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A TWO-NODE JACKSON NETWORK WITH INFINITE SUPPLY OF WORK

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We consider a Jackson network with two nodes, with no exogenous input, but instead an infinite supply of work at each of the nodes: Whenever a node is empty, it processes a job from this infinite supply. We obtain an explicit expression for the steady state distribution of this system, as an infinite sum of product forms.

1. INTRODUCTION

We consider a Jackson network with two nodes (Fig. 1), numbered $i = 1,2$. Processing times at the nodes are independent, and those at node i are exponentially distributed with rates μ_i, and jobs completing processing at node i move to node $3 - i$ with probability p_i and leave the system otherwise. There is no exogenous input to the system. However, whenever one of the nodes is empty, it will process a job from an infinite supply of jobs. This system can be described by a two-dimensional Markov jump process, $X(t) = (X_1(t), X_2(t))$, the state space of which consists of the pairs of nonnegative integers (n_1, n_2), where n_1 indicates the number of jobs at node 1 and n_2 indicates the number of jobs at node 2. Whenever $n_i > 0$, node i will process one of the jobs at the node. This introduces the transitions

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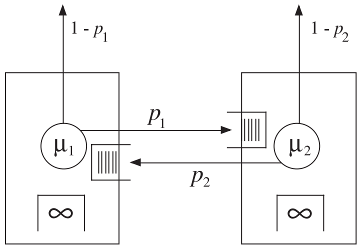

FIGURE 1. A two-node Jackson network with infinite supply of work.

$$
\begin{aligned}
&(n_1, n_2) \to (n_1 - 1, n_2 + 1) \text{ at rate } \mu_1 p_1, && n_1 > 0,\\
&(n_1, n_2) \to (n_1 - 1, n_2) \text{ at rate } \mu_1 (1 - p_1), && n_1 > 0,\\
&(n_1, n_2) \to (n_1 + 1, n_2 - 1) \text{ at rate } \mu_2 p_2, && n_2 > 0,\\
&(n_1, n_2) \to (n_1, n_2 - 1) \text{ at rate } \mu_2 (1 - p_2), && n_2 > 0.
\end{aligned}
\tag{1.1}
$$

Whenever node i is empty, it will process a job from its infinite supply, at the same rate μ_i, and upon completion, this job will move to the other node with probability p_i and leave the system with probability $1 - p_i$. This introduces the additional transitions:

$$
\begin{aligned}
(0, n_2) &\to (0, n_2 + 1) \text{ at rate } \mu_1 p_1,\\
(n_1, 0) &\to (n_1 + 1, 0) \text{ at rate } \mu_2 p_2.
\end{aligned}
\tag{1.2}
$$

Note that jobs from the infinite supply of each buffer are indistinguishable from jobs queued at the nodes, but queued jobs have preemptive priority over jobs in the infinite supply. The transitions (1.2) constitute arrivals into the system.

The two nodes in this system are processing jobs all the time. Hence, there are four independent Poisson streams in this system: Jobs depart the system in two Poisson streams with rates $\mu_1(1 - p_1)$ and $\mu_2(1 - p_2)$, and jobs arrive at the two nodes in two Poisson streams, with rates $\mu_1 p_1$ and $\mu_2 p_2$. The queue at node i therefore behaves as an $M/M/1$ queue, with arrival rate $\mu_{3-i} p_{3-i}$ and service rate μ_i. The system is stable if

$$
\rho_i = \frac{\mu_{3-i} p_{3-i}}{\mu_i} < 1, \qquad i = 1, 2,
$$

with marginal steady state distributions

$$
P_i(n) = \lim_{t \to \infty} \mathbb{P}(X_i(t) = n) = (1 - \rho_i)\rho_i^n, \qquad n \ge 0, \qquad i = 1, 2. \tag{1.3}
$$

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However, the queue lengths at the two nodes in steady state are not independent; the joint steady state distribution is not a product form:

$$P(n_1, n_2) = \lim_{t\to\infty} \mathbb{P}((X_1(t), X_2(t)) = (n_1, n_2)) \neq P_1(n_1)P_2(n_2), \qquad n_1, n_2 \geq 0.$$

In this note, we derive explicit expressions for the joint steady state distribution of the two-node system. We use the compensation approach, developed by Adan, Wessels, and Zijm [2] to obtain an expression that is an infinite sum of product forms.

This two-node Jackson network with an infinite supply of work describes quite a useful model of cooperative service by two servers. Consider jobs that require a sequence of tasks; the first task is performed by one of the servers and the remaining tasks are performed by alternating servers. Server i performs tasks at rate μ_i, and the job then requires an additional task with probability p_i, or else it is complete and leaves the system. We assume that each of these servers has an infinite supply of jobs to start. However, each server gives preemptive priority to tasks that it received from the other server. Each server then has a queue of jobs that are "in process" and the analysis of these queues tells us how much storage for WIP (work in process) is needed and what is the cycle time of a job from first task to completion.

The concept of infinite supply of work, in contrast to the usual queuing assumption that jobs arrive randomly, is in fact very common in many systems: Whenever a server is expensive and it is desired not to keep it idle, one tries to monitor the server and control the inputs, so that the server never runs out of work. This is the case for an expensive machine, a highly trained server, or a high-performance communication link. In each case, work is shunted to such servers to prevent them from idling.

