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The structure of the bounded trajectories set of a scalar convex differential equation

A. I. Alonso and R. Obaya
ETSII Departamento de Matemática Aplicada a la Ingeniería,
Paseo del Cauce s/n. 47011, Universidad de Valladolid, Spain
(anaalo@wmatem.eis.uva.es; rafoba@wmatem.eis.uva.es)

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The present paper describes the topological and ergodic structure of the set of bounded trajectories of the flow defined by a scalar convex differential equation. We characterize the minimal subsets, the ergodic measures concentrated on them, and study the longtime behaviour of the bounded trajectories in terms of the Lyapunov exponents of the linearized equations. In particular, we obtain conditions that guarantee the existence of almost-periodic, almost-automorphic and recurrent solutions.

1. Introduction

Let (Ω, σ) be a minimal flow defined on a compact metric space, and write $\omega \cdot t = \sigma(t, \omega)$ for every $t \in \mathbb{R}$, $\omega \in \Omega$. We consider a continuous map $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

(i) $g(\omega, \lambda x_1 + (1-\lambda)x_2) \leqslant \lambda g(\omega, x_1) + (1-\lambda) g(\omega, x_2)$ for every $0 \leqslant \lambda \leqslant 1$ and $\omega \in \Omega, x_1, x_2 \in \mathbb{R}$, i.e. g is convex in the x component;

(ii) g is differentiable with respect to x and $\partial g/\partial x : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous.

In this paper we study the topological and ergodic structure of the bounded trajectories set of the family of differential equations

$$u' = g(\omega \cdot t, u), \quad \omega \in \Omega. \tag{1.1}$$

Equations (1.1) give rise to a local skew-product flow

$$\tau : U \subset \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}, \qquad \tau(t, \omega, x) = (\omega \cdot t, u(t, \omega, x)),$$

where $u(t, \omega, x)$ is the solution of (1.1) evaluated along the trajectory of ω with initial value x and t belongs to its maximal interval of definition. It is well known that if $u(t, \omega, x)$ remains bounded, then it is defined for every $t \in \mathbb{R}$. Let us consider

$$B = \Big\{ (\omega, x) \in \Omega \times \mathbb{R} \Big/ \sup_{t \in \mathbb{R}} |u(t, \omega, x)| < \infty \Big\}.$$

The map τ defines a global flow on B. Throughout this paper we assume $B \neq \emptyset$. We investigate the qualitative properties of the trajectories of B using techniques of random differential equations.

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Frequently, we get to the collective formulation (1.1) from a single differential equation. Let g_0, $\partial g_0/\partial x \in \mathcal{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and assume that they are bounded and uniformly continuous on the sets $\mathbb{R}\times I$, where I is a compact subset of $\mathbb{R}$. We consider the scalar equation

$$u' = g_0(t,u). \tag{1.2}$$

The function g_0 generates a family $\{g_{0,s}/s\in\mathbb{R}\}$ in $\mathcal{C}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, where $g_{0,s}(t,u) = g_0(t+s,u)$ denotes the time translation. Let Ω be the hull of g_0, namely the closure of $\{g_{0,s}/s\in\mathbb{R}\}$ in the compact open topology. It is easy to check that Ω is compact and, in fact, metrizable; moreover, the time translation $\omega\cdot t$, where $\omega\cdot t(s,u) = \omega(t+s,u)$, induces a natural flow on Ω. Notice that every $\omega\in\Omega$ inherits from g_0 the differentiable character in the second component. The function g_0 has a unique extension to a continuous function $g:\Omega\times\mathbb{R}\to\mathbb{R}$, $(\omega,u)\mapsto\omega(0,u)$; in particular, if $g_0=\omega_0$, then $g_0(t,u) = g(\omega_0\cdot t,u)$. Thus we obtain (1.2) by evaluating (1.1) along the trajectory of ω_0. Periodic, uniformly almost-periodic and uniformly almost-automorphic differential equations are included in our setting.

The previous collective formulation allows us to apply strong techniques of ergodic and topological nature to determine the minimal sets, the ergodic measures concentrated on them and the asymptotic behaviour of the trajectories of B. In this paper we only assume the recurrence of the trajectories of (Ω,σ). In some cases, additional properties of this flow can be directly translated to the minimal subsets of (B,τ). Roughly speaking, we can say that the behaviour of the bounded solutions of all the convex equations (1.1) whose restriction on B is not linear is close to the one exhibited by the corresponding solutions of the quadratic equations.

Many results have been devoted to the existence of periodic solutions for (1.2). It is interesting to study the dependence on a parameter of the number of periodic solutions for some Riccati-type equations, related also to some population models with logistic growth (see [15, 17]). For higher orders, this kind of theorem has been motivated by Ambrosetti-Prodi [2] (see [18] for a recent review of this theory).

Linear and quadratic almost-periodic differential equations have also been extensively studied in the literature from a topological and ergodic point of view (see [1, 7, 9, 10, 13, 19–22] and many others). A systematic study of the bounded trajectories set of the Riccati equation obtained from the $2n$-dimensional disconjugate linear Hamiltonian systems can be found in [11, 12]. We now extend some of the qualitative properties described in the above references to the solutions of the scalar convex differential equation.

The paper is organized as follows. In § 2 we recall basic results on topological dynamics and ergodic theory, which allow us to formulate and develop the main contents. Section 3 contains essential properties of scalar convex differential equations based on the monotone behaviour of the variational equation. We obtain a complete description of the ergodic structure of B, assuming the existence of an ergodic measure ν with $\int_B \partial g/\partial x \,\mathrm{d}\nu = 0$.

Section 4 is devoted to the analysis of the qualitative behaviour of the trajectories of B when its section at every point of the base is an interval of positive length. When B is bounded, and hence compact, we distinguish two cases in our analysis. In the hyperbolic case, i.e. when the Lyapunov exponents do not vanish, the upper and lower tops of B determine the only minimal subsets of the flow. The rest of the trajectories move from the upper to the lower top as t goes from $-\infty$ to $+\infty$.

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In the parabolic case, i.e. with null Lyapunov exponents, the restriction on B of the convex differential equation (1.1) is linear. The flow (B,τ) is a distal extension of the base (Ω,σ) and decomposes into a complete collection of minimal subsets, all of them being a 1-cover of the base. When B is unbounded, all the Lyapunov exponents vanish, the restriction of g on B is linear and so the above description, of the parabolic case, remains valid.

Section 5 studies the topological and ergodic structure of B when its section at some point of Ω reduces to a single element. This means that (B,τ) contains a unique minimal set, M, which is an almost-automorphic extension of (Ω,σ). We now fix an ergodic measure m on Ω. If B is a 1-cover of the base, even if its section is a single element almost everywhere, then its ergodic structure is easy to obtain. In consequence, we proceed to analyse this question when the section is, almost everywhere, a non-degenerate interval. When B is essentially bounded and the Lyapunov exponents with respect to m are different from zero, then the upper and lower tops of B determine the only ergodic measures projecting onto m. There is an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that the intermediate trajectories of $B \cap \Pi^{-1}(\Omega_1)$ move from the upper to the lower top as t goes from $-\infty$ to $+\infty$. When B is essentially bounded and the Lyapunov exponents with respect to m vanish, then there is an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that the restriction of the convex differential equation (1.1) on $B \cap \Pi^{-1}(\Omega_1)$ is linear and its coefficients have a continuous representation on M. Besides, (B,τ) admits absolutely continuous invariant measures and decomposes into a complete collection of ergodic sheets. On the other hand, if B is not essentially bounded, then all the Lyapunov exponents with respect to m vanish and the above dynamical description remains valid.

Finally, it is interesting to remark that if g, $\partial g/\partial x : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous and g is concave in the second component, then $v(t,\omega,x) = u(-t,\omega,x)$ is a solution of the differential equation

$$v' = -g(\omega \cdot (-t), v),$$

which satisfies the properties of the family (1.1). The conclusions of this paper carry over to a concave differential equation.

2. Preliminaries

In this section we recall some basic notions of ergodic theory and topological dynamics that will be essential in the description of the main results of the paper.

Let Ω be a compact metric space. A *real continuous flow* on Ω is defined by a continuous mapping $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t,\omega) \mapsto \sigma(t,\omega)$ satisfying $\sigma_0 = \mathrm{Id}$ and $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for all $t, s \in \mathbb{R}$, where $\sigma_t(\omega) = \sigma(t,\omega)$. The *orbit* or *trajectory* of a point $\omega \in \Omega$ is given by the set $\{\sigma_t(\omega)/t \in \mathbb{R}\}$; a subset $\Delta \subset \Omega$ is said to be *invariant* if it contains the orbit of all its points. A compact invariant subset $\Delta \subset \Omega$ is *minimal* if it contains no non-empty proper closed invariant subset. The flow (Ω,σ) is *recurrent* or *minimal* if Ω itself is a minimal subset, i.e. all its orbits are dense.

Let $\omega_0 \in \Omega$. The sets

$$\omega(\omega_0) = \bigcap_{s \geqslant 0} \mathrm{cls}\{\sigma(t+s,\omega_0)/t \geqslant 0\},$$

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$$\alpha(\omega_0) = \bigcap_{s \leqslant 0} \mathrm{cls}\{\sigma(t+s,\omega_0)/t \leqslant 0\}$$

are called the ω-limit set and the α-limit set of ω_0, respectively. Both sets are compact invariant subsets of Ω.

Most of the concepts and results we state below can be found in [4] and [30]. Let d be a metric on Ω. We say that a subset $A \subset \mathbb{R}$ is *relatively dense* if there is a compact $K \subset \mathbb{R}$ such that $\mathbb{R} = A + K$. A point $\omega \in \Omega$ is called *almost periodic* if, for every $\varepsilon > 0$, the subset $A_\varepsilon = \{t \in \mathbb{R}/d(\omega, \sigma(t,\omega)) < \varepsilon\}$ is relatively dense. A point ω is *almost periodic* if and only if its trajectory is recurrent, i.e. $\Delta = \mathrm{cls}\{\sigma(t,\omega)/t \in \mathbb{R}\}$ is minimal. A flow (Ω,σ) decomposes into minimal subsets if and only if all their points are almost periodic.

Two points $\omega_1, \omega_2 \in \Omega$ are said to be a *distal pair* if

$$\inf_{t\in\mathbb{R}} d(\sigma(t,\omega_1), \sigma(t,\omega_2)) > 0.$$

If ω_1, ω_2 are not distal, then they are a *proximal pair*. A point $\omega \in \Omega$ is said to be a *distal point* if it is only proximal to itself. The flow (Ω,σ) is *distal* when every point in Ω is distal. We say that the flow (Ω,σ) is *almost periodic* when, for every $\varepsilon > 0$, there is $\delta > 0$ such that, if $\omega_1, \omega_2 \in \Omega$ with $d(\omega_1,\omega_2) < \delta$, then $d(\sigma(t,\omega_1), \sigma(t,\omega_2)) < \varepsilon$ for every $t \in \mathbb{R}$.

If (Γ,τ) is another continuous flow, a *flow homomorphism* from (Γ,τ) to (Ω,σ) is a continuous mapping $\Pi : \Gamma \to \Omega$ such that $\Pi(\tau(t,\gamma)) = \sigma(t,\Pi(\gamma))$ for all $\gamma \in \Gamma$ and $t \in \mathbb{R}$. An onto flow homomorphism is called a *flow epimorphism*; in addition, if $k \in \mathbb{N}$ and $\mathrm{card}\, \Pi^{-1}(\omega) = k$ for every $\omega \in \Omega$, we say that (Γ,τ) is a k*-cover* of (Ω,σ). If $k = 1$, the flows (Γ,τ) and (Ω,σ) are isomorphic. If Π is an epimorphism, then (Ω,σ) is called a *factor* of (Γ,τ) and (Γ,τ) is called an *extension* of (Ω,σ).

Let $\Pi : (\Gamma,\tau) \to (\Omega,\sigma)$ be a flow epimorphism and suppose that (Ω,σ) is minimal. We say that (Γ,τ) is a *distal extension* of (Ω,σ) if each pair $\gamma_1, \gamma_2 \in \Gamma$ with $\Pi(\gamma_1) = \Pi(\gamma_2)$ is a distal pair. We say that (Γ,τ) is an *almost-automorphic extension* of (Ω,σ) if it is minimal and there is a $\omega \in \Omega$ such that $\mathrm{card}\, \Pi^{-1}(\omega) = 1$. A minimal flow (Γ,τ) is *almost automorphic* if it is an almost-automorphic extension of an almost-periodic minimal flow (Ω,σ) (see [29]).

