

Analytical shock solutions at large and small Prandtl number

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Exact one-dimensional solutions to the equations of fluid dynamics are derived in the $Pr \rightarrow \infty$ and $Pr \rightarrow 0$ limits (where Pr is the Prandtl number). The solutions are analogous to the $Pr = 3/4$ solution discovered by Becker and analytically capture the profile of shock fronts in ideal gases. The large- Pr solution is very similar to Becker's solution, differing only by a scale factor. The small- Pr solution is qualitatively different, with an embedded isothermal shock occurring above a critical Mach number. Solutions are derived for constant viscosity and conductivity as well as for the case in which conduction is provided by a radiation field. For a completely general density- and temperature-dependent viscosity and conductivity, the system of equations in all three limits can be reduced to quadrature. The maximum error in the analytical solutions when compared to a numerical integration of the finite- Pr equations is $O(Pr^{-1})$ as $Pr \rightarrow \infty$ and $O(Pr)$ as $Pr \rightarrow 0$.

Key words: compressible flows, Navier–Stokes equations, shock waves

1. Introduction

Although the power of numerical techniques makes them indispensable for solving the equations of fluid dynamics, analytical solutions, while difficult to find, remain useful, for several reasons: they build physical intuition, they can be quickly evaluated over a wide dynamic range, and they can be used to verify numerical algorithms. One such solution was discovered by Becker (1922) under the assumptions of a steady state, one planar dimension, constant viscosity, an ideal gas equation of state, and a fluid Prandtl number of $3/4$. It consists of implicit, closed-form expressions for the fluid variables and analytically captures the behaviour of shocks in ideal gases with $Pr = 3/4$. Thomas (1944), Morduchow & Libby (1949), Hayes (1960) and Iannelli (2013) extended Becker's solution to non-constant viscosity and conductivity, for both a power-law variation with temperature and a Sutherland viscosity model (White 1991). This is a more realistic assumption for gases, whose viscosity typically varies with

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temperature (White 1991). Approximate solutions for $Pr \neq 3/4$ have also been derived by Khidr & Mahmoud (1985).

It is shown here that analogous solutions can be derived in both the $Pr \rightarrow \infty$ and $Pr \rightarrow 0$ limits, for both constant and non-constant viscosity and thermal conductivity. The transport properties of large-Prandtl-number fluids are dominated by momentum diffusion, whereas those of small-Prandtl-number fluids are dominated by thermal diffusion. Becker's (1922) solution applies to air and many other gases, which have $Pr \sim 0.75$. Examples at the other extremes include mercury ($Pr \sim 10^{-2}$), gas mixtures ($Pr \sim 10^{-1}$), engine oil ($Pr \sim 10^2$ – 10^5) and the Earth's mantle ($Pr > 10^{23}$) (Clay 1973; Kaminski & Jaupart 2003; Bejan 2004; Hogg 2012). In addition, plasmas behave as small- Pr fluids due to the importance of heat conduction by both electrons and radiation (Zel'dovich & Raizer 2002). A proton–electron plasma, for example, has $Pr = 0.065$ (Chapman & Cowling 1939). It should be noted that not all of these fluids obey an ideal gas equation of state, and other physics may need to be taken into account; see the discussion in § 4. Taken together with Becker's solution, the solutions derived here yield analytical profiles of shock fronts in ideal gases over a wide range of parameter space. The basic equations are outlined in § 2, § 3 gives the derivation of the solutions, and § 4 discusses some implications.

2. Basic equations

For a fluid with mass density ρ , velocity magnitude v , pressure p , internal energy e , temperature T , viscosity μ (this can be regarded as either the dynamic viscosity in the limit of negligible bulk viscosity, or the dynamic viscosity plus three-quarters of the bulk viscosity) and thermal conductivity κ ($Pr = \mu C_p / \kappa$, where C_p is the specific heat at constant pressure), the equations of fluid dynamics in planar geometry are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x} \left(\rho v^2 + p - \frac{4\mu}{3} \frac{\partial v}{\partial x} \right) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho e \right) + \frac{\partial}{\partial x} \left[\rho v \left(\frac{1}{2} v^2 + h \right) - \frac{4\mu}{3} v \frac{\partial v}{\partial x} - \kappa \frac{\partial T}{\partial x} \right] = 0, \tag{2.3}$$

where $h = e + p/\rho$ is the fluid enthalpy (Landau & Lifshitz 1987). It will be assumed throughout that the fluid obeys an ideal gas equation of state,

$$p = (\gamma - 1)\rho e, \tag{2.4}$$

so that $h = \gamma e = C_p T$ with $C_p = \gamma C_v$, where C_v is the specific heat at constant volume. Under this assumption and the assumption of a steady state, (2.1)–(2.3) can be integrated from $-\infty$ to x to give

