] \models \forall x \in F_G(a)\ \exists y\ \langle x, y \rangle \in F_G(A)$. We need to find $b \in M$ such that $\mathcal{M}[G] \models \forall x \in F_G(a)\ \exists y \in F_G(b)\ \langle x, y \rangle \in F_G(A)$.

By axiom (4), we may pick some $p \in G$ such that $p \Vdash \forall x \in a\ \exists y\ \langle x, y \rangle \in A$. By axiom (6), $X = \{\langle c, r \rangle \mid c \in_r a\} \in M$. For each $\langle c, r \rangle \in X$, let

$$D_{\langle c,r \rangle} = \{s \in P \mid [s \le p, r \ \wedge\ \exists d \in M\ s \Vdash \langle c, d \rangle \in A]\ \vee\ s \perp r\} \in \mathcal{C}.$$

By Lemma 4.5, it follows that each $D_{\langle c,r \rangle}$ is dense below $p$ in $\mathbb{P}$. Using that $\mathbb{P}$ is pretame by axiom (8), there is $q \in G$ below $p$ and there is a sequence $\langle d_{\langle c,r \rangle} \mid \langle c, r \rangle \in X \rangle \in M$ such that $d_{\langle c,r \rangle} \subseteq D_{\langle c,r \rangle}$ and $d_{\langle c,r \rangle}$ is predense below $q$ in $\mathbb{P}$ for each $\langle c, r \rangle \in X$.

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[17]Let us remark that axiom (****) is not a consequence of axioms (1)–(8). If we additionally assumed axiom (****), this use of the standard setup for class forcing could as well be avoided, for we could then directly invoke Theorem 5.1 at this point.

Using collection in $M$, there is a set $Y \in M$ such that for each $\langle c, r \rangle \in X$ and for each $s \in d_{\langle c,r \rangle}$ that is compatible with $r$, there is $d \in Y$ such that $s \Vdash \varphi(c, d, A)$. Using axiom (7), let $b \in M$ be such that $d \in_s b$ if and only if $d \in Y$ and there is $\langle c, r \rangle \in X$ such that $s \in d_{\langle c,r \rangle}$ and $s \Vdash \varphi(c, d, A)$. Then, by construction, $\mathcal{M}[G] \models \forall x \in F_G(a) \, \exists y \in F_G(b) \, \langle x, y \rangle \in F_G(A)$, as desired.

Let us next show that $\mathcal{M}[G]$ satisfies separation. Let $a \in M$ and $A \in \mathcal{C}$. We need to find $b \in M$ such that $\mathcal{M}[G] \models F_G(b) = F_G(a) \cap F_G(A)$. By axiom (6), $X = \{\langle c, r \rangle \mid c \in_r a\} \in M$. For each $\langle c, r \rangle \in X$, let $D_{\langle c,r \rangle} = \{q \in P \mid q \leq r$ decides $c \in A \vee q \perp r\} \in \mathcal{C}$. By an easy application of Lemma 4.1, it follows that each $D_{\langle c,r \rangle}$ is dense in $\mathbb{P}$. Using that $\mathbb{P}$ is pretame by axiom (8), we may pick $p \in G$ and a sequence $\langle d_{\langle c,r \rangle} \mid \langle c, r \rangle \in X \rangle \in M$ such that $d_{\langle c,r \rangle} \subseteq D_{\langle c,r \rangle}$ and $d_{\langle c,r \rangle}$ is predense below $p$ in $\mathbb{P}$ for each $\langle c, r \rangle \in X$. Using axiom (7), let $b \in M$ be such that $c \in_q b$ if and only if there is $r \in P$ such that $q \leq r$, $\langle c, r \rangle \in X$, $q \in d_{\langle c,r \rangle}$, and $q \Vdash c \in A$. It clearly follows that $b$ is as desired.

Let us observe at this point that we have by now verified the axioms of $\mathbf{GB}^-$ in $\mathcal{M}[G]$. By our absoluteness assumptions on the $\mathbb{P}$-membership relation, this allows us to perform the relevant recursion to obtain $F_G \in \mathcal{C}[G]$.

Let us argue that the axiom of choice is preserved. Let $a \in M$. We have to find a well-order $\lhd$ of $F_G(a)$ in $M[G]$. Using axiom $(\bar{6})$, let $X = \{c \mid \exists r \in P \, c \in_r a\} \in M$, and let $\prec$ be a well-order of $X$, using the axiom of choice in $M$. Now given $x$ and $y$ in $F_G(a)$, making use of our above observation, we simply let $x \lhd y$ if and only if there is $\dot{x} \in X$ such that $x = F_G(\dot{x})$ and for all $\dot{y} \in X$ such that $y = F_G(\dot{y})$, we have $\dot{x} \prec \dot{y}$. Using that separation holds in $\mathcal{M}[G]$, we thus obtain $\lhd \in M[G]$.