A Borel measure on Ω will be a finite regular measure defined on the Borel sets. Let m be a normalized Borel measure on Ω. m is *invariant* under σ if $m(\sigma_t(\Delta)) = m(\Delta)$ for every Borel subset $\Delta \subset \Omega$ and every $t \in \mathbb{R}$. It is said that m is *ergodic* if it is invariant and $m(\Delta) = 0$ or $m(\Delta) = 1$ for every invariant subset $\Delta \subset \Omega$.

We will denote by $\mathcal{M}_{\mathrm{inv}}(\Omega,\sigma)$ the set of positive and normalized invariant measures on Ω; its extremal points are ergodic measures. Phelps [24] and Mañé [16] describe different ways of reconstruction of invariant measures from the ergodic ones. A flow is said to be *uniquely ergodic* if it admits a unique invariant measure. We write $\mathcal{C}_0(\Omega)$ for the set of continuous functions, f, such that $\int_\Omega f\,\mathrm{d}m = 0$ for every $m \in \mathcal{M}_{\mathrm{inv}}(\Omega,\sigma)$.

We next assume that (Ω,σ) is minimal and fix an ergodic measure m on Ω. We consider a skew-product flow τ defined on a compact subset $B \subset \Omega \times \mathbb{R}$, i.e. a continuous map $\tau : \mathbb{R} \times B \to B$, $(t,(\omega,x)) \mapsto (\sigma(t,\omega), u(t,\omega,x))$ satisfying $\tau_0 = \mathrm{Id}_B$ and $\tau_{t+s} = \tau_t \circ \tau_s$ for every $s,t \in \mathbb{R}$. We also assume that B has a simple

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geometry, u admits partial derivative with respect to x and the map $\mathbb{R} \times B \to \mathbb{R}$, $(t, \omega, x) \mapsto u_x(t, \omega, x)$ is continuous.

The projection $\Pi : B \to \Omega$, $(\omega, x) \mapsto \omega$ defines a flow epimorphism $\Pi : (B, \tau) \to (\Omega, \sigma)$. Let $M \subset B$ be a compact minimal subset. It is well known that (M, τ) is an almost-automorphic extension of the base (Ω, σ) ([27] includes a more general version of this result, valid for strongly monotone semiflows). A minimal subset $M \subset B$ is *hyperbolic* if there exists $\alpha \in \mathbb{R}$ such that, for every $(\omega, x) \in M$, either $\limsup_{|t| \to \infty} \ln u_x(t, \omega, x)/t \leqslant \alpha < 0$ or $0 < \alpha \leqslant \liminf_{|t| \to \infty} \ln u_x(t, \omega, x)/t$. Every hyperbolic minimal subset (M, τ) is a 1-cover of the base (Ω, σ).

Let $\mu \in \mathcal{M}_{\mathrm{inv}}(B, \tau)$. We say that μ projects onto m if $\mu(\Pi^{-1}(\Delta)) = m(\Delta)$ for every Borel subset $\Delta \subset \Omega$. We denote by $\mathcal{M}_{\mathrm{inv},m}(B, \tau)$ the set of positive and normalized invariant measures on B projecting onto m.

Let μ be an invariant measure on B projecting onto m. We say that μ disintegrates into the family of real Borel measures $(\mu_\omega)_{\omega \in \Omega}$ if, for every $\varepsilon > 0$, there is a compact subset $\Omega_\varepsilon \subset \Omega$ with $m(\Omega_\varepsilon) > 1 - \varepsilon$ such that the restriction $(\mu_\omega)_{\omega \in \Omega_\varepsilon}$ is weakly continuous and

$$\int_B f \, \mathrm{d}\mu = \int_\Omega \left(\int_{\mathbb{R}} f(\omega, x) \, \mathrm{d}\mu_\omega \right) \mathrm{d}m \quad \text{for every } f \in \mathcal{C}(B).$$

We refer to μ as an *absolutely continuous* invariant measure (respectively, *singular continuous* or *purely discontinuous*) when μ_ω is absolutely continuous with respect to the Lebesgue measure l on $\mathbb{R}$ (respectively, singular continuous or purely discontinuous) for almost every $\omega \in \Omega$. Let $r = m \times l$ be the product measure on B. Notice that μ is absolutely continuous if and only if it is absolutely continuous with respect to r.

Assume that the map $x : \Omega \to \mathbb{R}$ is measurable and there is an invariant subset $\Omega_0 \subset \Omega$ with $m(\Omega_0) = 1$ such that $x(\omega \cdot t) = u(t, \omega, x(\omega))$ for every $\omega \in \Omega_0, t \in \mathbb{R}$. Then $N = \{(\omega, x(\omega))/\omega \in \Omega\}$ is an ergodic sheet of (B, τ). A suitable interpretation of [5, theorem 4.1] shows that every ergodic measure of (B, τ) is purely discontinuous and it is concentrated on an ergodic sheet (see also [3]).

Let us consider $r(B) > 0$ and take $p \in L^1(B, r)$. The measure $\mathrm{d}\mu = p \, \mathrm{d}r$ is invariant under τ if and only if its density function satisfies

$$p((\omega \cdot t, u(t, \omega, x)) \cdot u_x(t, \omega, x) = p(\omega, x)$$

for almost every $(\omega, x) \in B$ and $t \in \mathbb{R}$. The existence of absolutely continuous invariant measures depends on the behaviour of the functions $(p_T)_{T>0}$ defined by

$$p_T(\omega, x) = \frac{1}{2T} \int_{-T}^{T} u_x(t, \omega, x) \, \mathrm{d}t.$$

Theorem 5.4 of [22] asserts that (B, τ) admits an absolutely continuous invariant measure if and only if there is an invariant subset $D \subset B$ with $r(D) > 0$ such that the limit $p(\omega, x) = \lim_{T \to \infty} p_T(\omega, x)$ exists and is a positive real number for every $(\omega, x) \in D$. (The same conclusion is valid if we define p_T by means of the positive or negative averages $(1/T) \int_0^T u_x(t, \omega, x) \, \mathrm{d}t$ or $(1/T) \int_{-T}^0 u_x(t, \omega, x) \, \mathrm{d}t$, respectively.)

3. Some remarks on the convex differential equation

Throughout the paper, (Ω, σ) stands for a minimal flow and we consider a continuous function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ differentiable and convex in the second component with $\partial g/\partial x : \Omega \times \mathbb{R} \to \mathbb{R}$ continuous. We study the behaviour of the bounded trajectories of the family of differential equations

$$u' = g(\omega \cdot t, u), \quad \omega \in \Omega.$$

Maintaining the notation we have detailed before, we introduce

$$B = \Big\{(\omega, x) \in \Omega \times \mathbb{R} \Big/ \sup_{t \in \mathbb{R}} |u(t, \omega, x)| < \infty\Big\}.$$

We assume that $B \neq \emptyset$; if $(\omega_0, x_0) \in B$, then $\operatorname{cls}\{(\omega_0 \cdot t, u(t, \omega_0, x_0))/t \in \mathbb{R}\} \subset B$, which means that $\Pi^{-1}(\omega) \cap B \neq \emptyset$ for every $\omega \in \Omega$. The map $\tau : \mathbb{R} \times B \to B$, $(t, \omega, x) \mapsto (\omega \cdot t, u(t, \omega, x))$ defines a skew-product flow on B. In this section, we fix an ergodic measure m on Ω.

We say that α is a *Lyapunov exponent with respect to* m if there exists an ergodic measure ν on B projecting onto m such that $\alpha = \lim_{|t| \to \infty} (1/t) \ln u_x(t, \omega, x)$ for almost every $(\omega, x) \in B$ with respect to ν. Birkhoff's ergodic theorem leads us to $\alpha = \int_B \partial g/\partial x \, d\nu$.

Our first objective is to obtain some basic relations which are a direct consequence of the monotonicity of the variational equation.

Let $u(t, \omega, x_1)$ and $u(t, \omega, x_2)$ be two different bounded solutions of (1.1) with $x_1 < x_2$. Then

$$\begin{aligned} u(t, \omega, x_2) - u(t, \omega, x_1) &= (u(0, \omega, x_2) - u(0, \omega, x_1)) \frac{\partial u}{\partial x}(t, \omega, x^*) \\ &= (x_2 - x_1) \exp\bigg(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x^*)) \, ds\bigg) \end{aligned}$$

where $x_1 < x^*(t, \omega) < x_2$. Since the function $\partial g/\partial x$ is increasing in the x component, we deduce that

$$\frac{u(t, \omega, x_2) - u(t, \omega, x_1)}{x_2 - x_1} \geqslant \exp\bigg(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x_1)) \, ds\bigg) \quad \text{for } t > 0, \tag{3.1}$$

$$\frac{u(t, \omega, x_2) - u(t, \omega, x_1)}{x_2 - x_1} \leqslant \exp\bigg(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x_1)) \, ds\bigg) \quad \text{for } t < 0. \tag{3.2}$$

Similarly,

$$\frac{u(t, \omega, x_2) - u(t, \omega, x_1)}{x_2 - x_1} \leqslant \exp\bigg(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x_2)) \, ds\bigg) \quad \text{for } t > 0, \tag{3.3}$$

$$\frac{u(t, \omega, x_2) - u(t, \omega, x_1)}{x_2 - x_1} \geqslant \exp\bigg(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x_2)) \, ds\bigg) \quad \text{for } t < 0. \tag{3.4}$$

In the extended real line, we define

$$\begin{aligned} x_1(\omega) &= \operatorname{ext\,inf}\{x/(\omega, x) \in B\} \in \mathbb{R} \cup \{-\infty\}, \\ x_2(\omega) &= \operatorname{ext\,sup}\{x/(\omega, x) \in B\} \in \mathbb{R} \cup \{+\infty\}. \end{aligned}$$

A function is called *essentially bounded* with respect to m if it is bounded almost everywhere. We say that B is *essentially bounded* with respect to m when the functions $\omega \mapsto x_i(\omega)$, $i = 1, 2$, are essentially bounded, i.e. there exists a positive finite constant c and an invariant subset $\Omega_0 \subset \Omega$ with $m(\Omega_0) = 1$ such that $|x_i(\omega)| \leqslant c$ for every $\omega \in \Omega_0$, $i = 1, 2$.

If B is essentially bounded, then $(\omega, x_i(\omega)) \in B$, $i = 1, 2$, for almost every $\omega \in \Omega$. The invariant measures ν_1, ν_2 defined by $\int_B f \, \mathrm{d}\nu_i = \int_\Omega f(\omega, x_i(\omega)) \, \mathrm{d}m$, $i = 1, 2$, for every $f \in \mathcal{C}(B)$ are ergodic measures concentrated on B and projecting onto m.

We now analyse the measurable structure of B when zero is a Lyapunov exponent with respect to m.

THEOREM 3.1. *Let us assume that B is essentially bounded, $r(B) > 0$ and there exists an ergodic measure ν_0 on B projecting onto m with $\int_B \partial g/\partial x(\omega, x) \, \mathrm{d}\nu_0 = 0$. Then (B, τ) admits a finite invariant measure μ equivalent to the Lebesgue measure and decomposes into a complete collection $(N_j)_{j \in J}$ of ergodic sheets. Besides, one has $\int_B \partial g/\partial x(\omega, x) \, \mathrm{d}\nu = 0$ for every $\nu \in \mathcal{M}_{\mathrm{inv},m}(B, \tau)$.*

Proof. Let ν_0 be an ergodic measure concentrated on B with

$$\int_B \frac{\partial g}{\partial x}(\omega, x) \, \mathrm{d}\nu_0 = 0.$$

It follows from [5, theorem 4.1] that ν_0 is purely discontinuous and it is concentrated on an ergodic sheet $N = \{(\omega, n(\omega))/\omega \in \Omega\}$.