$$\rho v = \rho_0 v_0 \equiv m_0, \tag{2.5}$$

$$v^2 + \frac{\gamma - 1}{\gamma} h - \frac{4\mu}{3m_0} v \frac{dv}{dx} = \left(v_0^2 + \frac{\gamma - 1}{\gamma} h_0 \right) \frac{\rho_0}{\rho}, \tag{2.6}$$

$$\frac{1}{2} v^2 + h - \frac{4\mu}{3\rho} \frac{dv}{dx} - \frac{\kappa}{\rho v C_p} \frac{dh}{dx} = \left(\frac{1}{2} v_0^2 + h_0 \right) \frac{\rho_0 v_0}{\rho v}, \tag{2.7}$$

where the zero-slope boundary conditions appropriate for a shock are assumed to hold at $x = \pm\infty$. A subscript '0' here denotes a fluid quantity in the ambient (pre-shock) state. These equations can be combined into two ordinary differential equations

governing the spatial profile of the shock front:

$$\frac{4\mu}{3m_0} v \frac{dv}{dx} = v^2 + \frac{\gamma - 1}{\gamma} h - \frac{\gamma + 1}{2\gamma} (v_0 + v_1)v, \quad (2.8)$$

$$\frac{\kappa}{m_0 C_p} \frac{dh}{dx} = \frac{h}{\gamma} - \frac{v^2}{2} + \frac{\gamma + 1}{2\gamma} (v_0 + v_1)v - \frac{\gamma + 1}{\gamma - 1} \frac{v_0 v_1}{2}. \quad (2.9)$$

Here the integration constants have been expressed in terms of both pre-shock and post-shock (denoted by a subscript ‘1’) velocities via the Rankine–Hugoniot jump conditions,

$$\frac{v_1}{v_0} = \frac{\gamma - 1 + 2/M_0^2}{\gamma + 1}, \quad (2.10)$$

where $M_0^2 = v_0^2/c_0^2$ is the shock Mach number and $c_0 = \sqrt{\gamma p_0/\rho_0}$ is the adiabatic sound speed in the ambient fluid (Landau & Lifshitz 1987).

3. Solutions

The derivation of the Becker (1922) solution is outlined in §3.1, followed by a derivation of the $Pr \rightarrow \infty$ and $Pr \rightarrow 0$ solutions in §§3.2 and 3.3, respectively. These are all derived for constant viscosity and conductivity. Section 3.4 shows how the solutions can be extended to non-constant viscosity and conductivity, using radiation heat conduction as an example. General expressions for the shock profiles in all three Pr limits under the assumption of a viscosity and conductivity that vary as powers of the density and temperature are derived in §3.5.

3.1. Becker’s ($Pr = 3/4$) solution

Becker (1922) noticed that, for $Pr = 3/4$, equation (2.7) for the energy flux,

$$\frac{v^2}{2} + h - \frac{\kappa}{m_0 C_p} \frac{d}{dx} \left(\frac{v^2}{2} + h \right) = \frac{v_0^2}{2} + h_0, \quad (3.1)$$

is linear and has the finite solution

$$\frac{v^2}{2} + h = \frac{v_0^2}{2} + h_0 = \frac{\gamma + 1}{\gamma - 1} \frac{v_0 v_1}{2}, \quad (3.2)$$

where the second equality follows from the Rankine–Hugoniot conditions. Solving this equation for h and inserting it into (2.6) for the momentum flux leads to

$$v L_\kappa \frac{\kappa}{\kappa_0} \frac{dv}{dx} = \frac{\gamma + 1}{2} (v - v_0)(v - v_1), \quad (3.3)$$

where

$$L_\kappa \equiv \frac{\kappa_0}{m_0 C_v}. \quad (3.4)$$

Equation (3.3) can be rewritten as an integral over the velocity,

$$x = \frac{2L_\kappa}{\gamma + 1} \int \frac{(\kappa/\kappa_0)v}{(v - v_0)(v - v_1)} dv. \quad (3.5)$$

