RELATIVE DEFINABILITY OF n-GENERICS

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Abstract. A set $G \subseteq \omega$ is n-generic for a positive integer n if and only if every Σ_n^0 formula of G is decided by a finite initial segment of G in the sense of Cohen forcing. It is shown here that every n-generic set G is properly Σ_n^0 in some G-recursive X. As a corollary, we also prove that for every $n > 1$ and every n-generic set G there exists a G-recursive X which is generalized low$_n$ but not generalized low$_{n-1}$. Thus we confirm two conjectures of Jockusch [4].

§1. Introduction. In recursion theory, Cohen generics have been intensively studied and are usually just called *generics*. In [4] Jockusch presented the first systematic study of the Turing degrees of generics and introduced the notion of *n-genericity*. A set G is n-generic for a positive n if and only if every Σ_n^0 or Π_n^0 fact of G is forced by a finite initial segment of G. For recursion theorists, it is natural to investigate which properties of generics hold for n-generics and which techniques applied to generics could also be applied to n-generics. On the other hand, many recursion theoretic techniques are more easily applied to n-generics. So n-generics, in particular 1-generics, are even more intensively studied and play important roles in recursion theory.

An interesting recursion theoretic property of 1-generics is their relative recursive enumerability. Recall that X is *recursively enumerable in and above* (r.e.a. for short) Y if X is Y-r.e. and $Y <_T X$. In terms of definability, if X is r.e.a. Y then X is properly Σ_1^0 in Y. An *r.e.a.* set is one which is r.e. some other set, and an *r.e.a. degree* is a Turing degree containing an r.e.a. set. In the theory of Turing degrees, the r.e.a. degrees have been proven important. Jockusch [4] proved that every 1-generic set G is r.e.a. and thus the 1-generic degrees form a subclass of the r.e.a. degrees. In the same article, it was also shown that every 2-generic G is properly Σ_2^0 in some G-recursive X. The first result connects 1-generic degrees with recursively enumerable degrees. For example, combining the relative recursive enumerability of 1-generics and a theorem of r.e. degrees (Yates [8]) Jockusch concluded that the Turing degrees below a 1-generic degree are not dense ([4, Corollary 5.3]). Moreover, as the r.e.a. degrees form a large and well-known class (e.g., see [2]), the first result seems more well-known. However, the proofs of the two results are similar. Both proofs were done by cleverly defining some functionals which produce the desired X for sufficient

generic G. But these clever functionals do not seem to admit easy generalizations to a large n. So Jockusch posted a general conjecture that every n-generic G is properly Σ_n^0 in some G-recursive X for every positive integer n ([4, Conjecture 5.13]).

The main purpose of this article is to confirm the above conjecture of Jockusch. We provide a proof in Section 2. Then we present some consequences of our main theorem in Section 3, including a confirmation of [4, Conjecture 5.14]. We also post some open questions in Section 3.

Our terminology is rather standard. A set $X \subseteq \omega$ is usually identified with its characteristic function in 2^ω. We write $\sigma \subseteq \tau$ if σ is an initial segment of $\tau \in 2^{<\omega}$ and $\sigma \subset \tau$ if $\sigma \subseteq \tau$ and $\sigma \neq \tau$. The concatenation of two finite sequences s and t is simply denoted by st. If $x_0, \ldots, x_{n-1} \in \omega$ then $\langle x_0 \ldots x_{n-1} \rangle$ denotes the finite sequence of length n with x_i being its i-th element for each i. A set $D \subseteq 2^{<\omega}$ is *dense* if every $\sigma \in 2^{<\omega}$ is extended by some $\tau \in D$. For a fixed $\rho \in 2^{<\omega}$, a set $D \subseteq 2^{<\omega}$ is *dense below ρ* if every extension of ρ is extended by some element of D. For a fixed $G \subseteq \omega$, a set $D \subseteq 2^{<\omega}$ is *dense along G* if every finite initial segment of G is extended by some element of D. A (Turing) functional is an oracle Turing machine. We write $\Gamma(\sigma; x) \downarrow$, if Γ is a functional, $x < |\sigma|$ and the computation halts in $|\sigma|$ steps; $\Gamma(\sigma; x) \uparrow$ denotes $\neg(\Gamma(\sigma; x) \downarrow)$. We also use $\Gamma(\sigma)$ with the parameter x omitted to denote the partial function sending x to i iff $\Gamma(\sigma; x) \downarrow = i$. Throughout the article, forcing always means Cohen forcing.

Besides [4], we also recommend Kumabe [6] for background on n-generic degrees.

§2. The main theorem.
In this section we prove the following theorem.

THEOREM 2.1. *For each $n > 0$, there exists a partial recursive functional Γ such that if G is n-generic then $\Gamma(G)$ is total and G is properly Σ_n^0 in $\Gamma(G)$.*

To prove Theorem 2.1, we recursively construct two functionals Γ and Δ such that the following hold for every n-generic G:

(A1) $X = \Gamma(G)$ is total.
(A2) $\omega - G$ is not Σ_n^X.
(A3) $\Delta_k(X; x_0, \ldots, x_{n-k})$ is total for each $k < n-1$, where

$$\Delta_0(X; x_0, \ldots, x_{n-1}) = \Delta(X; x_0, \ldots, x_{n-1}),$$
$$\Delta_{k+1}(X; x_0, \ldots, x_{n-k-2}) = \lim_{x_{n-k-1}} \Delta_k(X; x_0, \ldots, x_{n-k-2}, x_{n-k-1}).$$

(A4) For every x, if $x \in G$ then $\Delta_{n-1}(X; x) \downarrow = 1$, otherwise either $\Delta_{n-1}(X; x) \uparrow$ or $\Delta_{n-1}(X; x) \downarrow \neq 1$.

By (A3–A4), G is Σ_n^X.

In the construction, we refine (A1–A4) to countably many requirements and recursively assign to each requirement countably many stages so that every stage is assigned to exactly one requirement. A stage assigned to a requirement R is called an *R-stage*. At each stage we construct finite approximations of Γ and Δ. If stage s is an R-stage then *the R-module* performs a finite set of actions for the sake of meeting requirement R.

First, we sketch the basic strategies for (A1–A4). For convenience, we write $\vec{x}$ for a tuple $(x_0, \ldots, x_{k-1})$ where k is the length of $\vec{x}$.

2.1. Meeting (A1). We view Γ as a partial recursive function $\Gamma : 2^{<\omega} \to 2^{<\omega}$ which is continuous, i.e., $\Gamma(\sigma) \subseteq \Gamma(\tau)$ whenever $\sigma \subseteq \tau$. To meet (A1), we require that

$$\{\sigma : |\Gamma(\sigma)| > l\} \text{ is dense in } 2^{<\omega} \text{ for each } l. \tag{2.1}$$

So we have the following requirements:

$$P(\sigma, l) : (\exists \tau \supset \sigma)|\Gamma(\tau)| > l.$$

A $P(\sigma, l)$-module picks some $\tau \supset \sigma$ and defines $\Gamma(\tau)$ as some ξ of length at least $l + 1$. The set in (2.1) will be not only dense but also recursive. Hence $\Gamma(G)$ is total for every 1-generic G.

2.2. Meeting (A2). Suppose that $X = \Gamma(G)$ is total. We replace (A2) with the following requirement for each Σ_n^0 formula θ:

$$(\forall x \notin G)(\exists \sigma \supset G \upharpoonright x)\sigma \Vdash \theta(\Gamma(G), x) \to$$
$$\{\sigma : (\exists x)\sigma(x) = 1 \wedge \sigma \Vdash \theta(\Gamma(G), x)\} \text{ is dense along } G. \tag{2.2}$$

As $n > 0$ and Γ satisfies (2.1), $\sigma \Vdash \theta(\Gamma(G), x)$ is a Σ_n^0 predicate of G, σ and x. Assume that $G = \{x : \neg \theta(\Gamma(G), x)\}$ for a Σ_n^0 formula θ. By the n-genericity of G, the premise of (2.2) holds. Applying (2.2) and the n-genericity of G again, there exist $\sigma \subset G$ and x such that $\sigma(x) = 1$ and $\sigma \Vdash \theta(\Gamma(G), x)$ and thus $\theta(\Gamma(G), x)$ holds. So we have a desired contradiction. This proves that (2.2) implies (A2).

The strategy to meet (2.2) is as follows: For each ρ, we construct a recursive set $D \subset 2^{<\omega}$ dense below $\rho\langle 0 \rangle$, and for each $\sigma \in D$ assign some $\tau \supseteq \rho\langle 1 \rangle$ and guarantee that if σ forces a Π_{n-1}^0 formula φ of $\Gamma(G)$ then τ forces φ too. We call this τ the *deputy* of σ. Assume that every deputy performs its task and the premise of (2.2) holds. We write each Σ_n^0 formula θ as $(\exists y)\varphi$ where φ is Π_{n-1}^0. Fix $x \notin G$ and let $\rho = G \upharpoonright x$, and assume that $\sigma \Vdash \theta(\Gamma(G), x)$ for some $\sigma \supset \rho$. Then either $\sigma(x) = 1$ or some $\sigma' \supseteq \sigma$ forces $\varphi(\Gamma(G), x, y)$ and has a deputy $\tau \supset \rho\langle 1 \rangle$. Hence $\tau(x) = 1$ and $\tau \Vdash \varphi(\Gamma(G), x, y)$ as well. So the above strategy guarantees (2.2).

Suppose that $\rho \subset \rho\langle 0 \rangle \subset \sigma$ and τ is the deputy of σ. In order to describe how τ can perform its duty as the deputy of σ, let us begin with small n's.

