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COMMUTE TIMES AND THE EFFECTIVE RESISTANCES OF RANDOM TREES

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In this article we study the commute and hitting times of simple random walks on spherically symmetric random trees in which every vertex of level n has outdegree 1 with probability $1 - q_n$ and outdegree 2 with probability q_n. Our argument relies on the link between the commute times and the effective resistances of the associated electric networks when 1 unit of resistance is assigned to each edge of the tree.

1. INTRODUCTION

The random walk on a graph $\mathcal{G} = (V, E)$ is a Markov chain defined on its set of vertices V, and from vertex x, it moves to a neighbor y chosen with uniform probability. The access time (hitting time) H_{xy} of a vertex y starting from a vertex x is defined to be the mean number of time units required to reach y for the first time. Obviously, H_{xy} is not necessarily equal to H_{yx} unless the graph has a vertex-transitive automorphism group; see [7]. The commute time τ_{xy} between x and y is the mean time units to go from x to y

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and then back to x; that is, $\tau_{xy} = H_{xy} + H_{yx} = \tau_{yx}$. An electrical network is a weighted graph for which the weights are called conductances and their reciprocals are called resistances. If a finite network is considered and two vertex a (source) and b (sink) are distinguished and a battery is hooked up between a and b so that the potential at a is 1 and the potential at b is zero, then a potential is established at each vertices and a current flows through the edges. These two functions are defined and uniquely determined by two laws:

- Ohm's law: If xy is an edge of the graph, the current flowing from x to y satisfies $v_x - v_y = i_{xy} r_{xy}$, where v, i, and r stand respectively for potential, current, and resistance.
- Kirchhoff's law: The sum of the currents flowing out of any vertex different from a and b is zero.

We now consider $\mathcal{G}$ as an electric network by assigning 1-unit resistance to each edge $xy \in E$. The effective resistance $\mathcal{R}_{xy}$ between x and y is the voltage that develops at x if a 1 unit of current is injected into x and y is grounded. Equivalently, it is the potential difference between x and y required to send 1 unit current from x to y. The effective resistance between two vertices x and y of distance n apart satisfies the inequality $\mathcal{R}_{xy} \leq n$, where the equality holds true if and only if there is a unique path connecting x and y; see [2]. As such, if x_n is a vertex at level n of a tree rooted at a vertex r, then $\mathcal{R}_{rx_n} = n$. A tree Γ is called a spherically symmetric tree (SST) if the degree of a vertex (number of its neighbors) depends only on its distance from the root. In this case, all vertices of the same level have the same degree. In this article we are interested in studying the hitting and the commute time between two vertices of an SST. Let d_n denote the outdegree (degree -1) of each vertex at level n and Z_n denote the size at that level. Obviously, $Z_n = \prod_{k=0}^{n-1} d_k$. Throughout this article, d_n, $n = 0, 1, 2, \ldots$, are assumed to be independent random variables and the corresponding tree is called spherically symmetric random tree (SSRT). Shorting (gathering) some vertices in one vertex will not increase the effective resistance between any two vertices in the sense that shorting vertices of the same potential will keep the effective resistance unchanged, whereas shorting vertices of different potentials will decrease the effective resistance; see [3]. Because of the symmetric structure of the spherically symmetric tree, shorting vertices of the same level will not change its effective resistance. This entails that the effective resistance between the root r and the level n, shorted in one node s, of a spherically symmetric tree equals

$$\mathcal{R}_{rs} = \sum_{k=1}^{n} \frac{1}{Z_k}.$$

For instance, for a degree-2 homogeneous tree with two paths of length n each dangling out of the root, the effective resistance $\mathcal{R}_{rs} = n/2$, whereas for the binary tree where each vertex has two children, $\mathcal{R}_{rs} = 1 - (1/2)^n$.

This article is organized as follows. In Section 2 we give some auxiliary lemmas concerning some limit results. In Section 3 we introduce lemmas concerning commute times in terms of the effective resistances of the associated electric networks. Commute times of random trees are studied in Section 4; Section 5 is devoted to hitting times.