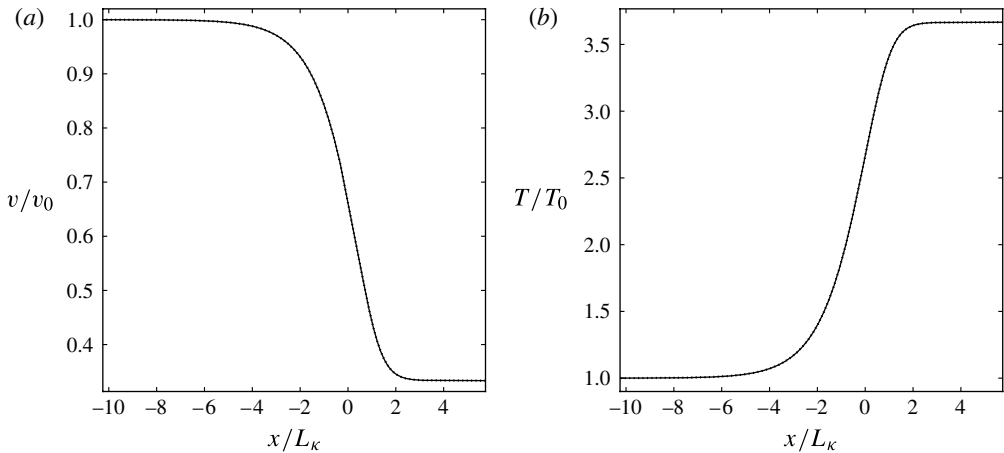


FIGURE 1. (a) Velocity and (b) temperature for the Becker solution ($Pr = 3/4$) with $M_0 = 3$. No distinction is visible between the analytical (solid) and numerical (dotted) results.

For constant $\kappa = \kappa_0$, this integral is given (to within an arbitrary constant) by

$$x = \frac{2L_\kappa}{\gamma + 1} \ln[(v_0 - v)^{v_0/(v_0-v_1)}(v - v_1)^{-v_1/(v_0-v_1)}]. \tag{3.6}$$

Physical notation has been retained here as an aid to intuition; notice that $x = \pm\infty$ at $v = v_1$ and $v = v_0$, respectively. Defining the origin at the adiabatic sonic point ($v = \sqrt{v_0 v_1}$) and using $\eta \equiv v/v_0 = \rho_0/\rho$ (the specific volume relative to its ambient value) rather than v yields the expression given in Zel'dovich & Raizer (2002). From (3.2), the temperature in this limit is given by

$$T = \frac{R_\infty v_0 v_1 - v^2}{2C_p}, \tag{3.7}$$

where

$$R_\infty \equiv \frac{\gamma + 1}{\gamma - 1} \tag{3.8}$$

is the maximum compression ratio.

Figure 1 shows the velocity and temperature for this solution, using expressions (3.6) and (3.7). For comparison, results from a numerical integration of (2.8) and (2.9) are shown in figure 1 as well. The numerical results here and in the following sections were obtained via a shooting method using the `odeint` differential equation solver in `scipy`. An important practical note here is that it is necessary to shoot from the post-shock state in order to obtain the desired solution. Equation (3.1) admits an exponential solution in addition to the constant solution, representing an additional energy flux at the boundary of arbitrary magnitude (Zel'dovich & Raizer 2002). For an integration from the pre- to post-shock state, this solution is exponentially growing, bounded only by the end point of the integration, and can quickly dominate the numerical results. For an integration from the post- to pre-shock state, the exponential solution is decaying and therefore unproblematic.

Analytical shock solutions

3.2. Large-Pr solution

In the limit $Pr \rightarrow \infty$ ($\kappa \rightarrow 0$), equations (2.6) and (2.7) become

$$v^2 + \frac{\gamma - 1}{\gamma} h - \frac{4\mu}{3m_0} v \frac{dv}{dx} = \frac{\gamma + 1}{2\gamma} (v_0 + v_1)v, \quad (3.9)$$

$$\frac{1}{2}v^2 + h - \frac{4\mu}{3m_0} v \frac{dv}{dx} = \frac{\gamma + 1}{\gamma - 1} \frac{v_0 v_1}{2}, \quad (3.10)$$

which can be combined to give

$$vL_\mu \frac{\mu}{\mu_0} \frac{dv}{dx} = \frac{\gamma + 1}{2} (v - v_0)(v - v_1), \quad (3.11)$$

where

$$L_\mu \equiv \frac{4\mu_0}{3m_0} = \frac{4Pr}{3\gamma} L_\kappa. \quad (3.12)$$

This can again be expressed as an integral over velocity,

$$x = \frac{2L_\mu}{\gamma + 1} \int \frac{(\mu/\mu_0)v}{(v - v_0)(v - v_1)} dv, \quad (3.13)$$

with the solution (for constant $\mu = \mu_0$) given by

$$x = \frac{2L_\mu}{\gamma + 1} \ln[(v_0 - v)^{v_0/(v_0 - v_1)} (v - v_1)^{-v_1/(v_0 - v_1)}]. \quad (3.14)$$

Comparing expression (3.14) with (3.6), it can be seen that the velocity profile in the large-Pr solution differs from that of the Becker (1922) solution only by the scale factor $L_\mu/L_\kappa = 4Pr/(3\gamma)$ (assuming constant Pr). The difference between the temperature profiles is more complicated, since solving (3.9) and (3.10) for the temperature in this limit yields an expression that differs from expression (3.7):

$$T = \frac{v^2 - 4v_i v + R_\infty v_0 v_1}{2C_v}, \quad (3.15)$$

where

$$v_i \equiv \frac{\gamma + 1}{4\gamma} (v_0 + v_1). \quad (3.16)$$

Figure 2 shows the velocity and temperature for the large-Pr solution with $M_0 = 3$ and constant viscosity. A value of $Pr = 10^3$ was used to generate the numerical results in this figure.

