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A non-existence result for nonlinear parabolic equations with singular measures as data

Francesco Petitta
CMA, University of Oslo, PO Box 1053 Blindern, 0316 Oslo, Norway
(francesco.petitta@cma.uio.no)

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In this paper we prove a non-existence result for nonlinear parabolic problems with zero lower-order terms whose model is

$$\begin{aligned} u_t - \Delta_p u + |u|^{q-1}u &= \lambda \quad \text{in } (0,T)\times\Omega, \\ u(0,x) &= 0 \quad \text{in } \Omega, \\ u(t,x) &= 0 \quad \text{on } (0,T)\times\partial\Omega, \end{aligned}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-laplace operator, λ is measure concentrated on a set of zero parabolic r-capacity ($1<p<r$) and q is large enough.

1. Introduction

The question of whether or not a solution should exist for semilinear problems has been largely studied in the elliptic framework; in a pioneering paper by Brézis [4], the author proved the following.

THEOREM 1.1. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N>2$, with $0\in\Omega$, let f be a function in $L^1(\Omega)$ and let f_n be a sequence of $L^\infty(\Omega)$ functions such that*

$$\lim_{n\to+\infty}\int_{\Omega\setminus B_\rho(0)} |f_n - f|\,\mathrm{d}x = 0 \quad \text{for all } \rho>0. \tag{1.1}$$

Let u_n be the sequence of solutions of the following nonlinear elliptic problems:

$$\left.\begin{aligned} -\Delta u_n + |u_n|^{q-1}u_n &= f_n \quad \text{in } \Omega, \\ u_n &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

with $q \geqslant N/(N-2)$. Then u_n converges to the unique solution u of the equation $-\Delta u + |u|^{q-1}u = f$.

If $f=0$, an example of functions f_n satisfying condition (1.1) is that of a sequence of non-negative $L^\infty(\Omega)$ functions converging in the weak-$*$ topology of measures to δ_0, the Dirac mass concentrated at the origin. In this case, u_n converges to 0, which is not a solution of the equation with δ_0 as datum. The result of theorem 1.1 is strongly connected with a theorem by Bénilan and Brézis [1], which states that the problem $-\Delta u + |u|^{q-1}u = \delta_0$ has no distributional solution if $q \geqslant N/(N-2)$.

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On the other hand (see [2,4]), if $q < N/(N-2)$, then there exists a unique solution of

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= \delta_0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The *threshold* $N/(N-2)$ essentially depends on the linearity of the Laplacian operator and on the fact that the Dirac mass is a measure which is concentrated on a point: a set of zero elliptic N-capacity.

In [9] this result was improved to the nonlinear framework; there the authors actually proved that, if λ is a measure concentrated on a set of zero elliptic r-capacity, $r < q$, and q is large enough, then the problem

$$\begin{aligned} -\Delta_p u + |u|^{q-1}u &= \lambda \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

has no solutions in a very *strong sense*; that is, if we approximate λ with smooth functions in the narrow topology of measures, then the approximating solutions u_n converge to 0. In [9] the result is proved for more general Leray–Lions-type nonlinear operators (see [8]).

We will combine an idea of [9] with a suitable parabolic *cut-off lemma* to prove a general non-existence result in the framework of nonlinear parabolic problems with singular measures as data.

If Ω is an open bounded subset of $\mathbb{R}^N$, $N > 2$, and $T > 0$, we denote by Q the parabolic cylinder $(0,T) \times \Omega$. If λ is a bounded Radon measure on Q, then we will say that λ is concentrated on a Borel set B and write $\lambda = \lambda_{\llcorner B}$, if $\lambda(E) = \lambda(B \cap E)$, for any measurable subset E of Q.

Our main result (see theorem 2.3, below) states the non-existence of solutions for parabolic problems in the sense of approximating sequences; as a particular case, we will obtain the following.