- If $n = 1$ and φ is a Π_0^0 formula then there is a recursive subset P of $2^{<\omega}$ such that $\sigma \Vdash \varphi(\Gamma(G))$ if and only if $\Gamma(\sigma)$ extends an element of P. So we define $\Gamma(\tau) \supset \Gamma(\sigma)$. Thus every element of P extended by $\Gamma(\sigma)$ is also extended by $\Gamma(\tau)$ and if $\sigma \Vdash \varphi(\Gamma(G))$ then $\tau \Vdash \varphi(\Gamma(G))$ as well. Note that this is the key ingredient of Lemma 5.2 in Jockusch [4].

- Suppose that $n = 2$. We need two layers of deputies, one to handle Π_0^0 formulae and another to handle Π_1^0 formulae. Assume that τ above is supposed to handle Π_1^0 formulae, i.e., if $\sigma \Vdash \varphi(\Gamma(G))$ for some Π_1^0 formula φ then τ should also force φ. We call τ the 1-deputy of σ. We pick 0-deputies $\mu \supset \sigma$ for v densely below τ and make these deputies act as in the above paragraph. To ensure that the above paragraph could be applied to 0-deputies, we certainly need that $\Gamma(\tau) \supseteq \Gamma(\sigma)$. With this strategy, if $\varphi = (\forall y)\psi$ where ψ is Σ_0^0 and $\tau \not\Vdash \varphi(\Gamma(G))$, then some $v \supset \tau$ forces some $\neg\psi(\Gamma(G), k)$ and so does the 0-deputy $\mu \supset \sigma$ of some extension of v. Hence $\sigma \not\Vdash \varphi(\Gamma(G))$ either. The reader may have noticed the similarity between the above argument and that following the proof of [4, Lemma 5.10].

Note that each σ may extend more than one $\rho\langle 0\rangle$. So to be precise, the corresponding $(n-1)$-deputy should be called something like the $(\rho, n-1)$-deputy of σ. For simplicity, we only define deputies for sequences of fresh lengths at each stage. We also require that each σ has at most one deputy and thus it cannot have deputies at different layers. So the deputy of σ is unambiguous. But we may still use the terms k-deputy and (ρ, k)-deputy occasionally. For bookkeeping purposes, we simultaneously construct a partial recursive injection d such that if $d(\sigma)$ is defined then it is the deputy of σ.

Based on these simple examples, for a fixed positive integer n and every ρ and $x = |\rho|$, we define a $(\rho, n-1)$-deputy extending $\rho\langle 1\rangle$ for each σ in a recursive set dense below $\rho\langle 0\rangle$. If $0 < k < n$ and τ is defined as the k-deputy of σ then for ν in a recursive set dense below τ we define its $(\tau, k-1)$-deputy $\mu = d(\nu)$ to be some extension of σ. As in the above examples, if $\mu = d(\nu)$ then we always ensure that $\Gamma(\mu) \supset \Gamma(\nu)$.

We present the above strategies in a more formal fashion as follows. First we design a hierarchy of requirements:

$$D(0, \sigma, \tau) : \Gamma(\sigma) \subseteq \Gamma(\tau),$$

$$D(k, \sigma, \tau) : \{\nu : (\exists\mu \supset \sigma)(D(k-1, \nu, \mu) \text{ is satisfied})\} \text{ is dense below } \tau$$

for $0 < k < n$, and

$$D(\rho) : \{\sigma : (\exists\tau \supset \rho\langle 1\rangle)(D(n-1, \sigma, \tau) \text{ is satisfied})\} \text{ is dense below } \rho\langle 0\rangle.$$

A $D(k, \sigma, \tau)$ applies only if τ is defined as the k-deputy of σ and requires that τ fulfills its job as a deputy. The $D(\rho)$ requires that $(n-1)$-deputies are defined for elements in a set dense below $\rho\langle 0\rangle$.

A $D(\rho)$-module acts as below:

1. Pick the least $\mu \supset \rho\langle 0\rangle$ such that no $\sigma \supseteq \mu$ has a $(\rho, n-1)$-deputy defined.
2. Pick $\sigma \supset \mu$ with fresh length and define $d(\sigma)$, the $(n-1)$-deputy of σ, as some $\tau \supset \rho\langle 1\rangle$ also with fresh length.
3. Extend the approximation of Γ so that $\Gamma(\sigma) \subset \Gamma(\tau)$.

For $D(k, \sigma, \tau)$ with $0 < k < n$, if at a $D(k, \sigma, \tau)$-stage τ is already defined as the k-deputy of σ then the $D(k, \sigma, \tau)$-module acts like a $D(\rho)$-module:

1. Pick the least $\bar{\nu} \supset \tau$ such that no $\nu \supseteq \bar{\nu}$ has its $(\tau, k-1)$-deputy defined.
2. Pick $\nu \supset \bar{\nu}$ with fresh length and define $d(\nu)$, the $(k-1)$-deputy of ν, as some $\mu \supset \sigma$ with fresh length.
3. Extend the approximation of Γ so that $\Gamma(\mu) \supset \Gamma(\nu)$.

A $D(0, \sigma, \tau)$-module simply does nothing, because if $\tau = d(\sigma)$ then $D(0, \sigma, \tau)$ is automatically satisfied when $d(\sigma)$ is defined by a $D(\rho)$- or $D(1, -)$-module.

Figure 1 shows a partial sample of the construction of deputies for $n = 2$, where τ_i's are the 1-deputies of σ_i's and μ_i's the 0-deputies of ν_i's. It might be worth noticing that in the sample given in Figure 1 the 0-deputies of ν_i's extending τ_0 need not form a set dense below σ_0 though the ν_i's having deputies are required to be dense below τ_0. In general, if τ is the k-deputy of σ and $k > 0$, then the extensions of τ having $(k-1)$-deputies are necessarily dense below τ, but their deputies are not necessarily dense below σ.
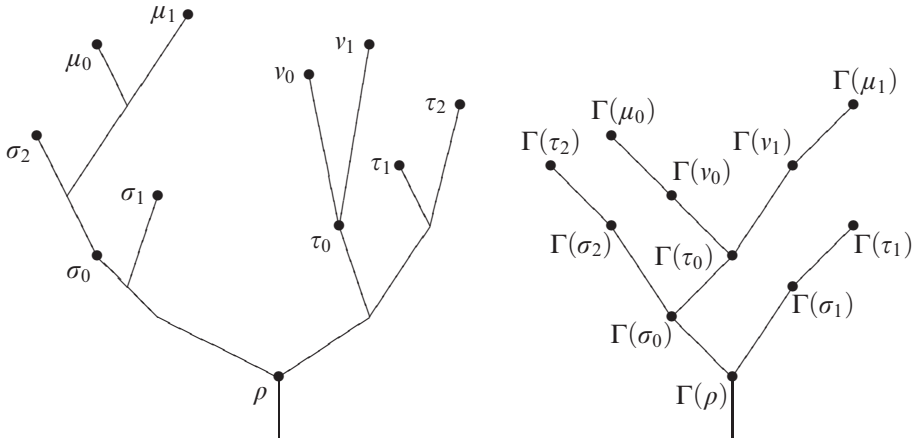
FIGURE 1. The construction of deputies for $n = 2$.

LEMMA 2.2. *If $D(n - k, \sigma, \tau)$ is satisfied and $\sigma \Vdash \varphi(\Gamma(G))$ for some Π^0_{n-k} formula φ then $\tau \Vdash \varphi(\Gamma(G))$.*

PROOF. The case for $n - k = 0$ holds as in the above examples.

Suppose that $n - k > 0$, $D(n - k, \sigma, \tau)$ is satisfied and $\tau \nVdash \varphi(\Gamma(G))$ for some Π^0_{n-k} formula φ. Write φ as $(\forall y)\psi$ where ψ is Σ^0_{n-k-1}. Then there exist j and $\bar{v} \supset \tau$ such that $\bar{v} \Vdash \neg\psi(\Gamma(G), j)$. Pick $v \supseteq \bar{v}$ and $\mu \supset \sigma$ such that $D(n - k - 1, v, \mu)$ is satisfied. By the induction hypothesis, $\mu \Vdash \neg\psi(\Gamma(G), j)$ too. So $\sigma \nVdash \varphi(\Gamma(G))$. ⊣

2.3. Meeting (A3–A4). To meet (A3–A4), we have requirements demanding the convergence of $\Delta_k(\Gamma(G); \vec{x})$ for appropriate $\vec{x}$'s. Each module of these requirements is assigned a specific triple $(k, \sigma, \vec{x})$ and picks a pair (τ, i) such that $\tau \supset \sigma$ and $i < 2$, then it tries to guarantee that $\tau \Vdash \Delta_k(\Gamma(G); \vec{x}) \downarrow= i$. To achieve their goals, these modules must collaborate. Suppose that $k > 0$ and $\tau \Vdash \Delta_k(\Gamma(G); \vec{x}) \downarrow= i$ is to be ensured. The module M (at the k-th layer) will pick a threshold $\hat{x}$ and demand that if a $(k - 1)$-th layer module M' works to guarantee $\xi \Vdash \Delta_{k-1}(\Gamma(G); \vec{x}, x) \downarrow= j$ for ξ extending τ and $x > \hat{x}$ then j must equal i. Such a demand by the k-th layer module M is called a $(k-1)$-th layer *constraint*. For $k = 1$ the constraint means that if $\Delta(\Gamma(\xi); \vec{x}, x)$ is defined for $\xi \supset \tau$ and $x > \hat{x}$ then its value must be i. For $k > 1$, the module M' imposes a $(k - 2)$-th layer constraint similarly. So the collaboration is carried out by imposing cascading constraints and obeying constraints imposed by modules at higher layers.