3.3. Small-Pr solution

In the limit $Pr \rightarrow 0$ ($\mu \rightarrow 0$), equations (2.6) and (2.7) become

$$v^2 + \frac{\gamma - 1}{\gamma} h = \frac{\gamma + 1}{2\gamma} (v_0 + v_1)v, \quad (3.17)$$

$$\frac{v^2}{2} + h - \frac{\kappa}{m_0 C_p} \frac{dh}{dx} = \frac{\gamma + 1}{\gamma - 1} \frac{v_0 v_1}{2}. \quad (3.18)$$

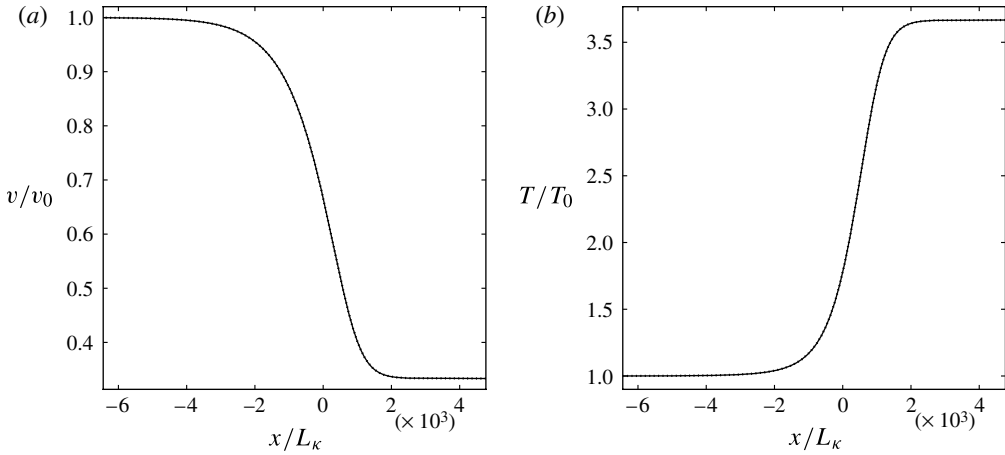


FIGURE 2. (a) Velocity and (b) temperature for the $Pr \rightarrow \infty$ solution with $M_0 = 3$ and constant viscosity. No distinction is visible between the analytical (solid) and numerical (dotted) results.

Taking the spatial derivative of (3.17) and eliminating the enthalpy derivative via (3.18) and the enthalpy via (3.17) gives

$$2(v - v_i)L_\kappa \frac{\kappa}{\kappa_0} \frac{dv}{dx} = \frac{\gamma + 1}{2}(v - v_0)(v - v_1). \tag{3.19}$$

Notice that, unlike (3.3) and (3.11), equation (3.19) is singular at $v = v_i$ (v_i is the isothermal sonic point for this solution). Expressed as an integral over the velocity,

$$x = \frac{4L_\kappa}{\gamma + 1} \int \frac{(\kappa/\kappa_0)(v - v_i)}{(v - v_0)(v - v_1)} dv, \tag{3.20}$$

a solution can be obtained for $x(v)$ (again assuming constant κ):

$$x = \frac{4L_\kappa}{\gamma(\gamma + 1)} \ln[(v_0 - v)^{(\beta v_0 - v_1)/(v_0 - v_1)}(v - v_1)^{(v_0 - \beta v_1)/(v_0 - v_1)}], \tag{3.21}$$

where

$$\beta \equiv \frac{3\gamma - 1}{\gamma + 1}. \tag{3.22}$$

From (3.17), the temperature in this limit is given by

$$T = \frac{v(2v_i - v)}{(\gamma - 1)C_v}. \tag{3.23}$$

As discussed in Zel'dovich & Raizer (2002), the small- Pr solution can be either discontinuous or continuous depending upon whether the isothermal sonic point lies inside or outside the shock region. The function $T(v)$ given by expression (3.23) passes through a maximum at $v = v_i$ and is monotonically increasing ($dT/dv > 0$) for $v_1 < v < v_i$ (see figure 7.7 of Zel'dovich & Raizer (2002) for a graphical representation). The velocity in the frame of the shock (or, equivalently, the specific volume) must decrease in this region, as it has not yet reached its final value, i.e. $dv/dx < 0$. This implies that the temperature also decreases in this

