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HANKEL DETERMINANTS OF FACTORIAL FRACTIONS

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Abstract

By making use of the Cauchy double alternant and the Laplace expansion formula, we establish two closed formulae for the determinants of factorial fractions that are then utilised to evaluate several determinants of binomial coefficients and Catalan numbers, including those obtained recently by Chammam [‘Generalized harmonic numbers, Jacobi numbers and a Hankel determinant evaluation’, *Integral Transforms Spec. Funct.* **30**(7) (2019), 581–593].

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1. Introduction and motivation

Let $\mathbb{N}$ be the set of natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For an indeterminate x and $n \in \mathbb{N}_0$, the rising and falling factorials are defined respectively by

$$\begin{aligned} (x)_0 &\equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for } n \in \mathbb{N}, \\ \langle x\rangle_0 &\equiv 1 \quad \text{and} \quad \langle x\rangle_n = x(x-1)\cdots(x-n+1) \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

They can be expressed as the quotients

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad \langle x\rangle_n = \frac{\Gamma(1+x)}{\Gamma(1+x-n)},$$

where the Γ-function is given by the Euler integral

$$\Gamma(x) = \int_0^\infty y^{x-1}e^{-y}\,dy \quad \text{with } \mathrm{Re}(x) > 0.$$

By employing Jacobi polynomials, Chammam [5] recently evaluated two interesting Hankel determinants. After removing the superfluous variable ξ, replacing α by $\alpha - 1$

and making simplifications, we can equivalently express them as follows:

$$\det_{0\le i,j\le n}\left[\frac{(\alpha)_{i+j}}{(1+\alpha+\beta)_{i+j}}\right]=\prod_{k=0}^{n}\frac{k!\,(\alpha)_k(1+\beta)_k}{(\alpha+\beta+k)_k(1+\alpha+\beta)_{2k}}, \tag{1.1}$$

$$\det_{0\le i,j\le n}\left[\sum_{k=0}^{i+j}\frac{\alpha}{\alpha+k}\right]=(-1)^n\prod_{k=1}^{n}\frac{k!\,(k-1)!\,(\alpha)_{k+1}}{(\alpha+k)_k(1+\alpha)_{2k}}\left\{1+\alpha\sum_{\ell=1}^{n}\frac{(\alpha+2\ell)}{\ell(\alpha+\ell)}\right\}. \tag{1.2}$$

Motivated by these two determinants and the binomial determinants appearing in [2, 3, 8, 11], we shall investigate two general determinants with many free parameters. In the next section, we shall prove a closed formula (Theorem 2.1) for a determinant of rational functions that not only extends (1.1) substantially, but also contains several interesting determinant identities of binomial coefficients and Catalan numbers as special cases. Then, in Section 3, we shall examine another determinant whose entries are partial sums of factorial fractions. The result turns out to be a determinant evaluation (Theorem 3.1) that unifies both (1.2) and some other Hankel determinants. Finally, an amazing binomial identity (Corollary 3.8) follows as a consequence.

2. Determinants of factorial fractions

Let $\{y_n\}_{n\ge 0}$ be a sequence of distinct elements. For $n\in\mathbb{N}_0$, suppose that $P_n(x)$ is a polynomial of degree less than or equal to n and $Q_n(x)$ is defined by the product

$$Q_n(x)=\prod_{k=1}^{n}(x+y_k)\quad\text{with } Q_0(x)=1.$$

Then we have the following general theorem.

THEOREM 2.1 (Determinant evaluation).

$$\det_{0\le i,j\le n}\left[\frac{P_j(x_i)}{Q_j(x_i)}\right]=\prod_{0\le i<j\le n}(x_i-x_j)\prod_{k=0}^{n}\frac{P_k(-y_k)}{Q_n(x_k)}.$$

PROOF. By means of the partial fractions

$$\frac{P_j(x_i)}{(x_i+y_0)Q_j(x_i)}=\sum_{k=0}^{j}\frac{\Omega(k,j)}{x_i+y_k},\qquad\text{where } \Omega(k,j)=\frac{P_j(-y_k)}{\prod_{\jmath=0,\ \jmath\ne k}^{j}(y_\jmath-y_k)},$$

we have the matrix decomposition

$$\left[\frac{P_j(x_i)}{(x_i+y_0)Q_j(x_i)}\right]_{0\le i,j\le n}=\left[\frac{1}{x_i+y_k}\right]_{0\le i,k\le n}\times[\Omega(k,j)]_{0\le k,j\le n}.$$