Assume ν_0 does not coincide either with ν_1 or ν_2. Thus there is an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ and $x_1(\omega) < n(\omega) < x_2(\omega)$ for every $\omega \in \Omega_1$. We know that

$$\lim_{|t| \to \infty} \frac{1}{t} \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, n(\omega \cdot s)) \, \mathrm{d}s = 0 \quad \text{for almost every } \omega \in \Omega.$$

We now analyse the oscillations of the integrals $\int_0^t \partial g/\partial x(\omega \cdot s, n(\omega \cdot s)) \, \mathrm{d}s$. A careful study of the behaviour of these integrals and their consequences can be found, for instance, in [1, 6–8]. Let $\omega \in \Omega_1$. We deduce from (3.1) that

$$\frac{x_2(\omega \cdot t) - n(\omega \cdot t)}{x_2(\omega) - n(\omega)} \geqslant \exp\left(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, n(\omega \cdot s)) \, \mathrm{d}s\right) \quad \text{for } t > 0.$$

Notice that the condition $\limsup_{t \to +\infty} \int_0^t \partial g/\partial x(\omega \cdot s, n(\omega \cdot s)) \, \mathrm{d}s = +\infty$ implies that $\limsup_{t \to +\infty}(x_2(\omega \cdot t) - n(\omega \cdot t)) = +\infty$, in contradiction with the boundedness of the trajectories. We conclude that $\sup_{t \geqslant 0} \int_0^t \partial g/\partial x(\omega \cdot s, n(\omega \cdot s)) \, \mathrm{d}s < \infty$, and hence we can apply [1, theorem 6.3] to deduce the existence of a positive measurable function $h : \Omega \to \mathbb{R}$ satisfying

$$h(\omega \cdot t) = h(\omega) \exp\left(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, n(\omega \cdot s)) \, \mathrm{d}s\right) \tag{3.5}$$

almost everywhere. We can admit that the relation (3.5) holds for every $\omega \in \Omega_1$ and $t \in \mathbb{R}$. From Birkhoff's ergodic theorem, we deduce the existence of $0 < h^* \leqslant \infty$ such that

$$h^* = \lim_{|T| \to \infty} \frac{1}{T} \int_0^T h(\omega \cdot s) \, \mathrm{d}s \tag{3.6}$$

for almost every $\omega \in \Omega$; moreover, $h^* = \int_\Omega h(\omega) \, \mathrm{d}m$.

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We follow arguments taken from [22] to construct an invariant measure μ concentrated on B and equivalent to the Lebesgue measure. We consider the invariant subset $C_2 = \{(\omega, x)/n(\omega) \leqslant x \leqslant x_2(\omega)\}$ and introduce on C_2 the family of functions $(p_T)_{T>0}$ defined by

$$p_T(\omega, x) = \frac{1}{T}\int_0^T u_x(t, \omega, x)\,\mathrm{d}t = \frac{1}{T}\int_0^T \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr)\,\mathrm{d}t \tag{3.7}$$

for every $(\omega, x) \in C_2$. It is easy to check that

$$\begin{aligned} p_T(\omega\cdot l, u(l, \omega, x)) \cdot \exp\biggl(\int_0^l \frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr) \\ = \frac{T+l}{T} p_{T+l}(\omega, x) - \frac{1}{T}\int_0^l \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr)\,\mathrm{d}t. \end{aligned} \tag{3.8}$$

We take $p^* = \liminf_{T\to\infty} p_T$ on C_2. Notice that

$$p^*(\omega, n(\omega)) = \frac{1}{h(\omega)} \liminf_{T\to\infty} \frac{1}{T}\int_0^T h(\omega\cdot s)\,\mathrm{d}s = \frac{h^*}{h(\omega)}$$

for almost every $\omega \in \Omega$. Since the function $\partial g/\partial x(\omega, x)$ is increasing in x, there is an invariant subset $\Omega_2 \subset \Omega_1$ with $m(\Omega_2) = 1$ such that $p^*(\omega, x) \geqslant p^*(\omega, n(\omega)) > 0$ for every $\omega \in \Omega_2$ and $n(\omega) \leqslant x \leqslant x_2(\omega)$. We deduce, according to Fatou's lemma, that

$$0 < \lambda_2 = \int_{C_2} p^*\,\mathrm{d}r \leqslant \liminf_{T\to\infty} \int_{C_2} p_T\,\mathrm{d}r = r(C_2) < +\infty.$$

From (3.8), we derive

$$p^*(\omega\cdot t, u(t, \omega, x)) \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr) = p^*(\omega, x).$$

In consequence, $\mathrm{d}\mu^* = p^*\chi_{C_2}\,\mathrm{d}r$ is an invariant measure equivalent to the Lebesgue measure on C_2. Note that we also conclude that $h \in L^1(\Omega, m)$.

We now consider the invariant subset $C_1 = \{(\omega, x)/x_1(\omega) \leqslant x \leqslant n(\omega)\}$ and introduce on C_1 the family $(p_T)_{T<0}$ defined by the relation (3.7) with $T < 0$. Note that

$$p_T(\omega, x) = -\frac{1}{T}\int_T^0 u_x(t, \omega, x)\,\mathrm{d}t = -\frac{1}{T}\int_T^0 \exp\biggl(\int_t^0 -\frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr)\,\mathrm{d}t$$

for every $(\omega, x) \in C_1$. We take $p_* = \liminf_{T\to-\infty} p_T$ on C_1. We have that

$$p_*(\omega, n(\omega)) = \frac{1}{h(\omega)} \liminf_{T\to-\infty} \frac{1}{T}\int_0^T h(\omega\cdot s)\,\mathrm{d}s = \frac{h^*}{h(\omega)}$$

for almost every $\omega \in \Omega$. Since the function $-\partial g/\partial x(\omega, x)$ is decreasing in x, there is an invariant subset (that we represent again by Ω_2) with $m(\Omega_2) = 1$ such that $p_*(\omega, x) \geqslant p_*(\omega, n(\omega)) > 0$ for every $\omega \in \Omega_2$ and $x_1(\omega) < x < n(\omega)$. We know that

$$\lambda_1 = \int_{C_1} p_*\,\mathrm{d}r \leqslant \liminf_{T\to-\infty} \int_{C_1} p_T\,\mathrm{d}r < \infty.$$

Besides,

$$p_*(\omega\cdot t, u(t,\omega,x))\exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s,\omega,x))\,\mathrm{d}s\biggr) = p_*(\omega,x).$$

In consequence, $\mathrm{d}\mu_* = p_*\chi_{C_1}\,\mathrm{d}r$ is an invariant measure equivalent to the Lebesgue measure on C_1. Finally, if

$$p = \frac{1}{\lambda_1+\lambda_2}(p_*\chi_{C_1} + p^*\chi_{C_2}),$$

then the measure $\mathrm{d}\mu = p\,\mathrm{d}r$ is normalized invariant and equivalent to r on B.

On the other hand, in the case that ν_0 coincides with one of the ergodic measures ν_1 or ν_2, it suffices to apply these arguments in one of the sectors C_1 or C_2 previously defined to construct the measure μ equivalent to r.

Now we describe the ergodic measures concentrated on B. Note that if $\omega\in\Omega_2$, then $p(\omega,x)\geqslant p(\omega,n(\omega))>0$ for every $(\omega,x)\in B$. This implies that the measure $\mathrm{d}\mu = p(\omega,x)\,\mathrm{d}x$ is equivalent to the Lebesgue measure in the interval $[x_1(\omega),x_2(\omega)]$. Moreover,

$$\int_{x_1(\omega)}^{x_2(\omega)} p(\omega,x)\,\mathrm{d}x = 1$$

for almost every $\omega\in\Omega$ and we can admit that it holds for every $\omega\in\Omega_2$. Let $J=[1,2]$. For each $\omega\in\Omega_2$ and $j\in J$, we represent by $x_j(\omega)$ the real number verifying

$$\int_{x_1(\omega)}^{x_j(\omega)} p(\omega,x)\,\mathrm{d}x = j-1.$$

The map $x_j:\Omega\to\mathbb{R}$, $\omega\mapsto x_j(\omega)$ is measurable and $x_j(\omega\cdot t) = u(t,\omega,x_j(\omega))$ for every $\omega\in\Omega_2$ and $t\in\mathbb{R}$. In consequence, $N_j = \{(\omega,x_j(\omega))/\omega\in\Omega_2\}$ defines an ergodic sheet for every $j\in J$. Besides, $B\cap(\Omega_2\times\mathbb{R}) = \bigcup_{j\in J} N_j$. Therefore, $B=\bigcup_{j\in J} N_j$ almost everywhere with respect to every ergodic measure projecting onto m, i.e. B decomposes into ergodic sheets. Arguments based on the inequalities (3.1) and (3.4) allow us to conclude that

$$0 = \int_\Omega \frac{\partial g}{\partial x}(\omega,x_j(\omega))\,\mathrm{d}m \quad \text{if } 1<j<2.$$

Moreover,

$$\begin{aligned}\alpha_1 &= \int_\Omega \frac{\partial g}{\partial x}(\omega,x_1(\omega))\,\mathrm{d}m = \lim_{j\to 1^+}\int_\Omega \frac{\partial g}{\partial x}(\omega,x_j(\omega))\,\mathrm{d}m = 0,\\ \alpha_2 &= \int_\Omega \frac{\partial g}{\partial x}(\omega,x_2(\omega))\,\mathrm{d}m = \lim_{j\to 2^-}\int_\Omega \frac{\partial g}{\partial x}(\omega,x_j(\omega))\,\mathrm{d}m = 0.\end{aligned}$$

Therefore, we have verified that $\int_B \partial g/\partial x(\omega,x)\,\mathrm{d}\nu = 0$ for every ergodic measure concentrated on B and projecting onto m. Since every invariant measure is the limit of a convex combination of ergodic measures in the weak topology, we also obtain $\int_B \partial g/\partial x(\omega,x)\,\mathrm{d}\nu = 0$ for every $\nu\in\mathcal{M}_{\mathrm{inv},m}(B,\tau)$. □

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4. Case I: int(B) is not void

In this section we proceed with the study of the dynamical structure of B assuming that, for every $\omega \in \Omega$, the section $\{x \in \mathbb{R}/(\omega, x) \in B\}$ has strictly positive measure. We determine the minimal subsets, the ergodic measures and the longtime behaviour of the trajectories in terms of the Lyapunov exponents of the flow.

THEOREM 4.1. *Let us assume that B is bounded and $x_1(\omega) \neq x_2(\omega)$ for every $\omega \in \Omega$. Then B contains at least two minimal subsets M_1, M_2, given by*

$$M_i = \{(\omega, x_i(\omega))/\omega \in \Omega\}, \quad i = 1, 2.$$

For each $m \in \mathcal{M}_{\rm inv}(\Omega, \sigma)$, the invariant measures $\nu_1 = \nu_1(m)$, $\nu_2 = \nu_2(m)$ defined by $\int_B f \, \mathrm{d}\nu_i(m) = \int_\Omega f(\omega, x_i(\omega)) \, \mathrm{d}m$, $i = 1, 2$, for every $f \in \mathcal{C}(B)$, are ergodic measures concentrated on B and projecting onto m. There exist real numbers $\alpha_1(m) \leqslant 0 \leqslant \alpha_2(m)$ such that

$$\alpha_i(m) = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_i(\omega)) \, \mathrm{d}m, \quad i = 1, 2.$$

Let us fix $m_0 \in \mathcal{M}_{\rm inv}(\Omega, \sigma)$. One of the two following assertions holds.

(i) *If $\alpha_1(m_0) = \alpha_2(m_0) = 0$, there are functions $a \in \mathcal{C}_0(\Omega)$, $b \in \mathcal{C}(\Omega)$ such that $g(\omega, x) = a(\omega)x + b(\omega)$ for every $(\omega, x) \in B$ and $\alpha_1(m) = \alpha_2(m) = 0$ for every $m \in \mathcal{M}_{\rm inv}(\Omega, \sigma)$. In addition, B decomposes into a complete collection $(M_j)_{j \in J}$ of 1-cover minimal subsets and every bounded solution of (1.1) is recurrent.*

(ii) *If $\alpha_1^2(m_0) + \alpha_2^2(m_0) > 0$, then $\alpha_1(m) < 0 < \alpha_2(m)$ for every $m \in \mathcal{M}_{\rm inv}(\Omega, \sigma)$. Furthermore, M_1, M_2 are the only minimal subsets of B and $\nu_1(m)$, $\nu_2(m)$ are the only ergodic measures concentrated on B projecting onto m. In addition, if $(\omega, x) \in B$ with $x_1(\omega) < x < x_2(\omega)$, then*

$$\lim_{t \to \infty} (u(t, \omega, x) - x_1(\omega \cdot t)) = \lim_{t \to -\infty} (x_2(\omega \cdot t) - u(t, \omega, x)) = 0.$$

Proof. Since B is bounded, and hence compact, one has that $(\omega, x_i(\omega)) \in B$ for every $\omega \in \Omega$, $i = 1, 2$, and we can write $B = \bigcup_{\omega \in \Omega} \{\omega\} \times [x_1(\omega), x_2(\omega)]$. Notice that the maps $x_i : \Omega \to \mathbb{R}$, $\omega \mapsto x_i(\omega)$, $i = 1, 2$, are semicontinuous, so there exists a residual invariant subset $\Omega_0 \subset \Omega$ of points of continuity. Let us define

$$M_i = \operatorname{cls}\{(\omega, x_i(\omega))/\omega \in \Omega_0\}, \quad i = 1, 2,$$

and consider the projections $\Pi_{M_i} : M_i \to \Omega$, $(\omega, x) \mapsto \omega$, $i = 1, 2$. The minimal subsets M_1, M_2 are almost-automorphic extensions of the base Ω; in fact, one has $\operatorname{card} \Pi_{M_1}^{-1}(\omega) = \operatorname{card} \Pi_{M_2}^{-1}(\omega) = 1$ for every $\omega \in \Omega_0$ and consequently $M_1 \cap M_2 = \emptyset$. It is obvious that the measures $\nu_1(m)$, $\nu_2(m)$, defined for each $m \in \mathcal{M}_{\rm inv}(\Omega, \sigma)$ in the statement of the theorem, are ergodic measures concentrated on B and projecting onto m.