Below we refine the above strategy. We shall have n-layers of requirements. But by (A2), Δ_{n-1} cannot be total. So the $(n - 1)$-th layer is different. At the $(n - 1)$-th layer, for each σ and x such that $|\sigma| > x$, we have the following requirement

$$Q(n - 1, \sigma, x) : \sigma(x) = 1 \leftrightarrow (\exists \tau \supseteq \sigma)\tau \Vdash \Delta_{n-1}(\Gamma(G); x) \downarrow= 1.$$

The $Q(n-1, \sigma, x)$-module acts according to the first of the following that applies:

- If $n = 1$, $\sigma(x) = 1$ and $\Delta(\Gamma(\tau); x) \uparrow$ for all $\tau \supseteq \sigma$ then pick $\tau \supset \sigma$ and extend Γ and Δ so that $\Delta(\Gamma(\tau); x) = 1$. If $n = 1$ and $\sigma(x) = 0$ then do nothing.

- Assume that $n > 1$ and $\sigma(x) = 1$. Pick $\tau \supset \sigma$ of fresh length and a fresh *threshold* $\hat{x}$ and impose a *Q-constraint* at the $(n-2)$-th layer denoted by $c(n-2, \tau, x, \hat{x}, 1)$, unless there is already some $c(n-2, \zeta, x, -)$ such that ζ and σ are comparable (in this case either σ or an extension of σ is expected to force $\Delta_{n-1}(\Gamma(G); x) \downarrow$). If ρ extends τ then we say that $c(n-2, \tau, -)$ is *active at* ρ, and in this case all $Q(n-2, \rho, -)$-modules will obey the above constraint. As we shall see, obeying $c(n-2, \tau, x, \hat{x}, 1)$ guarantees the desired forcing relation, i.e., $\tau \Vdash \Delta_{n-1}(\Gamma(G); x) \downarrow = 1$ or more precisely $\tau \Vdash (\forall x_1 > \hat{x})\Delta_{n-2}(\Gamma(G); x, x_1) \downarrow = 1$. In the next paragraph we shall see how a Q-module at a lower layer obeys constraints imposed by Q-modules at higher layers.

- Suppose that $n > 1$ and $\sigma(x) = 0$. Pick τ and $\hat{x}$ as above and also a fresh x_1. If $n = 2$ then extend Δ and Γ so that $\Delta(\Gamma(\tau); x, x_1) \downarrow = 0$, and if $n > 2$ then impose a constraint $c(n-3, \tau, \langle xx_1 \rangle, \hat{x}, 0)$, so that $\tau \Vdash \Delta_{n-2}(\Gamma(G); x, x_1) \downarrow = 0$. In this way, the $Q(n-1, \rho, x)$-modules for $\rho \supset \sigma$ together prevent σ forcing $\Delta_{n-1}(\Gamma(G); x) \downarrow = 1$.

If $0 < k < n - 1$, for every σ and every $\vec{x}$ such that $|\vec{x}| = n - k$ and $\max \vec{x} < |\sigma|$, we have the following requirement:

$$Q(k, \sigma, \vec{x}) : (\exists \tau \supset \sigma, i < 2)\tau \Vdash \Delta_k(\Gamma(G); \vec{x}) \downarrow = i.$$

The $Q(k, \sigma, \vec{x})$-module first picks $\tau \supset \sigma$ of fresh length and a fresh threshold $\hat{x}$. Then it acts according to the first of the following three cases that applies:

- If there exists $c(k-1, \xi, \vec{x}, -)$ such that ξ and σ are comparable then it does nothing, since for τ being the longer of σ and ξ we can expect τ to witness the satisfaction of the requirement.

- If some $c = c(k, \xi, \vec{x} \restriction (n-k-1), \hat{y}, i)$ is active at σ (i.e., $\xi \subseteq \sigma$) and $\hat{y} < x_{n-k-1}$, x_{n-k-1} being the last element of $\vec{x}$, then this Q-module obeys the constraint c by imposing a new constraint $c(k-1, \tau, \vec{x}, \hat{x}, i)$. In other words, if a $Q(k+1, -)$-module demands that $\xi \Vdash (\forall x > \hat{y})\Delta_k(\Gamma(G); \vec{x} \restriction (n-k-1), x) \downarrow = i$ then the $Q(k, \sigma, \vec{x})$-module demands that $\tau \Vdash \Delta_k(\Gamma(G); \vec{x}) \downarrow = i$.

- If none of the above cases applies then the $Q(k, \sigma, \vec{x})$-modules imposes a constraint $c(k-1, \tau, \vec{x}, \hat{x}, \sigma(x_0))$.

The $Q(0, \sigma, x)$'s for $n = 1$ are instances of $Q(n-1, -)$'s. Suppose that $n > 1$. For each σ and $\vec{x}$ such that $|\vec{x}| = n$ and $|\sigma| > \max \vec{x}$, we have the following requirement:

$$Q(0, \sigma, \vec{x}) : (\exists \tau \supset \sigma, i < 2)\Delta(\Gamma(\tau); \vec{x}) \downarrow = i.$$

A $Q(0, \sigma, \vec{x})$-module acts only if $\Delta(\Gamma(\tau); \vec{x})$ is undefined for any τ extending σ. If it acts, the $Q(0, \sigma, \vec{x})$-module first picks $\tau \supset \sigma$ of fresh length. If a constraint $c(0, \xi, \vec{x} \restriction n-1, \hat{y}, i)$ is active at σ and $\hat{y} < x_{n-1}$, x_{n-1} being the last element of $\vec{x}$, then this Q-module defines $\Delta(\Gamma(\tau); \vec{x}) \downarrow = i$ so that it obeys the constraint above; otherwise it defines $\Delta(\Gamma(\tau); \vec{x}) \downarrow = \sigma(x_0)$, x_0 being the first element of $\vec{x}$.

If the construction were carried out only by Q-modules, $\Delta_k(\Gamma(G))$ would be total for all $k < n - 1$ and all 1-generic G and the expected forcing relations would be secured. It seems that we would even have the totality of $\Delta_{n-1}(\Gamma(G))$ and

$$G(x_0) = \Delta_{n-1}(\Gamma(G); x_0). \tag{2.3}$$

However, (2.3) contradicts (A2), since the totality of each $k < n - 1$ implies that $\Delta_k(\Gamma(G); x_0, \ldots, x_{n-k-1}) \downarrow = i$ is a Σ^0_{k+1} predicate in $\Gamma(G)$ for each $k < n$. This contradiction will be dissolved in the next subsection when we try to bring the D- and Q-modules together.

2.4. Conflicts. To integrate the modules, we must resolve the following issues.

1. The functionals Γ and Δ are defined by various modules. For the consistency of the two functionals, all modules should follow some rules when defining Γ and Δ.

2. From Section 2.3, we can see that the Q-modules are designed to collaborate with each other, i.e., lower layer modules obey constraints imposed by higher layer ones. However, the construction of deputies by D-modules could introduce some problems. For, if a $D(k, \sigma, \tau)$-module defines a deputy $\mu = d(v)$ for $v \supset \tau$, then $\Gamma(\mu)$ and $\Delta(\Gamma(\mu))$ extend $\Gamma(v)$ and $\Delta(\Gamma(v))$, respectively, and it is not obvious that $\Delta(\Gamma(v))$ is submitted to Q-constraints active at μ.

3. The problem discussed at the end of the previous Section 2.3.

For (1), we setup two rules to be followed by every module at every stage:

(C1) $\Gamma(\sigma) = \Gamma(\sigma\langle 1^k \rangle)$ for all σ and k, meaning that $\Gamma(\sigma)$ and $\Gamma(\sigma\langle 1^k \rangle)$ are identical as partial functions.

(C2) $\Delta(\xi) = \Delta(\xi\langle 1^k \rangle)$ for all ξ and k as identical partial functions again.

If these rules are followed then we can build valid functionals in some appropriate way as shown in Lemmas 2.4 and 2.8. To follow (C1–C2), we explicitly define $\Gamma(\sigma)$ and $\Delta(\sigma)$ only for σ ending with 0.

To examine issue (2) concretely, let us consider the guiding example below (illustrated by Figure 2): $n = 3$, $\tau \supset \rho\langle 1 \rangle$ is the 2-deputy of $\sigma \supset \rho\langle 0 \rangle$,
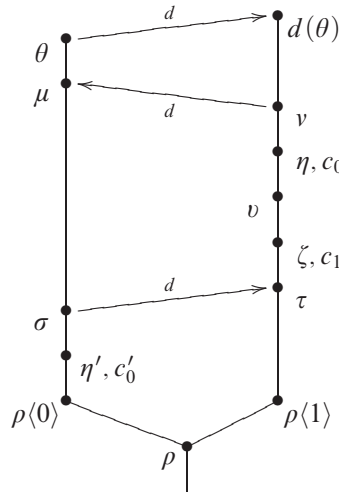

FIGURE 2. The guiding example.

and some v extending τ has a 1-deputy μ extending σ. Let $x = |\rho|$. Suppose that the $Q(2, \rho\langle 0\rangle, x)$-module imposes a constraint $c_0' = c(0, \eta', \langle xx_1\rangle, \hat{x}_1', 0)$ as $(\rho\langle 0\rangle)(x) = 0$. Since $\tau(x) = 1$, naturally the $Q(2, \tau, x)$-module may impose a constraint $c_1 = c(1, \zeta, x, \hat{x}, 1)$. Moreover, suppose that later a $Q(1, v, \langle xx_1\rangle)$-module with $v \supset \zeta$ obeys c_1 and imposes another $c_0 = c(0, \eta, \langle xx_1\rangle, \hat{x}_1, 1)$. It could be that c_0' is active at σ and c_0 active at v, since 2-deputies are defined for densely many extensions of $\rho\langle 0\rangle$ and 1-deputies for densely many extensions of τ. At some $\theta \supset \mu$, suppose that for $\vec{x} = \langle xx_1x_2\rangle$ with $x_2 > \max\{\hat{x}_1, \hat{x}_1'\}$, c_0' is obeyed and $\Delta(\Gamma(\theta); \vec{x}) = 0$ is defined. Later the $D(1, v, \mu)$-module defines a 0-deputy $d(\theta) \supset v$. Since $\Gamma(d(\theta)) \supset \Gamma(\theta)$, $\Delta(\Gamma(d(\theta)); \vec{x}) = 0$ too. So c_0 and the forcing relation $\eta \Vdash (\forall x_2 > \hat{x}_1)\Delta(\Gamma(G); x, x_1, x_2) = 1$ are broken though c_0 is active at $d(\theta)$.