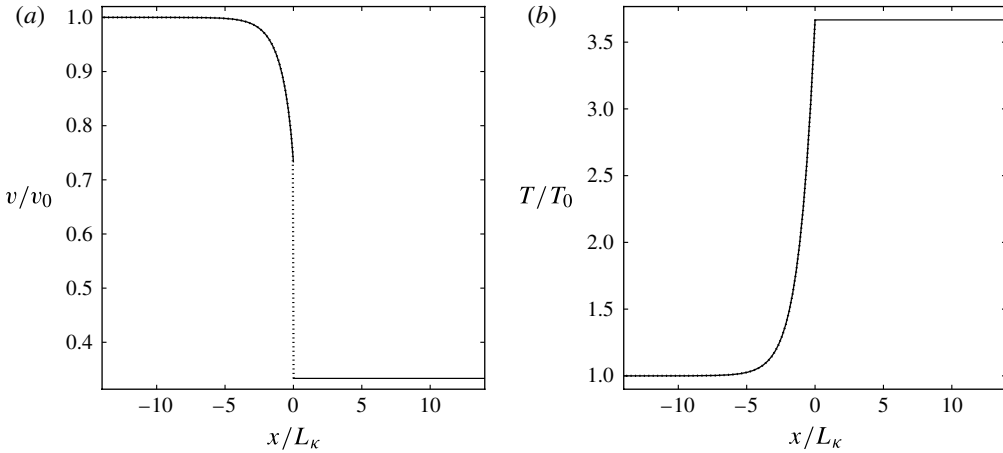


FIGURE 3. (a) Velocity and (b) temperature for the $Pr \rightarrow 0$ solution with $M_0 = 3$ and constant conductivity. No distinction is visible between the analytical (solid) and numerical (dotted) results. The isothermal shock is located at $x = 0$. The material in front of the shock is heated to temperatures above the ambient temperature because heat is being conducted from the hotter post-shock region to the colder pre-shock region.

region: $dT/dx = (dT/dv)(dv/dx) < 0$. However, this contradicts

$$\frac{dT}{dx} = \frac{(\gamma + 1)\rho_0 v_0}{2(\gamma - 1)\kappa} (v_0 - v)(v - v_1) > 0, \tag{3.24}$$

i.e. the temperature monotonically increases throughout the shock. The region $v_1 < v < v_i$ is thus excluded as unphysical. Since the presence of heat conduction also implies a continuous temperature, the only possibility is for the velocity to drop immediately to v_1 as soon as the temperature reaches T_1 , i.e. an isothermal shock occurs. From (3.23), $T = T_1$ for $v(2v_i - v) = v_1(2v_i - v_1)$, or $(v - v_1)(v - 2v_i + v_1) = 0$, i.e. the embedded discontinuity occurs at

$$v = 2v_i - v_1. \tag{3.25}$$

If the singularity lies within the shock region, $v_i > v_1$, the small- Pr solution is given by expression (3.21) for $2v_i - v_1 < v < v_0$, followed by an isothermal shock from $v = 2v_i - v_1$ to $v = v_1$. If the singularity falls outside the shock region, $v_i < v_1$ or

$$M_0 < \sqrt{\frac{3\gamma - 1}{\gamma(3 - \gamma)}}, \tag{3.26}$$

the solution is continuous and given by expression (3.21) throughout the shock region. Figure 3 shows the velocity and temperature for a discontinuous small- Pr solution with $M_0 = 3$ and constant conductivity. A value of $Pr = 10^{-3}$ was used to generate the numerical results in this figure.

3.4. Radiation heat conduction

In an opaque gas, thermal radiation is in local thermodynamic equilibrium with the gas and diffuses from high- to low-temperature regions, thus acting as a form of heat conduction. For a constant opacity, radiation gives rise to a thermal conductivity with

a T^3 dependence,

$$\kappa = \frac{16\sigma}{3\chi} T^3, \tag{3.27}$$

where σ is the Stefan–Boltzmann constant and χ is the opacity in units of inverse length (Zel’dovich & Raizer 2002). Equation (3.20) in that case can be expressed as

$$x = \frac{4L_\kappa}{\gamma + 1} \int \left(\frac{T}{T_0}\right)^3 \frac{v - v_i}{(v - v_0)(v - v_1)} dv, \tag{3.28}$$