Let's argue that if $\mathcal{M}$ has a set-like global well-order $\prec \in \mathcal{C}$, then we can find a set-like global well-order $\lhd$ of $M[G]$ in $\mathcal{C}[G]$. Using our above observation, we simply let $x \lhd y$ if and only if there is $\dot{x} \in M$ such that $x = F_G(\dot{x})$ and for all $\dot{y} \in M$ such that $y = F_G(\dot{y})$, we have $\dot{x} \prec \dot{y}$. Using that first order class comprehension holds in $\mathcal{M}[G]$, we thus obtain $\lhd \in \mathcal{C}[G]$.

Finally, we argue that assuming $\mathbb{P}$ to be tame, the powerset axiom is preserved. Working in $\mathcal{M}[G]$, it suffices to show that $\mathcal{P}(a)$ exists in $M[G]$ for every $a \in M$: If $b \in M[G]$, $b = F_G(\sigma)$, using axiom $(\bar{6})$, let $X = \{c \mid \exists r \in P \, c \in_r \sigma\} \in M$. Using that $F_G \in \mathcal{C}[G]$, and using that replacement holds in $\mathcal{M}[G]$, there is a surjection from $X$ onto $b$ in $M[G]$, and this surjection naturally induces a surjection from $\mathcal{P}(X)$ onto $\mathcal{P}(b)$ in $\mathcal{C}[G]$. Using replacement in $\mathcal{M}[G]$ once again, if $\mathcal{P}(X) \in M[G]$, it follows that $\mathcal{P}(b) \in M[G]$.

Therefore, let $a \in M$. By Lemma 4.2, we may pick $q \in G$ and $\alpha \in \mathrm{Ord}(M)$ witnessing the tameness of $\mathbb{P}$ with respect to $a$. Now for any $\sigma \in M$ such that $q \Vdash \sigma \subseteq \check{a}$, consider the sequence of classes $\vec{D} = \langle \langle D_0^i, D_1^i \rangle \mid i \in a \rangle$ defined by letting $D_0^i = \{r \leq q \mid r \Vdash \check{i} \notin \sigma\}$ and $D_1^i = \{r \leq q \mid r \Vdash \check{i} \in \sigma\}$. For every $i \in a$ and $j \in \{0, 1\}$, let $E_j^i$ be the dense set obtained as the downward closure of $D_j^i$. By pretameness, for every $r \leq q$, we obtain $s \leq r$ and a predense $\leq s$ partition $\vec{d} = \langle \langle d_0^i, d_1^i \rangle \mid i \in a \rangle$ such that $d_0^i \subseteq E_0^i$ and $d_1^i \subseteq E_1^i$ for every $i \in a$. By our choice of $q$ and $\alpha$, we find $\vec{e} \in V_\alpha^M$ and $t \leq s$ such

that $\vec{d}$ and $\vec{e}$ are equivalent $\leq s$.[18] Since this yields a dense set of conditions $t$, we may pick such $t \in G$. Thus,

$$i \in F_G(\sigma) \iff G \cap e_1^i \neq \emptyset,$$

and therefore using axiom (7), letting $\sigma_0 \in M$ be such that $c \in_s \sigma_0$ if and only if $c = \check{i}$ for some $i \in a$ and $s \in e_1^i$, it follows that $F_G(\sigma) = F_G(\sigma_0)$. Making use of the map $\Gamma$ described in axiom (7), we have a definable choice of such $\sigma_0$'s in $M$ given a predense $\leq r$ partition $\vec{e} \in V_\alpha^M$, and using replacement in $M$, we thus obtain a set $\Sigma \in M$ containing a suitable $\sigma_0$ for every predense $\leq r$ partition $\vec{e} \in V_\alpha^M$. Using axiom (7) once again, let $\pi \in M$ be such that $c \in_s \pi$ if and only if $c \in \Sigma$ and $s = 1$. Using separation in $\mathcal{M}[G]$, this clearly yields $\{x \in F_G(\pi) \mid x \subseteq a\} \in M[G]$ to be the powerset of $a$ in $\mathcal{M}[G]$, as desired.