As we will see in Section 3, infinite-supply Jackson nodes provide much better performance than standard Jackson nodes.

Multiclass queuing networks with infinite supplies of jobs in some of the classes, also called infinite virtual queues, were introduced by Adan and Weiss [1], Kopzon and Weiss [8,9], and Weiss [12–14]; see also Levy and Yechiali [10]. They represent monitored control over job arrivals, as it often exists in manufacturing and communication systems. Jackson networks are described by Jackson [6] and Kelly [7]. Weiss [14] has discussed Jackson networks with virtual infinite buffers: He has derived flow rates, stability conditions, and partial steady state distributions. This work is also closely related to the results of Goodman and Massey [5]. The analysis in the current article provides one example of such networks, which is highly tractable.

2. MAIN THEOREM

The two-node Jackson network with an infinite supply of work is described by a Markov jump process moving on the two-dimensional nonnegative integer grid. The Markov process performs a two-dimensional simple random walk on the positive-integer grid, with transitions only to neighboring states, and with reflecting barriers on the horizontal and vertical axes. Furthermore, in the interior of the positive quadrant, the random walk has no transitions to the north, the northeast, and the east directions. The transition rates for this random walk are described in Figure 2.

For such Markov jump processes, it is possible to obtain a closed-form expression of the steady state distribution, by the compensation method developed in the work of Adan et al. [2]. The random walk in Figure 2 has the property that the transition rates at the vertical boundary $n_1 = 0, n_2 > 0$ and the horizontal boundary $n_1 > 0, n_2 = 0$ are projections of the ones in the interior $n_1, n_2 > 0$. Boxma and van Houtum [3] showed that this property considerably simplifies the expression of the steady state distribution. See also [11].

The main steps in the derivation of the steady state probabilities are as follows: The balance equations for the interior are satisfied by product form expressions $\alpha^{n_1}\beta^{n_2}$, where α and β are solutions of a quadratic equation $Q(\alpha,\beta) = 0$. Solutions of this form do not as a rule satisfy the equations for the horizontal or vertical boundaries. However, it is possible to find compensating product forms such that the linear combination $\alpha^{n_1}\beta^{n_2} + c\tilde{\alpha}^{n_1}\beta^{n_2}$ satisfies the balance equations for the interior and the vertical boundary of the quadrant. Similarly, it is possible to find compensating product forms such that $\alpha^{n_1}\beta^{n_2} + d\alpha^{n_1}\tilde{\beta}^{n_2}$ satisfies the balance equations for the interior and the horizontal boundary of the quadrant. Then, starting with a product form $\alpha^{n_1}\beta^{n_2}$ with α and β satisfying $Q(\alpha,\beta) = 0$, one can construct an infinite linear combination by adding product forms to alternately compensate for the horizontal and vertical boundary. The resulting solution formally satisfies all of the balance equations. One then needs to choose the parameters of the product forms and their coefficients such that the solution is absolutely convergent. This method does indeed work for our system.

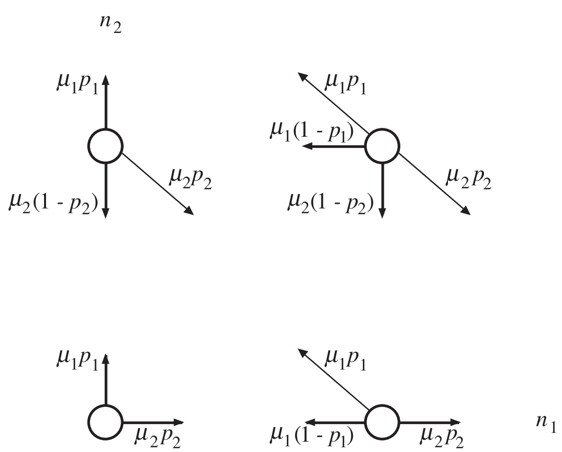

Figure 2. Transition rates for the two-node system.

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In Section 4, we will present the detailed derivation of the steady state distribution, without invoking the results in [2,3]. In the derivation, we make use of the steady state marginal distributions (1.3). This yields a particularly elegant and simple expression:

THEOREM 2.1: *The steady state distribution of the two-node Jackson network with infinite supply of work, when $\rho_1, \rho_2 < 1$, is given, for all $(n_1, n_2) \neq (0,0)$, by*

$$P(n_1,n_2) = \sum_{k=1}^{\infty} (-1)^{k+1}[(1-\alpha_k)\alpha_k^{n_1}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}(1-\beta_k)\beta_k^{n_2}], \tag{2.1}$$

where, for $k \geq 1$,

$$\begin{aligned} \alpha_{k+1}^{-1} &= \frac{\mu_1+\mu_2}{\mu_2 p_2}\beta_k^{-1} - \alpha_{k-1}^{-1} - \frac{1-p_2}{p_2}, \\ \beta_{k+1}^{-1} &= \frac{\mu_1+\mu_2}{\mu_1 p_1}\alpha_k^{-1} - \beta_{k-1}^{-1} - \frac{1-p_1}{p_1}, \end{aligned} \tag{2.2}$$

with initially $\alpha_0 = \beta_0 = 1, \alpha_1 = \rho_1$, and $\beta_1 = \rho_2$. The steady state probability $P(0,0)$ is

$$P(0,0) = 1 - \rho_1 - \rho_2 + \sum_{k=1}^{\infty} (-1)^{k+1}(\alpha_k\beta_{k+1} + \alpha_{k+1}\beta_k). \tag{2.3}$$