Let us fix $m_0 \in \mathcal{M}_{\rm inv}(\Omega, \sigma)$. We are going to prove the existence of a unique ergodic measure concentrated on M_1 and projecting onto m_0. Suppose, contrary to

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our claim, that there exists an invariant subset $\Omega_1 \subset \Omega$ with $m_0(\Omega_1) = 1$ such that $\operatorname{card} \Pi^{-1}_{M_1}(\omega) \geqslant 2$ for every $\omega \in \Omega_1$, and define

$$u_2(\omega) = \sup\{x/(\omega, x) \in M_1\}, \qquad u_1(\omega) = \inf\{x/(\omega, x) \in M_1\}$$

for each $\omega \in \Omega$. Clearly, $u_1(\omega) < u_2(\omega) < x_2(\omega)$ for every $\omega \in \Omega_1$. Set

$$\beta = \int_\Omega \frac{\partial g}{\partial x}(\omega, u_2(\omega))\, \mathrm{d}m_0.$$

From Birkhoff's ergodic theorem, we conclude the existence of an invariant subset $\Omega_1^* \subset \Omega_1$ with $m_0(\Omega_1^*) = 1$ such that, if $\omega \in \Omega_1^*$, one has

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s = \beta.$$

Assume first that $\beta > 0$. If $\omega \in \Omega_1^*$, then

$$\frac{x_2(\omega \cdot t) - u_2(\omega \cdot t)}{x_2(\omega) - u_2(\omega)} \geqslant \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s\biggr) \quad \text{for } t > 0.$$

Hence $\lim_{t\to\infty}(x_2(\omega \cdot t) - u_2(\omega \cdot t)) = +\infty$, which contradicts the condition that both trajectories $(\omega \cdot t, u_2(\omega \cdot t))$, $(\omega \cdot t, x_2(\omega \cdot t))$ remain in B.

Consider now that $\beta = 0$. It follows from (3.1) that there is a constant c such that, for every $\omega \in \Omega_1^*$ and each pair of real numbers $t_1 < t_2$, one has

$$\int_{t_1}^{t_2} \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s \leqslant c. \tag{4.1}$$

Since $\int_\Omega \partial g/\partial x(\omega, u_2(\omega))\, \mathrm{d}m_0 = 0$, we deduce, from the recurrence property stated in [28], that, for almost every $\omega \in \Omega_1^*$, we can find a sequence $(t_n)_{n=1}^\infty$ with $\liminf_{n\to\infty} t_n = -\infty$ and $\limsup_{n\to\infty} t_n = +\infty$ such that

$$\int_0^{t_n} \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s = 0 \quad \text{for every } n \in \mathbb{N}.$$

We select one of these elements $\omega \in \Omega_1^*$ and the corresponding sequence $(t_n)_{n=1}^\infty$. Let us assume $t^* > 0$ with

$$\int_0^{t^*} \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s < -c$$

and fix $n_0 \in \mathbb{N}$ with $t_{n_0} > t^*$. Then

$$\int_{t^*}^{t_{n_0}} \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s > c,$$

which contradicts (4.1). Let us consider the existence of $t^* < 0$ with

$$\int_{t^*}^0 \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s < -c$$

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and fix $n_0 \in \mathbb{N}$ with $t_{n_0} < t^*$. Then

$$\int_{t_{n_0}}^{t^*} \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s > c,$$

which again contradicts (4.1). Thus we conclude that, for almost every $\omega \in \Omega_1^*$, one has the inequality

$$\left| \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s \right| \leqslant c$$

for every $t \in \mathbb{R}$. Since the trajectory $(\omega \cdot t, u_2(\omega \cdot t))_{t \in \mathbb{R}}$ is dense in M_1, we obtain that

$$\left| \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x))\, \mathrm{d}s \right| \leqslant c \tag{4.2}$$

for every $(\omega, x) \in M_1$ and $t \in \mathbb{R}$. From (4.2), we deduce that the flow (M_1, τ) is a distal extension of (Ω, σ). Hence there is $\delta > 0$ such that $|u_2(\omega \cdot t) - u_1(\omega \cdot t)| \geqslant \delta$ for every $\omega \in \Omega_1$ and $t \in \mathbb{R}$. We conclude that $|u_2(\omega) - u_1(\omega)| \geqslant \delta$ for every $\omega \in \Omega$, which contradicts the condition that M_1 is an almost-automorphic extension of the base (Ω, σ). Therefore, the option $\beta = 0$ can not occur. (This result is also consequence of the theory of Johnson [7] and Sacker-Sell [25].)

Finally, assume that $\beta < 0$. If $\omega \in \Omega_1^*$, then

$$\frac{u_2(\omega \cdot t) - u_1(\omega \cdot t)}{u_2(\omega) - u_1(\omega)} \geqslant \exp\left(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u_2(\omega \cdot s))\, \mathrm{d}s \right) \quad \text{for } t < 0.$$

Hence $\lim_{t \to -\infty}(u_2(\omega \cdot t) - u_1(\omega \cdot t)) = +\infty$, which contradicts the condition that both trajectories $(\omega \cdot t, u_1(\omega \cdot t))$, $(\omega \cdot t, u_2(\omega \cdot t))$ remain in B.

Consequently, we have actually proved that $\operatorname{card} \Pi_{M_1}^{-1}(\omega) = 1$ on a set of complete measure. The same arguments and conclusion apply to (M_2, τ).

For each measure $m \in \mathcal{M}_{\mathrm{inv}}(\Omega, \sigma)$, let us introduce the Lyapunov exponent

$$\alpha_i(m) = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_i(\omega))\, \mathrm{d}m.$$

From inequalities (3.1) and (3.4), we deduce that α_1 and α_2 do not have the same sign, i.e. $\alpha_1(m) \leqslant 0 \leqslant \alpha_2(m)$.

Case 1. We first assume that $\alpha_1 \cdot \alpha_2 = 0$. In this case, theorem 3.1 leads to $\alpha_1 = \alpha_2 = 0$, i.e. both Lyapunov exponents vanish. As we have detailed above, there are positive constants c_1, c_2 such that

$$\left| \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x_1))\, \mathrm{d}s \right| \leqslant c_1, \tag{4.3}$$

$$\left| \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x_2))\, \mathrm{d}s \right| \leqslant c_2 \tag{4.4}$$

for every $(\omega, x_1) \in M_1$, $(\omega, x_2) \in M_2$ and $t \in \mathbb{R}$, which implies that M_1, M_2 are 1-covers of the base (Ω, σ). We can write $M_i = \{(\omega, u_i(\omega))/\omega \in \Omega\}$, $i = 1, 2$. Note that $u_i(\omega) = x_i(\omega)$ for every $\omega \in \Omega_0$, $i = 1, 2$. Let us assume that there is $\omega_1 \in \Omega$ with $x_2(\omega_1) > u_2(\omega_1)$. Since (Ω, σ) is minimal, we can fix $\omega_0 \in \Omega_0$ and a sequence of

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real numbers $(t_n)_{n=1}^{\infty}$ verifying that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \omega_1 \cdot t_n = \omega_0$. The inequality (3.1) provides the existence of $\delta > 0$ such that $x_2(\omega_1 \cdot t_n) - u_2(\omega_1 \cdot t_n) > \delta$ for every $n \in \mathbb{N}$. Taking limits as $n \to \infty$, we deduce that $x_2(\omega_0) - u_2(\omega_0) \geqslant \delta$, which is impossible. The same arguments could be applied to M_1; consequently, we can identify $M_i = \{(\omega, x_i(\omega))/\omega \in \Omega\}$, $i = 1, 2$.

In addition, since the function $\partial g/\partial x$ is increasing in x, we deduce from (4.3) and (4.4) that if $c = \max(c_1, c_2)$, then

$$\left| \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x))\,\mathrm{d}s \right| \leqslant c$$

for every $(\omega, x) \in B$, $t \in \mathbb{R}$, which shows that (B, τ) is a distal extension of (Ω, σ).

Let $\omega \in \Omega$ and take a sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers. The family of real maps $\tau_{t_n}(\omega) : B \cap \Pi^{-1}(\omega) \to B \cap \Pi^{-1}(\omega \cdot t_n)$, $x \mapsto u(t_n, \omega, x)$ is equicontinuous and uniformly bounded. Then it follows from Arzela–Ascoli's theorem that they admit a subsequence that converges in the uniform topology to an injective limit. Let us consider $(\omega_0, x_0) \in B$ and take $M_0 = \mathrm{cls}\{(\omega_0 \cdot t, u(t, \omega_0, x_0))/t \in \mathbb{R}\}$. From the above discussion, it is easy to check that there is $k \in \mathbb{N} \cup \{\infty\}$ such that $\mathrm{card}(M_0 \cap \Pi^{-1}(\omega)) = k$ for every $\omega \in \Omega$. We set

$$x_2 = \sup\{x/(\omega_0, x) \in M_0\}, \qquad x_1 = \inf\{x/(\omega_0, x) \in M_0\}$$

and assume that $k > 1$, $x_2 > x_1$. There exists a sequence $(t_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} (\omega_0 \cdot t_n, u(t_n, \omega_0, x_1)) = (\omega_0, x_2);$$

besides, $u(t_n, \omega_0, x_2) - u(t_n, \omega_0, x_0) \geqslant \delta$ for some $\delta > 0$ and every $n \in \mathbb{N}$, which, letting $n \to \infty$, contradicts the definition of x_2. Thus we deduce that $k = 1$ and so every point of B is almost periodic and (B, τ) decomposes into a complete collection $(M_j)_{j\in J}$ of minimal subsets, being all of them a 1-cover of the base (Ω, σ).

Finally, since the function $\partial g/\partial x$ is increasing in x, we obtain

$$\frac{\partial g}{\partial x}(\omega, x_2(\omega)) - \frac{\partial g}{\partial x}(\omega, x_1(\omega)) \geqslant 0$$

for every $\omega \in \Omega$. Besides,

$$\int_\Omega \frac{\partial g}{\partial x}(\omega, x_2(\omega))\,\mathrm{d}m_0 = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_1(\omega))\,\mathrm{d}m_0 = 0,$$

and hence

$$\int_\Omega \left(\frac{\partial g}{\partial x}(\omega, x_2(\omega)) - \frac{\partial g}{\partial x}(\omega, x_1(\omega)) \right) \mathrm{d}m_0 = 0,$$

which allows us to state that $\partial g/\partial x(\omega, x_2(\omega)) = \partial g/\partial x(\omega, x_1(\omega))$ for almost every $\omega \in \Omega$. Any two continuous functions which coincide on a dense coincide everywhere, so $\partial g/\partial x(\omega, x_2(\omega)) = \partial g/\partial x(\omega, x_1(\omega))$ for every $\omega \in \Omega$. It follows that there exists $a \in \mathcal{C}_0(\Omega)$, $b \in \mathcal{C}(\Omega)$ with

$$\frac{\partial g}{\partial x}(\omega, x) = a(\omega) \quad \text{and} \quad g(\omega, x) = a(\omega)x + b(\omega)$$

for every $\omega \in \Omega$. We have now that $\alpha_1(m) = \alpha_2(m) = \int_\Omega a(\omega)\,\mathrm{d}m = 0$ for every $m \in \mathcal{M}_{\mathrm{inv}}(\Omega, \sigma)$.