2. AUXILIARY LEMMAS

LEMMA 1: *Consider two sequences of nonnegative random variables X_n and Y_n such that $\sum_n Y_n$ is divergent a.s. If $\lim_n X_n/Y_n = L$ a.s., then*

$$\lim_n \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n Y_k} = L \quad a.s.$$

PROOF: Let $t_n = X_n/Y_n$. Then as $n \longrightarrow \infty$, $t_n \longrightarrow L$ on some event A such that $p(A) = 1$. Additionally, $\sum_n Y_n = \infty$ on an event B such that $p(B) = 1$. Note that $p(A \cap B) = 1$. Pick $\omega \in A \cap B$ and $\varepsilon > 0$; then there exists sufficiently large $n_0(\omega)$ such that for $k > n_0(\omega)$, we get

$$L - \varepsilon \leq t_k(\omega) \leq L + \varepsilon.$$

For the rest of the proof we will use n_0 for $n_0(\omega)$. Now, for $n > n_0$,

$$\sum_{k=n_0+1}^{n} (L-\varepsilon) Y_k(\omega) \leq \sum_{k=n_0+1}^{n} X_k(\omega) \leq \sum_{k=n_0+1}^{n} (L+\varepsilon) Y_k(\omega)$$

and, hence,

$$\left(\sum_{k=1}^{n} Y_k(\omega) - \sum_{k=1}^{n_0} Y_k(\omega)\right)(L-\varepsilon) \leq \sum_{k=1}^{n} Y_k(\omega) - \sum_{k=1}^{n_0} Y_k(\omega)$$
$$\leq \left(\sum_{k=1}^{n} Y_k(\omega) - \sum_{k=1}^{n_0} Y_k(\omega)\right)(L+\varepsilon).$$

As such,

$$\left(1 - \frac{\sum_{k=1}^{n_0} Y_k(\omega)}{\sum_{k=1}^{n} Y_k(\omega)}\right)(L-\varepsilon) \leq \frac{\sum_{k=1}^{n} X_k(\omega)}{\sum_{k=1}^{n} Y_k(\omega)} - \frac{\sum_{k=1}^{n_0} X_k(\omega)}{\sum_{k=1}^{n} Y_k(\omega)}$$
$$\leq \left(1 - \frac{\sum_{k=1}^{n_0} Y_k(\omega)}{\sum_{k=1}^{n} Y_k(\omega)}\right)(L+\varepsilon).$$

Since $\sum_n Y_n = \infty$,

$$(L-\varepsilon) \leq \lim_n \frac{\sum_{k=1}^n X_k(\omega)}{\sum_{k=1}^n Y_k(\omega)} \leq (L+\varepsilon).$$

Since ε is arbitrary small,

$$\lim_n \frac{\sum_{k=1}^n X_k(\omega)}{\sum_{k=1}^n Y_k(\omega)} = L.$$

This entails that

$$\lim_n \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n Y_k} = L \quad \text{a.s. on} \quad A \cap B.$$

The proof is complete. ■

For a realization of the two sequences X_n and Y_n we obtain the following.

LEMMA 2 [5]: *Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two positive sequences such that $\sum_k b_k = \infty$. If $a_k/b_k \longrightarrow L$, then $\sum_{k=1}^n a_k / \sum_{k=1}^n b_k \longrightarrow L$.*

LEMMA 3: *Consider a positive increasing function f and define a sequence $a_k, k = 1, 2, \ldots$ such that $f(t) = a_t$. Let $I_n = \int_1^n f(t)\, dt$ and $S_n = \sum_{k=1}^n a_k$. If $I_{n+1}/I_n \longrightarrow b$ then*

$$1 \leq \lim_n \frac{S_n}{I_n} \leq b.$$

PROOF: Since f is increasing,

$$\int_{k-1}^{k} f(t)\, dt \leq a_k \leq \int_k^{k+1} f(t)\, dt,$$

and, hence,

$$\sum_{k=2}^n \int_{k-1}^{k} f(t)\, dt \leq \sum_{k=2}^n a_k \leq \sum_{k=2}^n \int_k^{k+1} f(t)\, dt,$$

from which we obtain

$$\int_1^n f(t)\, dt \leq S_n - a_1 \leq \int_2^{n+1} f(t)\, dt,$$

that is,

$$I_n \leq S_n - a_1 \leq I_{n+1} - I_2. \tag{1}$$

This entails that

$$1 \leq \frac{S_n}{I_n} - \frac{a_1}{I_n} \leq \frac{I_{n+1}}{I_n} - \frac{I_2}{I_n}.$$