The closed-form expression in Theorem 2.1 immediately leads to similar expressions for the distribution of the total number in the system and for the (factorial) moments of the queue lengths at node 1 and 2. Let X_i denote the queue length of node i in steady state. Then we have the following:

COROLLARY 2.2:

(i) For all $n > 0$,

$$\begin{aligned} &\mathbb{P}(X_1 + X_2 = n) \\ &\quad = \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 0}} P(n_1,n_2) \\ &\quad = \sum_{k=1}^{\infty} (-1)^{k+1}\left[(1-\alpha_k)(1-\beta_{k+1})\frac{\alpha_k^{n+1} - \beta_{k+1}^{n+1}}{\alpha_k - \beta_{k+1}}\right. \\ &\qquad\qquad \left. + (1-\alpha_{k+1})(1-\beta_k)\frac{\alpha_{k+1}^{n+1} - \beta_k^{n+1}}{\alpha_{k+1} - \beta_k}\right]. \end{aligned} \tag{2.4}$$

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(ii) For all $m, n \geq 0$ and $m + n > 0$,

$$\mathbb{E}\left(\binom{X_1}{m}\binom{X_2}{n}\right) = \sum_{k=1}^{\infty}(-1)^{k+1}\left[\frac{\alpha_k^m}{(1-\alpha_k)^m}\frac{\beta_{k+1}^n}{(1-\beta_{k+1})^n} + \frac{\alpha_{k+1}^m}{(1-\alpha_{k+1})^m}\frac{\beta_k^n}{(1-\beta_k)^n}\right]. \tag{2.5}$$

Note that exact formulas for α_k and β_k can be obtained from the difference equation (2.2) but are not particularly illuminating. The asymptotic behavior of α_k and β_k is derived in Proposition 4.14.

3. COMPARISON WITH STANDARD JACKSON NETWORK

We compare our two-node system with an infinite supply of work and a standard Jackson network, with exogenous random inputs. Throughout this section we label our system as "∞-supply" and the Jackson network with random exogenous input as "standard." For the comparison, we consider two nodes in a standard Jackson network as shown in Figure 3. Here, we have two nodes with the same processing rates μ_i, and with the same probabilities $1 - p_i$ to complete a job, which then departs the system. The total inputs into the nodes are at rates λ_i, and they consist of both exogenous arrivals and feedback from other nodes. Recall that the *input streams are not Poisson*. The outflow in steady state is also at rate λ_i and includes a *Poisson output stream of departures from the system* at rate $\lambda_i(1 - p_i)$. As is well known, the steady state joint distribution of the jobs in the two nodes is the product form

$$P(n_1, n_2|\text{standard}) = \left(1 - \frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_1}{\mu_1}\right)^{n_1}\left(1 - \frac{\lambda_2}{\mu_2}\right)\left(\frac{\lambda_2}{\mu_2}\right)^{n_2}.$$

This is only stable if $\lambda_i < \mu_i$; therefore, the output rate of the standard nodes is always less than the rate $(1 - p_i)\mu_i$ achieved by the ∞-supply system, and if one tries to approach this rate, the queue length explodes.

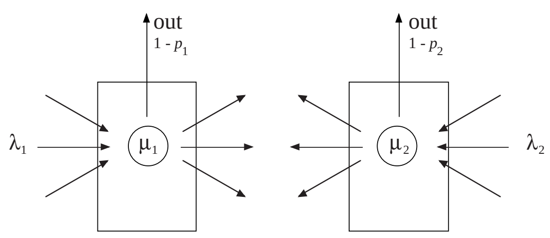

FIGURE 3. Standard Jackson network nodes.

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It is interesting to compare the two systems in the case that both have the same traffic intensities. For the remainder of this section, we take $\rho_i = \lambda_i/\mu_i = \mu_{3-i}p_{3-i}/\mu_i$. We compare the total number in the two nodes for the two systems. The marginal steady state distributions in the nodes of the two systems are the same, namely Geometric, with $P(X_i \geq n) = \rho_i^n$. In particular, it follows that the mean number in the system is the same for both networks. However, the steady state distribution of the total number in the system is different.

In Figure 4, we show the distribution of the total number in the system for $\mu_1 = 2$, $\mu_2 = 3$, $p_1 = 0.8$, $p_2 = 0.5$ (top) and $\mu_1 = \mu_2 = 1$, $p_1 = p_2 = 0.8$ (bottom). We also plot the standard product form probabilities for comparison.