THEOREM 1.2. *Let f_n be a sequence of functions in $L^\infty(Q)$ such that*

$$\lim_{n\to\infty} \int_Q \varphi f_n \,\mathrm{d}x = \int_Q \varphi \,\mathrm{d}\lambda \quad \textit{for all } \varphi \in C(\bar{Q}),$$

where λ is a bounded Radon measure on Q concentrated on a set of zero parabolic r-capacity, and let

$$q > \frac{r}{r-2}. \tag{1.3}$$

Then the solutions of

$$\left.\begin{aligned} (u_n)_t - \Delta u_n + |u_n|^{q-1}u_n &= f_n \quad \textit{in } (0,T) \times \Omega, \\ u_n(0,x) &= 0 \quad \textit{in } \Omega, \\ u_n(t,x) &= 0 \quad \textit{on } (0,T) \times \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

are such that both u_n and $|\nabla u_n|$ converge to 0 in $L^1(Q)$.

Moreover,

$$\lim_{n\to\infty} \int_Q |u_n|^{q-1}u_n \varphi \,\mathrm{d}x = \int_Q \varphi \,\mathrm{d}\lambda \quad \textit{for all } \varphi \in C_0(Q).$$

REMARK 1.3. Theorem 1.2 states that in fact the sets of zero r-capacity are in some sense *removable singularities* for the problem

$$\left.\begin{aligned} u_t - \Delta u + |u|^{q-1}u = f \quad & \text{in } (0,T)\times\Omega, \\ u(0,x) = 0 \quad & \text{in } \Omega, \\ u(t,x) = 0 \quad & \text{on } (0,T)\times\partial\Omega, \end{aligned}\right\} \tag{1.5}$$

with large q, since the approximation *does not see these sets*. In fact, the singular measure λ turns out to be *cancelled out* by the zero-order terms of the approximating problems in the weakly-$*$ sense of the measures.

Moreover, as we shall prove, the convergence is actually stronger than the one stated in theorem 1.2.

Let us finally explicitly remark that the choice of the homogeneous initial datum is not restrictive; indeed, since the result is obtained for measures on Q which do not charge the set $\{0\}\times\Omega$, our argument is, as we will see, essentially independent of the initial datum.

2. Basic assumptions and tools

Let $p > 1$; we recall the notion of *parabolic p-capacity* associated to our problem (for further details see [7, 12]).

DEFINITION 2.1. Let $Q = Q_T = (0,T)\times\Omega$ for any fixed $T > 0$, and let us define $V = W_0^{1,p}(\Omega)\cap L^2(\Omega)$, endowed with its natural norm $\|\cdot\|_{W_0^{1,p}(\Omega)} + \|\cdot\|_{L^2(\Omega)}$ and

$$W = \{u \in L^p(0,T;V),\ u_t \in L^{p'}(0,T;V')\}, \tag{2.1}$$

endowed with its natural norm $\|u\|_W = \|u\|_{L^p(0,T;V)} + \|u_t\|_{L^{p'}(0,T;V')}$. If $U \subseteq Q$ is an open set, we define the *parabolic p-capacity* of U as

$$\operatorname{cap}_p(U) = \inf\{\|u\|_W : u \in W,\ u \geqslant \chi_U \text{ a.e. in } Q\},$$

where as usual we set $\inf\emptyset = +\infty$; we then define for any Borel set $B \subseteq Q$

$$\operatorname{cap}_p(B) = \inf\{\operatorname{cap}_p(U), U \text{ an open set of } Q, B \subseteq U\}.$$

Let us state our basic assumptions: let Ω be a bounded, open subset of $\mathbb{R}^N$, let T be a positive number and let $Q = (0,T)\times\Omega$. Let $a : (0,T)\times\Omega\times\mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (i.e. $a(\cdot,\cdot,\xi)$ is measurable on Q for every ξ in $\mathbb{R}^N$, and $a(t,x,\cdot)$ is continuous on $\mathbb{R}^N$ for almost every (t,x) in Q) such that the following hold:

$$a(t,x,\xi)\cdot\xi \geqslant \alpha|\xi|^p, \quad p > 1, \tag{2.2}$$

$$|a(t,x,\xi)| \leqslant \beta[b(t,x) + |\xi|^{p-1}], \tag{2.3}$$

$$[a(t,x,\xi) - a(t,x,\eta)]\cdot(\xi - \eta) > 0 \tag{2.4}$$

for almost every (t,x) in Q, for every ξ, η in $\mathbb{R}^N$, with $\xi \neq \eta$, where α and β are two positive constants and b is a non-negative function in $L^{p'}(Q)$.