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Case 2. Let us analyse the case $\alpha_1(m_0) < 0 < \alpha_2(m_0)$. We know that $\alpha_1(m) < 0 < \alpha_2(m)$ for every $m \in \mathcal{M}_{\text{inv}}(\Omega, \sigma)$. Notice that there are constants $\alpha' > 0$ and $t_0 > 0$ such that

$$\frac{1}{t}\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, x_2(\omega\cdot s))\,\mathrm{d}s > \alpha'$$

for every $\omega \in \Omega$ and $t \geqslant t_0$ (otherwise we would find $m \in \mathcal{M}_{\text{inv}}(\Omega, \sigma)$ with $\int_\Omega \partial g/\partial x(\omega, x_2(\omega))\,\mathrm{d}m \leqslant 0$). From (3.1), we obtain $\lim_{t\to\infty} u(t, \omega, x) = +\infty$ for every $x > x_2(\omega)$, thus we can identify $M_2 = \{(\omega, x_2(\omega))/\omega \in \Omega\}$. Similarly, $\lim_{t\to-\infty} u(t, \omega, x) = -\infty$ for every $x < x_1(\omega)$ and we can therefore also write $M_1 = \{(\omega, x_1(\omega))/\omega \in \Omega\}$.

We verify that M_1, M_2 are the only minimal subsets contained in B. We argue by contradiction and assume the existence of a third minimal subset M_3. From inequalities (3.1) and (3.4), we obtain that $\int_{M_3} \partial g/\partial x(\omega, x)\,\mathrm{d}\nu = 0$ for every invariant measure concentrated on M_3. Then theorem 3.1 leads us to the case $\alpha_1(m) = \alpha_2(m) = 0$, which is a contradiction.

Both minimal sets M_1, M_2 are hyperbolic. Besides, a simply analysis of the variational linear equation shows that if $(\omega, x) \in M_1$ (respectively, $(\omega, x) \in M_2$), then $u(t, \omega, x)$ is uniformly and asymptotically stable when $t \to +\infty$ (respectively, $t \to -\infty$). It follows that M_1 and M_2 are attractors when $t \to +\infty$ and $t \to -\infty$, respectively.

Let us fix $\omega \in \Omega$, $x \in \mathbb{R}$, with $x_1(\omega) < x < x_2(\omega)$. The ω-limit set of (ω, x) contains a minimal subset different from M_2; therefore, it agrees with M_1. Analogously, the α-limit set of (ω, x) agrees with M_2. Hence

$$\lim_{t\to+\infty}(u(t, \omega, x) - x_1(\omega\cdot t)) = \lim_{t\to-\infty}(x_2(\omega\cdot t) - u(t, \omega, x)) = 0.$$

In consequence,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (f(\omega\cdot s, u(s, \omega, x)) - f(\omega\cdot s, x_1(\omega\cdot s)))\,\mathrm{d}s = 0$$

and

$$\lim_{t\to-\infty}\frac{1}{t}\int_0^t (f(\omega\cdot s, x_2(\omega\cdot s)) - f(\omega\cdot s, u(s, \omega, x)))\,\mathrm{d}s = 0$$

for every $f \in \mathcal{C}(B)$. For each ergodic measure m on Ω, we see that $\nu_1(m)$, $\nu_2(m)$ are the only ergodic measures on B projecting on m.

Finally, notice that if $\omega \in \Omega$ and $x_2(\omega) < x$, then

$$\lim_{t\to+\infty} u(t, \omega, x) = +\infty \quad\text{and}\quad \lim_{t\to-\infty}(u(t, \omega, x) - x_2(\omega\cdot t)) = 0,$$

and if $\omega \in \Omega$, $x < x_1(\omega)$, then

$$\lim_{t\to+\infty}(x_1(\omega\cdot t) - u(t, \omega, x)) = 0 \quad\text{and}\quad \lim_{t\to-\infty} u(t, \omega, x) = -\infty.$$

Consequently, in the hyperbolic case, M_1, M_2 contain the only recurrent trajectories of B. □

We now analyse the same question when B is not bounded. The topological and ergodic structure admit a simpler description.

THEOREM 4.2. *Let us assume that B is not bounded and $x_1(\omega) \neq x_2(\omega)$ for every $\omega \in \Omega$. There are functions $a \in \mathcal{C}_0(\Omega)$, $b \in \mathcal{C}(\Omega)$ such that $g(\omega, x) = a(\omega)x + b(\omega)$ for every $(\omega, x) \in B$. In consequence, B decomposes into a complete collection $(M_j)_{j\in J}$ of 1-cover minimal subsets.*

Proof. The procedure is to construct a sequence of differential equations that agree with the original (1.1) on wide sectors of $\Omega \times \mathbb{R}$ and allows us to apply theorem 4.1.

For each $j \in \mathbb{N}$, we define

$$h_j(x) = \begin{cases} 1 + \sup\left\{\left|\dfrac{\partial g}{\partial x}(\omega, s)\right| \Big/ \omega \in \Omega,\ |s| = |x|\right\} & \text{if } x \geqslant j+1, \\ \lambda h_j(j+1) & \text{if } x = j + \lambda \text{ and } \lambda \in [0,1], \\ 0 & \text{if } 0 \leqslant x \leqslant j, \\ -h_j(-x) & \text{if } x \leqslant 0. \end{cases}$$

Notice that $h_j(x)$ is a continuous and increasing real function. For each $j \in \mathbb{N}$, we take

$$g_j(\omega, x) = g(\omega, x) + \int_0^x h_j(s)\,\mathrm{d}s.$$

It is easy to check that $g_j(\omega, x) = g(\omega, x)$ if $-j \leqslant x \leqslant j$. Besides, g_j is still convex in the x component and, for every $j \in \mathbb{N}$, there is an $n_j \in \mathbb{N}$, $n_j > j$, such that $g_j(\omega, x) \geqslant 1$ whenever $|x| \geqslant n_j$. To see this last statement, notice that for $x > j+1$ we can write

$$\begin{aligned} \int_0^x h_j(s)\,\mathrm{d}s &= \int_j^{j+1} h_j(s)\,\mathrm{d}s + \int_{j+1}^x h_j(s)\,\mathrm{d}s \\ &\geqslant c_j + \int_{j+1}^x \left(1 - \frac{\partial g}{\partial x}(\omega, s)\right)\mathrm{d}s \\ &= c_j + x - j - 1 - g(\omega, x) + g(\omega, j+1), \end{aligned}$$

where $c_j = \int_j^{j+1} h_j(s)\,\mathrm{d}s$ is a positive constant. In this way, we have

$$g_j(\omega, x) = g(\omega, x) + \int_0^x h_j(s)\,\mathrm{d}s \geqslant c_j + x - j - 1 + g(\omega, j+1)$$

and, letting $x \to +\infty$, we obtain $g_j(\omega, x) \geqslant 1$ for x large enough. Analogously, letting $x \to -\infty$, we get $g_j(\omega, x) \geqslant 1$ for negative x with $|x|$ large enough. Altogether, we can find $n_j > j$ such that $g_j(\omega, x) \geqslant 1$ if $|x| > n_j$.

We now consider the family of differential equations

$$u' = g_j(\omega \cdot t, u), \quad \omega \in \Omega, \tag{4.5}$$

and the skew-product flow defined by (4.5) on $\Omega \times \mathbb{R}$. Let $u_j(t, \omega, x)$ be the solution of (4.5) along the trajectory of ω with initial value x and

$$B_j = \left\{(\omega, x) \in \Omega \times \mathbb{R} \Big/ \sup_{t\in\mathbb{R}} |u_j(t, \omega, x)| < +\infty\right\}.$$

Notice that, given $(\omega, x) \in B$, there exists a $j_0 \in \mathbb{N}$ such that $|u(t, \omega, x)| \leqslant j_0$ for every $t \in \mathbb{R}$, and then, for every $j \geqslant j_0$, $u(t, \omega, x)$ is a bounded solution of (4.5),

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that is, $(\omega,x)\in B_j$. In other words, we can write

$$B\subset\bigcup_{j=1}^{\infty}\bigcap_{m\geqslant j}B_m=\liminf_{j\to\infty}B_j. \tag{4.6}$$

In particular, we obtain that B_j is not empty for $j\geqslant j_0$. Furthermore, the sets B_j are bounded; in fact, it is easy to check that $B_j\subset\Omega\times[-n_j,n_j]$. As a consequence, we can write $B_j=\bigcup_{\omega\in\Omega}\{\omega\}\times[x_{j,1}(\omega),x_{j,2}(\omega)]$, as we did in the previous theorem.

Let us assume that, for every $j\in\mathbb{N}$, there is $k_j>j$ and $\omega_j\in\Omega$ with $x_{k_j,1}(\omega_j)=x_{k_j,2}(\omega_j)$. Define $\Omega_j=\{\omega\in\Omega/x_{k_j,1}(\omega)=x_{k_j,2}(\omega)\}$, which are residual invariant sets. Consider the set $\Omega_0=\bigcap_{j=1}^{\infty}\Omega_j$, which is also residual. If we take $\omega\in\Omega_0$, then $x_{k_j,1}(\omega)=x_{k_j,2}(\omega)$ for every $j\in\mathbb{N}$, which proves, according to (4.6), that $B\cap\Pi^{-1}(\omega)$ is at most a countable set, contradicting the fact that the section of B at every point $\omega\in\Omega$ is a non-degenerate interval. We can conclude the existence of an index, we assume that it agrees with j_0, such that, if $j\geqslant j_0$, we have that $x_{j,1}(\omega)\neq x_{j,2}(\omega)$ for every $\omega\in\Omega$.

We are now in a position to apply theorem 4.1 to the sets B_j, $j\geqslant j_0$. Actually, we are going to see that all the sets are in the parabolic case. Remember that in the hyperbolic case there are exactly two minimal subsets in B (the tops), which are attractors, one at $+\infty$ and the other at $-\infty$, in a way that every intermediate trajectory should connect the lower and the upper top of the set of bounded trajectories.

Begin by taking $(\omega_0,x_0)\in B$ with $x_1(\omega_0)<x_0<x_2(\omega_0)$ and consider $M_0=\text{cls}\{(\omega_0\cdot t,u(t,\omega_0,x_0)/t\in\mathbb{R}\}$. Let us assume, without loss of generality, that B is not bounded above. Then we can take $(\omega_1,x_1)\in B$, satisfying $x_1>x$ for every $(\omega,x)\in M_0$, and we can further consider $M_1=\text{cls}\{(\omega_1\cdot t,u(t,\omega_1,x_1))/t\in\mathbb{R}\}$, which is a compact set such that $M_1\not\subset M_0$. Under the assumption made in B, we can still take $(\omega_2,x_2)\in B$ with $x_2>x$ for every $(\omega,x)\in M_1$, and consider its orbit $o(\omega_2,x_2)=\{(\omega_2\cdot t,u(t,\omega_2,x_2))/t\in\mathbb{R}\}$, which is bounded.

Let us fix $j_1\geqslant j_0$ such that, if $(\omega,x)\in M_0\cup M_1\cup o(\omega_2,x_2)$, then $|x|\leqslant j_1$, and consequently $M_0\cup M_1\cup o(\omega_2,x_2)\subset B_j$ for every $j\geqslant j_1$. According to the remarks just mentioned, this can only happen if B_j is in the parabolic case, so B_j decomposes into a complete collection of 1-cover minimal subsets. Furthermore, the equations are linear over B_j, i.e. $g_j(\omega,x)=a_j(\omega)x+b_j(\omega)$ for every $(\omega,x)\in B_j$. Besides, if we represent $M_0=\{(\omega,x_0(\omega))/\omega\in\Omega\}$, then

$$a_j(\omega)=\frac{\partial g_j}{\partial x}(\omega,x_0(\omega))=\frac{\partial g}{\partial x}(\omega,x_0(\omega)),$$
$$b_j(\omega)=g(\omega,x_0(\omega))-a_j(\omega)x_0(\omega),$$

and it turns out that the coefficients $a_j(\omega)$, $b_j(\omega)$ only depend on the function g, that is, $g_j(\omega,x)=a(\omega)x+b(\omega)$ for every $j\geqslant j_1$.