But if we examine the motivation of deputies then we can see a natural solution. Recall that if τ is the k-deputy of σ and σ forces a Π_k^0 predicate $\varphi(\Gamma(G))$ of $\Gamma(G)$ then τ should force $\varphi(\Gamma(G))$ as well. In the above example, the Q-constraint c_0' demands that η' forces the Π_2^0 predicate

$$(\forall x_2 > \hat{x}_1')(\Delta(\Gamma(G); x, x_1, x_2) \downarrow = 0). \tag{2.4}$$

Since the Q-modules make $\Delta(\Gamma(G))$ a total 2-valued function, (2.4) is equivalent to the following Π_1^0 predicate of $\Gamma(G)$:

$$(\forall x_2 > \hat{x}_1')\neg(\Delta(\Gamma(G); x, x_1, x_2) \downarrow = 1). \tag{2.5}$$

As c_0' is active at σ and τ is the 2-deputy of σ, τ should also be demanded to force (2.5). When the $D(\rho)$-module defines $\tau = d(\sigma)$, it should impose a Q-constraint $c(0, \tau, \langle xx_1\rangle, \hat{x}_1', 0)$, which has all its parameters except τ copied from c_0' and thus is called a *replica* of c_0'. This action is called *replication* or *replication of constraints*. For the consistency of this action, we need two new rules:

(C3) If a D-module defines a deputy $\tau = d(\sigma)$ then $\tau = \rho\langle 1^k 0\rangle$ with a fresh k and an appropriate ρ.

(C4) If a Q-constraint is active at any $\rho\langle 1^k\rangle$ then it is active at ρ.

With (C3–C4), when the $D(\rho)$-module in the guiding example defines $d(\sigma)$, it can define the deputy to be some $\tau = \rho\langle 1^l 0\rangle$ such that every constraint active at $\rho\langle 1^l\rangle$ is active at ρ. Hence if the constraints active at σ are consistent then after replication the constraints active at τ are consistent as well. In general, if a D-module defines a k-deputy $\tau = d(\sigma)$ and a Q-constraint $c = c(j, \zeta, -)$ is active at σ with $j < k$, then this D-module should impose a new Q-constraint like $c(j, \tau, -)$ which is called a *replica* of $c(j, \zeta, -)$ and has its parameters except τ identical with those of c. Actually, it suffices to replicate $c(j, \zeta, -)$ as above and satisfying an additional condition that $\rho \subset \zeta \subseteq \sigma$ if τ is defined as the (ρ, k)-deputy of σ.

Assume that the replication strategy is implemented and (C3–C4) are followed. Let us examine the guiding example again. When the $D(\rho)$-module acts, it can pick τ so that no constraints like c_1 or c_0' are active at τ and then it replicates constraints active at σ to τ. So if c_0' were already active at σ then a replica of c_0' would be active at τ and thus no constraint like c_0 would ever appear above τ. Hence the definition of $d(\theta)$ later would not break any constraint. Suppose that no constraint like c_0' is active at σ but c_0 is active at v. When the $D(2, \sigma, \tau)$-module acts, it defines $d(v)$ to be some $\mu \supset \sigma$ and replicates c_0 to $c_0'' = c(0, \mu, \langle xx_1\rangle, \hat{x}_1, 1)$. Hence if later $\Delta(\Gamma(\theta); \vec{x})$

is defined it must equal 1. Thus c_0 will not be broken by the definition of $d(\theta)$. Thus we solve the conflict in the guiding example.

Notice that as in Section 2.3, no $c(n-2, \sigma, x, \hat{x}, 0)$ would ever be imposed by Q-modules. So the replication strategy will never produce any $c(n-2, \tau, x, \hat{x}, 0)$. Furthermore, it is necessary that a Q-constraint $c(j, -)$ with $k \leq j$ active at v will not be replicated to the k-deputy μ of v. As an illustration, in the example above, if c_1 were replicated to μ when μ is defined as the 1-deputy of v then we would have $\mu \Vdash \Delta_2(\Gamma(G); x) = 1$ while $\mu(x) = 0$.

Replication of constraints also solves the issue (3) above. Let us go back to the guiding example and Figure 2. Recall that $x = |\rho|$. By the $Q(2, -)$-modules, the following set is dense below $\rho\langle 0 \rangle$:

$$\{\xi : \text{there exists some } c(0, \xi, \langle xy_1 \rangle, \hat{y}_1, 0)\}.$$

So we may assume that some $c(0, \xi, \langle xy_1 \rangle, \hat{y}_1, 0)$ above is active at σ, meaning that $\sigma \Vdash \Delta_1(\Gamma(G); x, y_1) = 0$. Also recall that the $Q(2, \tau, x)$-module may impose a constraint $c_1 = c(1, \zeta, x, \hat{x}, 1)$ as $\tau(x) = 1$. By the $Q(1, -)$-modules, the following set is dense below ζ:

$$\{\eta : \text{there exists some } c(0, \eta, \langle xx_1 \rangle, \hat{x}_1, 1)\}.$$

As $(\tau, 1)$-deputies are defined for densely many extensions of ζ, some $c(0, \eta, -)$ as above will have a replica active above σ, implying that some extension λ of σ will force $\Delta_1(\Gamma(G); x, x_1) = 1$ for some x_1. Hence, for a generic G extending λ, we have $\Delta_1(\Gamma(G); x, y_1) = 0$ and $\Delta_1(\Gamma(G); x, x_1) = 1$. Due to the construction of deputies and replication of constraints, there are infinitely many such oscillations along G, causing $\Delta_2(\Gamma(G); x)$ undefined. On the other hand, no $\pi \supseteq \rho\langle 1 \rangle$ could have a 2-deputy above $\rho\langle 0 \rangle$. So such oscillations cannot happen infinitely often to any generic $H \supseteq \rho\langle 1 \rangle$ and $\Delta_2(\Gamma(H); x)$ would equal 1.

Moreover, we must guarantee that

(C5) $\tau(y) = 1$ for all $y < |\sigma|$ with $\sigma(y) = 1$ if τ is the $(n-1)$-deputy of σ.

For, if $\sigma(y) = 1$ then there could be some $c(n-2, \zeta, y, 1)$ active at σ which demands σ forcing $\Delta_{n-1}(\Gamma(G); y) \downarrow = 1$ and would be replicated when $\tau = d(\sigma)$ is defined. Certainly we do not want to have any $c(n-2, \eta, y, 1)$ active at τ if $\tau(y) = 0$ as we cannot allow τ forcing $\Delta_{n-1}(\Gamma(G); y) \downarrow = 1 \neq \tau(y)$. But the above condition can be easily met by choosing $\tau = \rho\langle 1^p 0 \rangle$ with p fresh when a $D(\rho)$-module defines a $(n-1)$-deputy.

2.5. The construction.

Here we formally present how the construction proceeds by stages. To help the reader recall the motivations, some explanations are given in brackets. As mentioned, we recursively assign countably many stages to each requirement so that every stage is assigned to exactly one requirement. At an R-stage s, the R-module acts as below.

(1) $P(\sigma, m)$-module: If there does not exist any $\tau \supseteq \sigma$ with $|\Gamma(\tau)| > m$ then pick a fresh number l and let $\tau = \sigma\langle 1^l 0 \rangle$ and $\Gamma(\tau) = \Gamma(\sigma)\langle 1^l 0 \rangle$.

(2) $D(0, \sigma, \tau)$-modules simply do nothing.

(3) $D(k, \sigma, \tau)$-module where $k > 0$: If τ is not yet the k-deputy of σ then proceed to stage $s + 1$. Suppose that τ is the k-deputy of σ. Pick the least $\bar{v} \supseteq \tau$ such that no $v \supseteq \bar{v}$ has its $(\tau, k-1)$-deputy defined. Pick a fresh number l and let $v = \bar{v}\langle 1^l 0 \rangle$.

Let the $(\tau, k-1)$-deputy $d(v)$ of v be $\sigma\langle 1^{|v|}0\rangle$ and let $\Gamma(d(v)) = \Gamma(v)\langle 1^{|v|}0\rangle$. Moreover, for every Q-constraint $c(j, \zeta, \vec{x}, \hat{x}, i)$ with $j < k-1$ and $\tau \subset \zeta \subset v$, impose a new Q-constraint $c(j, d(v), \vec{x}, \hat{x}, i)$. [The first half defines $(k-1)$-deputies for v in a set dense below τ as extensions of σ. The second half replicates Q-constraints active at v to $d(v)$, except those active at τ, so that if v is expected to force $\Delta_{k-1}(\Gamma(G); \vec{x}) \downarrow= i$ then so is $d(v)$.]

(4) $D(\rho)$-module: Pick the least $\bar{\sigma} \supseteq \rho\langle 0\rangle$ such that *no* $\sigma \supseteq \bar{\sigma}$ has its $(\rho, n-1)$-deputy defined. Pick a fresh number l and let $\sigma = \bar{\sigma}\langle 1^l 0\rangle$. Let the $(\rho, n-1)$-deputy $d(\sigma)$ of σ be $\rho\langle 1^{|\sigma|}0\rangle$ and let $\Gamma(d(\sigma)) = \Gamma(\sigma)\langle 1^{|\sigma|}0\rangle$. Moreover, for every Q-constraint $c(j, \zeta, \vec{x}, \hat{x}, i)$ such that $\rho \subset \zeta \subset \sigma$, impose a new Q-constraint $c(j, d(\sigma), \vec{x}, \hat{x}, i)$. [It is similar to $D(k, -)$-modules.]