We define the differential operator

$$A(u) = -\operatorname{div}(a(t,x,\nabla u)), \quad u \in L^p(0,T;W_0^{1,p}(\Omega)).$$

Under assumptions (2.2)–(2.4), A is a coercive and pseudo-monotone operator acting from the space $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

We now deal with the problem

$$\left.\begin{aligned} u_t - \operatorname{div}(a(t,x,\nabla u)) + |u|^{q-1}u &= g + \lambda && \text{in } (0,T)\times\Omega,\\ u(0,x) &= 0 && \text{in } \Omega,\\ u(t,x) &= 0 && \text{on } (0,T)\times\partial\Omega, \end{aligned}\right\} \tag{2.5}$$

with $g \in L^1(Q)$, $q > 1$, a satisfying (2.2)–(2.4), and where $\lambda = \lambda^+ - \lambda^-$ is a bounded measure concentrated on a set $E = E^+ \cup E^-$, such that $\bigcap_r(E) = 0$.

Note that the existence of renormalized solutions (which in particular turn out to be distributional solutions for problem (2.5)) is one of the results proved in a forthcoming paper (see [11]) in the case of diffuse measures as data, i.e. measures which do not charge the sets of zero parabolic p-capacity.

Let us recall that a sequence of bounded measures λ_n on an open set $D \subset \mathbb{R}^N$ narrowly converges to a measure λ if

$$\lim_{n\to\infty} \int_D \varphi \,\mathrm{d}\lambda_n = \int_D \varphi \,\mathrm{d}\lambda \quad \text{for all } \varphi \in C(\bar{D}).$$

We approximate the data with smooth g_n which converge to g in $L^1(Q)$ and smooth $f_n = f_n^{\oplus} - f_n^{\ominus}$, with $f_n^{\oplus}$ and $f_n^{\ominus}$ converging, respectively, to λ^+ and λ^- in the narrow topology of measures. We consider the solutions u_n of

$$\left.\begin{aligned} (u_n)_t - \operatorname{div}(a(t,x,\nabla u_n)) + |u_n|^{q-1}u_n &= g_n + f_n && \text{in } (0,T)\times\Omega,\\ u_n(0,x) &= 0 && \text{in } \Omega,\\ u_n(t,x) &= 0 && \text{on } (0,T)\times\partial\Omega. \end{aligned}\right\} \tag{2.6}$$

Let us give the notion of entropy solution for parabolic problem (2.5) with a general $g \in L^1(\Omega)$, recalling that

$$S^p = \{u \in L^p(0,T;W_0^{1,p}(\Omega));\ u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\},$$

that $T_k(s) = \max(-k, \min(k,s))$ for any $k > 0$, and that

$$\Theta_k(z) = \int_0^z T_k(s)\,\mathrm{d}s$$

is the primitive of the truncation function.

Definition 2.2. Let $g \in L^1(\Omega)$ and $\lambda = 0$. A measurable function u is an *entropy solution* of (2.5) if

$$T_k(u-g) \in L^p(0,T;W_0^{1,p}(\Omega)) \text{ for every } k > 0, \tag{2.7}$$

$$t \in [0,T] \mapsto \int_\Omega \Theta_k(u-g-\varphi)(t,x)\,\mathrm{d}x \tag{2.8}$$

is a continuous function for all $k \geqslant 0$ and all $\varphi \in S^p \cap L^\infty(Q)$ and, moreover,

$$\int_\Omega \Theta_k(u-g-\varphi)(T,x)\,\mathrm{d}x - \int_\Omega \Theta_k(u-g-\varphi)(0,x)\,\mathrm{d}x + \int_0^T \langle \varphi_t, T_k(u-g-\varphi)\rangle\,\mathrm{d}t + \int_Q a(t,x,\nabla u)\cdot\nabla T_k(u-g-\varphi)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_Q g T_k(u-g-\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.9}$$

for all $k \geqslant 0$ and all $\varphi \in S^p \cap L^\infty(Q)$.

Recall that, by a result of [6], the unique entropy solution of problem (2.5) (with $\lambda = 0$) coincides with the renormalized solution of the same problem in [13] (see also [7, 10]).

Our main result, which we state as follows, concerns the non-existence of solutions for problem (2.5) in the sense of approximating sequences.