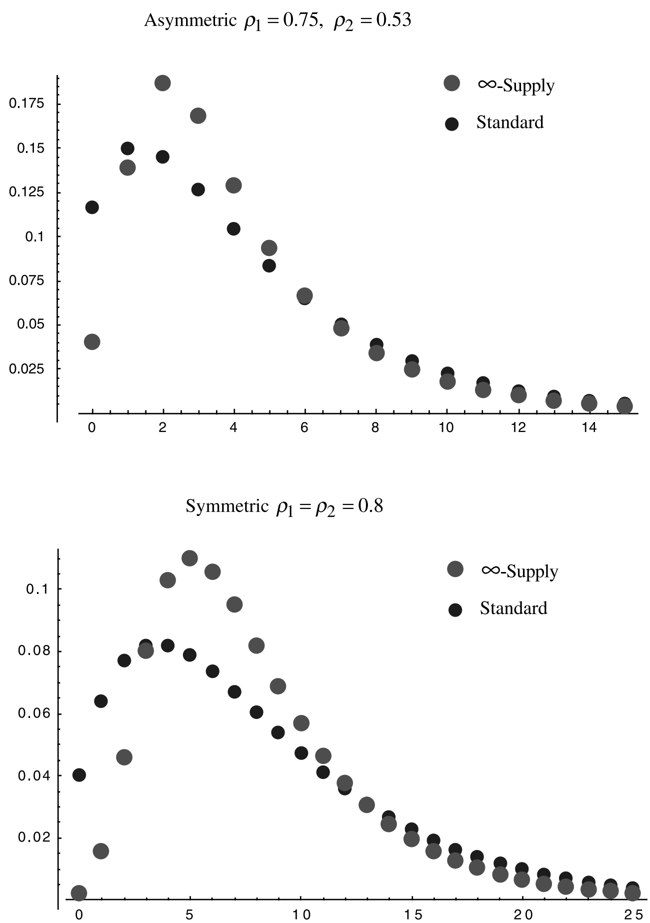

Figure 4. Probabilities of total number in the system.

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The correlation between X_1 and X_2, calculated from formula (2.5), equals -0.2976 for the first example of an asymmetric system and -0.3873 for the second example of a symmetric system. In fact, it can be shown that the correlation is always negative; see Section 4.8. Negative correlation reduces the variance of the total number in the system compared to independent nodes. In Figure 5, we show the correlation between X_1 and X_2 for the symmetric system $\mu_1 = \mu_2$ and $p_1 = p_2 = p$. Clearly, the negative correlation gets stronger as p tends to 1; the limiting value for $p = 1$ is equal to $\frac{2}{3}\pi^2 - 7$ (see Section 4.8).

We can also get the asymptotic tail probabilities of $X_1 + X_2$, from (2.4). We will show that the sum is absolutely convergent and that the parameters α_k and β_k monotonously decrease. The values for large n therefore behave like the largest geometric term,

$$\mathbb{P}(X_1 + X_2 \geq n|\infty\text{-supply}) \sim \begin{cases} \dfrac{1-\beta_2}{1-\beta_2/\rho_1}\rho_1^n, & \rho_1 > \rho_2 \\ \left(\dfrac{1-\beta_2}{1-\beta_2/\rho_1} + \dfrac{1-\alpha_2}{1-\alpha_2/\rho_1}\right)\rho_1^n, & \rho_1 = \rho_2. \end{cases} \tag{3.1}$$

The corresponding asymptotics for the product form standard Jackson network are

$$\mathbb{P}(X_1 + X_2 \geq n|\text{standard}) \sim \begin{cases} \dfrac{1-\rho_2}{1-\rho_2/\rho_1}\rho_1^n, & \rho_1 > \rho_2 \\ n(1-\rho_1)\rho_1^n, & \rho_1 = \rho_2. \end{cases}$$

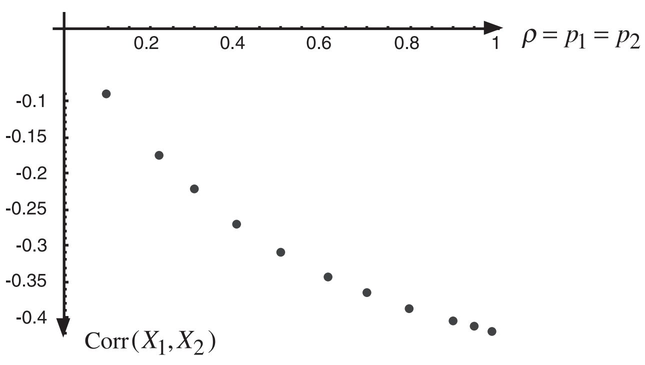

FIGURE 5. Correlation between X_1 and X_2 for the symmetric system $\mu_1 = \mu_2$ and $p_1 = p_2 = p$.

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Hence, the asymptotic ratio of the two is

$$\frac{\mathbb{P}(X_1+X_2\geq n|\text{standard})}{\mathbb{P}(X_1+X_2\geq n|\infty\text{-supply})}\sim\begin{cases}\dfrac{\dfrac{1-\rho_2}{1-\rho_2/\rho_1}}{\dfrac{1-\beta_2}{1-\beta_2/\rho_1}}, & \rho_1>\rho_2\\[2ex] \left[\dfrac{1-\rho_1}{\dfrac{1-\beta_2}{1-\beta_2/\rho_1}+\dfrac{1-\alpha_2}{1-\alpha_2/\rho_1}}\right]n, & \rho_1=\rho_2.\end{cases} \tag{3.2}$$

In Table 1, we summarize various quantities for the two examples and the comparison of the total number in the system for the ∞-supply and standard systems.