Now, given $(\omega,x)\in B$, we take $j\geqslant j_1$ such that $|u(t,\omega,x)|\leqslant j$ for every $t\in\mathbb{R}$, so that $(\omega,x)\in B_j$ and $g(\omega,x)=g_j(\omega,x)=a(\omega)x+b(\omega)$. In this way, we conclude that the equation is linear over B, as stated. Besides, it is also immediate that B decomposes into a complete collection of minimal subsets. ☐

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REMARK 4.3. Maintaining the hypothesis and the notation of theorem 4.2, we will show that the representation (4.6) is, in fact, an equality, that is,

$$B = \bigcup_{j=1}^{\infty} \bigcap_{m\geqslant j} B_m = \liminf_{j\to\infty} B_j.$$

We know that $B \subset \bigcup_{j=1}^{\infty} \bigcap_{m\geqslant j} B_m$, thus it is only necessary to prove the opposite inclusion. Let $M_0 = \{(\omega, x_0(\omega))/\omega \in \mathbb{R}\}$ be a minimal subset of B and take an index $j_0 \in \mathbb{N}$ such that $j_0 > |x_0(\omega)|$ for every $\omega \in \Omega$. This implies that $\bigcap_{j\geqslant j_0} B_j \neq \emptyset$.

We write

$$B' = \bigcup_{j=1}^{\infty} \bigcap_{m\geqslant j} B_m$$

and take $(\omega_0, x_0) \in B'$ with $x_0 \geqslant x_0(\omega_0)$. There are integers $j_2 \geqslant j_1 \geqslant j_0$ such that $(\omega_0, x_0) \in \bigcap_{j=j_1}^{\infty} B_j$ and $j_2 > x_{j_1,2}(\omega)$ for every $\omega \in \Omega$. Then

$$g(\omega_0, x_0) = g_{j_2}(\omega_0, x_0) = a(\omega_0)x_0 + b(\omega_0),$$

which proves that $(\omega_0, x_0) \in B$. When $(\omega_0, x_0) \in B'$ and $x_0 \leqslant x_0(\omega)$, we can use similar arguments to conclude that $(\omega_0, x_0) \in B$.

5. Case II: int(B) is the empty set

In this section we proceed with the study of the dynamical behaviour of B assuming the existence of an element $\omega_0 \in \Omega$, such that the section $\{x \in \mathbb{R}/(\omega_0, x) \in B\}$ becomes a single point. The structure found is a weak version of the topological one described in § 4.

THEOREM 5.1. *Let us assume that there exists $\omega_0 \in \Omega$ such that $x_1(\omega_0) = x_2(\omega_0)$. Then B contains a unique minimal subset M and there is a residual invariant subset $\Omega_0 \subset \Omega$ with $x_1(\omega) = x_2(\omega)$ for every $\omega \in \Omega_0$. Moreover, under one of the following hypothesis,*

(i) $\sup_{t\in\mathbb{R}} |\int_0^t \partial g/\partial x(\omega\cdot s, u(s,\omega,x))\,\mathrm{d}s| < \infty$ *for some $(\omega, x) \in B$; or*

(ii) *M is an hyperbolic minimal subset*

one has that $\Omega_0 = \Omega$ and $B = M$ is a 1-cover minimal subset.

Proof. The closure of every trajectory of B is a compact invariant subset and contains a minimal subset. In these conditions, it is obvious that B contains a unique minimal subset M and (M, τ) is an almost-automorphic extension of the base (Ω, σ).

We write $\Omega_0 = \{\omega \in \Omega/x_1(\omega) = x_2(\omega)\}$. Certainly, Ω_0 is an invariant subset. Let us first assume that hypothesis (i) holds. We accept the existence of an element $(\omega_1, x_1) \in B$ such that

$$c = \sup_{t\in\mathbb{R}} \left| \int_0^t \frac{\partial g}{\partial x}(\omega_1 \cdot s, u(s, \omega_1, x_1))\,\mathrm{d}s \right| < \infty.$$

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This implies that

$$\left|\int_{t_1}^{t_2} \frac{\partial g}{\partial x}(\omega_1 \cdot s, u(s, \omega_1, x_1))\,\mathrm{d}s\right| \leqslant 2c$$

for every $t_1, t_2 \in \mathbb{R}$ and, since the ω-limit set of (ω_1, x_1) contains M, we obtain that

$$\left|\int_{0}^{t} \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x))\,\mathrm{d}s\right| \leqslant 2c$$

for every $(\omega, x) \in M$ and $t \in \mathbb{R}$. Accordingly, we conclude that M is a 1-cover minimal subset, i.e. $M = \{(\omega, x(\omega))/\omega \in \Omega\}$. We show that $M = B$. We argue by contradiction and assume the existence of $\omega_1 \in \Omega$ with $x(\omega_1) < x_1 < x_2(\omega_1)$. We can find a sequence $(t_n)_{n=1}^{\infty}$ of real numbers with $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \omega_1 \cdot t_n = \omega_0$. From the inequality (3.1), it follows that

$$x_2(\omega_0) - x_1(\omega_0) \geqslant \liminf_{n\to\infty}(u(t_n, \omega_1, x_1) - x(\omega_1 \cdot t_n)) \geqslant \delta > 0,$$

which is impossible. The relation $x_1(\omega_1) < x < x(\omega_1)$ leads us to the same contradiction. Thus we conclude that $M = B$ and $x_1(\omega) = x_2(\omega)$ for every $\omega \in \Omega$.

We now analyse the structure of B when M is an hyperbolic minimal subset. We assume that M is attractor as $t \to +\infty$ and there exists an element $(\omega_1, x_1) \in B - M$. Then the compact subset N defined by the α-limit points of (ω_1, x_1) is a compact invariant subset with $N \cap M = \emptyset$, which is a contradiction. We again conclude that $B = M$ and $x_1(\omega) = x_2(\omega)$ for every $\omega \in \Omega$.

We next study the case where Ω_0 is a proper invariant subset of Ω. Consider the family of differential equations

$$u' = g_j(\omega \cdot t, u), \quad \omega \in \Omega,$$

defined by (4.5) in the previous section, and the skew-product flow induced by them on $\Omega \times \mathbb{R}$. We write

$$B_j = \left\{(\omega, x) \in \Omega \times \mathbb{R} \Big/ \sup_{t\in\mathbb{R}} |u_j(t, \omega, x)| < \infty\right\}$$

and let $j_0 \in \mathbb{N}$ be such that $j_0 > |x|$ for every $(\omega, x) \in M$, then $B \subset \cup_{j\geqslant j_0} B_j$. Notice that $\partial g/\partial x(\omega, x) = \partial g_j/\partial x(\omega, x)$ for every $(\omega, x) \in M$, $j \geqslant j_0$. We write $B_j = \bigcup_{\omega\in\Omega}\{\omega\} \times [x_{j,1}(\omega), x_{j,2}(\omega)]$. Let us assume that there is $j \geqslant j_0$ such that $x_{j,1}(\omega) \neq x_{j,2}(\omega)$ for every $\omega \in \Omega$. We conclude from theorem 4.1 that M satisfies hypotheses (i) or (ii) of the statement, which implies that $\Omega_0 = \Omega$, contrary to the assumption. Hence we deduce that, for every $j \geqslant j_0$, there is $\omega_j \in \Omega$ with $x_{j,1}(\omega_j) = x_{j,2}(\omega_j)$. Let us denote by Ω_j the sets of points of continuity of the maps $x_{j,1}, x_{j,2} : \Omega \to \mathbb{R}$. Notice that $x_{j,1}(\omega) = x_{j,2}(\omega)$ for every $\omega \in \Omega_j$, $j \geqslant j_0$. The set $\Omega_0' = \bigcap_{j\geqslant j_0} \Omega_j$ is invariant and residual; moreover, if $\omega \in \Omega_0'$, then

$$\{x_{j,1}(\omega)\} = \{x_{j,2}(\omega)\} = \Pi^{-1}(\omega) \cap M$$

for every $j \geqslant j_0$. Thus we conclude that $x_1(\omega) = x_2(\omega)$ and $\Omega_0' \subset \Omega_0$. This proves that Ω_0 is a residual invariant subset of Ω. □

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It is well known that $M \subset B$ is an almost-automorphic extension, but not necessarily a 1-copy of the base (Ω, σ). We consider the minimal compact flow (M, τ) and define the map $\tilde{g} : M \times \mathbb{R} \to \mathbb{R}$, $(\tilde{\omega}, x) \mapsto g(\Pi_M(\tilde{\omega}), x)$. The family of differential equations

$$u' = \tilde{g}(\tilde{\omega} \cdot t, u), \quad \tilde{\omega} \in M, \tag{5.1}$$

give rise to a skew-product flow

$$\tilde{\tau} : \mathbb{R} \times M \times \mathbb{R} \to M \times \mathbb{R}, \tilde{\tau}(t, \tilde{\omega}, x) = (\tilde{\omega} \cdot t, u(t, \Pi_M(\tilde{\omega}), x)),$$

which allows us to study the trajectories of (1.1) on the base (M, τ). Notice that

$$\begin{aligned}\tilde{B} &= \Big\{(\tilde{\omega}, x) \in M \times \mathbb{R} \Big/ \sup_{t \in \mathbb{R}} |u(t, \Pi_M(\tilde{\omega}), x)| < \infty\Big\} \\ &= \{(\tilde{\omega}, x) \in M \times \mathbb{R} / (\Pi_M(\tilde{\omega}), x) \in B\}.\end{aligned}$$

In particular, $x_1(\tilde{\omega}) = x_2(\tilde{\omega})$ if and only if $x_1(\Pi_M(\tilde{\omega})) = x_2(\Pi_M(\tilde{\omega}))$ and $\tilde{B}$ contains a unique minimal subset $\tilde{M} = \{(\omega, x, x) \in \tilde{B} / (\omega, x) \in M\}$, which is a 1-cover of the base.

It is interesting to remark the advantages that, in some cases, presents the topological structure of the new flow $(\tilde{B}, \tilde{\tau})$.

Throughout this section, we fix an ergodic measure m on Ω and we write $\Omega_0 = \{\omega \in \Omega / x_1(\omega) = x_2(\omega)\}$. We discuss the ergodic structure of B depending on the measure of the subset Ω_0.

THEOREM 5.2. *Let us assume that $x_1(\omega) = x_2(\omega)$ for almost every $\omega \in \Omega$. Then B is essentially bounded and the normalized invariant measure ν defined by $\int_B f \, \mathrm{d}\nu = \int_\Omega f(\omega, x_1(\omega)) \, \mathrm{d}m$ for every $f \in \mathcal{C}(B)$ is the only ergodic measure concentrated on B and projecting onto m.*

Moreover, if (Ω, B) is uniquely ergodic and, for almost every $\omega \in \Omega$, one has $\limsup_{t \to \infty} u(t, \omega, x) = \infty$ when $x > x_2(\omega)$ and $\liminf_{t \to -\infty} u(t, \omega, x) = -\infty$ when $x < x_1(\omega)$, then $\int_B \partial g / \partial x \, \mathrm{d}\nu = 0$.

Proof. We know that, in this situation, B contains a unique minimal subset M and $B \cap \Pi^{-1}(\omega) = M \cap \Pi^{-1}(\omega)$ for every $\omega \in \Omega_0$. This shows that B is essentially bounded and it is immediate that ν is the only ergodic measure concentrated on B and projecting onto m.

We now assume that (Ω, σ) is uniquely ergodic. We first consider the case $\lim_{|x| \to \infty} g(\omega, x) = +\infty$ for every $\omega \in \Omega$. Let us fix $\omega_1 \in \Omega$. There are real numbers $x_1 \in \mathbb{R}$ and $\delta_1 > 0$ with $\partial g / \partial x(\omega_1, x_1) > \delta_1 > 0$. We can find a neighbourhood $V(\omega_1)$ of ω_1 such that $\partial g / \partial x(\omega, x_1) \geqslant \delta_1$ for every $\omega \in V(\omega_1)$ and hence $\partial g / \partial x(\omega, x) \geqslant \delta_1$ for every $\omega \in V(\omega_1)$ and $x \geqslant x_1$. Notice that $\Omega = \bigcup_{\omega_1 \in \Omega} V(\omega_1)$ and it admits a finite covering. This provides $x' \in \mathbb{R}$, $\delta' > 0$, such that $\partial g / \partial x(\omega, x) \geqslant \delta'$ for every $\omega \in \Omega$, $x \geqslant x'$. The same arguments are also valid as $x \to -\infty$. In consequence, we conclude that $\lim_{|x| \to \infty} g(\omega, x) = +\infty$ uniformly on Ω. If $\alpha = \int_B \partial g / \partial x \, \mathrm{d}\nu$ is negative, i.e. $\alpha < 0$, then M is an hyperbolic minimal subset of B; in addition, M is an attractor as $t \to +\infty$. We take $\omega_1 \in \Omega$, $x_1 \in \mathbb{R}$, with $x_2(\omega_1) < x_1$ and $g(\omega, x) > 0$ for every $\omega \in \Omega$, $x \geqslant x_1$; then we have $u(t, \omega_1, x_1) \leqslant x_1$ for every $t \leqslant 0$ and $u(t, \omega_1, x_1) \geqslant x_1$ for every $t \geqslant 0$. Under these conditions, the compact invariant subset N defined by the α-limit points of

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ω_1 satisfies $N \cap M = \emptyset$, which is impossible. We obtain the same contradiction when $\alpha > 0$, and hence we conclude that $\alpha = 0$.