Observing that the last matrix is upper triangular,

$$\det_{0\le k,j\le n}[\Omega(k,j)]=\prod_{k=0}^{n}\Omega(k,k)=\frac{\prod_{k=0}^{n}P_k(-y_k)}{\prod_{i<j}(y_i-y_j)}.$$

Then the product formula in the theorem follows from the Cauchy double alternant (see Chu [9, 10]):

$$\det_{0\le i,k\le n}\left[\frac{1}{x_i+y_k}\right]=\frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{0\le i,j\le n}(x_i+y_j)}. \qquad \square$$

In Theorem 2.1, letting $P_k(x)=(x-\sigma)_k$ and $y_k=y+k-1$ yields the following formula.

PROPOSITION 2.2 (Determinant evaluation).

$$\det_{0\le i,j\le n}\left[\frac{(x_i-\sigma)_j}{(x_i+y)_j}\right]=\prod_{0\le j<i\le n}(x_i-x_j)\prod_{k=0}^{n}\frac{(\sigma+y)_k}{(x_k+y)_n}.$$

In particular, when $\sigma=0$ and $x_k=x+k$, multiplying the ith row of the above matrix by $(x)_i/(x+y)_i$, we recover the following formula for a special Hankel determinant, which is equivalent to (1.1), rediscovered recently by Chammam [5, Theorem 3.1].

COROLLARY 2.3 (Burchnall [4]).

$$\det_{0\le i,j\le n}\left[\frac{(x)_{i+j}}{(x+y)_{i+j}}\right]=\prod_{k=0}^{n}\frac{k!\,(x)_k(y)_k}{(x+y)_{n+k}}.$$

The Catalan number defined by

$$C_n=\frac{1}{n+1}\binom{2n}{n}$$

can be expressed in terms of factorial quotients by

$$C_{i+j+\lambda}=4^{i+j+\lambda}\frac{(\frac{1}{2})_{i+j+\lambda}}{(2)_{i+j+\lambda}}.$$

By pulling out the common row and column factors and then applying Proposition 2.2, we can evaluate the determinant

$$\begin{aligned}\det_{0\le i,j\le n}[C_{i+j+\lambda}]&=4^{(n+1)(n+\lambda)}\det_{0\le i,j\le n}\left[\frac{(\frac{1}{2}+i+\lambda)_j}{(2+i+\lambda)_j}\right]\prod_{k=0}^{n}\frac{(\frac{1}{2})_{k+\lambda}}{(2)_{k+\lambda}}\\&=4^{(n+1)(n+\lambda)}\prod_{k=0}^{n}\frac{(\frac{1}{2})_{k+\lambda}(\frac{3}{2})_k}{(2)_{n+k+\lambda}}\prod_{0\le j<i\le n}(i-j),\end{aligned}$$

which simplifies into the following formula.

COROLLARY 2.4 (Tamm [15]).

$$\det_{0\le i,j\le n}[C_{i+j+\lambda}]=\prod_{k=0}^{n}\frac{(2k+1)!\,(2k+2\lambda)!}{(k+\lambda)!\,(k+\lambda+n+1)!}=\prod_{1\le i\le j<\lambda}\frac{2n+i+j+2}{i+j}.$$

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When $\lambda = 0, 1, 2$, there exist simpler expressions (see Aigner [1]):

$$\det_{0\le i,j\le n}[C_{i+j}] = 1, \quad \det_{0\le i,j\le n}[C_{i+j+1}] = 1, \quad \det_{0\le i,j\le n}[C_{i+j+2}] = n+2.$$

Analogously, for the central binomial coefficients, we can make similar operations:

$$\begin{aligned}\det_{0\le i,j\le n}\left[\binom{2i+2j+2\lambda+\delta}{i+j+\lambda}\right] &= 2^{(n+1)(2n+2\lambda+\delta)}\det_{0\le i,j\le n}\left[\frac{(\frac{1}{2}+i+\lambda+\delta)_j}{(1+i+\lambda+\delta)_j}\right]\prod_{k=0}^{n}\frac{(\frac{1}{2})_{k+\lambda+\delta}}{(1)_{k+\lambda+\delta}}\\ &= 2^{(n+1)(2n+2\lambda+\delta)}\prod_{k=0}^{n}\frac{(\frac{1}{2})_k(\frac{1}{2})_{k+\lambda+\delta}}{(1)_{n+k+\lambda+\delta}}\prod_{0\le j<i\le n}(i-j),\end{aligned}$$

which give rise to the following determinant identities.