The most interesting part here is the strong form of variance reduction and tail probability (i.e., overflow probabilities in practice) obtained in the infinite-supply network, when the two nodes are symmetric.

TABLE 1. Comparison of Infinite-Supply and Standard Jackson Networks

	Asymmetric Example	Symmetric Example
Data		
μ_1	2	1
μ_2	3	1
p_1	0.8	0.8
p_2	0.5	0.8
ρ_1	0.75	0.8
ρ_2	0.533	0.8
Moments		
$E(X_1)$	3	4
$E(X_2)$	1.14	4
$V(X_1)$	12	20
$V(X_2)$	2.45	20
$\mathrm{Cov}(X_1,X_2)$	−1.61	−7.75
$\mathrm{Corr}(X_1,X_2)$	−0.2976	−0.3873
$\dfrac{V(X_1+X_2\|\text{standard})}{V(X_1+X_2\|\infty\text{-supply})}$	1.287	1.633
$\sim\dfrac{P(X_1+X_2\geq n\|\text{standard})}{P(X_1+X_2\geq n\|\infty\text{-supply})}$	1.33445	0.07143 n

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4. DERIVATION OF STEADY STATE DISTRIBUTION

In this section, we prove Theorem 2.1. We first derive the expression as a formal solution to the balance equations and then prove that this solution is absolutely convergent.

4.1. Balance Equations

The balance equations for the steady state probabilities in this system are obtained by equating the flow out and into each state, yielding the following:

$$\begin{aligned}(\mu_1+\mu_2)P(n_1,n_2) &= \mu_1 p_1 P(n_1+1,n_2-1)+\mu_1(1-p_1)P(n_1+1,n_2)\\ &\quad + \mu_2 p_2 P(n_1-1,n_2+1)+\mu_2(1-p_2)P(n_1,n_2+1),\\ &\qquad n_1,n_2>0, \end{aligned} \tag{4.1}$$

$$\begin{aligned}(\mu_1+\mu_2 p_2)P(n_1,0) &= \mu_2 p_2 P(n_1-1,0)+\mu_1(1-p_1)P(n_1+1,0)\\ &\quad + \mu_2 p_2 P(n_1-1,1)+\mu_2(1-p_2)P(n_1,1),\\ &\qquad n_1>0, \end{aligned} \tag{4.2}$$

$$\begin{aligned}(\mu_1 p_1+\mu_2)P(0,n_2) &= \mu_1 p_1 P(0,n_2-1)+\mu_2(1-p_2)P(0,n_2+1)\\ &\quad + \mu_1 p_1 P(1,n_2-1)+\mu_1(1-p_1)P(1,n_2),\\ &\qquad n_2>0, \end{aligned} \tag{4.3}$$

$$(\mu_1 p_1+\mu_2 p_2)P(0,0) = \mu_1(1-p_1)P(1,0)+\mu_2(1-p_2)P(0,1). \tag{4.4}$$

In the next section, we will characterize the product forms $\alpha^{n_1}\beta^{n_2}$ satisfying the balance equations in the interior of the quadrant.

4.2. Product Form Trial Solutions in the Interior of the Quadrant

Consider first (4.1) in the interior of the quadrant and a product form trial solution $\alpha^{n_1}\beta^{n_2}$. Substituting this trial solution in (4.1) and canceling $\alpha^{n_1-1}\beta^{n_2-1}$, we see immediately the following:

PROPOSITION 4.1: *The product form $\alpha^{n_1}\beta^{n_2}$ solves (4.1) for every $n_1,n_2=0,\pm1,\pm2,\ldots,$ if and only if α and β are on the curve:*

$$(\mu_1+\mu_2)\alpha\beta = \alpha^2(\mu_1 p_1+\mu_1(1-p_1)\beta)+\beta^2(\mu_2 p_2+\mu_2(1-p_2)\alpha). \tag{4.5}$$

Curve (4.5) is shown in Figure 6. The pairs of values $(\alpha,\beta)=(0,0)$ and $(\alpha,\beta)=(1,1)$ are on this curve. We also illustrate on the curve how the special roots that appear in the solution (2.1,2.2), α_k,β_k, are calculated, for $k=0,1,2,3$.

For every fixed value of $0<\alpha\le 1$, (4.5) yields a quadratic equation for β:

$$[\mu_2 p_2+\mu_2(1-p_2)\alpha]\beta^2-[(\mu_1+\mu_2)\alpha-\mu_1(1-p_1)\alpha^2]\beta+[\mu_1 p_1\alpha^2]=0. \tag{4.6}$$

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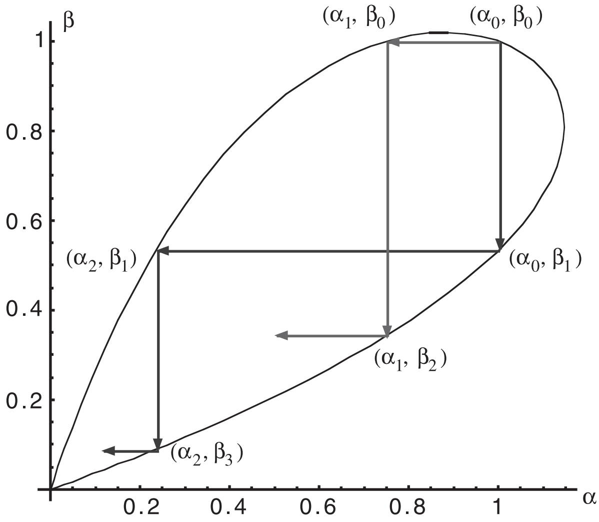

FIGURE 6. Curve (4.5) for $\mu_1 = 2, \mu_2 = 3$ and $p_1 = 0.8, p_2 = 0.5$.