For almost every $\omega \in \Omega$, one has $\limsup_{t\to+\infty} u(t,\omega,x) = +\infty$ when $x > x_2(\omega)$ and $\liminf_{t\to-\infty} u(t,\omega,x) = -\infty$ when $x < x_1(\omega)$. Let $j_0 \in \mathbb{N}$ be such that $j_0 > |x|$ for every $(\omega,x) \in M$. We analyse the family of differential equations

$$u' = g_{j_0}(\omega\cdot t, u), \quad \omega \in \Omega,$$

defined by (4.5) and the skew-product flow induced by them on $\Omega \times \mathbb{R}$. Notice that $\lim_{|x|\to\infty} g_{j_0}(\omega,x) = +\infty$; besides, it is easy to check that $B \cap \Pi^{-1}(\omega) = B_{j_0} \cap \Pi^{-1}(\omega)$ for almost every $\omega \in \Omega$. We conclude, from our previous discussion, that

$$\int_\Omega \frac{\partial g_{j_0}}{\partial x}(\omega, x_1(\omega))\,\mathrm{d}m = \int_B \frac{\partial g}{\partial x}\,\mathrm{d}\nu = 0.$$

Finally, we want to mention that if (Ω,σ) is uniquely ergodic and $\int_B \partial g/\partial x\,\mathrm{d}\nu = \alpha \neq 0$, then M is an hyperbolic minimal subset. In these conditions, theorem 5.1 confirms that $\Omega_0 = \Omega$ and $B = M$. □

THEOREM 5.3. *Let us assume that Ω_0 is not void, B is essentially bounded and $x_1(\omega) \neq x_2(\omega)$ for almost every $\omega \in \Omega$. Then the invariant measures ν_1, ν_2 defined by $\int_B f\,\mathrm{d}\nu_i = \int_\Omega f(\omega, x_i(\omega))\,\mathrm{d}m$, $i = 1,2$, for every $f \in \mathcal{C}(B)$ are ergodic measures concentrated on B and projecting onto m. There exist real numbers $\alpha_1 \leqslant 0 \leqslant \alpha_2$ such that*

$$\alpha_i = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_i(\omega))\,\mathrm{d}m, \quad i = 1,2.$$

Besides

(i) *If $\alpha_1 = \alpha_2 = 0$, then there exist functions $a, b \in \mathcal{C}(M)$ and an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that $\tilde g(\tilde\omega, x) = a(\tilde\omega)x + b(\tilde\omega)$ in* $\mathrm{cls}\{(\tilde\omega,x) \in \tilde B / \Pi_M(\tilde\omega) \in \Omega_1\}$. *In addition, (B,τ) admits a normalized invariant measure μ equivalent to the Lebesgue measure and decomposes into a complete collection $(N_j)_{j\in J}$ of ergodic sheets.*

(ii) *If $\alpha_1^2 + \alpha_2^2 > 0$, then $\alpha_1 < 0 < \alpha_2$ and ν_1 and ν_2 are the only ergodic measures concentrated on B and projecting onto m. There is an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that if $(\omega,x) \in B \cap \Pi^{-1}(\Omega_1)$ and $x_1(\omega) < x < x_2(\omega)$, then*

$$\lim_{t\to\infty} (u(t,\omega,x) - x_1(\omega\cdot t)) = \lim_{t\to-\infty} (x_2(\omega\cdot t) - u(t,\omega,x)) = 0.$$

In addition, when (Ω,σ) is uniquely ergodic, one has $\nu_1(M) = \nu_2(M) = 1$.

Proof. It is obvious that ν_1, ν_2 are ergodic measures concentrated on B and projecting onto m. We recall that the Lyapunov exponents α_1, α_2 vanish simultaneously or are both different from zero.

We first consider the case $\alpha_1 = \alpha_2 = 0$. It follows from theorem 3.1 that (B,τ) admits a finite invariant measure equivalent to the Lebesgue measure and decomposes into a complete collection $(N_j)_{j\in J}$ of ergodic sheets. Since the function $\partial g/\partial x$

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is increasing in x, we obtain

$$\frac{\partial g}{\partial x}(\omega, x_2(\omega)) - \frac{\partial g}{\partial x}(\omega, x_1(\omega)) \geqslant 0$$

almost everywhere. Besides,

$$\int_\Omega \frac{\partial g}{\partial x}(\omega, x_2(\omega))\,\mathrm{d}m = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_1(\omega))\,\mathrm{d}m = 0,$$

and hence

$$\int_\Omega \left(\frac{\partial g}{\partial x}(\omega, x_2(\omega)) - \frac{\partial g}{\partial x}(\omega, x_1(\omega))\right) \mathrm{d}m = 0.$$

This assures the existence of an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that $\partial g/\partial x(\omega, x_2(\omega)) = \partial g/\partial x(\omega, x_1(\omega))$ for every $\omega \in \Omega_1$.

We now consider the family of differential equations defined on the new base (M, τ) by (5.1). The only minimal subset $\tilde{M} \subset \tilde{B}$ is a 1-cover of the base. For every $\tilde{\omega} \in M$, there is a unique $x(\tilde{\omega}) \in \mathbb{R}$ such that $(\tilde{\omega}, x(\tilde{\omega})) \in \tilde{M}$ (notice that if $\tilde{\omega} = (\omega, x)$, then $x(\tilde{\omega}) = x$). We define the functions

$$a(\tilde{\omega}) = \frac{\partial \tilde{g}}{\partial x}(\tilde{\omega}, x(\tilde{\omega})) \quad \text{and} \quad b(\tilde{\omega}) = \tilde{g}(\tilde{\omega}, x(\tilde{\omega})) - a(\tilde{\omega}) \cdot x(\tilde{\omega}).$$

Thus $a, b \in \mathcal{C}(M)$ and $\int_M a(\tilde{\omega}) d\nu = 0$ for every $\nu \in \mathcal{M}_{\mathrm{inv},m}(M, \tau)$; besides, $\tilde{g}(\tilde{\omega}, x) = a(\tilde{\omega})x + b(\tilde{\omega})$ for every $(\tilde{\omega}, x) \in \tilde{B}$ with $\Pi_M(\tilde{\omega}) \in \Omega_1$. This representation remains valid in the closure.

We now assume that the Lyapunov exponents satisfy $\alpha_1 < 0 < \alpha_2$. For every $T \in \mathbb{R}$, $T \neq 0$, we consider the function

$$p_T(\omega, x) = \frac{1}{T} \int_0^T u_x(t, \omega, x)\,\mathrm{d}t.$$

We first introduce

$$\begin{aligned} q^*(\omega, x) &= \limsup_{T \to \infty} p_T(\omega, x) \\ &= \limsup_{T \to \infty} \frac{1}{T} \int_0^T \exp\left(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x))\,\mathrm{d}s\right) \mathrm{d}t. \end{aligned}$$

We recall that, for every $(\omega, x) \in B$ with $x_1(\omega) < x < x_2(\omega)$, inequality (3.1) makes it obvious that

$$\limsup_{t \to \infty} \int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x))\,\mathrm{d}s < \infty,$$

and hence $0 \leqslant q^*(\omega, x) < \infty$. Besides, it is easy to check that

$$q^*(\omega \cdot t, u(t, \omega, x)) \exp\left(\int_0^t \frac{\partial g}{\partial x}(\omega \cdot s, u(s, \omega, x))\,\mathrm{d}s\right) = q^*(\omega, x).$$

Let

$$C = \{(\omega, x) \in B / 0 < q^*(\omega, x) < \infty\}$$

and assume that $r(C) > 0$. Then $\mathrm{d}\mu^* = q^*\,\mathrm{d}r$ is a σ-finite invariant measure; besides, $1/q^* \in L^1(C, \mathrm{d}\mu^*)$. The existence of $\lim_{T \to \infty} (1/T) \int_0^T 1/q^*(\omega \cdot s, u(s, \omega, x))\,\mathrm{d}s$ for almost every $(\omega, x) \in C$ with respect to μ^* is now a consequence of Birkoff's ergodic theorem (see [14, theorem 2.3]). This provides the existence of

$$\begin{aligned}\lim_{T\to\infty} p_T(\omega, x) &= q^*(\omega, x) \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{q^*}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s \\ &= q^*(\omega, x)\end{aligned}$$

for almost every $(\omega, x) \in C$ with respect to r. Theorem 5.4 of [22] allows us to state that μ^* is a finite measure on C equivalent to the Lebesgue measure, which is impossible. We conclude that $r(C) = 0$ and $q^*(\omega, x) = 0$ almost everywhere with respect to r on B. In addition, taking into account that the function $\partial g/\partial x$ is increasing in the x component, we deduce that there is a proper invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that, if $(\omega, x) \in B \cap \Pi^{-1}(\Omega_1)$ and $x_1(\omega) < x < x_2(\omega)$, then $q^*(\omega, x) = 0$. We can also admit that

$$\lim_{|t|\to\infty} \frac{1}{t}\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, x_i(\omega\cdot s))\,\mathrm{d}s = \alpha_i, \quad i = 1, 2,$$

for $\omega \in \Omega_1$.

Notice that if $(\omega, x) \in B \cap \Pi^{-1}(\Omega_1)$ and $x_1(\omega) < x < x_2(\omega)$, then

$$u(t, \omega, x) - x_1(\omega\cdot t) \leqslant (x - x_1(\omega)) \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr)$$

for every $t > 0$, according to (3.3). Thus we obtain that

$$\begin{aligned}&\frac{1}{T}\int_0^T (u(t, \omega, x) - x_1(\omega\cdot t))\,\mathrm{d}t \\ &\qquad \leqslant (x - x_1(\omega))\frac{1}{T}\int_0^T \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s, \omega, x))\,\mathrm{d}s\biggr)\,\mathrm{d}t,\end{aligned}$$

and hence

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T (u(t, \omega, x) - x_1(\omega\cdot t))\,\mathrm{d}t = 0. \tag{5.2}$$

This means that

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas}\{t \in [0, T]/u(t, \omega, x) - x_1(\omega\cdot t) \geqslant \varepsilon\} = 0$$

for all $\varepsilon > 0$, which also provides

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\omega\cdot t, u(t, \omega, x))\,\mathrm{d}t = \int_\Omega f(\omega, x_1(\omega))\,\mathrm{d}m$$

for every $f \in \mathcal{C}(B)$. In particular,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\partial g}{\partial x}(\omega\cdot t, u(t, \omega, x))\,\mathrm{d}t = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_1(\omega))\,\mathrm{d}m = \alpha_1 < 0.$$

In consequence,

$$\lim_{t\to\infty} \int_0^t \frac{\partial g}{\partial x}(\omega\cdot t, u(t, \omega, x))\,\mathrm{d}t = -\infty,$$

and hence $\lim_{t\to\infty}(u(t, \omega, x) - x_1(\omega\cdot t)) = 0$.

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We continue in this fashion, introducing

$$q_*(\omega,x) = \limsup_{T\to-\infty} p_T(\omega,x) = \limsup_{T\to-\infty} \frac{1}{T}\int_0^T \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s,\omega,x))\,\mathrm{d}s\biggr)\,\mathrm{d}t.$$

The same previous argument allows us to assure the existence of an invariant subset, we assume that it agrees with Ω_1, such that if $(\omega,x)\in B\cap \Pi^{-1}(\Omega_1)$ and $x_1(\omega)<x<x_2(\omega)$, then $q_*(\omega,x)=0$.