Having verified all axioms of $T$ in $\mathcal{M}[G]$, there are two ways to end the proof: In case axiom (∗∗∗∗) holds, the statement of our theorem is a direct consequence of Theorem 5.1, and we have not made any use of the standard setup for class forcing in this case. Without axiom (∗∗∗∗), we use that all of our axioms hold in the standard setup for class forcing with $\mathbb{P}$ over $\mathcal{M}$, and that $\mathcal{M}[G]$ is just the standard class forcing extension of $\mathcal{M}$ by $G$ by Lemma 7.2, and therefore $\mathcal{M}[G]$ is the $\subseteq$-smallest model of $T$ that contains $\mathcal{C} \cup \{G\}$ as a subset of its collection of classes by Theorem 5.1 used within the standard setup.                                                                                    ⊣

Our next proposition shows in particular that notions of forcing that preserve GB need to be tame (in the sense of Definition 6.2), thus strengthening [5, Proposition 2.20]. Its proof is similar to the proof that is provided for [5, Proposition 2.20].

PROPOSITION 7.4. *Axioms* (1)–(7) *together with axiom* (∗∗∗) *imply axiom* (8).

PROOF. If $\mathbb{P}$ is not pretame, then there is a sequence $\vec{D} = \langle D_i \mid i \in I \rangle \in \mathcal{C}$ of dense subclasses of $\mathbb{P}$ with $I \in M$ and a condition $p \in P$ witnessing that pretameness fails with respect to $\vec{D}$. Using axiom (1), let $G$ be a $\mathbb{P}$-generic filter over $\mathcal{M}$ with $p \in G$. In the extension $\mathcal{M}[G]$, consider the (class) function $F \colon I \to \mathrm{Ord}^M$ defined by letting

$$F(i) = \min\{\alpha \in \mathrm{Ord}^M \mid G \cap D_i \cap V_\alpha^M \neq \emptyset\},$$

which is an element of $\mathcal{C}[G]$ by first order class comprehension in $\mathcal{M}[G]$. Since replacement holds in $\mathcal{M}[G]$, there is some $x \in M[G]$ with $\mathrm{ran}\, F \subseteq x$. Let $\gamma = {}^x$. Using axiom (4), let $q \leq p$ be a condition in $G$ that forces this property of $\check{\gamma}$. Using our assumption on $p$ and on $\vec{D}$ however,

$$\exists i \in I\, \forall r \in D_i \cap V_\gamma^M\ q \perp r,$$

yielding $G \cap D_i \cap V_\gamma^M = \emptyset$, a contradiction.

---

[18]This step of the argument does not seem possible with the original definition of tameness from [5], for we were only able to apply its instances to predense $\leq q$ partitions; however $\vec{d}$ is a predense $\leq s$ partition and $s \leq q$.

If $T$ contains the powerset axiom however $\mathbb{P}$ is not tame, we may thus choose $a \in M$ and $p \in P$ such that for every $\alpha \in \mathrm{Ord}(M)$ and $q \leq p$, there is $r \leq q$ and a sequence $\vec{d} \in M$ of predense $\leq r$ partitions of $\mathbb{P}$ with domain $a$ such that whenever $s \leq r$, then $\vec{d}$ is not equivalent $\leq s$ to any $\vec{e}$ in $V_\alpha^M$. Using axiom (1), let $G$ be a $\mathbb{P}$-generic filter over $\mathcal{M}$ with $p \in G$. Since the above yields a dense set of such conditions $r$ for every $\alpha \in \mathrm{Ord}(M)$, by Lemma 4.2, there are $r_\alpha \in G$ and a sequence of predense $\leq r_\alpha$ partitions $\vec{d}_\alpha = \langle \langle d_{\alpha,0}^i, d_{\alpha,1}^i \rangle \mid i \in a \rangle \in M$ for every $\alpha \in \mathrm{Ord}(M)$, witnessing that $\mathbb{P}$ is not tame.[19] Whenever $\vec{d} = \langle \langle d_0^i, d_1^i \rangle \mid i \in a \rangle \in M$ is a sequence of predense $\leq t$ partitions for some $t \in G$, we use axiom (7) to define $\dot{f}_{\vec{d}} \in M$, letting $x \in_u \dot{f}_{\vec{d}}$ if and only if

$$\left[ x = \langle \check{i}, \check{0} \rangle \wedge u \in d_0^i \right] \vee \left[ x = \langle \check{i}, \check{1} \rangle \wedge u \in d_1^i \right].$$