with $\kappa_0 = 16\sigma T_0^3/(3\chi)$. Using (3.23), this can be rewritten as

$$x = \frac{4L_\kappa \gamma^3 M_0^6}{\gamma + 1} \int \frac{\eta^3 (2\eta_i - \eta)^3 (\eta - \eta_i)}{(\eta - 1)(\eta - \eta_1)} d\eta, \tag{3.29}$$

where $\eta \equiv v/v_0$ and $\eta_i \equiv v_i/v_0$. The integrand in (3.29) can be expanded into

$$\frac{\eta^3 (2\eta_i - \eta)^3 (\eta - \eta_i)}{(\eta - 1)(\eta - \eta_1)} = \frac{\eta^6 + c_1 \eta^5 + c_2 \eta^3}{\eta - 1} + \frac{-2\eta^6 + c_3 \eta^5 + c_4 \eta^3}{\eta - \eta_1}, \tag{3.30}$$

where

$$c_1 \equiv \frac{7\eta_i + \eta_1 - 2 - 18\eta_i^2}{1 - \eta_1}, \quad c_2 \equiv \frac{4\eta_i^3 (2\eta_i - 5)}{\eta_1 - 1}, \tag{3.31}$$

$$c_3 \equiv \frac{-7\eta_i \eta_1 - \eta_1^2 + 2\eta_1 + 18\eta_i^2}{1 - \eta_1}, \quad c_4 \equiv \frac{4\eta_i^3 (5\eta_1 - 2\eta_i)}{\eta_1 - 1}. \tag{3.32}$$

Using the result (for integer m)

$$\int \frac{z^n}{z - c} dz = c^n \ln(z - c) + \sum_{m=1}^n c^{n-m} \frac{z^m}{m}, \tag{3.33}$$

the integral in (3.29) is given by

$$\begin{aligned} & \int \frac{\eta^3 (2\eta_i - \eta)^3 (\eta - \eta_i)}{(1 - \eta)(\eta - \eta_1)} d\eta \\ &= \ln(1 - \eta)^{\alpha_1} + \ln(\eta - \eta_1)^{-\alpha_2} + \sum_{m=1}^6 (1 - 2\eta_1^{6-m}) \frac{\eta^m}{m} \\ & \quad + \sum_{m=1}^5 (c_1 + c_3 \eta_1^{5-m}) \frac{\eta^m}{m} + \sum_{m=1}^3 (c_2 + c_4 \eta_1^{3-m}) \frac{\eta^m}{m}, \end{aligned} \tag{3.34}$$

where

$$\alpha_1 \equiv \frac{(\eta_i - 1)(2\eta_i - 1)^3}{\eta_1 - 1}, \quad \alpha_2 \equiv \frac{\eta_1^3 (\eta_i - \eta_1)(2\eta_i - \eta_1)^3}{\eta_1 - 1}. \tag{3.35}$$

Inserting this result into expression (3.29) gives a closed-form expression for $x(v)$.

Figures 4 and 5 show the velocity and temperature for the solution described in this section with $M_0 = 10$ and $M_0 = 1.2$, respectively. A value of $Pr = 10^{-4}$ was used to generate the numerical results in these figures. Incidentally, this is an analytical

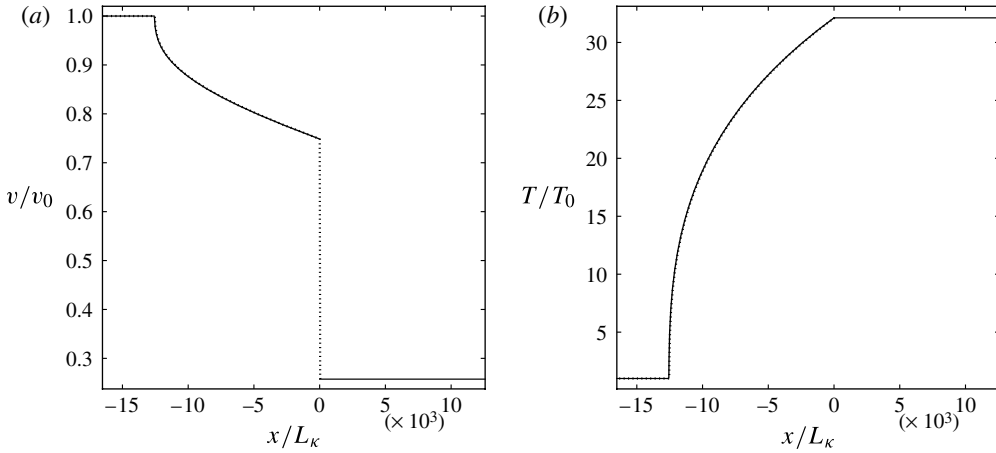


FIGURE 4. (a) Velocity and (b) temperature for the $Pr \rightarrow 0$ solution with $M_0 = 10$ and $\kappa \sim T^3$. No distinction is visible between the analytical (solid) and numerical (dotted) results. The isothermal shock is located at $x = 0$. The material in front of the shock is heated to temperatures above the ambient temperature because heat is being conducted via radiation from the hotter post-shock region to the colder pre-shock region.