(5) $Q(0, \sigma, \vec{x})$-module for $n > 1$: If $\Delta(\Gamma(\tau); \vec{x})$ is defined for some τ comparable with σ then proceed to stage $s + 1$. Suppose that $\Delta(\Gamma(\tau); \vec{x})$ is undefined for all τ comparable with σ. Pick a fresh number l and let $\xi = \Gamma(\sigma\langle 1^l 0\rangle) = \Gamma(\sigma)\langle 1^l 0\rangle$. If some Q-constraint $c(0, \rho, \vec{x} \restriction n-1, \hat{x}, i)$ is active at σ and $x_{n-1} > \hat{x}$, x_{n-1} being the last element of $\vec{x}$, then let $\Delta(\xi; \vec{x}) = i$. Otherwise let $\Delta(\xi; \vec{x}) = \sigma(x_0)$. [If some Q-constraint active at σ demands $\Delta_1(\Gamma(G); \vec{x} \restriction n-1) \downarrow= i$ being forced then the current Q-module obeys this constraint by making $\xi \Vdash \Delta(\Gamma(G); \vec{x}) \downarrow= i$.]

(6) $Q(k, \sigma, \vec{x})$-module where $n - 1 > k > 0$: If some $c(k-1, \rho, \vec{x}, \hat{x}, i)$ is active with ρ and σ comparable then proceed to stage $s + 1$. Suppose that there is *no* Q-constraint as above. Pick a fresh number l and let $\tau = \sigma\langle 1^l 0\rangle$. If some Q-constraint $c(k, \rho, \vec{x} \restriction (n-k-1), \hat{x}, j)$ is active at σ and $x_{n-k-1} > \hat{x}$, x_{n-k-1} being the last element of $\vec{x}$, then let $i = j$, otherwise let $i = \sigma(x_0)$. Impose a new Q-constraint $c(k-1, \tau, \vec{x}, l, i)$. [If σ is expected to force $\Delta_{k+1}(\Gamma(G); \vec{x} \restriction (n-k-1)) \downarrow= j$ then the $Q(k, \sigma, \vec{x})$-module obeys the corresponding constraint by imposing a lower layer constraint so that eventually τ forces $\Delta_k(\Gamma(G); \vec{x}) \downarrow= j$.]

(7) $Q(n-1, \sigma, x)$-module: There are two cases below.

(7a) $n = 1$. If $\sigma(x) = 0$ or $\Delta(\Gamma(\rho); x) \downarrow$ for some ρ comparable with σ then proceed to stage $s + 1$. Suppose that $\sigma(x) = 1$ and $\Delta(\Gamma(\rho); x) \uparrow$ for any ρ comparable with σ. Pick a fresh number l. Define $\xi = \Gamma(\sigma\langle 1^l 0\rangle) = \Gamma(\sigma)\langle 1^l 0\rangle$ and $\Delta(\xi; x) = 1$.

(7b) $n > 1$. First pick a fresh number l. Act according to the first of the followings that applies

- If there already exists a constraint $c(n-2, \zeta, x, -)$ such that ζ and σ are comparable then proceed to stage $s + 1$.
- If $\sigma(x) = 1$ then impose a constraint $c(n-2, \sigma\langle 1^l 0\rangle, x, l, 1)$. [So the collaboration of Q-modules will make $\sigma\langle 1^l 0\rangle$ forcing $(\forall x_1 > l)\Delta_{n-2}(\Gamma(G); x, x_1) \downarrow= 1$.]
- If $\sigma(x) = 0$ and $n = 2$ then let $\xi = \Gamma(\sigma\langle 1^l 0\rangle) = \Gamma(\sigma)\langle 1^l 0\rangle$ and let $\Delta(\xi; x, l) = 0$.
- If $\sigma(x) = 0$ and $n > 2$ then impose a constraint $c(n-3, \sigma\langle 1^l 0\rangle, \langle xl\rangle, l, 0)$. [It is expected that $\sigma\langle 1^l 0\rangle \Vdash \Delta_{n-2}(\Gamma(G); x, l) \downarrow= 0$ and thus σ does not force $\Delta_{n-1}(\Gamma(G); x) \downarrow= 1$.]

2.6. The verification. We verify that the construction produces the desired functionals by proving a series of lemmas.

LEMMA 2.3. $(C1–C5)$ *hold.*

PROOF. It can be proved by an easy induction on stages. ⊣

It is rather easy to see that the P's are satisfied.

LEMMA 2.4. Γ *is well-defined and* $\Gamma(G)$ *is total for every* 1-*generic* G.

PROOF. Suppose that at the beginning of stage s we have

1. Γ is well-defined and
2. $\Gamma(\tau) \supset \Gamma(\sigma)$ whenever $\tau = d(\sigma)$.

Note that Γ is extended at stage s only if s is a P-, $D(k, -)$- $(k > 0)$, $D(\rho)$- or $Q(0, -)$-stage.

If s is a $P(\sigma, m)$-stage then (1) holds at the end of stage s, by the induction hypothesis and (C1). Also (2) holds at the end of stage s as well, since d is not changed at stage s.

Suppose that $k > 0$ and s is a $D(k, \sigma, \tau)$-stage. Then $\tau = d(\sigma)$. Suppose that the $D(k, \sigma, \tau)$-module picks $v \supset \tau$ and defines $d(v) = \mu \supset \sigma$. By the induction hypothesis,

$$\Gamma(\mu) \supset \Gamma(v) \supseteq \Gamma(\tau) \supset \Gamma(\sigma).$$

So both (1) and (2) hold at the end of stage s.

The case for $D(\rho)$-stage is similar to that of $D(k, -)$-stage. And that of $Q(0, -)$-stage is similar to that of P-stage. So Γ is well-defined.

Clearly every $P(\sigma, m)$ is satisfied and the set

$$\{\sigma : |\Gamma(\sigma)| > m\}$$

is recursive and dense for every m. Hence $\Gamma(G)$ is total for every 1-generic G. $\dashv$

By the proof of Lemma 2.4, we have the following fact of deputies.

LEMMA 2.5. *If* $\tau = d(\sigma)$ *then* $\Gamma(\tau) \supset \Gamma(\sigma)$.

Next we turn to the Q's. It is important to learn how deputies and Q-constraints interact.

LEMMA 2.6. *Suppose that* τ_k *is the* k-*deputy of* σ_k *and* ρ *is the greatest common initial segment of* σ_k *and* τ_k.

1. *If* $k = n - 1$ *then every* Q-*constraint* $c(m, \eta, -)$ *active at* τ_k *is either active at* ρ *or a replica of some* $c(m, \zeta, -)$ *with* $\rho \subset \zeta \subset \sigma_k$.
2. *If* $k < n - 1$ *then there exist* σ_{k+1} *and* τ_{k+1} *such that*
    (a) $\sigma_{k+1} \subset \sigma_k$ *and* $\tau_{k+1} \subset \tau_k$;
    (b) σ_{k+1} *is the* $(k + 1)$-*deputy of* τ_{k+1} *and* τ_k *is the* (σ_{k+1}, k)-*deputy of* σ_k;
    (c) *every* Q-*constraint* $c(m, \eta, -)$ *active at* τ_k *is either active at* τ_{k+1} *or a replica of some* $c(m, \zeta, -)$ *with* $\sigma_{k+1} \subset \zeta \subset \sigma_k$.
3. *If* $m < k$ *then for every* $c(m, \eta, \vec{x}, \hat{x}, i)$ *active at* τ_k *there exists* $c(m, \zeta, \vec{x}, \hat{x}, i)$ *active at* σ_k *and for each* $c(m, \mu, \vec{y}, \hat{y}, j)$ *active at* σ_k *there exists* $c(m, v, \vec{y}, \hat{y}, j)$ *active at* τ_k.

PROOF. (1) and (2) are trivial facts of the construction.

For $k = n - 1$, (3) follows from the fact that $\tau_k = d(\sigma_k)$ is defined by the $D(\rho)$-module and every Q-constraint active at σ_k either is active at ρ or has a replica active at τ_k. Suppose that $n - 1 > k > m$. There exist σ_{k+1} and τ_{k+1} as in (2). If $c(m, \eta, -)$ is active at τ_k then by (2c) above it is either active at τ_{k+1} or a replica of some $c(m, \zeta, -)$ active at σ_k. So the conclusion either follows from the induction hypothesis or holds trivially. For $c(m, \mu, -)$ active at σ_k, if $\sigma_{k+1} \subset \mu$ then it has a

replica $c(m, \tau_k, -)$; otherwise it is active at σ_{k+1} and the conclusion follows from the induction hypothesis. $\dashv$

The following fact implies that Q-constraints are arranged consistently.

LEMMA 2.7. *For each* $k, \vec{x}, \mu$ *there exists at most one* $c(k, \zeta, \vec{x}, \hat{x}, i)$ *active at* μ.

PROOF. Suppose that the lemma holds at the end of stage $s - 1$ and at least one new constraint is imposed at stage s. We prove that the lemma also holds at the end of stage s. Since new Q-constraints are imposed only at Q- or D-stages, there are three cases.

(i) s is a Q-stage. The conclusion is guaranteed by the construction.

(ii) s is a $D(k', \sigma, \tau)$-stage. Then τ is the k'-deputy of σ. Suppose that the D-module picks $\bar{v} \supset \tau$ and $v = \bar{v}\langle 1^p 0\rangle$ and defines $\mu = d(v) = \sigma\langle 1^q 0\rangle$. It suffices to prove that if $c = c(k, \zeta, \vec{x}, \hat{x}, i)$ and $c' = c(k, \eta, \vec{x}, \hat{y}, j)$ are active at μ then $\zeta = \eta$, $\hat{x} = \hat{y}$ and $i = j$. If neither ζ nor η equals μ then by the construction both constraints are imposed before stage s and thus coincide by the induction hypothesis. Assume that $\zeta = \mu$ and c is imposed at stage s. So $k' - 1 > k$ and c is a replica of some $c_0 = c(k, \zeta_0, -)$ active at v and imposed before stage s. If c' is active at σ, then by (3) of Lemma 2.6 and that $k' > k$ there exists $c_1' = c(k, \rho, \vec{x}, \hat{y}, j)$ active at τ. So both c_0 and c_1' are active at v and imposed before stage s. Hence c_0 and c_1' coincide by the induction hypothesis. However, this coincidence implies that c_0 is active at τ and has no replica imposed at stage s by the $D(k', \sigma, \tau)$-module and thus contradicts c being a replica of c_0. If c' is not active at σ then c' is a replica of some $c_0' = c(k, \eta_0, -)$ by (2b) of Lemma 2.6. So both c_0 and c_0' are active at v and imposed before stage s. By the induction hypothesis again, c_0 and c_0' coincide. Then c and c' are both replicas of the same constraint imposed at stage s and thus coincide as well.