Corollary 2.5 (Binomial determinants: $\delta = 0, 1$).

$$\det_{0\le i,j\le n}\left[\binom{2i+2j+2\lambda+\delta}{i+j+\lambda}\right] = \prod_{k=0}^{n}\frac{(2k)!\,(2k+2\lambda+\delta)!}{(k+\lambda)!\,(k+\lambda+n+\delta)!}.$$

For $\delta = 1$, Tamm [15] derived the equivalent formula

$$\det_{0\le i,j\le n}\left[\binom{1+2i+2j+2\lambda}{i+j+\lambda}\right] = \prod_{1\le i\le j\le\lambda}\frac{2n+i+j+1}{i+j-1}.$$

The first few values are recorded as examples:

$$\begin{aligned}\det_{0\le i,j\le n}\left[\binom{2i+2j}{i+j}\right] &= 2^n,\\ \det_{0\le i,j\le n}\left[\binom{2i+2j+1}{i+j}\right] &= 1,\\ \det_{0\le i,j\le n}\left[\binom{2i+2j+2}{i+j+1}\right] &= 2^{n+1},\\ \det_{0\le i,j\le n}\left[\binom{2i+2j+3}{i+j+1}\right] &= 3+2n,\\ \det_{0\le i,j\le n}\left[\binom{2i+2j+4}{i+j+2}\right] &= 2^{n+1}(3+2n).\end{aligned}$$

By carrying out the same procedure, we can also evaluate the corresponding determinants when the matrix entries in Corollaries 2.4 and 2.5 are inverted.

Corollary 2.6 (Determinant evaluations: $\delta = 0, 1$).

$$\det_{0\le i,j\le n}[C^{-1}_{i+j+\lambda}] = \frac{12^n n!^2\,(1-2n)}{(2n)!^2}\prod_{k=0}^{n}\frac{(2k)!\,(k+\lambda+1)!\,(k+\lambda+n)!}{(2k+2\lambda+2n)!},$$

$$\det_{0\le i,j\le n}\left[\binom{2i+2j+2\lambda+\delta}{i+j+\lambda}^{-1}\right] = \frac{(-2)^n n!}{(2n)!}\prod_{k=0}^{n}\frac{(2k)!\,(k+\lambda+\delta)!\,(k+\lambda+n)!}{(2k+2\lambda+2n+\delta)!}.$$

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By writing the binomial coefficients in terms of shifted factorial quotients, it is not hard to derive from Proposition 2.2 the following further determinant identities.

COROLLARY 2.7 (Ostrowski [14]).

$$\det_{0\le i,j\le n}\left[\binom{x}{\lambda_i+j}\right]=\prod_{0\le i<j\le n}(\lambda_i-\lambda_j)\prod_{k=0}^{n}\frac{\langle x+k\rangle_{\lambda_k+k}}{(n+\lambda_k)!},$$

$$\det_{0\le i,j\le n}\left[\binom{x+\lambda_i+j}{\lambda_i+j}\right]=\prod_{0\le i<j\le n}(\lambda_i-\lambda_j)\prod_{k=0}^{n}\frac{\langle x+\lambda_k\rangle_{\lambda_k+k}}{(n+\lambda_k)!}.$$

When the matrix entries are turned upside down, there are similar results.

COROLLARY 2.8 (Binomial determinants).

$$\det_{0\le i,j\le n}\left[\binom{x}{\lambda_i+j}^{-1}\right]=\prod_{0\le j<i\le n}(\lambda_i-\lambda_j)\prod_{k=0}^{n}\frac{\lambda_k!\,\langle 1+x\rangle_k}{\langle x\rangle_{n+\lambda_k}},$$

$$\det_{0\le i,j\le n}\left[\binom{x+\lambda_i+j}{\lambda_i+j}^{-1}\right]=\prod_{0\le j<i\le n}(\lambda_i-\lambda_j)\prod_{k=0}^{n}\frac{\lambda_k!\,(x)_k}{(1+x)_{n+\lambda_k}}.$$

3. Determinants of Hankel matrices

According to Proposition 2.2, we may construct the determinant

$$H_n(\xi):=\det_{0\le i,j\le n}\left[\sum_{k=0}^{i+j}\frac{(x_i-\sigma)_k}{(x_i+y)_k}\xi^k\right].$$