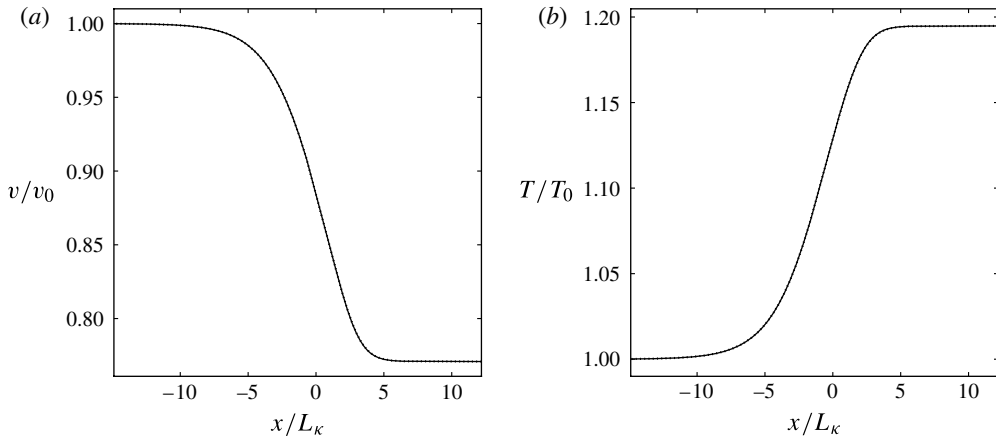


FIGURE 5. (a) Velocity and (b) temperature for the $Pr \rightarrow 0$ solution with $M_0 = 1.2$ and $\kappa \sim T^3$. No distinction is visible between the analytical (solid) and numerical (dotted) results.

solution for radiative shocks (Zel'dovich & Raizer 2002; Lowrie & Rauenzahn 2007) in the limit of constant opacity and a radiation energy much lower than the gas internal energy. In the notation of Lowrie & Rauenzahn (2007), this solution applies to the $\mathcal{P}_0 \rightarrow 0$ limit, where \mathcal{P}_0 is approximately the ratio of radiation to gas pressures. Compare figures 4 and 5 with figures 3 and 5 of Lowrie & Rauenzahn (2007).

3.5. General viscosity and conductivity

Equations (3.5), (3.13) and (3.20) can be solved numerically for any $\kappa(\rho, T)$ and $\mu(\rho, T)$, whether analytical or tabular, using (2.5) and either (3.7), (3.15) or (3.23) to express ρ and T as functions of v . The problem can thus be reduced to quadrature

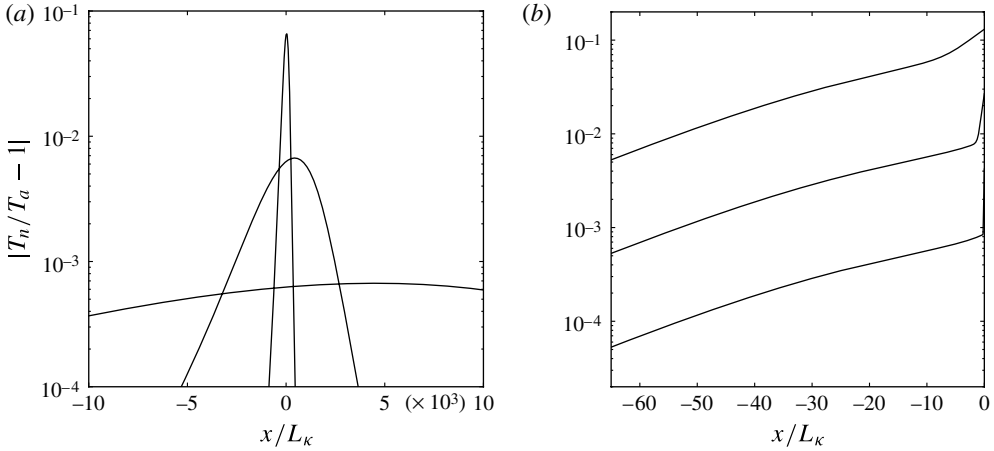


FIGURE 6. Temperature errors in the (a) large- Pr and (b) small- Pr solutions with constant viscosity and conductivity (figures 2 and 3), for (from top to bottom) $Pr = 10, 100, 1000$ in panel (a) and $Pr = 0.1, 0.01, 0.001$ in panel (b).