Since $I_n \longrightarrow \infty$ and $I_{n+1}/I_n \longrightarrow b$,

$$1 \leq \lim_n \frac{S_n}{I_n} \leq b.$$

and the lemma is proved. ■

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A similar argument shows, for positive decreasing function f and I_n and S_n are defined as above, that

$$a \leq \lim_n \frac{S_n}{I_n} \leq 1. \quad \textbf{(2)}$$

LEMMA 4 [5]: *For* $\beta > -1, \lim_n \left[\left(\sum_{k=1}^n k^\beta\right)/(n^{\beta+1})\right] = 1/(\beta+1)$.

3. COMMUTE TIME AND ELECTRICAL NETWORKS

One important link between random walks on finite connected graphs and electrical networks is the relationship concerning the commute time between two vertices x and y and the effective resistance $\mathcal{R}_{xy}$. In the rest of the article 1 ohm of resistance is assigned to each edge of the graph under consideration.

LEMMA 5 [2]: *In a graph, the effective resistance between two vertices x and y of distance n apart satisfies the inequality $\mathcal{R}_{xy} \leq n$, where the equality holds true if and only if there a unique path connecting x and y.*

LEMMA 6 [2,6]: *The commute time between two vertices x and y of a finite connected graph $\mathcal{G}$ is $\tau_{xy} = 2m\mathcal{R}_{xy}$, where m is the number of edges of $\mathcal{G}$ and $\mathcal{R}_{xy}$ is the effective resistance between x and y.*

For instance, for a degree-2 homogeneous tree, $\tau_{rs} = 2(2n)(n/2) = 2n^2$, whereas for the binary tree, $\tau_{rs} = 4(2^n - 1)(1 - (1/2)^n)$. We notice that for the former tree, τ_{rs} grows in a polynomial rate, whereas for the latter one, it grows exponentially. This motivates us to study the phase transition between polynomial and exponential rates of the commute times of random walks on random trees. Logically, we will consider trees of mixed degrees 2 and 3. It seems intuitive that if the degree-3 vertices become sparser as the tree grows, it will be more likely that we get commute time of polynomial type.

CUTTING PRINCIPLE [3]: Cutting certain branches can only increase the effective resistance between two given nodes.

The following lemma is an immediate consequence of the cutting principle.

LEMMA 7: *Let $\mathcal{G}$ be a connected finite random graph. Then the number of edges m and effective resistance between two given vertices are negatively correlated.*

LEMMA 8: *The commute time between two randomly chosen vertices x and y of a finite connected random graph $\mathcal{G}$ satisfies $\tau_{xy} \leq 2E(m)E(\mathcal{R}_{xy})$.*

PROOF: We may assume, without loss of generality, that $\mathcal{G}$ is a random tree Γ of height n and r is the root of Γ and the level n is shorted in one node s. Let ζ_n denote the σ-field generated by the degrees of vertices of Γ. If $Z_k, k = 0, 1, 2, \ldots, n$, denotes the sizes of the successive levels of Γ and T_{rs} is the random variable representing the time to

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reach s from r and then back to r, then $\tau_{rs} = E(T_{rs})$. Now, $\tau_{rs} = E(E(T_{rs}|\zeta_n))$. From Lemma 6,

$$E(T_{rs}|Z_1 = z_1, \ldots, Z_n = z_n) = 2\left(\sum_{k=1}^{n} z_k\right)\mathcal{R}^*_{rs},$$

where R^*_{rs} denotes the effective resistance of a realization of Γ that corresponds to $z_1, z_2, \ldots, z_n$. This entails that

$$E(T_{rs} \mid \zeta_n) = 2\left(\sum_{k=1}^{n} Z_k\right)\mathcal{R}_{rs}.$$

Hence, Lemma 7 assures that

$$\tau_{rs} = 2E\left[\left(\sum_{k=1}^{n} Z_k\right)(\mathcal{R}_{rs})\right] \leq 2E(m)E(\mathcal{R}_{rs}).$$

■

A similar argument along with Lemma 5 can be used to show that if x_n denotes, as usual, a leaf of level n of a tree, then

$$\tau_{rx_n} = 2E(m)n. \tag{3}$$

4. SPHERICALLY SYMMETRIC RANDOM TREES

For the rest of the article we will use the notation $a_n = \theta(b_n)$if $\lim_n(a_n/b_n) = K \in (0, \infty)$. As mentioned earlier, we will assume that all of the nodes of level n are shorted in one node s.