THEOREM 2.3. *Let* $1 < p < r$ *and*

$$q > \frac{r(p-1)}{r-p} \tag{2.10}$$

and let u_n *be the unique solution of problem (2.6). Then* $|\nabla u_n|^{p-1}$ *converges strongly to* $|\nabla u|^{p-1}$ *in* $L^\sigma(Q)$ *with* $\sigma < pq/(q+1)(p-1)$*, where* u *is the unique entropy (renormalized) solution of the following problem:*

$$\left.\begin{aligned} u_t - \operatorname{div}(a(t,x,\nabla u)) + |u|^{q-1}u &= g \quad \text{in } (0,T)\times\Omega,\\ u(0,x) &= 0 \quad \text{in } \Omega,\\ u(t,x) &= 0 \quad \text{on } (0,T)\times\partial\Omega. \end{aligned}\right\} \tag{2.11}$$

Moreover,

$$\lim_{n\to\infty}\int_Q |u_n|^{q-1}u_n\varphi\,\mathrm{d}x = \int_Q |u|^{q-1}u\varphi\,\mathrm{d}x + \int_Q \varphi\,\mathrm{d}\lambda \quad \text{for all } \varphi \in C_0(Q). \tag{2.12}$$

3. Proof of theorem 2.3

From here on ω will indicate any quantity that vanishes as the parameters in its argument go to their (obvious, if not explicitly stressed) limit point with the same order in which they appear, that is, for example,

$$\lim_{\delta\to 0^+}\limsup_{m\to+\infty}\limsup_{n\to\infty}|\omega(n,m,\delta)| = 0.$$

Moreover, for the sake of simplicity, in what follows, the convergences, even if not explicitly stressed, may be understood to be taken possibly up to a suitable subsequence extraction.

To prove theorem 2.3 we will use the following lemma, proved in [10].

LEMMA 3.1. *Let* $\mu = \lambda_s^+ - \lambda_s^-$ *be a bounded Radon measure on* Q*, where* λ_s^+ *and* λ_s^- *are non-negative and concentrated, respectively, on two disjoint sets* E^+ *and* E^- *of*

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zero r-capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that

$$\lambda_s^+(E^+ \setminus K_\delta^+) \leqslant \delta, \qquad \lambda_s^-(E^- \setminus K_\delta^-) \leqslant \delta, \tag{3.1}$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+, \psi_\delta^- \equiv 1 \quad \text{respectively on } K_\delta^+, K_\delta^-, \tag{3.2}$$

$$0 \leqslant \psi_\delta^+, \psi_\delta^- \leqslant 1, \tag{3.3}$$

$$\operatorname{supp}(\psi_\delta^+) \cap \operatorname{supp}(\psi_\delta^-) \equiv \emptyset. \tag{3.4}$$

Moreover,

$$\|\psi_\delta^+\|_{S^r} \leqslant \delta, \qquad \|\psi_\delta^-\|_{S^r} \leqslant \delta \tag{3.5}$$

and, in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{r'}(0,T;W^{-1,r'}(\Omega))} \leqslant \delta, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \leqslant \delta, \tag{3.6}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{r'}(0,T;W^{-1,r'}(\Omega))} \leqslant \delta, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \leqslant \delta, \tag{3.7}$$

and both ψ_δ^+ and ψ_δ^- converge to 0 weakly-$*$ in $L^\infty(Q)$, in $L^1(Q)$, and, up to subsequences, almost everywhere as δ vanishes.

Moreover, if $f_n = f_n^+ - f_n^-$ is as in (2.6), we have

$$\int_Q \psi_\delta^- f_n^+ = \omega(n,\delta), \qquad \int_Q \psi_\delta^- \,\mathrm{d}\lambda_s^+ \leqslant \delta, \tag{3.8}$$

$$\int_Q \psi_\delta^+ f_n^- = \omega(n,\delta), \qquad \int_Q \psi_\delta^+ \,\mathrm{d}\lambda_s^- \leqslant \delta, \tag{3.9}$$

$$\int_Q (1-\psi_\delta^+) f_n^+ = \omega(n,\delta), \qquad \int_Q (1-\psi_\delta^+) \,\mathrm{d}\lambda_s^+ \leqslant \delta, \tag{3.10}$$

$$\int_Q (1-\psi_\delta^-) f_n^- = \omega(n,\delta), \qquad \int_Q (1-\psi_\delta^-) \,\mathrm{d}\lambda_s^- \leqslant \delta. \tag{3.11}$$

For the convenience of the reader we will now split the proof of theorem 2.3 into three steps. In the first step we prove some basic estimates on the approximating solutions, while the second step is devoted to checking how the zero-order term behaves far from the support of λ; in the last step we conclude the proof by showing that the limit function u is an entropy solution of problem (2.11) and that (2.12) holds true.