That is, $\dot{f}_{\vec{d}}$ can be seen as a canonical name for the function $f_{\vec{d}} \in M[G]$, $f_{\vec{d}} \colon a \to 2$ that is defined by letting $f_{\vec{d}}(i) = 0 \iff G \cap d_0^i \neq \emptyset$ for every $i \in a$. Note that, using axioms (4) and (5), two sequences $\vec{d}$ and $\vec{d}'$ of predense $\leq t$ partitions with domain $a$ of $\mathbb{P}$ in $\mathcal{C}$ are equivalent $\leq t$ if and only if $t \Vdash \dot{f}_{\vec{d}} = \dot{f}_{\vec{d}'}$. Thus, for any $\alpha \in \mathrm{Ord}^M$, $r_\alpha$ forces that $\dot{f}_{\vec{d}_\alpha} \neq \dot{f}_{\vec{e}}$ for any predense $\leq r_\alpha$ partition $\vec{e} \in V_\alpha^M$ with domain $a$. This implies that in $\mathcal{M}[G]$, we obtain a cofinal partial function $F$ from $\{ f \in M[G] \mid f \colon a \to 2 \}$ to $\mathrm{Ord}$, by letting $F(f)$ be the least $\alpha$ such that there is a condition $r \in G$ and a predense $\leq r$ partition $\vec{e} \in V_\alpha^M$ with domain $a$ such that $f = f_{\vec{e}}$. By replacement in $\mathcal{M}[G]$ thus, the powerset of $a$ cannot exist as a set in $M[G]$, and hence $\mathcal{M}[G]$ does not satisfy the powerset axiom. $\dashv$

**§8. Set forcing.** The axioms for set forcing, which are essentially as in [3], can easily be derived from the case of class forcing over models of GBc that was treated in this paper. In addition, we will also consider the base theories $\mathrm{ZF}^-$, $\mathrm{ZFC}^-$, and $\mathrm{ZF}$, which correspond to the second order theories $\mathrm{GB}^-$, $\mathrm{GBc}^-$, and $\mathrm{GB}$. Let $T$ thus denote our base theory, which will either be $\mathrm{ZF}^-$, $\mathrm{ZFC}^-$, $\mathrm{ZF}$, or $\mathrm{ZFC}$. Our terminology will be slightly different from that of [3]. In this section, let $\mathcal{L}(\in)$ denote the collection of first order formulas.

DEFINITION 8.1. A *set forcing generic framework* is a tuple of the form

$$\left\langle M, \mathbb{P}, R, (\Vdash_\varphi)_{\varphi \in \mathcal{L}(\in)}, \mathfrak{G} \right\rangle \text{ with the following properties.}$$

- $M$ is a transitive set-size model of $T$.
- $\mathbb{P} = \langle P, \leq \rangle \in M$ is a preorder.
- The $\mathbb{P}$-*membership relation* $R$ is a relation on $P \times M^2$ that is definable over $M$ by an $\mathcal{L}(\in)$-formula. We denote the property $R(p, a, b)$ as $a \in_p b$.[20]

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[19]Making a cautious remark here, we do not claim that the sequences of $r_\alpha$'s or $\vec{D}_\alpha$'s are in any way internal to $\mathcal{M}$.

[20]In a standard forcing setup, this would correspond to the property that $\langle a, p \rangle \in b$.

- $\mathfrak{G}$ is a second order unary predicate on $P$, i.e., a unary predicate on $\mathcal{P}(P)$, and we require that $\mathfrak{G}(G)$ implies that $G \subseteq \mathbb{P}$ is a filter. If $\mathfrak{G}(G)$ holds, we say that $G$ is a *generic filter*, or a $\mathbb{P}$-*generic filter on $M$*. Whenever we quantify over $G$ in the following, we assume that we quantify over $G$'s such that $\mathfrak{G}(G)$ holds.
- For every $\varphi \in \mathcal{L}(\in)$, $\Vdash_\varphi \in \mathcal{C}$ is a predicate (which we also call a *forcing relation for* $\varphi$) on $P \times M^m$, where $m$ denotes the number of free variables of $\varphi$. If $\langle q, a_0, \dots, a_{m-1} \rangle \in \Vdash_\varphi$, we also write $q \Vdash \varphi(a_0, \dots, a_{m-1})$.

Let us briefly provide the list of corresponding axioms for set forcing.

(S1) **Existence of generic filters:** $\forall p \in P \; \exists G \; p \in G$.

(S2) **Well-Foundedness:** The binary relation $\exists p \in \mathbb{P} \; a \in_p b$ on $M$ is well-founded.

(S3) **Growth of Information:** For all $\varphi \in \mathcal{L}(\in)$, for all $\vec{a} \in M$, and $p, q \in P$, if $p \Vdash \varphi(\vec{a})$ and $q \leq p$, then $q \Vdash \varphi(\vec{a})$.