(iii) s is a $D(\rho)$-stage. The proof for this case is similar to that for (ii). $\dashv$

LEMMA 2.8. Δ *is well-defined.*

PROOF. Lemma 2.7 implies that Q-modules encounter no conflicting $c(0, -)$'s. So the validity of Δ follows from (C2) and the construction. $\dashv$

A constraint imposed before or at stage s is a constraint *active at stage* s. *An active constraint* is one active at some stage. If some $c(k, \zeta, \vec{x}, \hat{x}, i)$ is active then we expect that ζ forces $\Delta_k(\Gamma(G); \vec{x}, x) \downarrow = i$ for all $x > \hat{x}$. The next lemma is important for this end.

LEMMA 2.9. *Suppose that* $c(k, \zeta, \vec{x}, \hat{x}, i)$ *is active,* ζ *and* η *are comparable and* $x > \hat{x}$.

1. *If there also exists an active* $c(k - 1, \eta, \vec{x}\langle x\rangle, \hat{y}, j)$ *then* $\eta \supseteq \zeta$.
2. *If* $k = 0$, $\Delta(\Gamma(\eta); \vec{x}, x) \downarrow$ *but* $\Delta(\Gamma(\eta \restriction (|\eta| - 1)); \vec{x}, x) \uparrow$ *then* $\eta \supseteq \zeta$.

PROOF. (1) Suppose that ζ is the shortest witnessing the failure of clause (1), i.e., there exist active $c_k = c(k, \zeta, \vec{x}, \hat{x}, i)$ and $c_{k-1} = c(k - 1, \eta, \vec{x}\langle x\rangle, \hat{y}, j)$ and $x > \hat{x}$ but $\eta \subset \zeta$. Let s and s_0 be the stages at which c_k and c_{k-1} are imposed, respectively. Then $s \geq s_0$, since constraints are always imposed at strings of fresh length. If $s = s_0$ then s must be a D-stage as at a Q-stage at most one constraint is imposed. Then $\eta = \zeta$ by the D-modules, contradicting the choice of ζ. Hence $s > s_0$.

If s is a Q-stage then $\hat{x}$ is greater than any number appeared in the construction before stage s. In particular $\hat{x} > x$, contradicting the premise.

Suppose that s is a $D(k', \sigma, \tau)$-stage. Then τ is the k'-deputy of σ and ζ is the $(k' - 1)$-deputy of some $v \supset \tau$. Moreover, $k' - 1 > k$ and c_k is a replica of some $c_k' = c(k, \zeta', \vec{x}, \hat{x}, i)$ such that $\tau \subset \zeta' \subset v$. As c_{k-1} is imposed at stage $s_0 < s$, c_{k-1} is active at σ by (2) of Lemma 2.6. By (3) of Lemma 2.6, there exists $c_{k-1}' = c(k-1, \eta', \vec{x}\langle x\rangle, \hat{y}, j)$ active at τ. But then ζ' is shorter than ζ and witnesses the failure of clause (1). So we have a desired contradiction.

The case that s is a $D(\rho)$-stage is similar.

(2) Assume that ζ is the shortest witnessing the failure of clause (2). Let c_0 denote $c(0, \zeta, \vec{x}, \hat{x}, i)$. Let s be the stage when c_0 is imposed. By the premise of (2), $\Gamma(\eta)$ is explicitly defined at some stage s_0. By the construction, η is longer than any ξ appeared before stage s_0. As we assume that $\zeta \supset \eta$, $s_0 \leq s$.

If s is a Q-stage then $s_0 < s$, since a Q-module cannot extend Γ when it imposes a constraint. So $\hat{x}$ is greater than any number appeared before stage s, in particular $\hat{x} > x$, contradicting the premise.

Suppose that s is a $D(k', \sigma, \tau)$-stage. Then τ is the k'-deputy of σ and ζ being the $(k' - 1)$-deputy of some $v \supset \tau$ is of the form $\sigma\langle 1^l 0\rangle$. Moreover, c_0 is a replica of some $c_0' = c(0, \zeta', -)$ such that $\tau \subset \zeta' \subset v$. By the construction, $|\zeta| > |\zeta'|$. As $\sigma \subset \rho \subset \zeta$ implies that $\Gamma(\rho) = \Gamma(\sigma)$ by (C1), $\eta \subseteq \sigma$. Since $\Gamma(\tau) \supset \Gamma(\sigma) \supseteq \Gamma(\eta)$, $\Delta(\Gamma(\tau); \vec{x}, x) \downarrow$. Let η' be the shortest initial segment of τ such that $\Delta(\Gamma(\eta'); \vec{x}, x) \downarrow$. Then ζ' witnesses the failure of clause (2) as well, contradicting ζ being the shortest.

By a similar argument, it is impossible that s is a $D(\rho)$-stage. $\dashv$

Below we show that every active constraint is obeyed.

LEMMA 2.10. *Suppose that $c(k, \xi, \vec{x}, \hat{x}, i)$ is active, ξ is comparable with μ and $x > \hat{x}$.*

1. *If $k > 0$ and $c(k - 1, \mu, \vec{x}\langle x\rangle, \hat{y}, j)$ is also active then $j = i$.*
2. *If $k = 0$ and $\Delta(\Gamma(\mu); \vec{x}, x) \downarrow= j$ then $j = i$.*

PROOF. Assume that both (1) and (2) hold before stage s. Fix parameters as in the premises. Let $c_k = c(k, \xi, \vec{x}, \hat{x}, i)$. By Lemma 2.9, we may assume that $\mu \supseteq \xi$.

(1) Assume that c_k is imposed at stage s_0 and $c_{k-1} = c(k - 1, \mu, \langle \vec{x} x\rangle, \hat{y}, j)$ at stage s. Then $s \geq s_0$ and s is a Q- or D-stage.

If s is a Q-stage then $\xi \subset \mu$ as at each Q-stage at most one constraint is imposed. If c_{k-1} is imposed by a $Q(k, \sigma, -)$-module then μ is of the form $\sigma\langle 1^l 0\rangle$ and c_k is active at σ by (C4). So $j = i$ by the $Q(k, \sigma, -)$-module and Lemma 2.7.

Suppose that s is a $D(k', \sigma, \tau)$-stage. Then τ is the k'-deputy of σ, $\sigma \subset \mu$ and μ is the $(k' - 1)$-deputy of some $v \supset \tau$. Moreover, $k' - 1 > k - 1$ and c_{k-1} is a replica of some $c_{k-1}' = c(k - 1, \eta, -)$ such that $\tau \subset \eta \subset v$. If $\xi \subset \mu$, $\xi \subseteq \sigma$. By (3) of Lemma 2.6, there exists $c_k' = c(k, \zeta, \vec{x}, \hat{x}, i)$ active at τ. If $\xi = \mu$ then c_k is also a replica of some $c_k'' = c(k, \zeta', -)$ active at v. In either case, $j = i$ by the induction hypothesis.

The case where s is a $D(\rho)$-stage is similar.

(2) We may assume that $\Delta(\Gamma(\mu \restriction (|\mu| - 1)); \vec{x}, x) \uparrow$. So $\Gamma(\mu)$ is explicitly defined in the construction at some stage s.

If s is a P-stage then $\Delta(\Gamma(\mu); \vec{x}, x)$ diverges too as $\Gamma(\mu)$ is of a fresh length and P-modules never change Δ.

If s a $Q(0, \sigma, -)$-stage then μ is of the form $\sigma\langle 1^l 0\rangle$ with some fresh l. As $Q(0, -)$-modules never impose constraints, $\xi \subset \mu$. By (C4), c_k is active at σ. So by Lemma 2.7 and the $Q(0, -)$-module, $j = i$.

Suppose that s is a $D(k', \sigma, \tau)$-stage. Then $k' > 0 = k$ and $\mu = d(v) = v\langle 1^l 0 \rangle$ for some l and $v \supset \tau$. So $\Gamma(\mu) \supset \Gamma(v)$. As the $D(k', \sigma, \tau)$-module does not extend Δ explicitly, $\Delta(\Gamma(v); \vec{x}, x) \downarrow= j$ as well. If c_k is active at σ, then there exists $c_k' = c(k, \zeta, \vec{x}, \hat{x}, i)$ active at τ by (3) of Lemma 2.6. If c_k is not active at σ then c_k is a replica of some c_k'' active at v. In either case, $j = i$ by the induction hypothesis applying to c_k' and v or c_k'' and v.

The case where s is a $D(\rho)$-stage is similar. $\dashv$

Now we can prove the satisfaction of the Q's for $n = 1$.

LEMMA 2.11. *If $n = 1$ and $x < |\sigma|$ then $Q(n - 1, \sigma, x)$ is satisfied. Hence*

$$(\forall x)(x \in G \leftrightarrow \Delta(\Gamma(G); x) \downarrow= 1)$$

for every 1-generic G.

PROOF. Suppose that $x < |\sigma|$ and $\sigma(x) = 1$. By the $Q(0, \sigma, x)$-module there exists $\tau \supset \sigma$ with $\Delta(\Gamma(\tau); x) \downarrow= 1$.