THEOREM 9: *Consider a SSRT, Γ, with outdegree sequence $\{d_n; n \geq 0\}$ such that*

$$d_n = \begin{cases} 1 & \text{with probability } 1 - q_n \\ 2 & \text{with probability } q_n, \end{cases}$$

where $q_n \downarrow 0$ and $\sum_n q_n < \infty$. Then, $\tau_{rs} \leq \theta(n^2)$ and $\tau_{rx_n} = \theta(n^2)$.

PROOF: It follows from the independence assumption of the d_i's that the mean number of edges of Γ is

$$m = \sum_{k=1}^{n} E(Z_k) = \sum_{k=1}^{n}\prod_{j=0}^{k-1} E(d_j)$$
$$= \sum_{k=1}^{n}\prod_{j=0}^{k-1}(1 + q_j).$$

Since $\sum_n q_n < \infty$,

$$a_n = \prod_{j=1}^{n}(1+q_j) \longrightarrow K < \infty.$$

Choosing $b_n = 1$ in Lemma 2, we obtain

$$\lim_n \frac{\sum_{k=1}^{n} a_k}{n} = K$$

and, hence,

$$\sum_{k=1}^{n}\prod_{j=0}^{k-1}(1+q_j) = \theta(n). \qquad \textbf{(4)}$$

On the other hand, the mean effective resistance is

$$\begin{aligned}\sum_{k=1}^{n} E(1/Z_k) &= \sum_{k=1}^{n}\prod_{j=0}^{k-1} E(1/d_j)\\ &= \sum_{k=1}^{n}\prod_{j=0}^{k-1}(1-\frac{1}{2}q_j).\end{aligned}$$

Once again, $\sum_n q_n < \infty$ implies that

$$\prod_{j=0}^{n}\left(1-\frac{1}{2}q_j\right) \longrightarrow L > 0,$$

and Lemma 2 assures that

$$\sum_{k=1}^{n}\prod_{j=0}^{k-1}\left(1-\frac{1}{2}q_j\right) = \theta(n). \qquad \textbf{(5)}$$

It follows now from (4), (5), and Lemma 8 that $\tau_{rs} \le \theta(n^2)$. The result $\tau_{rx_n} = \theta(n^2)$ follows from (3), (4), and Lemma 5. ■

Note: This result is not surprising, since the Borel–Cantelli lemma assures that $d_n = 1$ eventually a.s. and the random walk in such case is not much different from the random walk on $\{0, 1, 2, \ldots, n\}$, for which the commute between zero and n is $2n^2$. This is also consistent with the fact that the Brownian motion takes you a distance $\sqrt{n}$ in n time units.

THEOREM 10: *Consider a* ***SSRT*** *with outdegree sequence* $\{d_n; n \ge 0\}$ *such that*

$$d_n = \begin{cases} 1 & \textit{with probability } 1-q_n\\ 2 & \textit{with probability } q_n,\end{cases}$$

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where $q_n = min(1, c/n), c > 0$. *Then the following hold:*

- *for* $c = 2, \tau_{rs} \le \theta(n^3 \log n)$,
- *for* $c > 2, \tau_{rs} \le \theta(n^{c+1})$,
- *for* $c < 2, \tau_{rs} \le \theta(n^{2+c/2})$.

Moreover, for any $c > 0, \tau_{rx_n} = \theta(n^{c+2})$.