Define the relation $\in_G$ on $M$ by letting $a \in_G b$ if $\exists p \in G \; a \in_p b$. Recursively define our *evaluation function* $F_G$ along the relation $\in_G$, letting $F_G(a) = \{F_G(b) \mid b \in_G a\}$ for each $a \in M$. Let $M[G]$ denote the $\in$-structure on the transitive set $F_G[M]$.

(S4)     **Truth Lemma:** For all $\varphi \in \mathcal{L}(\in)$, all $\vec{a} \in M$, and all $G$,

$$M[G] \models \varphi(F_G(\vec{a})) \text{ iff } \exists p \in G \; p \Vdash \varphi(\vec{a}).$$

(S5)     **Definability Lemma:** For all $\varphi \in \mathcal{L}(\in)$, all $\vec{a} \in M$, and $p \in P$,

$$p \Vdash \varphi(\vec{a}) \text{ iff } \forall G \ni p \; M[G] \models \varphi(F_G(\vec{a})).$$

(S6)     **Set-Likeness:** If $b \in M$, then $\{a \mid \exists p \in P \; a \in_p b\} \in M$.

(S∗)     **Names for ground model objects:** $\forall a \in M \; \exists \check{a} \in M \; \forall G \; F_G(\check{a}) = a$.

(S∗∗)     **Name for generic filters:** $\exists \dot{G} \in M \; \forall G \; F_G(\dot{G}) = G$.

(S∗∗∗)     **Preservation of axioms:** $\forall G \; M[G] \models T$.

(S∗∗∗∗) **Absoluteness of $\mathbb{P}$-membership:** The formula $\varphi$ defining the $\mathbb{P}$-membership relation $R$ is absolute for transitive models of $T$ containing $M$.

We can use these axioms to deduce the following version of Theorem 5.1 (by essentially using the same argument). A *set forcing generic extension* is a set forcing generic framework together with a particular choice of generic filter $G$.

THEOREM 8.2. *Given a particular set forcing generic extension, assuming axioms* (S1)–(S6), *and also axioms* (S∗), (S∗∗), (S∗∗∗), *and* (S∗∗∗∗), *to hold, $M[G]$ is actually well-defined in the sense that it depends only on $M$ and on $G$, and is in fact the $\subseteq$-smallest model $N$ of $T$ that contains $M \cup \{G\}$ as a subset.*

Proceeding toward a set forcing analogue of the results of Section 7, we introduce one further axiom.

(S7) **Universality:** If $s \in M$ is a relation on $M \times P$, then there exists $b \in \mathcal{M}$ such that

$$s(a, p) \text{ if and only if } a \in_p b.$$

We additionally assume that there is a map $\Gamma$ that is first order definable over $M$ which on input $s$ yields one such witnessing $b \in M$.[21]

The arguments from Section 7 then also yield the following.

THEOREM 8.3. *Axioms* $(S1)$–$(S7)$ *imply axioms* $(S*)$, $(S**)$, *and* $(S***)$, *and also that* $M[G]$ *is the* $\subseteq$-*smallest model of $T$ that contains $M \cup \{G\}$ as a subset.*

Another way of verifying Theorem 8.3 is to observe that a set forcing generic framework satisfying axioms (S1)–(S7) corresponds exactly to a class forcing generic framework over a model $\mathcal{M} = \langle M, \mathcal{C} \rangle$ satisfying axioms (1)–(8) for which $\mathcal{C}$ is just the collection of subsets of $M$ that are first order definable over $M$, and to then apply Theorem 7.3:[22]

In one direction, it should be pretty clear that starting from a class forcing generic framework in this setting, when $\mathbb{P} \in M$ however is a set-sized partial order, we can easily isolate a corresponding set forcing generic framework with respect to $\mathbb{P}$ and verify our set forcing axioms for that framework from their class forcing counterparts.

Let us provide a few more details on the other direction, which is relevant for verifying Theorem 8.3. We start with a set forcing generic framework

$$\left\langle M, \mathbb{P}, R, (\Vdash_\varphi)_{\varphi \in \mathcal{L}(\in)}, \mathfrak{G} \right\rangle.$$