By subtracting the precedent column from the jth one, we can rewrite $H_n(\xi)$ as

$$H_n(\xi)=\det_{0\le i\le n}\left[\begin{array}{c|c}\displaystyle\sum_{k=0}^{i}\frac{(x_i-\sigma)_k}{(x_i+y)_k}\xi^k & \displaystyle\frac{(x_i-\sigma)_{i+j}}{(x_i+y)_{i+j}}\xi^{i+j}\\ \cdots\cdots & \cdots\cdots\\ j=0 & 1\le j\le n\end{array}\right].$$

Then the Laplace expansion along the first column gives the expression

$$H_n(\xi)=\sum_{k=0}^{n}(-1)^k\sum_{\imath=0}^{k}\xi^{\imath}\frac{(x_k-\sigma)_{\imath}}{(x_k+y)_{\imath}}H_n^k(\xi),$$

where $H_n^k(\xi)$ is the minor of $H_n(\xi)$ with the first column and the kth row being crossed out:

$$H_n^k(\xi)=\det_{\substack{0\le i\le n\\1\le j\le n}}\left[\frac{(x_i-\sigma)_{i+j}}{(x_i+y)_{i+j}}\xi^{i+j}\right]_{i\ne k}.$$

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By extracting the common factors from rows and columns and then appealing to Proposition 2.2, we can evaluate the minor as follows:

$$\begin{aligned}
H_n^k(\xi) &= \xi^{n^2+n-k} \prod_{\imath \neq k} \frac{(x_\imath - \sigma)_{\imath+1}}{(x_\imath + y)_{\imath+1}} \det_{\substack{0\le i\le n\\ 1\le j\le n}} \left[\frac{(1+x_i+i-\sigma)_{j-1}}{(1+x_i+i+y)_{j-1}} \right]_{i\neq k} \\
&= \xi^{n^2+n-k} \prod_{\imath \neq k} \frac{(x_\imath - \sigma)_{\imath+1}}{(x_\imath + y)_{\imath+1}} \frac{\prod_{\ell=1}^n (\sigma+y)_{\ell-1}}{\prod_{\jmath\neq k}(1+x_\jmath+\jmath+y)_{n-1}} \prod_{\substack{i>j\\ i,j\neq k}} (x_i+i-x_j-j) \\
&= \xi^{n^2+n-k} \prod_{\ell=0}^n \frac{(x_\ell-\sigma)_{\ell+1}(\sigma+y)_{\ell-1}}{(x_\ell+y)_{\ell+n}} \prod_{i>j}(x_i+i-x_j-j) \\
&\quad \times (-1)^k \frac{(x_k+y)_{k+n}}{(x_k-\sigma)_{k+1}} \frac{\sigma+y-1}{\prod_{\substack{\jmath=0\\ \jmath\neq k}}^n (x_\jmath+\jmath-x_k-k)}.
\end{aligned}$$

Therefore, $H_n(\xi)$ can be expressed as a double sum

$$\begin{aligned}
H_n(\xi) &= \sum_{k=0}^n (-1)^k \sum_{\imath=0}^k \xi^\imath \frac{(x_k-\sigma)_\imath}{(x_k+y)_\imath} H_n^k(\xi) \\
&= \xi^{n^2+n} \prod_{\ell=0}^n \frac{(x_\ell-\sigma)_{\ell+1}(\sigma+y)_{\ell-1}}{(x_\ell+y)_{\ell+n}} \prod_{i>j}(x_i+i-x_j-j) \\
&\quad \times \sum_{k=0}^n \sum_{\imath=0}^k \frac{(x_k-\sigma)_\imath}{(x_k+y)_\imath} \frac{(x_k+y)_{k+n}}{(x_k-\sigma)_{k+1}} \frac{(\sigma+y-1)\xi^{\imath-k}}{\prod_{\substack{\jmath=0\\ \jmath\neq k}}^n (x_\jmath+\jmath-x_k-k)}.
\end{aligned}$$

THEOREM 3.1 (Determinant evaluation).