PROPOSITION 4.2: *The quadratic equation (4.6) has two real roots for all* $0 < \alpha \leq 1$. *For* $\alpha = 1$, *the roots are* $\bar{\beta} = 1$ *and* $\underline{\beta} = \rho_2$. *For* $0 < \alpha < 1$, *the larger root is* $\bar{\beta} > \alpha$ *and the smaller root is* $0 < \underline{\beta} < \alpha$.

PROOF: For the fixed value $\alpha = \alpha_0 = 1$, the quadratic equation (4.6) for β is

$$\mu_2 \beta^2 - (\mu_1 p_1 + \mu_2)\beta + \mu_1 p_1 = 0,$$

with the two roots $\bar{\beta} = 1$ and $\underline{\beta} = \beta_1 = \mu_1 p_1/\mu_2 = \rho_2$. For $0 < \alpha < 1$, if we substitute $\beta = \alpha$ in the right-hand side of the quadratic equation (4.6), we get

$$\alpha^2(\alpha - 1)(\mu_1(1 - p_1) + \mu_2(1 - p_2)) < 0.$$

Hence, the quadratic equation (4.6) has two roots, one of them larger and the other smaller than α. The product of the two roots is $\mu_1 p_1/\mu_2$; hence, both are positive. ■

Similarly, for every fixed value $0 < \beta \leq 1$, (4.5) yields a quadratic equation for α:

$$[\mu_1 p_1 + \mu_1(1 - p_1)\beta]\,\alpha^2 - [(\mu_1 + \mu_2)\beta - \mu_2(1 - p_2)\beta^2]\,\alpha + [\mu_2 p_2 \beta^2] = 0, \tag{4.7}$$

PROPOSITION 4.3: *The quadratic equation (4.7) has two real roots for all* $0 < \beta \leq 1$. *For* $\beta = 1$, *the roots are* $\bar{\alpha} = 1$ *and* $\underline{\alpha} = \rho_1$. *For* $0 < \beta < 1$, *the larger root is* $\bar{\alpha} > \beta$ *and the smaller root is* $0 < \underline{\alpha} < \beta$.

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It is convenient to divide the quadratic equations (4.6) and (4.7) by $\alpha^2\beta^2$ and to consider quadratic equations for α^{-1}, β^{-1}:

$$[\mu_1 p_1](\beta^{-1})^2 - [(\mu_1+\mu_2)\alpha^{-1} - \mu_1(1-p_1)](\beta^{-1}) + [(\alpha^{-1})(\mu_2(1-p_2) + \mu_2 p_2 \alpha^{-1})] = 0 \quad \textbf{(4.8)}$$

$$[\mu_2 p_2](\alpha^{-1})^2 - [(\mu_1+\mu_2)\beta^{-1} - \mu_2(1-p_2)](\alpha^{-1}) + [(\beta^{-1})(\mu_1(1-p_1) + \mu_1 p_1 \beta^{-1})] = 0. \quad \textbf{(4.9)}$$

4.3. Compensating for the Vertical and Horizontal Boundaries

Let α and β satisfy (4.5), so that $\alpha^{n_1}\beta^{n_2}$ solves the balance equations (4.1) for all $n_1, n_2 = 0, \pm 1, \pm 2, \ldots$. We want to find a compensating term such that $\alpha^{n_1}\beta^{n_2} + c\tilde{\alpha}^{n_1}\tilde{\beta}^{n_2}$ will, in addition, solve the horizontal boundary equations (4.2).

We first subtract (4.2) from (4.1) to obtain

$$\mu_2(1-p_2)P(n_1,0) + \mu_2 p_2 P(n_1-1,0) = \mu_1 p_1 P(n_1+1,-1), \qquad n_1 > 0. \quad \textbf{(4.10)}$$

Since our trial solution $\alpha^{n_1}\beta^{n_2} + c\tilde{\alpha}^{n_1}\tilde{\beta}^{n_2}$ solves (4.1), it will solve (4.2) if and only if it solves (4.10). We substitute the trial solution in (4.10), yielding

$$[\mu_2(1-p_2) + \mu_2 p_2 \alpha^{-1} - \mu_1 p_1 \alpha\beta^{-1}]\alpha^{n_1} + c[\mu_2(1-p_2) + \mu_2 p_2 \tilde{\alpha}^{-1} - \mu_1 p_1 \tilde{\alpha}\tilde{\beta}^{-1}]\tilde{\alpha}^{n_1} = 0. \quad \textbf{(4.11)}$$