We first have to extend our $\mathbb{P}$-membership relation $R$. Using the same letter $R$ also for its extension, we simply do so by letting $R(p, a, B)$ if and only if $\langle a, p \rangle \in B$ in case $B \in \mathcal{C} \setminus M$. That is, at the top level, when $B$ is a proper class, we define our $\mathbb{P}$-membership relation to behave analogous to the relation $\langle a, p \rangle \in B$ from the classical setting. For every nonzero length $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}(\in)$, we have to define relations $\Vdash_\varphi^{\vec{A}}$; however we simply define them to equal the corresponding first order forcing relations from our set forcing generic framework for the formulas obtained by replacing the elements of $\vec{A}$ by their defining formulas. Using our extended $\mathbb{P}$-membership relation, we can now define $a \in_G B$ if $\exists p \in G \; a \in_p B$ in case $B$ is a proper class, and we can thus also define $F_G(B)$ for $B \in \mathcal{C} \setminus M$, letting $F_G(B) = \{F_G(a) \mid \exists p \in G \langle a, p \rangle \in B\}$. We then let $\mathcal{M}[G] = \langle F_G[M], F_G[\mathcal{C}] \rangle$, as we did in Section 3. It is then easy to see that each of axioms (S1)–(S6) implies

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[21]As in the case of axiom (7), this additional definability assumption seems not to be strictly necessary for our present purposes when our theory $T$ contains the powerset axiom.

[22]Let us remark however that an analogous result with respect to Theorem 8.2 should not hold—there seems to be no reason to expect that starting from axioms (S1)–(S6) together with $(S*)$, $(S**)$, $(S***)$, and $(S****)$, the $\mathbb{P}$-membership relation on sets could be extended to definable classes so that it satisfies the absoluteness requirement from axiom $(****)$.

its corresponding axiom from (1)–(6). In order to verify axiom (7), assume that $S \in \mathcal{C} \setminus M$ is a relation on $M \times P$. Let $B \in \mathcal{C}$ be such that $S(a, p)$ holds if and only if $\langle a, p \rangle \in B$. Then $B$ witnesses the relevant instance of axiom (7). Finally, it is well-known and easy to check that any notion of set forcing is pretame for models of $\mathbf{GB}^-$, and tame for models of $\mathbf{GB}$, thus yielding axiom (8).

**§9. Kelley–Morse set theory.** Kelley–Morse set theory KM extends the axioms of GBC by the scheme of second order class comprehension. Let $\mathcal{L}^2(\in)$ denote the collection of all second order formulas in the language with the $\in$-predicate.

DEFINITION 9.1. A *class forcing generic framework for* KM is a tuple of the form

$$\left\langle \mathcal{M}, \mathbb{P}, R, \left( \Vdash_\varphi^{\vec{A}} \right)_{\varphi \in \mathcal{L}^2(\in), \vec{A} \in \mathcal{C}}, \mathfrak{G} \right\rangle$$

with the same properties as those of a class forcing generic framework in Definition 2.1, except that the last item from that definition has to be changed to the following.

- For every $\vec{A} \in \mathcal{C}$ and $\varphi \in \mathcal{L}^2(\in)$ for which the number of free second order variables corresponds to the length of $\vec{A}$, $\Vdash_\varphi^{\vec{A}} \in \mathcal{C}$ is a predicate (which we also call a *forcing relation for* $\varphi$) on $P \times M^m$, where $m$ denotes the number of free first order variables of $\varphi$.

    If $\langle q, a_0, \dots, a_{m-1} \rangle \in \Vdash_\varphi^{\vec{A}}$, we also write $q \Vdash \varphi(a_0, \dots, a_{m-1}, \vec{A})$.

Axioms (1)–(5) are the same as in Section 3, except that $\mathcal{L}(\in)$ has to be replaced by $\mathcal{L}^2(\in)$ throughout, and that in axioms (4) and (5), we have to refer to the number of free second order variables, rather than just the number of second order variables. Axioms $(\bar{6})$, $(*)$, and $(**)$ are exactly as in Section 3.

We can then use the above axioms to show that the forcing predicates obey their usual defining clauses as in Section 4, except that we have to consider additional types of formulas. As in the case of GBC we can avoid atomic formulas expressing (in)equality for classes, for we may consider those to be defined in terms of the $\in$-relation. But we need to treat second order quantification, extending Lemma 4.5 by the following clause. It is verified essentially by the same argument as for first order quantification in [3]; however we would like to provide the argument for the benefit of our readers.

LEMMA 9.2. $p \Vdash \exists X \varphi(\vec{a}, \vec{A})$ *iff* $\forall r \leq p \, \exists q \leq r \, \exists X \in \mathcal{C} \, q \Vdash \varphi(\vec{a}, \vec{A})$.

PROOF. Let us assume that

(i) $p \Vdash \exists X \varphi(X, \vec{a}, \vec{A})$.

By axiom (5), this is equivalent to

(ii) $\forall G \ni p \, \mathcal{M}[G] \models \exists X \varphi(X, F_G(\vec{a}), F_G(\vec{A}))$.