under quite general conditions. For a viscosity and thermal conductivity that vary as a power law in density and temperature,

$$\mu = \mu_0 \left(\frac{\rho}{\rho_0}\right)^a \left(\frac{T}{T_0}\right)^b, \quad \kappa = \kappa_0 \left(\frac{\rho}{\rho_0}\right)^a \left(\frac{T}{T_0}\right)^b, \quad (3.36)$$

expressions (3.5), (3.13) and (3.20) become

$$x(Pr = 3/4) = \frac{2L_\kappa}{\gamma + 1} \left[\frac{(\gamma - 1)M_0^2}{2} \right]^b \int \frac{\eta^{1-a}(R_\infty \eta_1 - \eta^2)^b}{(\eta - 1)(\eta - \eta_1)} d\eta, \quad (3.37)$$

$$x(Pr = \infty) = \frac{2L_\mu}{\gamma + 1} \left[\frac{\gamma(\gamma - 1)M_0^2}{2} \right]^b \int \frac{\eta^{1-a}(\eta^2 - 4\eta_i \eta + R_\infty \eta_1)^b}{(\eta - 1)(\eta - \eta_1)} d\eta, \quad (3.38)$$

$$x(Pr = 0) = \frac{4L_\kappa}{\gamma + 1} (\gamma M_0^2)^b \int \frac{(\eta - \eta_i)\eta^{b-a}(2\eta_i - \eta)^b}{(\eta - 1)(\eta - \eta_1)} d\eta. \quad (3.39)$$

Analytical expressions in terms of elementary functions can be obtained for particular values of a and b (the solution in §3.4 is an example with $a = 0, b = 3$), although they can be quite lengthy. The expression for a Spitzer conductivity ($a = 0, b = 5/2$), for example, is even longer than expression (3.34) and is not reproduced here (Spitzer 1956). The best approach for general a and b is to perform the quadratures in (3.37)–(3.39) numerically. Notice that μ and κ have been assumed to have the same temperature and density dependence so that Pr is constant, for simplicity; this assumption is not necessary and is easily relaxed.

4. Discussion

Exact solutions to the equations of fluid dynamics have been derived in the $Pr \rightarrow \infty$ and $Pr \rightarrow 0$ limits, analogous to the $Pr \rightarrow 3/4$ solution derived by Becker (1922). As shown in figure 6, the solutions are accurate to within $O(Pr^{-1})$ for $Pr \rightarrow \infty$ and $O(Pr)$ for $Pr \rightarrow 0$. The derived solutions are given in their most general form by

expressions (3.5), (3.13) and (3.20), along with specific forms for a constant viscosity and conductivity (equations (3.6), (3.14) and (3.21)) and for a power-law temperature and density dependence (equations (3.37)–(3.39)). The applicability of these solutions to fluids in general is limited by the use of an ideal gas equation of state; the small- Pr solution is applicable to ideal gas mixtures and single-component ideal gases in which temperatures are high enough for radiation heat conduction to be important. Although plasmas behave as small- Pr ideal gases, the greater mobility of the electrons relative to the ions results in separate electron and ion temperatures, a physical effect not included in this analysis (Spitzer 1956; Zel'dovich & Raizer 2002). The large- Pr solution appears to be of mostly academic interest unless it can be extended to analytical equations of state appropriate for liquids and solids (Ohtani 2009; Mozaffari & Eslami 2013); it remains useful, however, for code verification.

A small- Pr solution with a T^3 dependence has also been derived (§ 3.4) that is equivalent to the semi-analytical radiative shock solutions of Lowrie & Rauenzahn (2007) in the limit of low radiation energy density and constant opacity. Expressions (3.29) and (3.39) provide a good estimate of the width of radiative shocks, the former for the constant-opacity case, and the latter for a power-law opacity. In the case of a power-law opacity, simply make the substitution $a = -a' - 1$ and $b = 3 - b'$, where a' and b' are the density and temperature power laws, respectively, for the opacity expressed in units of area per mass (Bell & Lin 1994). Notice that the width of a radiative shock can be quite sensitive to the shock Mach number ($x \sim M_0^6$ in the case of a constant opacity), although the applicability of the Navier–Stokes equations to large-Mach-number shocks is questionable (Mott-Smith 1951; Jukes 1957).

In addition to providing physical insight, the analytical solutions derived here are useful for quickly evaluating shock profiles over a wide range of parameter space. It is possible to comprehend at a glance the scaling of the solutions with various parameters without resorting to a comprehensive parameter survey via numerical integration. The solutions are also nonlinear, with the only assumptions behind their derivation being a steady state, one planar dimension, and an ideal gas equation of state. In particular, no terms in the evolution equations have been approximated, which makes these solutions an excellent verification test for numerical algorithms.

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