To satisfy (4.11) for all $n_1 > 0$, we are forced to take $\tilde{\alpha} = \alpha$; thus, to solve (4.1), we need to take $\tilde{\beta}$ as the second root of the quadratic equation (4.6). Using the quadratic equation (4.8), we get the second root $\tilde{\beta}^{-1}$ in terms of α and the first root β^{-1}:

$$\tilde{\beta}^{-1} = \frac{\mu_1+\mu_2}{\mu_1 p_1}\alpha^{-1} - \beta^{-1} - \frac{1-p_1}{p_1}.$$

We also get the product of the roots of (4.8):

$$\mu_1 p_1 \beta^{-1}\tilde{\beta}^{-1} = \alpha^{-1}(\mu_2(1-p_2) + \mu_2 p_2 \alpha^{-1}). \quad \textbf{(4.12)}$$

By canceling α^{n_1+1} in (4.11), we obtain an equation for c:

$$(1+c)\alpha^{-1}(\mu_2(1-p_2) + \mu_2 p_2 \alpha^{-1}) = \mu_1 p_1(\beta^{-1} + c\tilde{\beta}^{-1}). \quad \textbf{(4.13)}$$

We now use (4.12) to cancel $\mu_1 p_1 \beta^{-1}\tilde{\beta}^{-1}$ on both sides and obtain

$$c = -\frac{1-\tilde{\beta}}{1-\beta}.$$

Multiplying the linear combination by the constant $(1-\alpha)(1-\beta)$, we may conclude that $(1-\alpha)\alpha^{n_1}(1-\beta)\beta^{n_2} - (1-\alpha)\alpha^{n_1}(1-\tilde{\beta})\tilde{\beta}^{n_2}$ solves the balance equa-

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tions (4.1) and (4.2). The procedure to compensate for the vertical boundary equations is symmetric.

PROPOSITION 4.4: *Let* α *and* β *satisfy (4.5). Then* $(1-\alpha)\alpha^{n_1}(1-\beta)\beta^{n_2} - (1-\alpha)\alpha^{n_1}(1-\tilde{\beta})\tilde{\beta}^{n_2}$ *solves the balance equations* (4.1) *and* (4.2) *in the interior and the horizontal boundary if we take*

$$\tilde{\beta}^{-1} = \frac{\mu_1+\mu_2}{\mu_1 p_1}\alpha^{-1} - \beta^{-1} - \frac{1-p_1}{p_1}. \tag{4.14}$$

Similarly, $(1-\alpha)\alpha^{n_1}(1-\beta)\beta^{n_2} + (1-\tilde{\alpha})\tilde{\alpha}^{n_1}(1-\beta)\beta^{n_2}$ *solves the balance equations* (4.1) *and* (4.3) *in the interior and the vertical boundary if we take*

$$\tilde{\alpha}^{-1} = \frac{\mu_1+\mu_2}{\mu_2 p_2}\beta^{-1} - \alpha^{-1} - \frac{1-p_2}{p_2}. \tag{4.15}$$

4.4. Infinite Sequences of Compensations

Motivated by the marginal distribution (1.3), we start from a product form solution with $\alpha_1 = \rho_1$. The roots of (4.6) are $\beta_0 = 1$ and $\beta_2 < \rho_1$. Since we need convergence, we start from the trial solution $\alpha_1^{n_1}\beta_2^{n_2}$. To conform with our desired final form, we multiply this trial solution by a constant.

PROPOSITION 4.5: *The trial solution* $(1-\alpha_1)\alpha_1^{n_1}(1-\beta_2)\beta_2^{n_2}$ *with* $\alpha_1 = \rho_1$ *and* $\beta_2^{-1} = [(\mu_1+\mu_2)/\mu_1 p_1]\rho_1^{-1} - 1 - [(1-p_1)/p_1]$ *solves (4.1) and (4.2) for all* $n_1 > 0$ *and* $n_2 \geq 0$.

PROOF: In this case, the compensating term would have $\tilde{\beta} = 1$, but then $1-\tilde{\beta} = 0$, so the compensating term disappears. ■

We next add a compensating term to solve (4.1) and (4.3). According to (4.15), we choose

$$\alpha_3^{-1} = \frac{\mu_1+\mu_2}{\mu_2 p_2}\beta_2^{-1} - \alpha_1^{-1} - \frac{1-p_2}{p_2}$$

to obtain a two-term trial solution

$$(1-\alpha_1)\alpha_1^{n_1}(1-\beta_2)\beta_2^{n_2} - (1-\alpha_3)\alpha_3^{n_1}(1-\beta_2)\beta_2^{n_2}. \tag{4.16}$$

In this solution, the first term alone solves (4.1) and (4.2), and the two terms together solve (4.1) and (4.3).

From now on, we continue to add compensating terms to satisfy (4.2) and to satisfy (4.3) alternately. In the next step, we need to compensate the second term of (4.16) to solve (4.2) again, and we choose β_4 according to (4.14). We continue these compensating steps indefinitely.