$$\begin{aligned}
\det_{0\le i,j\le n} \left[\sum_{k=0}^{i+j} \frac{(x_i-\sigma)_k}{(x_i+y)_k} \xi^k \right] &= \xi^{n^2+n} \prod_{\ell=0}^n \frac{(x_\ell-\sigma)_{\ell+1}(\sigma+y)_{\ell-1}}{(x_\ell+y)_{n+\ell}} \prod_{i>j}(x_i+i-x_j-j) \\
&\quad \times \sum_{k=0}^n \frac{(x_k+y)_{k+n}}{(x_k-\sigma)_{k+1}} \frac{(\sigma+y-1)}{\prod_{\substack{\jmath=0\\ \jmath\neq k}}^n (x_\jmath+\jmath-x_k-k)} \sum_{\imath=0}^k \xi^{\imath-k} \frac{(x_k-\sigma)_\imath}{(x_k+y)_\imath}.
\end{aligned}$$

Now we are going to examine the particular case $\xi = 1$. Define the sequence τ_k and compute its difference by

$$\tau_k := \frac{(x-\sigma)_k}{(x+y-1)_k} \quad \text{and} \quad \tau_k - \tau_{k+1} = \frac{(\sigma+y-1)(x-\sigma)_k}{(x+y-1)(x+y)_k}.$$

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Then we can evaluate by telescoping (see Chu [12]) the partial sum

$$\begin{aligned}\sum_{k=0}^{m}\frac{(x-\sigma)_k}{(x+y)_k} &= \frac{(x+y-1)}{(\sigma+y-1)}\sum_{k=0}^{m}(\tau_k-\tau_{k+1})\\ &= \frac{(x+y-1)}{(\sigma+y-1)}(1-\tau_{m+1})\\ &= \frac{(x+y-1)}{(\sigma+y-1)}\Big\{1-\frac{(x-\sigma)_{m+1}}{(x+y-1)_{m+1}}\Big\}.\end{aligned}$$

Denote by $\Xi(\xi)$ the double sum in Theorem 3.1. By substitution, we can reduce it to the difference of two single sums

$$\begin{aligned}\Xi(1) &= \sum_{k=0}^{n}\frac{(x_k+y)_{k+n}}{(x_k-\sigma)_{k+1}}\frac{(\sigma+y-1)}{\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}\sum_{\imath=0}^{k}\frac{(x_k-\sigma)_\imath}{(x_k+y)_\imath}\\ &= \sum_{k=0}^{n}\frac{(x_k+y-1)_{n+k+1}}{(x_k-\sigma)_{k+1}\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}\Big\{1-\frac{(x_k-\sigma)_{k+1}}{(x_k+y-1)_{k+1}}\Big\}\\ &= \sum_{k=0}^{n}\frac{(x_k+y-1)_{n+k+1}}{(x_k-\sigma)_{k+1}\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}-\sum_{k=0}^{n}\frac{(x_k+k+y)_n}{\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}.\end{aligned}$$

The rightmost sum equals $(-1)^n$ because for a polynomial of degree n, its divided differences of order n (see Chu [7, 10]) coincide with its leading coefficients. Hence, we get the following simplified formula.

PROPOSITION 3.2 (Determinant evaluation).

$$\begin{aligned}\det_{0\le i,j\le n}\Big[\sum_{k=0}^{i+j}\frac{(x_i-\sigma)_k}{(x_i+y)_k}\Big] &= \prod_{\ell=0}^{n}\frac{(\sigma+y)_{\ell-1}(x_\ell-\sigma)_{\ell+1}}{(x_\ell+y)_{n+\ell}}\prod_{i>j}(x_i+i-x_j-j)\\ &\quad\times\Big\{\sum_{k=0}^{n}\frac{(x_k+y-1)_{n+k+1}}{(x_k-\sigma)_{k+1}\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}-(-1)^n\Big\}.\end{aligned}$$

By employing the partial sum expression appearing in the proof of the above proposition, we can evaluate an equivalent determinant.

PROPOSITION 3.3 (Determinant evaluation).

$$\begin{aligned}\det_{0\le i,j\le n}\Big[1-\frac{(x_i-\sigma)_{i+j+1}}{(x_i+y)_{i+j+1}}\Big] &= \prod_{\ell=0}^{n}\frac{(\sigma+y)_{\ell}(x_\ell-\sigma)_{\ell+1}}{(x_\ell+y)_{n+\ell+1}}\prod_{i>j}(x_i+i-x_j-j)\\ &\quad\times\Big\{\sum_{k=0}^{n}\frac{(x_k+y)_{n+k+1}}{(x_k-\sigma)_{k+1}\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}-(-1)^n\Big\}.\end{aligned}$$

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Letting $\sigma = 0$ and $y = 1$ in Proposition 3.3 yields the next determinant formula.