For the other direction, assume that σ is the shortest such that

$$(\exists x)(\sigma(x) = 0 \wedge \Delta(\Gamma(\sigma); x) \downarrow= 1). \tag{2.6}$$

Fix an x satisfying the matrix of (2.6). We may assume that $\Delta(\Gamma(\sigma \restriction (|\sigma| - 1)); x) \uparrow$. So $\Gamma(\sigma)$ is explicitly defined in the construction at some stage s. If s is a P-stage then $\sigma = \rho\langle 1^l 0 \rangle$ for some fresh l and no new computation of Δ is enumerated at stage s. Thus (2.6) cannot hold by the choice of σ. If s is a $Q(0, -, x)$-stage then by the construction nothing happens at stage s. Suppose that s is a $D(\rho)$-stage. Then σ is the deputy of some $\tau \supseteq \rho\langle 0 \rangle$, $\sigma = \rho\langle 1^{|\tau|} 0 \rangle$ and $\Delta(\Gamma(\sigma)) = \Delta(\Gamma(\tau))$. So $\Delta(\Gamma(\sigma); |\sigma| - 1)$ is undefined and $x < |\sigma| - 1$. As $\sigma(x) = 0$, $x < |\rho|$. So $\tau(x) = \rho(x) = 0$ and $\Delta(\Gamma(\tau); x) = 1$, contradicting the choice of σ. Since $n = 1$, there exist no $D(k, -)$-stages with $0 < k < n$.

So $Q(0, \sigma, x)$ is satisfied. The other half follows easily from the first half. $\dashv$

For $n > 1$, the satisfaction of the Q's follows from the next two lemmata.

LEMMA 2.12. *Suppose that $n - 1 > k$. Then every $Q(k, -)$ is satisfied, and if $c(k, \sigma, \vec{x}, \hat{x}, i)$ is active then $\Delta_{k+1}(\Gamma(G); \vec{x}) = i$ for each 1-generic $G \supset \sigma$ and thus*

$$\sigma \Vdash \Delta_{k+1}(\Gamma(G); \vec{x}) \downarrow= i.$$

PROOF. We prove the lemma by induction on k.

Suppose that $k = 0$. By the construction, for every $\vec{x}\langle x \rangle$ of length n the following set is recursive and dense

$$\{\sigma : \Delta(\Gamma(\sigma); \vec{x}, x) \downarrow\}.$$

So every $Q(0, \sigma, \vec{x}\langle x \rangle)$ is satisfied. Furthermore, assume that $c(0, \sigma, \vec{x}, \hat{x}, i)$ is active and $\Delta(\Gamma(\mu); \vec{x}, x) \downarrow= j$ where $x > \hat{x}$ and μ is comparable with σ. By (2) of Lemma 2.10, $j = i$. Hence

$$\sigma \Vdash \Delta_1(\Gamma(G); \vec{x}) \downarrow= i.$$

Suppose that $k > 0$. By the construction, for every $\vec{x}\langle x \rangle$ of length $n - k$ the following set is recursive and dense

$$\{\tau : \text{ there exists an active } c(k - 1, \tau, \vec{x}\langle x \rangle, \hat{x}, j)\}.$$

By the induction hypothesis,

$$\tau \Vdash \Delta_k(\Gamma(G); \vec{x}, x) \downarrow$$

for densely many τ. So the $Q(k, -)$'s are satisfied. Furthermore, assume that $c(k, \sigma, \vec{x}, \hat{x}, i)$ is active and G is a 1-generic extending σ. Then for every $x > \hat{x}$ there exists $\mu \subset G$ such that $c(k - 1, \mu, \vec{x}\langle x \rangle, \hat{y}, j)$ is active. By (1) of Lemma 2.10, $j = i$. By the induction hypothesis, $\Delta_k(\Gamma(G); \vec{x}, x) \downarrow= i$ and $\Delta_{k+1}(\Gamma(G); \vec{x}, x) = \lim_x \Delta_k(\Gamma(G); \vec{x}, x) = i$. Hence $\sigma \Vdash \Delta_{k+1}(\Gamma(G); \vec{x}) \downarrow= i$. $\dashv$

LEMMA 2.13. *If $n > 1$ and x and σ are such that $|\sigma| > x$ then*

$$\sigma(x) = 1 \to \sigma \Vdash \Delta_{n-1}(\Gamma(G); x) \downarrow= 1$$

and

$$\sigma(x) = 0 \to \sigma \Vdash \neg(\Delta_{n-1}(\Gamma(G); x) \downarrow= 1).$$

Hence for $n > 1$ every $Q(n - 1, \sigma, x)$ is satisfied.

PROOF. Suppose that $\sigma(x) = 1$. By the $Q(n - 1, -, x)$-modules, the following set is recursive and dense below σ:

$$\{\mu : \text{some } c(n - 2, \zeta, x, \hat{x}, i) \text{ is active at } \mu\}.$$

Since no module can ever impose any $c(n - 2, \xi, y, \hat{y}, 0)$, the dense set above equals to the set below

$$\{\mu : \text{some } c(n - 2, \zeta, x, \hat{x}, 1) \text{ is active at } \mu\}.$$

By Lemma 2.12,

$$\sigma \Vdash \Delta_{n-1}(\Gamma(G); x) \downarrow= 1$$

On the other hand, suppose that $\sigma(x) = 0$.

CLAIM 2.14. *There is no active $c(n - 2, \zeta, x, \hat{x}, i)$ such that ζ and σ are comparable.*

PROOF OF CLAIM 2.14. For a contradiction, assume that there exists such an active $c(n - 2, \zeta, x, \hat{x}, i)$. Since the construction never produces a constraint like $c(n - 2, -, 0)$, $i = 1$. By the construction, $|\zeta| > x$. So we may assume that $\zeta = \sigma$ and σ is the shortest with such a constraint. Clearly such a Q-constraint cannot be imposed by a Q-module. Neither can it be imposed by a $D(k, -)$-module with $k \leq n - 1$, since such a module can only replicate $c(j, -)$ with $j < k - 1 \leq n - 2$. Suppose that a $D(\rho)$-module imposes c. Then $\sigma \supset \rho\langle 1 \rangle$ and σ is the deputy of some $\tau \supset \rho\langle 0 \rangle$. By (3) of Lemma 2.6, there exists some $c(n - 2, \eta, x, \hat{x}, 1)$ active at τ. As $\tau(y) = 1$ implies $\sigma(y) = 1$ for all y, $\sigma(x) = 0$ implies $\eta(x) = \tau(x) = 0$ too, contradicting the choice of σ. $\dashv$

Let $\mu \supset \sigma$ be arbitrary. By Claim 2.14, no $c(n - 2, \zeta, x, \hat{x}, i)$ is active at μ. Note that there are at most finitely many Q-constraints active at μ. By the $Q(n - 1, \mu, x)$-module, there exist x_1 and $\zeta \supset \mu$, such that either $n = 2$ and $\Delta(\Gamma(\zeta); x, x_1) \downarrow= 0$ or $n > 2$ and there exists an active constraint $c(n - 3, \zeta, \langle xx_1 \rangle, \hat{x}, 0)$. By Lemma 2.12, either case implies that for each $\hat{x}$ the following set is dense below σ

$$\{\tau : (\exists x_1 > \hat{x})\tau \Vdash \Delta_{n-2}(\Gamma(G); x, x_1) \downarrow= 0\}.$$

Hence

$$\sigma \Vdash \neg(\Delta_{n-1}(\Gamma(G); x) \downarrow= 1).$$

It follows that $Q(n - 1, \sigma, x)$ is satisfied. $\dashv$

By Lemma 2.12 and that the following set is dense and recursive

$$\{\sigma : \text{some } c(n - k - 1, \sigma, \vec{x}, x, i) \text{ is active}\},$$

$\Delta_{n-k}(\Gamma(G))$ is G-recursive for $n \geq k > 1$ and every 1-generic G. Then by Lemma 2.13, every 1-generic G is Σ_k^0 in $\Delta_{n-k}(\Gamma(G))$.

Finally, we prove the satisfaction of the D's and also (A2).

LEMMA 2.15. *If τ is defined as the k-deputy of σ then $D(k, \sigma, \tau)$ is satisfied.*

PROOF. We prove the lemma by induction on k.

By Lemma 2.5, the lemma holds for $k = 0$.

Suppose that $k > 0$ and τ is defined as the k-deputy of σ. By the $D(k, \sigma, \tau)$-module, the set below is dense below τ

$$\{v : (\exists \mu \supset \sigma)(\mu \text{ is the } (k-1)\text{-deputy of } v)\}.$$

By the induction hypothesis, the following set is also dense below τ

$$\{v : (\exists \mu \supset \sigma)(D(k-1, v, \mu) \text{ is satisfied})\}.$$

So $D(k, \sigma, \tau)$ is satisfied. ⊣

LEMMA 2.16. *Every $D(\rho)$ is satisfied. Hence G is not Π_n^0 in $\Gamma(G)$ for every n-generic G.*

PROOF. By the $D(\rho)$-module, the set below is dense below $\rho\langle 0 \rangle$

$$\{\sigma : (\exists \tau \supset \rho\langle 1 \rangle)(\tau \text{ is the } (n-1)\text{-deputy of } \sigma)\}.$$

By Lemma 2.15, the following set is also dense below $\rho\langle 0 \rangle$

$$\{\sigma : (\exists \tau \supset \rho\langle 1 \rangle)(D(n-1, \sigma, \tau) \text{ is the satisfied})\}.$$

Hence $D(\rho)$ is satisfied.