By the definition of $\mathcal{M}[G]$, this is in turn equivalent to

(iii) $\forall G \ni p \; \exists \dot{X} \in \mathcal{C} \; \mathcal{M}[G] \models \varphi(F_G(\dot{X}), F_G(\vec{a}), F_G(\vec{A}))$.

Making use of axiom (4), we equivalently obtain

(iv) $\forall G \ni p \; \exists \dot{X} \in \mathcal{C} \; \exists q \in G \; q \Vdash \varphi(\dot{X}, \vec{a}, \vec{A})$.

Applying Lemma 4.3 yields equivalence to

(v) $\forall r \leq p \; \exists q \leq r \; \exists \dot{X} \in \mathcal{C} \; q \Vdash \varphi(\dot{X}, \vec{a}, \vec{A})$,

as desired.                                                                              ⊣

We can now let axioms ($\ast\ast\ast$) and ($\ast\ast\ast\ast$) be as in Section 5, and verify the analogue of Theorem 5.1 when $T = \mathrm{KM}$ exactly as in that section, showing that $\mathcal{M}[G]$ is the least model of KM that contains $\mathcal{C} \cup \{G\}$ as a subset of its collection of classes.

Let axiom (6) be exactly as in Section 7. Let axiom (7) be exactly as in Section 7, except that we can ignore its additional definability assumption, for the axioms of KM include the existence of a global well-order. Let axiom (8) be the assumption that $\mathbb{P}$ is tame (for $\mathcal{M}$), as in Section 7, noting that the axioms of KM include the power set axiom.

As in Section 7, we would like to show that our axioms (1)–(8) imply axioms ($\ast$), ($\ast\ast$), and ($\ast\ast\ast$), and that $\mathcal{M}[G]$ is the $\subseteq$-smallest model of $T$ that contains $\mathcal{C} \cup \{G\}$ as a subset of its collection of classes. For ($\ast$) and ($\ast\ast$), this is shown exactly as in Lemma 7.1. The verification of ($\ast\ast\ast$) and the minimality of $\mathcal{M}[G]$ proceeds exactly as in the proofs of Theorem 7.3; however we have to additionally verify second order class comprehension in $\mathcal{M}[G]$ in the second proof. This is done by essentially the same argument as for first order class comprehension from the second proof of Theorem 7.3.

PROPOSITION 9.3. $\mathcal{M}[G]$ *satisfies second order class comprehension.*

PROOF. If $\varphi \in \mathcal{L}^2(\in)$ and $B \in \mathcal{C}$, using axiom (7), let $A \in \mathcal{C}$ be such that $a \in_p A$ if and only if $p \Vdash \varphi(a, B)$. It then follows that $F_G(A) = \{x \in M[G] \mid \mathcal{M}[G] \models \varphi(x, F_G(B))\}$.                                                      ⊣

§10. Extensions of KM. The results of the previous section can also be extended to base theories beyond KM. As a sample result, let us finally investigate the case of the base theory $T$ that is KM together with the axiom of class choice (CC), which was first studied by Mostowski and Marek in [9].

DEFINITION 10.1.   • Given $Y \in \mathcal{C}$ and $\alpha \in \mathrm{Ord}$, we let $Y_\alpha = \{x \mid \langle \alpha, x \rangle \in Y\}$.
  • CC is the statement that whenever $\varphi \in \mathcal{L}^2(\in)$, $A \in \mathcal{C}$, and for every ordinal $\alpha$ there is some $X \in \mathcal{C}$ such that $\varphi(\alpha, X, A)$ holds, then there is $Y \in \mathcal{C}$ such that for every ordinal $\alpha$, we have $\varphi(\alpha, Y_\alpha, A)$.

Note that in the above, we could equivalently use arbitrary sets rather than just ordinals as indices, for our base theory KM provides us with a global well-order in order-type Ord. We will tacitly make use of this in the below.

It is shown in [2] that tame forcing preserves KM + CC. All results from the previous section thus apply to the stronger base theory $T = \text{KM} + \text{CC}$ as well—we only need to verify that axioms (1)–(8) from Section 9 imply the preservation of CC. Our proof of this is based on the argument from [2]. Given $x, y \in M$, let $\{\check{x}, \check{y}\}$ be defined by letting $a \in_p \{\check{x}, \check{y}\}$ if and only if $a \in \{x, y\}$ and $p = 1$. Let $\langle \check{x}, \check{y} \rangle$ be defined by letting $a \in_p \langle \check{x}, \check{y} \rangle$ if and only if $a \in \{\{\check{x}, \check{x}\}, \{\check{x}, \check{y}\}\}$ and $p = 1$. Thus, $\langle \check{x}, \check{y} \rangle$ is a canonical name for the ordered pair $\langle F_G(x), F_G(y) \rangle$ in our setup.