PROOF: Obviously, $E(d_k) = \min(2, 1 + c/k)$ and, hence,

$$E(Z_n) = \prod_{k=0}^{n-1} E(d_k) = \prod_{k=0}^{n-1} \min\left(2, 1 + \frac{c}{k}\right) = \theta(n^c).$$

This entails that

$$\sum_{k=1}^{n} E(Z_k) = \theta(n^{c+1}); \tag{6}$$

see [4]. Additionally,

$$E\left(\frac{1}{d_k}\right) = \max\left(\frac{1}{2}, 1 - \frac{c}{2k}\right).$$

Hence,

$$E\left(\frac{1}{Z_n}\right) = \prod_{k=0}^{n-1} E\left(\frac{1}{d_k}\right) = \prod_{k=0}^{n-1} \max\left(\frac{1}{2}, 1 - \frac{c}{2k}\right) = \theta(n^{-c/2}),$$

and for $c = 2$,

$$\sum_{k=1}^{n} E(1/Z_k) = \theta(\log n). \tag{7}$$

From Lemma 8, we get

$$\tau_{rs} \le \theta(n^3 \log n).$$

Whereas for $c > 2$, Lemma 3 gives

$$\sum_{k=1}^{n} E(1/Z_k) = \theta(1). \tag{8}$$

We conclude from Lemma 8 that

$$\tau_{rs} \le \theta(n^{c+1}).$$

Finally, for $c < 2$,

$$\sum_{k=1}^{n} E(1/Z_k) = \theta(n^{1-c/2}). \tag{9}$$

Hence,

$$\tau_{rs} \leq \theta(n^{2+c/2});$$

the results concerning τ_{rx_n} follow from (3) and (6). ■

We now consider a model of random trees in which the outdegree-1 vertices become sparser in a rate faster than that of the models considered earlier.

THEOREM 11: *Consider an **SSRT** with outdegree sequence* $\{d_n; n \geq 0\}$ *such that* $d_0 = 1$ *a.s. and, for* $n \geq 1$,

$$d_n = \begin{cases} 1 & \textit{with probability } 1 - \dfrac{1}{n^\lambda} \\ 2 & \textit{with probability } \dfrac{1}{n^\lambda}, \end{cases}$$

where $0 < \lambda < 1$. *Then the commute time* τ_{rx_n} *is such that*

$$\theta(n^{\lambda+1}2^{n^{1-\lambda}}) \leq \tau_{rx_n} \leq \theta(n^{\lambda+1}e^{n^{1-\lambda}}).$$

Note: The case where $\lambda > 1$ is included in Theorem 9 and the case $\lambda = 1$ is included in Theorem 10.

PROOF: It follows from Jensen's inequality that

$$\begin{aligned} \log E(Z_n) \geq E(\log Z_n) &= \sum_{k=0}^{n-1} E(\log d_k) \\ &= \sum_{k=1}^{n-1} (\log 2)\frac{1}{k^\lambda} \geq \alpha + (\log 2)\frac{n^{1-\lambda}}{1-\lambda}, \end{aligned}$$

for some constant α, where the last inequality follows from (2). Hence,

$$E(Z_n) \geq e^\alpha(2^{n^{1-\lambda}/(1-\lambda)}).$$

We now deduce from (1) that

$$\sum_{k=1}^{n} E(Z_k) \geq e^\alpha \sum_{k=1}^{n}(2^{k^{1-\lambda}/(1-\lambda)}) = \theta(n^\lambda 2^{n^{1-\lambda}}). \tag{10}$$

On the other hand,

$$\begin{aligned} \log E(Z_n) &= \sum_{k=1}^{n-1} \log E(d_k) = \sum_{k=1}^{n-1} \log\left(1 + \frac{1}{k^\lambda}\right) \\ &\leq \sum_{k=1}^{n-1} k^{-\lambda} \leq 1 + \frac{(n-1)^{1-\lambda}}{1-\lambda}, \end{aligned}$$

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where the last inequality follows from Lemma 4, in which case, (1) implies

$$\sum_{k=1}^{n} E(Z_k) \leq \sum_{k=1}^{n} e^{1+[(k-1)^{1-\lambda}/(1-\lambda)]} = \theta(n^{\lambda} e^{n^{1-\lambda}}). \tag{11}$$

It follows from (3), (10), and (11) that

$$\theta(n^{\lambda+1} 2^{n^{1-\lambda}}) \leq \tau_{rx_n} \leq \theta(n^{\lambda+1} e^{n^{1-\lambda}}).$$

■

5. HITTING TIMES

Our argument for calculating the hitting times of the random walks on trees relies on the essential edge lemma. An edge xy of a graph is called essential if its removal would disconnect the graph into two components $A(xy)$ and $A(yx)$ containing x and y, respectively. Let $e(xy)$ denote the set of edges of $A(xy)$. Obviously, each edge of a tree is an essential edge. The following is the essential edge lemma.