Suppose that G is n-generic. Then $\Gamma(G)$ is total by Lemma 2.4. Let $X = \Gamma(G)$. Assume that $\omega - G$ equals

$$S = \{x : (\exists y)\varphi(x, y)\}$$

for some φ which is Π_{n-1}^0 in X. For each $x \notin G$ let $\rho_x = G \restriction x$. By the n-genericity of G and the satisfaction of $D(\rho_x)$, there exists an initial segment σ_x of G such that $\rho_x \subset \sigma_x$, $\sigma_x \Vdash \varphi(x, y)$ for some y and $D(n-1, \sigma_x, \tau_x)$ is satisfied for some $\tau_x \supset \rho\langle 1 \rangle$. By Lemma 2.2, $\tau_x \Vdash \varphi(x, y)$. So the following Σ_n^0 set is dense along G

$$D = \{\tau : (\exists x < |\tau|)(\tau(x) = 1 \wedge \tau \Vdash (\exists y)\varphi(x, y))\}.$$

As G is n-generic, it has an initial segment σ in D. So $\omega - G \neq S$. ⊣

So we prove Theorem 2.1. Moreover, by the last two lemmata and the remark preceding Lemma 2.15, every n-generic G is properly Σ_k^0 in $\Delta_{n-k}(\Gamma(G))$ for $n \geq k > 1$.

§3. **Remarks.** Recall that for positive n a set X is *generalized* low$_n$ (GL_n for short) if

$$X^{(n)} \equiv_T (X \oplus \emptyset')^{(n-1)}.$$

Every n-generic G satisfies $G^{(n)} \equiv_T G \oplus \emptyset^{(n)}$ and thus is GL_n ([4, Lemma 2.6]). Jockusch proved that every 2-generic bounds some X which is GL_2 but not GL_1 as a corollary of the relative recursive enumerability of 1-generics ([4, Corollary 5.5]). Similarly he also proved that every 3-generic bounds a set in $GL_3 - GL_2$

([4, Corollary 5.11]). Then Jockusch conjectured that every $(n+1)$-generic bounds a set in $GL_{n+1} - GL_n$ for all positive n ([4, Conjecture 5.14]). Below we confirm this conjecture.

First, by relativizing the corresponding proof we have an analog of Lemma 2.2.

LEMMA 3.1. *Let Y be fixed. If $D(n-1, \sigma, \tau)$ is satisfied and $\sigma \Vdash \varphi$ for some φ which is Π_{n-1}^0 in $\Gamma(G) \oplus Y$ then $\tau \Vdash \varphi$ as well.*

COROLLARY 3.2. *For $n > 0$, every $(n+1)$-generic bounds a set in $GL_{n+1} - GL_n$.*

PROOF. Let Γ be the functional constructed in Section 2 for n and let G be an arbitrary $(n+1)$-generic. Then $\Gamma(G)$ is total and G is Σ_n^0 in $\Gamma(G)$.

Let $X = \Gamma(G)$. We claim that G is not Π_n^0 in $X \oplus \emptyset'$. For a contradiction, assume that φ is Π_{n-1}^0 in $X \oplus \emptyset'$ and the complement of G equals the following set

$$S = \{x : (\exists y)\varphi(x, y)\}.$$

For each $x \notin G$, let $\rho_x = G \upharpoonright x$ and let σ_x be an extension of $\rho_x \langle 0 \rangle$ forcing $\varphi(x, y)$ for some y. As φ is Π_{n-1}^0 in $X \oplus \emptyset'$, φ is Π_n^0 in G for $n > 1$ or Δ_2^0 in G for $n = 1$. By the $(n+1)$-genericity of G, σ_x exists. By the construction in Section 2, we may assume that $D(n-1, \sigma_x, \tau_x)$ is satisfied for some $\tau_x \supset \rho_x \langle 1 \rangle$. By Lemma 3.1, $\tau_x \Vdash \varphi(x, y)$ too. So the set below is dense along G

$$T = \{\tau : (\exists x < |\tau|)(\tau(x) = 1 \wedge \tau \Vdash (\exists y)\varphi(x, y))\}.$$

Note that T is Σ_{n+1}^0. By the $(n+1)$-genericity of G, there exists a finite initial segment of G in T. Thus $G \cap S \neq \emptyset$, giving us the desired contradiction.

So $G \leq_T X^{(n)}$ but $G \not\leq_T (X \oplus \emptyset')^{(n-1)}$. Hence $X \notin GL_n$. On the other hand,

$$X^{(n+1)} \leq_T G^{(n+1)} \equiv_T G \oplus \emptyset^{(n+1)} \leq_T (X \oplus \emptyset')^{(n)} \leq_T X^{(n+1)}.$$

Thus $X \in GL_{n+1}$.   ⊣

The above proof is an application of deputies. As another application, we obtain the following corollary strengthening Theorem 2.1 and also [1, Theorem 3] of Anderson. Let us recall some notions before stating the corollary. Let $(D_i : i \in \omega)$ be a fixed enumeration of all finite sets such that $x \in D_i$ and $|D_i|$ are uniformly recursive in both x and i. A *strong array* is a pairwise disjoint subsequence of $(D_i : i \in \omega)$. A set X is *hyperimmune* in another set Y if for any Y-recursive strong array $(D_{k_i} : i \in \omega)$ there exists i such that $X \cap D_{k_i} = \emptyset$.

COROLLARY 3.3. *If $n > 0$ and G is n-generic then there exists $X \leq_T G$ such that G is Σ_n^0 in X and the complement of G is hyperimmune in $X^{(n-1)}$.*

PROOF. Fix $n > 0$. Let Γ be the functional constructed in Section 2. Then $X = \Gamma(G)$ is recursive in G and G is Σ_n^0 in X.

For a contradiction, assume that Φ is a Turing machine such that $f = \Phi(X^{(n-1)})$ is total and $(D_{f(i)} : i \in \omega)$ is a strong array witnessing the complement of G being nonhyperimmune in $X^{(n-1)}$. Let $\varphi(i, x)$ be a formula asserting that $f(i) = x$ which is Σ_n^0 in X. Write $\varphi(i, x)$ as $(\exists y)\psi(i, x, y)$ where ψ is Π_{n-1}^0 in $\Gamma(G)$. For each m, let $\rho_m = G \upharpoonright m$ and let i_m and k_m be such that $f(i_m) = k_m$ and $m < \min D_{k_m}$. By the n-genericity of G, there exist $\sigma_m \subset G$ and y_m such that $\max D_{k_m} < |\sigma_m|$ and σ_m forces $\psi(i_m, k_m, y_m)$. Let l_m be the least l such that $l \in D_{k_m}$ and $\sigma_m(l) = 0$. As $m < \min D_{k_m}$, $l_m > m$ and $\rho_m \subset \sigma_m \upharpoonright l_m$. By the construction in Section 2,

we may assume the existence of some $\tau_m \supset (\sigma_m \restriction l_m)\langle 1 \rangle$ such that $D(n-1, \sigma_m, \tau_m)$ is satisfied and $\tau_m(x) = 1$ for all $x \in D_{k_m}$. By Lemma 2.2, $\tau_m \Vdash \psi(i_m, k_m, y_m)$ and thus $\tau_m \Vdash \varphi(i_m, k_m)$. So the following Σ_n^0 set is dense along G

$$T = \{\tau : (\exists i, k)\tau \Vdash (D_k \subset G \wedge \varphi(i, k))\}.$$

By the n-genericity of G, G meets T, i.e., there exists an initial segment of G in T. So $D_{f(i)} \subset G$ for some i and we have the desired contradiction. $\dashv$

Kumabe [5] obtained a strengthening of the relative recursive enumerability of 1-generics that for every positive n and every n-generic G there exists a G-recursive n-generic H such that G is recursively enumerable but not recursive in H. So it is natural to ask whether Theorem 2 allows such a stronger version. More precisely,

QUESTION 3.4. *For positive n, is every n-generic G properly Σ_n^0 in another n-generic H which is recursive in G?*

As mentioned, the r.e.a. degrees form a large class. Let $\mathbf{R}_1$ denote this class, and in general let $\mathbf{R}_n$ denote the class whose definition is obtained from replacing recursive enumerability by being Σ_n^0 in the definition of R. Kurtz [7] (see also [3, Theorem 8.21.8]) proved that $\mathbf{R}_1$ includes the degrees of 2-random reals. So we may ask whether a parallel property holds for n-randoms with $n > 2$.

QUESTION 3.5. *For $n > 2$, is every n-random X properly Σ_{n-1}^0 in some X-recursive Y? Does $\mathbf{R}_{n-1}$ include the n-random degrees?*

REFERENCES

[1] B. A. ANDERSON, *Relatively computably enumerable reals.* **Archive for Mathematical Logic**, vol. 50 (2011), no. 3–4, pp. 361–365.

[2] M. CAI and R. SHORE, *Domination, forcing, array nonrecursiveness and relative recursive enumerability*, this JOURNAL, vol. 77 (2012), pp. 226–239.

[3] R. DOWNEY and D. HIRSCHFELDT, **Algorithmic Randomness and Complexity**, Theory and Applications of Computability, Springer, 2010.

[4] C. G. JOCKUSCH, Jr., *Degrees of generic sets*, **Recursion Theory**: *Its Generalizations and Applications, Proceedings of Logic Colloquium '79* (F. R. Drake and S. S. Wainer, editors), Cambridge University Press, Cambridge, 1980, pp. 110–139.

[5] M. KUMABE, *Relative recursive enumerability of generic degrees*, this JOURNAL, vol. 56 (1991), no. 3, pp. 1075–1084.

[6] ———, *Degrees of generic sets*, **Computability, Enumerability, Unsolvability** (S. B. Cooper, T. A. Slaman, and S. S. Wainer, editors), London Mathematical Society, Lecture Notes Series, vol. 224, Cambridge University Press, Cambridge, 1996, pp. 167–183.

[7] S. KURTZ, **Randomness and genericty in the degrees of unsolvability**, Ph.D. thesis, University of Illinios at Urbana-Champaign, 1981.

[8] C. E. M. YATES, *Initial segments of the degrees of unsolvability, part II*: *Minimal degrees*, this JOURNAL, vol. 35 (1970), pp. 243–266.

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