PROPOSITION 10.2. $\mathcal{M}[G] \models \text{CC}$.

PROOF. Let $A \in \mathcal{C}$. As a first step, we will argue that a strong form of Lemma 9.2 holds, namely whenever $p \Vdash \exists X \varphi(X, A)$, then there is $X \in \mathcal{C}$ such that $p \Vdash \varphi(X, A)$. Using (second order) class comprehension in $\mathcal{M}$, let $D \in \mathcal{C}$ be the dense class of all $q \leq p$ for which there is some $X_q \in \mathcal{C}$ such that $q \Vdash \varphi(X_q, A)$. Let $A$ be a maximal antichain in $D$. Using CC in $\mathcal{M}$, for every $q \in A$, we may pick some $X_q \in \mathcal{C}$ such that $q \Vdash \varphi(X_q, A)$. Now we obtain our desired $X \in \mathcal{C}$ making use of axiom (7), letting $a \in_r X$ if and only if

$$r \leq q \ \wedge \ a \in \text{dom}(X_q) \ \wedge \ r \Vdash a \in X_q.$$

Now suppose that $\mathcal{M}[G] \models \forall \alpha \, \exists X \, \varphi(\alpha, X, F_G(A))$. By axiom (4), there is $p \in G$ such that $p \Vdash \forall \alpha \, \exists X \, \varphi(\alpha, X, A)$. For any $\alpha \in \text{Ord}^{\mathcal{M}}$, by the above, we find some $X^\alpha \in \mathcal{C}$ such that $p \Vdash \varphi(\alpha, X^\alpha, A)$. Using that our forcing relations are in $\mathcal{C}$, and using CC in $\mathcal{M}$, we may obtain $X \in \mathcal{C}$ such that $\forall \alpha \ p \Vdash \varphi(\alpha, X_\alpha, A)$. Making use of axiom (7), define $Y \in \mathcal{C}$ by letting $a \in_r Y$ if and only if for some $\alpha \in \text{Ord}^{\mathcal{M}}$, $a = \langle \check{\alpha}, x \rangle$ and $r$ is such that $x \in_r X_\alpha$. It follows that for all $\alpha \in \text{Ord}$, $\mathcal{M}[G] \models \varphi(\alpha, F_G(X)_\alpha, F_G(A))$, as desired. ⊣

§11. Open questions and possible future directions. One obvious question is whether our axiom system can be simplified, or whether some of its axioms could be weakened or even omitted. We have already remarked that the definability of the map $\Gamma$ provided by axiom (7) is not necessary in many cases, and it is not hard to see that for any of the axiom systems we consider, it would instead suffice, in the terminology of axiom (7), to have a map $\Gamma$ which provides a set of witnesses for each $S \in M$. Moreover, axiom (7) could in fact be replaced by the weaker Item (i) from Lemma 7.2, or some equivalent form of that item, which however seems less desirable as an axiom due to its reference to generic filters. All other requirements made by our axioms seem to be necessary as far as we know. Let us remark that Proposition 7.4 shows that axiom (8) cannot be omitted.

QUESTION 11.1. Are there possible improvements to our axiomatization of forcing, either by using a proper subset of our axioms, or perhaps by replacing some of our axioms by weaker or, in some other sense, better axioms?

In the present paper, we considered base theories containing at least the axioms of GB⁻, or the corresponding first order theory ZF⁻. The following question was posed to us by Ali Enayat.

QUESTION 11.2. Is there a similar axiom system for forcing over suitable base theories that are strictly weaker than ZF⁻ (and perhaps also their corresponding second order theories, when such exist), over which it is possible to develop a suitable machinery of forcing?

One such weaker base theory would be Kripke–Platek set theory; however in his [10], Mathias developed a weaker axiom system of provident set theory, and shows that models of this theory support a smooth theory of set forcing. It seems plausible that a careful adaption of our axioms could make them work in the context of such weaker models of set theory as well.

A topic that is somewhat implicit in our paper is touched by the following.

QUESTION 11.3. To what extent does our axiomatization shed light on the uniqueness of the forcing method as a means to extending models of set theory without giving up (ground model) control over the extension?

Another future direction of research would be the following.

QUESTION 11.4. Is it possible to provide natural and interesting axiomatizations for other technical concepts in set theory? For example, for symmetric submodels of forcing extensions, for (generic) ultrapowers or for (certain) canonical inner models of set theory?

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