Proof of theorem 2.3.

STEP 1 (basic estimates). Taking $T_k(u_n)$ as a test function in the weak formulation of (2.6), we readily have the following estimates on the approximating solutions:

$$\int_Q |\nabla T_k(u_n)|^p \leqslant Ck, \tag{3.12}$$

$$\sup_t \int_\Omega |u_n| \leqslant C \tag{3.13}$$

and, moreover, since

$$k \int_{\{|u_n| \geqslant k\}} |u_n|^q \leqslant \int_Q |u_n|^q |T_k(u_n)| \leqslant Ck,$$

so that

$$k^q \operatorname{meas}\{|u_n| \geqslant k\} \leqslant C$$

and

$$\int_{\{|u_n|<k\}} |u_n|^q \leqslant Ck^q,$$

we have that

$$|u_n|^q \quad \text{is bounded in } L^1(Q).$$

Because of this fact, using (3.12), by reasoning as in [3], one can prove that

$$|\nabla u_n|^{p-1} \quad \text{is bounded in } L^\rho(Q) \text{ for any } \rho < \frac{pq}{(q+1)(p-1)}.$$

Moreover, u_n (up to subsequences) converges almost everywhere to a function u, and, looking at the equation in (2.6), we have that

$$(u_n)_t - \operatorname{div}(a(t,x,\nabla u_n))$$

is bounded in $L^1(Q)$ and so by [3, theorem 3.3] we have that

$$\nabla u_n \to \nabla u \quad \text{a.e. on } Q.$$

Therefore, owing to the growth condition on a, we have that both

$$|\nabla u_n|^{p-1} \to |\nabla u|^{p-1} \qquad \text{strongly in } (L^\rho(Q))^N \tag{3.14}$$

and

$$a(t,x,\nabla u_n) \to a(t,x,\nabla u) \quad \text{strongly in } (L^\rho(Q))^N \tag{3.15}$$

for every $\rho < pq/(q+1)(p-1)$.

Step 2 (energy estimates). Let $\Psi_\delta = \psi_\delta^+ + \psi_\delta^-$, as in lemma 3.1; note that the use of these types of cut-off function to deal with, separately, the regular and the singular part of the data was first introduced in [5] in the elliptic framework.

Then, we want to show that

$$\int_{\{u_n>2m\}} |u_n|^q (1-\Psi_\delta)\,\mathrm{d}x = \omega(n,m,\delta) \tag{3.16}$$

and

$$\int_{\{u_n<-2m\}} |u_n|^q (1-\Psi_\delta)\,\mathrm{d}x = \omega(n,m,\delta). \tag{3.17}$$

We will prove (3.16) (the proof of (3.17) is analogous). Let us define

$$\beta_m(s) = \begin{cases} 1 & \text{if } s > 2m, \\ \dfrac{s}{m} - 1 & \text{if } m < s \leqslant 2m, \\ 0 & \text{if } s \leqslant m \end{cases} \tag{3.18}$$

and let us take $\beta_m(u_n)(1-\Psi_\delta)$ as a test function in (2.6); we obtain

$$\int_0^T \langle (u_n)_t, \beta_m(u_n)(1-\Psi_\delta)\rangle \,\mathrm{d}t \tag{A}$$

$$+ \frac{1}{m}\int_{\{m<u_n\leqslant 2m\}} a(t,x,\nabla u_n)\cdot\nabla u_n(1-\Psi_\delta) \tag{B}$$

$$- \int_Q a(t,x,\nabla u_n)\cdot\nabla\Psi_\delta\beta_m(u_n) \tag{C}$$

$$+ \int_Q |u_n|^{q-1}u_n\beta_m(u_n)(1-\Psi_\delta) \tag{D}$$

$$= \int_Q f_n^+\beta_m(u_n)(1-\Psi_\delta) \tag{E}$$

$$- \int_Q f_n^-\beta_m(u_n)(1-\Psi_\delta) \tag{F}$$

$$+ \int_Q g_n\beta_m(u_n)(1-\Psi_\delta). \tag{G}$$

Let us analyse all terms one by one. Using (3.15) and assumption (2.10), by means of the Egorov theorem we readily have