COROLLARY 3.4 (Determinant evaluation).

$$\det_{0\le i,j\le n}\left[\frac{1+i+j}{1+i+j+x_i}\right] = \prod_{\ell=0}^{n} \frac{\ell!\ x_\ell}{(x_\ell+\ell+1)_{n+1}} \prod_{i>j}(x_i+i-x_j-j)$$
$$\times\left\{\sum_{k=0}^{n} \frac{(x_k+k+1)_{n+1}}{x_k \prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)} - (-1)^n\right\}.$$

When $\{x_k \equiv x\}_{k\ge 0}$ is a constant sequence, we can make a further simplification for the product

$$\prod_{i=0}^{n} \frac{\ell!\ x_\ell}{(x_\ell+\ell+1)_{n+1}} \prod_{i>j}(x_i+i-x_j-j),$$

which becomes

$$\prod_{\ell=0}^{n} \frac{\ell!\ x}{(x+\ell+1)_{n+1}} \prod_{i>j}(i-j) = \prod_{\ell=0}^{n} \frac{\ell!^2\ x}{(x+\ell+1)_{n+1}}.$$

By means of the Chu–Vandermonde convolution formula, the sum

$$\sum_{k=0}^{n} \frac{(x_k+k+1)_{n+1}}{x_k \prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}$$

becomes

$$\sum_{k=0}^{n}(-1)^k \frac{(x+k+1)_{n+1}}{k!\,(n-k)!\ x} = \frac{x+n+1}{x}\sum_{k=0}^{n}\binom{-x-n-2}{k}\binom{x+n}{n-k} = (-1)^n\frac{x+n+1}{x}(n+1).$$

Consequently, we find a closed formula for the following Hankel determinant.

COROLLARY 3.5 (Hankel determinant).

$$\det_{0\le i,j\le n}\left[\frac{1+i+j}{1+i+j+x}\right] = (-x)^n\{nx+(n+1)^2\}\prod_{k=0}^{n}\frac{k!^2}{(x+k+1)_{n+1}}.$$

Moreover, for $\xi = y = 1$ and $\sigma \to 0$, Theorem 3.1 reduces to the determinant formula.

COROLLARY 3.6 (Determinant evaluation).

$$\det_{0\le i,j\le n}\left[\sum_{k=0}^{i+j}\frac{x_i}{x_i+k}\right] = \frac{1}{n!}\prod_{\ell=0}^{n}\frac{\ell!\ x_\ell}{(1+x_\ell+\ell)_n}\prod_{i>j}(x_i+i-x_j-j)$$
$$\times\sum_{k=0}^{n}\frac{(1+x_k+k)_n}{\prod_{\substack{\jmath=0\\ \jmath\neq k}}^{n}(x_\jmath+\jmath-x_k-k)}\sum_{\imath=0}^{k}\frac{1}{x_k+\imath}.$$

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When $\{x_k \equiv x\}_{k\geq 0}$ is a constant sequence, the last corollary evaluates the following Hankel determinant.

COROLLARY 3.7 (Hankel determinant).

$$\det_{0\leq i,j\leq n}\Big[\sum_{k=0}^{i+j}\frac{x}{x+k}\Big] = \frac{x^{n+1}}{n!^2}\prod_{\ell=0}^{n}\frac{\ell!^2}{(1+x+\ell)_n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{\imath=0}^{k}\frac{(1+x+k)_n}{x+\imath}.$$

This is equivalent to the Hankel determinant of the generalised harmonic numbers found recently by Chammam [5, Theorem 5.1]. By comparing this with (1.2), we find the following amazing binomial identity.

COROLLARY 3.8 (Binomial identity).

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{i=0}^{k}\frac{(1+x+k)_n}{x+i} = (-1)^n\frac{n!}{x}\Big\{1+\sum_{j=1}^{n}\frac{x(x+2j)}{j(x+j)}\Big\}.$$

In order to provide an independent proof for this unusual identity, we recall the inverse series relations discovered by Gould and Hsu [13]. For the two sequences $\{a_k, b_k\}_{k\geq 0}$, define the polynomials

$$\phi(y;0)\equiv 1 \quad \text{and} \quad \phi(y;n) = \prod_{k=0}^{n-1}(a_k + yb_k) \quad \text{for } n\in\mathbb{N}.$$