LEMMA 12 [1]: *If xy is an essential edge of a finite connected graph and H_{xy} stands for the hitting time of y starting from x, then*

$$H_{xy} = 2|e(xy)| + 1.$$

Note: We refer the reader to [1] for a generalized essential edge lemma for weighted random walks.

THEOREM 13: *Consider an **SSRT** with an outdegree sequence $\{d_n; n \geq 0\}$ such that*

$$d_n = \begin{cases} 1 & \text{with probability } 1 - q_n \\ 2 & \text{with probability } q_n, \end{cases}$$

where $q_n = \min(1, j/n)$ for some positive integer j. If H_{nr} is the hitting time of the root r from a leaf x_n, then for $j = 1$, $H_{nr} = \theta(n^2 \log n)$ and for $j > 1$, $H_{nr} = \theta(n^{j+1})$.

PROOF: Let $x_0, x_1, x_2, \ldots, x_n$ stand for the unique path connecting the root $r = x_0$ to the leaf x_n of level n and $H_{k+1,k}$ denote the mean time to hit x_k from x_{k+1}. The first passage times from x_{k+1} to x_k, $k = 0, 1, \ldots, n-1$ are stopping times. Applying the strong Markov property to these stopping times leads to

$$H_{nr} = \sum_{k=1}^{n} H_{k,k-1}.$$

Exploiting Lemma 12, we get

$$H_{nr} = \sum_{k=1}^{n} (2E(|e(x_k x_{k-1})|) + 1),$$

$$H_{nr} = n + 2\sum_{k=1}^{n} E(|e(x_k x_{k-1})|). \quad \textbf{(12)}$$

However, it follows from the assumption of independence that

$$E(|e(x_k x_{k-1})|) = E(d_k) + E(d_k)E(d_{k+1}) + \cdots + E(d_k)E(d_{k+1}) \cdots E(d_{n-1}). \quad \textbf{(13)}$$

It follows from the definition of q_n that for $k \le j$, $E(d_k) = 2$ and for $k > j$, $E(d_k) = 1 + j/k$. We only confine our investigation to the two cases $j = 1$ and $j = 2$.

Case 1: $j = 1$. Let $k \ge 2$, it follows from (13) that

$$\begin{aligned} E(|e(x_k x_{k-1})|) &= \left(1 + \frac{1}{k}\right) + \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k+1}\right) \\ &\quad + \cdots + \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k+1}\right) \cdots \left(1 + \frac{1}{n-1}\right) \\ &= \frac{1}{k} \sum_{j=k+1}^{n} j = \frac{n-k}{2k}(n+k+1). \end{aligned} \quad \textbf{(14)}$$

For $k = 1$, (13) gives

$$E(|e(x_1 x_0)|) = 2\left(1 + \frac{3}{2} + \frac{4}{2} + \cdots + \frac{n}{2}\right) = \frac{(n-1)(n+2)}{2}. \quad \textbf{(15)}$$

It can easily be verified, using (12)–(14), that

$$H_{nr} = \theta(n^2 \log n).$$

Case 2: $j = 2$. For $k \ge 3$ it can be shown that

$$\begin{aligned} &E(|e(x_k x_{k-1})|) \\ &\quad = \frac{1}{k(k+1)} \left[\sum_{i=k+1}^{n} i(i+1) \right] \\ &\quad = \frac{1}{k(k+1)} \left(\frac{n(n+1)(2n+1)}{6} - \frac{k(k+1)(2k+1)}{6} + \frac{n(n+1)}{2} - \frac{k(k+1)}{2} \right). \end{aligned} \quad \textbf{(16)}$$

If $k < 3$, then simple calculations show that both $E(|e(x_1 x_0)|)$ and $E(|e(x_k x_{k+1})|)$ are $\theta(n^3)$. Hence, carrying out summation over k for (16), we obtain $H_{nr} = \theta(n^3)$. Using similar reasoning, we get for $j \ge 3$ that $H_{nr} = \theta(n^{j+1})$. ■

Note: From Theorems 10 and 13 we can compute the hitting time H_{rn}.

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