$$-(\mathrm{C}) = \omega(n,m)$$

and

$$(\mathrm{G}) = \omega(n,m).$$

On the other hand, by lemma 3.1, we can write

$$\begin{aligned}
(\mathrm{E}) &\leqslant \int_Q f_n^+(1-\Psi_\delta)\,\mathrm{d}x \\
&= \int_Q f_n^+(1-\psi_\delta^+)\,\mathrm{d}x + \int_Q f_n^+\psi_\delta^-\,\mathrm{d}x \\
&= \int_Q (1-\psi_\delta^+)\,\mathrm{d}\lambda^+ + \int_Q \psi_\delta^-\,\mathrm{d}\lambda^- + \omega(n) \\
&= \omega(n,\delta).
\end{aligned}$$

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Moreover, we can drop both (B) and −(F) since they are non-negative, while if B_m is the primitive function of β_m, we can write

$$\begin{aligned}(\mathrm{A}) &= \int_Q B_m(u_n)_t(1-\Psi_\delta)\\ &= \int_Q B_m(u_n)(\Psi_\delta)_t + \int_\Omega B_m(u_n)(T)\\ &\geqslant \omega(n,m).\end{aligned}$$

Collecting together all these results, we obtain (3.16).

Step 3 (passing to the limit). Here, for technical reasons, we use the double cut-off function $\Psi_{\delta,\eta} = \psi_\delta^+\psi_\eta^+ + \psi_\delta^-\psi_\eta^-$, where ψ_δ^+, ψ_δ^-, ψ_η^+ and ψ_η^- are the functions constructed in lemma 3.1; the same trick was also used in [10] (see also [5]).

Let us define

$$h_m(s) = \begin{cases} 0 & \text{if } |s| > 2m,\\ 2 - \dfrac{|s|}{m} & \text{if } m < |s| \leqslant 2m,\\ 1 & \text{if } |s| \leqslant m.\end{cases} \tag{3.19}$$

We take $T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)$ in the weak formulation of (2.6), and we have

$$\int_0^T \langle (u_n)_t, T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\rangle\,\mathrm{d}t_t \tag{A}$$

$$+ \int_Q a(t,x,\nabla u_n)\cdot\nabla T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n) \tag{B}$$

$$- \int_Q a(t,x,\nabla u_n)\cdot\nabla\Psi_{\delta,\eta}T_k(u_n-\varphi)h_m(u_n) \tag{C}$$

$$+ \int_Q |u_n|^{q-1}u_nT_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n) \tag{D}$$

$$= \int_Q f_n^+T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n) \tag{E}$$

$$- \int_Q f_n^-T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n) \tag{F}$$

$$+ \int_Q g_nT_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n) \tag{G}$$

$$- \frac{1}{m}\int_{\{m<u_n\leqslant 2m\}} a(t,x,\nabla u_n)\cdot\nabla u_n(1-\Psi_{\delta,\eta})T_k(u_n-\varphi) \tag{H}$$

$$+ \frac{1}{m}\int_{\{-2m\leqslant u_n<-m\}} a(t,x,\nabla u_n)\cdot\nabla u_n(1-\Psi_{\delta,\eta})T_k(u_n-\varphi). \tag{I}$$

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Using lemma 3.1 and (3.15) we have (C) $=\omega(n,\eta)$, while

$$|(\mathrm{E})|+|(\mathrm{F})|\leqslant k\int_Q (f_n^+ + f_n^-)(1-\Psi_{\delta,\eta})\,\mathrm{d}x=\omega(n,\eta),$$

and we easily see that

$$(\mathrm{G})=\int_Q gT_k(u-\varphi)\,\mathrm{d}x+\omega(n,\eta).$$

On the other hand, using [10, lemma 6] we deduce that $|(\mathrm{H})|+|(\mathrm{I})|=\omega(n,m,\eta)$. Now let us look at (D):

$$\begin{aligned}(\mathrm{D})={}&\int_{\{-2m\leqslant u_n\leqslant 2m\}}|u_n|^{q-1}u_nT_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\\&+\int_{\{u_n>2m\}}u_n^qT_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\\&+\int_{\{u_n<-2m\}}|u_n|^qT_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n).\end{aligned}$$