Notice that if $(\omega,x)\in B\cap \Pi^{-1}(\Omega_1)$ and $x_1(\omega)<x<x_2(\omega)$, then

$$x_2(\omega\cdot t)-u(t,\omega,x) \leqslant (x_2(\omega)-x)\exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s,\omega,x))\,\mathrm{d}s\biggr)$$

for every $t<0$, according to (3.2). If $T<0$, we obtain

$$\frac{1}{T}\int_0^T (x_2(\omega\cdot t)-u(t,\omega,x))\,\mathrm{d}t \leqslant (x_2(\omega)-x)\frac{1}{T}\int_0^T \exp\biggl(\int_0^t \frac{\partial g}{\partial x}(\omega\cdot s, u(s,\omega,x))\,\mathrm{d}s\biggr)\,\mathrm{d}t,$$

and hence

$$\lim_{T\to-\infty}\frac{1}{T}\int_0^T (x_2(\omega\cdot t)-u(t,\omega,x))\,\mathrm{d}t = 0.$$

This provides

$$\lim_{T\to-\infty}\frac{1}{T}\int_0^T \frac{\partial g}{\partial x}(\omega\cdot t, u(t,\omega,x_0))\,\mathrm{d}t = \int_\Omega \frac{\partial g}{\partial x}(\omega, x_2(\omega))\,\mathrm{d}m = \alpha_2 > 0.$$

In consequence,

$$\lim_{t\to-\infty}\int_0^t \frac{\partial g}{\partial x}(\omega\cdot t, u(t,\omega,x))\,\mathrm{d}t = -\infty,$$

and hence $\lim_{t\to-\infty}(x_2(\omega\cdot t)-u(t,\omega,x))=0$. This completes this part of the proof.

Let us assume that (Ω,σ) is uniquely ergodic. If $\nu_1(M)=0$, then $\nu_2(M)=1$ and M should be an hyperbolic minimal subset, in contradiction with theorem 5.1. The same argument applies if $\nu_2(M)=0$. In consequence, $\nu_1(M)=\nu_2(M)=1$. ☐

The dynamical description of B becomes simpler when it is not essentially bounded. The following statement is a direct consequence of theorem 5.3.

Theorem 5.4. *Let us assume that Ω_0 is not void and B is not essentially bounded. There exist functions $a,b\in\mathcal{C}(M)$ and an invariant subset $\Omega_1\subset\Omega$ with $m(\Omega_1)=1$ such that $\tilde g(\tilde\omega,x)=a(\tilde\omega)x+b(\tilde\omega)$ in $\mathrm{cls}\{(\tilde\omega,x)\in\tilde B/\Pi_M(\tilde\omega)\in\Omega_1\}$. Besides, (B,τ) admits a normalized invariant measure equivalent to the Lebesgue measure and decomposes into a complete collection $(N_j)_{j\in J}$ of ergodic sheets.*

Proof. Recall that the subset Ω_0 is invariant and residual. Since B is not essentially bounded, we can also conclude that $m(\Omega_0)=0$.

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We consider the family of differential equations

$$u' = g_j(\omega \cdot t, u), \quad \omega \in \Omega, \quad j \in \mathbb{N},$$

defined by (4.5) in the previous section and the skew-product flow induced by them on $\Omega \times \mathbb{R}$. We write

$$B_j = \Big\{(\omega, x) \in \Omega \times \mathbb{R} \Big/ \sup_{t \in \mathbb{R}} |u_j(t, \omega, x)| < \infty\Big\}.$$

We know that

$$B \subset \bigcup_{j=1}^{\infty} \bigcap_{m \geqslant j} B_m = \liminf_{j \to \infty} B_j. \tag{5.3}$$

Let $j_0 \in \mathbb{N}$ be such that $j_0 > |x|$ for every $(\omega, x) \in M$. Then, for each $j \geqslant j_0$, one has $B_j \neq \emptyset$ and we can write $B_j = \bigcup_{\omega \in \Omega} \{\omega\} \times [x_{j,1}(\omega), x_{j,2}(\omega)]$.

Let us fix $j \geqslant j_0$ and assume $x_{j,1}(\omega) \neq x_{j,2}(\omega)$ for every $\omega \in \Omega$. Then M satisfies properties (i) or (ii) of theorem 5.1, and hence $\Omega_0 = \Omega$, which is not possible. In consequence, there is a residual invariant subset $\Omega_j \subset \Omega$ with $x_{j,1}(\omega) = x_{j,2}(\omega)$ for every $\omega \in \Omega_j$. Let us assume the existence of a sequence of integers $(j_k)_{k \geqslant 1}$ with $j_k \geqslant j_0, \lim_{k \to \infty} j_k = +\infty$ and $m(\Omega_{j_k}) = 1$ for every $k \geqslant 1$. Then it follows that $\bigcap_{k=1}^{\infty} \Omega_{j_k} \subset \Omega_0$, and hence $m(\Omega_0) = 1$, which is impossible. Thus we can admit that j_0 is large enough in order that $m(\Omega_j) = 0$ for every $j \geqslant j_0$.

We assume, for instance, that the map $\Omega \to \mathbb{R}$, $\omega \mapsto x_2(\omega)$ is not essentially bounded (replacing $x_2(\omega)$ by $x_1(\omega)$, we obtain an analogous case). For every $\omega \in \Omega$, we define

$$u_2(\omega) = \sup\{x \in \mathbb{R}/(\omega, x) \in M\}$$

and the subset

$$C = \{(\omega, x) \in B/x \geqslant u_2(\omega)\}.$$

We deduce, according to (5.3), the existence of $j_1 \in \mathbb{N}$, $j_1 \geqslant j_0$, such that $r(C \cap B_{j_1}) > 0$. From j_1, we construct a sequence $(j_k)_{k \geqslant 1}$ satisfying that if $C_k = \{(\omega, x) \in B/x \geqslant x_{j_k,2}(\omega)\}$, then $j_{k+1} \geqslant x_{j_k,2}(\omega)$ for every $\omega \in \Omega$ and $r(B_{j_{k+1}} \cap C_k) > 0$ for every $k \geqslant 1$. Let

$$\Theta_k = \{\omega \in \Omega/x_{j_{k+1},2}(\omega) > x_{j_k,2}(\omega)\}.$$

It is immediate that $m(\Theta_k) > 0$. If $\omega \in \Omega - \Theta_k$, then $x_{j_k,2}(\omega) \geqslant x_{j_{k+1},2}(\omega)$. Besides

$$\begin{aligned} x_{j_k,2}(\omega \cdot t) &= u_{j_k}(t, \omega, x_{j_k,2}(\omega)) \\ &\geqslant u(t, \omega, x_{j_k,2}(\omega)) \\ &\geqslant u(t, \omega, x_{j_{k+1},2}(\omega)) \\ &= x_{j_{k+1},2}(\omega \cdot t) \end{aligned}$$

for every $t \geqslant 0$. This shows that $\omega \cdot t \in \Omega - \Theta_k$ for every $t \geqslant 0$, which allows us to conclude that $m(\Omega - \Theta_k) = 0$ and $m(\Theta_k) = 1$.

We consider the Lyapunov exponents

$$\alpha_{j,2} = \int_{\Omega} \frac{\partial g_j}{\partial x}(\omega, x_{j,2}(\omega))\, \mathrm{d}m \quad \text{for } j \geqslant j_0.$$

Let us assume that there is $k_0 \in \mathbb{N}$ such that $\alpha_{j_k,2} > 0$ for every $k \geqslant k_0$. Then we can find an invariant subset $\Omega_1' \subset \Omega$ with $m(\Omega_1') = 1$ such that, if $\omega \in \Omega_1'$, $k \geqslant k_0$, and $x_{j_k,1}(\omega) < x < x_{j_k,2}(\omega)$, then

$$\lim_{t\to-\infty}(x_{j_k,2}(\omega\cdot t) - u_{j_k}(t,\omega,x)) = 0.$$

We take a compact subset $\Delta \subset \Omega_1'$ with $m(\Delta) > 0$ satisfying that

$$\delta_k = \inf_{\omega\in\Delta}(x_{j_{k+1},2}(\omega) - x_{j_k,2}(\omega)) > 0$$

for every $k \geqslant k_0$. Let $(\omega_1, x_1) \in C, k_1 \geqslant k_0$ and a sequence of real numbers $(t_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n = -\infty$, $\omega_1 \cdot t_n \in \Delta$ for every $n \in \mathbb{N}$ and $j_{k_1} > |u(t,\omega_1,x_1)|$ for every $t \in \mathbb{R}$. Then we obtain $u_{j_{k_1+1}}(t,\omega_1,x_1) = u_{j_{k_1}}(t,\omega_1,x_1)$ for every $t \in \mathbb{R}$, and hence

$$\begin{aligned}\limsup_{n\to\infty}&(x_{j_{k_1+1}}(\omega_1\cdot t_n) - u_{j_{k_1+1}}(t_n,\omega_1,x_1))\\ &= \limsup_{n\to\infty}((x_{j_{k_1+1},2}(\omega_1\cdot t_n) - x_{j_{k_1},2}(\omega_1\cdot t_n))\\ &\qquad + (x_{j_{k_1},2}(\omega_1\cdot t_n) - u_{j_{k_1}}(t_n,\omega_1,x_1))) \geqslant \delta_{k_1} > 0,\end{aligned}$$

contrary to our assumption. In consequence, we can find an integer $k_0 \in \mathbb{N}$ such that $\alpha_{j_k,2} = 0$ for every $k \geqslant k_0$ (otherwise, we could obtain a subsequence $(j_{k_n})_{n\in\mathbb{N}}$ with $\alpha_{j_{k_n},2} > 0$ for every $n \in \mathbb{N}$ and, repeating the above argument to the sequence $(\alpha_{j_{k_n}})_{n\in\mathbb{N}}$, we obtain a contradiction). The dynamical properties of the subsets $(B_{j_k})_{k\geqslant k_0}$ are described in statement (i) of theorem 5.3.

We consider the skew-product flow induced on $M \times \mathbb{R}$ by the family of differential equations

$$u' = \tilde{g}_j(\tilde{\omega}\cdot t, u), \quad \tilde{\omega} \in M, \quad j \geqslant j_0,$$

where $\tilde{g}_j(\tilde{\omega}, x) = g_j(\Pi_M(\tilde{\omega}), x)$ for every $(\tilde{\omega}, x) \in M \times \mathbb{R}$. Then

$$\tilde{M} = \{(\omega, x, x) \in \tilde{B}/(\omega, x) \in M\}$$

is the only minimal subset of $\tilde{B}_j$, besides

$$\tilde{g}_j(\tilde{\omega}, x) = \tilde{g}(\tilde{\omega}, x) \quad \text{and} \quad \frac{\partial \tilde{g}_j}{\partial x}(\tilde{\omega}, x) = \frac{\partial \tilde{g}}{\partial x}(\tilde{\omega}, x)$$

for every $(\tilde{\omega}, x) \in \tilde{M}$ and $j \geqslant j_0$. We deduce the existence of $a, b \in \mathcal{C}(M)$ and an invariant subset $\Omega_1 \subset \Omega$ with $m(\Omega_1) = 1$ such that $\int_M a(\tilde{\omega})\,\mathrm{d}\nu = 0$ for every $\nu \in \mathcal{M}_{\mathrm{inv},m}(M,\tau)$, and $\tilde{g}_j(\tilde{\omega}, x) = a(\tilde{\omega})x + b(\tilde{\omega})$ for every $(\tilde{\omega}, x) \in \tilde{B}_{j_k}$, with $\Pi_M(\tilde{\omega}) \in \Omega_1$ and $k \geqslant k_0$. If $(\tilde{\omega}, x) \in \tilde{B}$ with $\Pi_M(\tilde{\omega}) \in \Omega_1$, then there is $k \geqslant k_0$ such that $|u(t,\tilde{\omega},x)| < j_k$ for every $t \in \mathbb{R}$, so that $(\omega, x) \in \tilde{B}_{j_k}$ and

$$\tilde{g}(\tilde{\omega}, x) = \tilde{g}_{j_k}(\tilde{\omega}, x) = a(\tilde{\omega})x + b(\tilde{\omega}),$$

and this representation remains valid in the closure of $\tilde{B} \cap \Pi_M^{-1}(\Omega_1)$. Under these conditions, we know that (B, τ) admits an invariant measure equivalent to the Lebesgue measure and decomposes into a complete collection $(N_j)_{j\in J}$ of ergodic sheets. □

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Remark 5.5. Maintaining the hypotheses and the notation of theorem 5.4, it is easy to check that

$$B \cap \Pi^{-1}(\Omega_1) = \bigcup_{j=1}^{\infty} \bigcap_{m \geqslant j} (B_m \cap \Pi^{-1}(\Omega_1)).$$

Finally, we want to mention several almost-periodic differential equations studied in the literature that illustrate some of the situations described through the paper. A scalar linear equation with a unique almost-automorphic minimal subset is constructed in [9] (see [26] for a qualitative description of this property). The quadratic examples given by Millionščikov [19,20] and Vinograd [31] provide a subset of bounded trajectories, B, with not null Lyapunov exponents and a unique almost-automorphic minimal subset. Ortega and Tarallo [23] construct an example where B is bounded, decomposes into ergodic sheets and contains a unique almost-periodic minimal subset.

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