Then the Gould–Hsu inversions affirm that the system of linear equations

$$f(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\phi(k;n)g(k) \tag{3.1}$$

is equivalent to the system

$$g(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_k + kb_k}{\phi(n;k+1)}f(k). \tag{3.2}$$

This pair of inversions has been shown to be a powerful tool for proving binomial identities (see Chu [6]). If one binomial formula matches one of these two relations, then the dual binomial identity automatically corresponds to the other relation. To prove each is to prove both. Now, by specifying

$$\phi(y;n) = (1+x+y)_n \quad \text{and} \quad \begin{cases} f(n) = (-1)^n\dfrac{n!}{x}\Big\{1+\displaystyle\sum_{j=1}^{n}\frac{x(x+2j)}{j(x+j)}\Big\}, \\ g(k) = \displaystyle\sum_{i=0}^{k}\frac{1}{x+i}, \end{cases}$$

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we can see that (3.1) becomes exactly the binomial identity in Corollary 3.8. Hence, to prove it, it suffices to confirm its dual formula corresponding to (3.2):

$$\sum_{k=0}^{n}\binom{n}{k}\frac{1+x+2k}{(1+x+n)_{k+1}}\left\{1+\sum_{j=1}^{k}\frac{x(x+2j)}{j(x+j)}\right\}\frac{k!}{x}=\sum_{i=0}^{n}\frac{1}{x+i}.$$

Denote by Λ the above double sum on the left. We may reformulate it as

$$\Lambda=\sum_{k=0}^{n}\frac{(1+x+2k)\langle n\rangle_k}{(1+x+n)_{k+1}}\left\{\frac{1}{x}+\sum_{j=1}^{k}\frac{(x+2j)}{j(x+j)}\right\}.$$

Define further the sequence T_k and compute its difference by

$$T_k=\frac{\langle n\rangle_k}{(1+x+n)_k}\quad\text{and}\quad T_k-T_{k-1}=\frac{(1+x+2k)\langle n\rangle_k}{(1+x+n)_{k+1}}.$$

Then we can evaluate the double sum by telescoping:

$$\begin{aligned}\Lambda&=\sum_{k=0}^{n}(T_k-T_{k+1})\left\{\frac{1}{x}+\sum_{j=1}^{k}\frac{(x+2j)}{j(x+j)}\right\}\\&=\frac{1}{x}\sum_{k=0}^{n}(T_k-T_{k+1})+\sum_{j=1}^{n}\frac{(x+2j)}{j(x+j)}\sum_{k=j}^{n}(T_k-T_{k+1})\\&=\frac{T_0-T_{n+1}}{x}+\sum_{j=1}^{n}\frac{(x+2j)}{j(x+j)}(T_j-T_{n+1}).\end{aligned}$$

Since $T_{n+1}=0$, we get the expression

$$\Lambda=\frac{1}{x}+\sum_{j=1}^{n}\frac{(x+2j)\langle n\rangle_j}{j(x+j)(1+x+n)_j}.$$

Let $R(x)$ stand for the last sum with respect to j, which results in a rational function in x whose common denominator consists of distinct linear factors. Therefore, we can decompose it into partial fractions

$$R(x)=\sum_{i=1}^{n}\left\{\frac{A_i}{x+i}+\frac{B_i}{x+n+i}\right\},$$

where the connection coefficients are determined by

$$A_i=\lim_{x\to -i}(x+i)R(x)=\frac{\langle n\rangle_i}{(1+n-i)_i}=1$$

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and

$$\begin{aligned}
B_i &= \lim_{x\to -n-i}(x+n+i)R(x)\\
&= \lim_{x\to -n-i}\sum_{j=i}^{n}\frac{(x+2j)\langle n\rangle_j}{j(x+j)(1+x+n)_{i-1}(1+x+n+i)_{j-i}}\\
&= \sum_{j=i}^{n}(-1)^i\binom{n}{j}\binom{j-1}{i-1}\frac{2j-n-i}{j-n-i}\\
&= \sum_{j=i}^{n}(-1)^i\binom{n}{j}\binom{j}{i}\frac{i(2j-n-i)}{j(j-n-i)}\\
&= (-1)^i\binom{n}{i}\sum_{j=i}^{n}\binom{n-i}{j-i}\frac{i(2j-n-i)}{j(j-n-i)}.
\end{aligned}$$

This last sum turns out to be zero because its reversal under $j \to n-j+i$ results in the same sum, but with the opposite sign.

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