Using (3.16) and (3.17) we have that the last two terms in the right-hand side are $\omega(n,m,\eta)$, while

$$\begin{aligned}&\int_{\{-2m\leqslant u_n\leqslant 2m\}}|u_n|^{q-1}u_nT_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\\&\qquad=\int_{\{-2m\leqslant u\leqslant 2m\}}|u|^{q-1}uT_k(u-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)+\omega(n)\\&\qquad=\int_Q|u|^{q-1}uT_k(u-\varphi)(1-\Psi_{\delta,\eta})+\omega(n,m)\\&\qquad=\int_Q|u|^{q-1}uT_k(u-\varphi)+\omega(n,m,\eta).\end{aligned}$$

So that

$$(\mathrm{D})=\int_Q|u|^{q-1}uT_k(u-\varphi)+\omega(n,m,\eta).$$

Moreover,

$$\begin{aligned}(B)={}&\int_Q[a(t,x,\nabla u_n)-a(t,x,\nabla\varphi)]\cdot\nabla T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\\&+\int_Q a(t,x,\nabla\varphi)\cdot\nabla T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\end{aligned}$$

and

$$\begin{aligned}&\int_Q a(t,x,\nabla\varphi)\cdot\nabla T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\\&\qquad=\int_Q a(t,x,\nabla\varphi)\cdot\nabla T_k(u-\varphi)+\omega(n,m,\eta),\end{aligned}$$

while the first term can be handled by Fatou's lemma, finally obtaining

$$\int_Q a(t,x,\nabla u)\cdot\nabla T_k(u-\varphi) \leqslant \liminf_{\eta\to 0^+}\liminf_{m\to\infty}\liminf_{n\to\infty}\,(\mathrm{B}).$$

We now deal with (A). Let us define $\Theta_{k,m}(s)$ as the primitive function of $T_k(s)h_m(s)$, observe that $\Theta_{k,m}$ is a bounded function; so, by lemma 3.1, for any $\eta > 0$ there exists δ small enough such that

$$\begin{aligned}\left|\int_Q \Theta_{k,m}(u_n-\varphi)h_m(u_n)(\Psi_\delta)_t\right| &= \int_Q \Theta_k(u-\varphi)|(\Psi_\delta)_t| + \omega(n)\\ &\leqslant \eta + \omega(n) = \omega(n,\eta),\end{aligned}$$

and so, finally,

$$\begin{aligned}(\mathrm{A}) &= \int_0^T \langle (u_n-\varphi)_t, T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\rangle\,\mathrm{d}t\\ &\quad + \int_0^T \langle \varphi_t, T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\rangle\,\mathrm{d}t\\ &= \int_\Omega \Theta_{k,m}(u_n-\varphi)(T) - \int_\Omega \Theta_{k,m}(-\varphi)(0) + \int_Q \Theta_{k,m}(u_n-\varphi)(\Psi_\delta)_t\\ &\quad + \int_0^T \langle \varphi_t, T_k(u_n-\varphi)(1-\Psi_{\delta,\eta})h_m(u_n)\rangle\,\mathrm{d}t \geqslant \int_\Omega \Theta_k(u-\varphi)(T)\\ &\quad - \int_\Omega \Theta_k(-\varphi)(0) + \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle\,\mathrm{d}t + \omega(n,m,\eta),\end{aligned}$$

where in the last passage we used the fact that $r > p$ and Fatou's lemma, which can be applied for almost every $0 \leqslant T' \leqslant T$. Passing to the limit and gathering together all these facts we can conclude that u is an entropy solution of (2.11). Actually we proved this fact for almost every $0 \leqslant T' \leqslant T$ but thanks to uniqueness of the entropy solution one can easily show that u is the entropy solution for any $T > 0$.

To prove (2.12) take $\psi \in C_0^\infty(Q)$ in (2.6) to obtain

$$\int_Q |u_n|^{q-1}u_n\psi = -\int_Q a(t,x,\nabla u)\cdot\nabla\psi + \int_Q g\psi + \int_Q \psi\,\mathrm{d}\lambda + \omega(n),$$

which together with the fact that u is an entropy solution of problem (2.11) (and so a distributional one) yields (2.12) for ψ smooth. Finally, an easy density argument allows us to conclude the proof. □

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