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PROPOSITION 4.6: *For all $k \geq 1$, let*

$$\begin{aligned} \beta_{2k}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_1 p_1} \alpha_{2k-1}^{-1} - \beta_{2k-2}^{-1} - \frac{1-p_1}{p_1}, \\ \alpha_{2k+1}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_2 p_2} \beta_{2k}^{-1} - \alpha_{2k-1}^{-1} - \frac{1-p_2}{p_2}, \end{aligned} \tag{4.17}$$

with initially $\beta_0 = 1$ and $\alpha_1 = \rho_1$. Then trial solution

$$\sum_{k=1}^{\infty} [(1-\alpha_{2k-1})\alpha_{2k-1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2} - (1-\alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2}] \tag{4.18}$$

solves the balance equations for all $(n_1, n_2) \neq (0,1), (1,0), (0,0)$.

PROOF: We show in Section 4.7 that the infinite sum (4.18) is absolutely convergent for every $(n_1, n_2) \neq (0,0)$. We will also show that the summation of (4.18) over all the values of $(n_1, n_2) \neq (0,0)$ converges absolutely. In the rest of the proof, we take this statement as proved.

The pair (α_1, β_0) is on the curve (4.5). Hence, using (4.14) and (4.15) and induction, so are all the pairs $(\alpha_{2k-1}, \beta_{2k})$ and $(\alpha_{2k+1}, \beta_{2k})$. Hence, all the terms in (4.18) solve (4.1), and by absolute convergence, so does the infinite sum for $n_1 > 0$ and $n_2 > 0$.

In the sum (4.18), each negative term compensates the preceding positive term so that their sum solves (4.3); see Proposition 4.4. Hence, for all K,

$$\sum_{k=1}^{K} [(1-\alpha_{2k-1})\alpha_{2k-1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2} - (1-\alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2}]$$

solves (4.3). By absolute convergence, (4.18) solves (4.3), whenever the equations do not involve $(n_1, n_2) = (0,0)$. Hence, (4.18) solves (4.3) for all $(0, n_2), n_2 > 1$.

We saw that $(1-\alpha_1)\alpha_1^{n_1}(1-\beta_2)\beta_2^{n_2}$ solves (4.2). Each positive term $(1 - \alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k+2})\beta_{2k+2}^{n_2}$ compensates the preceding negative term $-(1 - \alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2}$ in the sum, so that their sum solves (4.2). Hence, for all K,

$$(1-\alpha_1)\alpha_1^{n_1}(1-\beta_2)\beta_2^{n_2} + \sum_{k=1}^{K} [-(1-\alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2} + (1-\alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k+2})\beta_{2k+2}^{n_2}]$$

solves (4.2). By absolute convergence, (4.18) solves (4.2), whenever the equations do not involve $(n_1, n_2) = (0,0)$. Hence, (4.18) solves (4.2) for all $(n_1, 0), n_1 > 1$. ∎

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Analogously, we can start from the $(1, \rho_2)$ on the curve (4.5) and get another solution:

PROPOSITION 4.7: *For all $k \geq 1$, let*

$$\alpha_{2k}^{-1} = \frac{\mu_1 + \mu_2}{\mu_2 p_2}\,\beta_{2k-1}^{-1} - \alpha_{2k-2}^{-1} - \frac{1-p_2}{p_2},$$
$$\beta_{2k+1}^{-1} = \frac{\mu_1 + \mu_2}{\mu_1 p_1}\,\alpha_{2k}^{-1} - \beta_{2k-1}^{-1} - \frac{1-p_1}{p_1}. \tag{4.19}$$

with initially $\alpha_0 = 1$ and $\beta_1 = \rho_2$. Then trial solution

$$\sum_{k=1}^{\infty}[(1-\alpha_{2k})\alpha_{2k}^{n_1}(1-\beta_{2k-1})\beta_{2k-1}^{n_2} - (1-\alpha_{2k})\alpha_{2k}^{n_1}(1-\beta_{2k+1})\beta_{2k+1}^{n_2}] \tag{4.20}$$

solves the balance equations for all $(n_1, n_2) \neq (0,1), (1,0), (0,0)$.

4.5. The Complete Solution

The two solutions in Propositions 4.6 and 4.7 were not defined for $(n_1, n_2) = (0,0)$. The reason is that, for $n_1 = n_2 = 0$, the sums are not (absolutely) convergent, and so they are meaningless. As a result, we could not check for $(n_1, n_2) = (1,0)$ or $(n_1, n_2) = (0,1)$.

To obtain a solution for all (n_1, n_2), we do the following: For all $(n_1, n_2) \neq (0,0)$, we take the sum of the two solutions (4.18) and (4.20). This yields

$$P(n_1, n_2) = \sum_{k=1}^{\infty}(-1)^{k+1}[(1-\alpha_k)\alpha_k^{n_1}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}(1-\beta_k)\beta_k^{n_2}]. \tag{4.21}$$

For $(n_1, n_2) = (0,0)$, we take

$$P(0,0) = 1 - \alpha_1 - \beta_1 + \sum_{k=1}^{\infty}(-1)^{k+1}(\alpha_k\beta_{k+1} + \alpha_{k+1}\beta_k). \tag{4.22}$$

PROPOSITION 4.8: *The expressions for $P(n_1, n_2)$ in (4.21) and (4.22) solve all of the balance equations.*

PROOF: We will show in Section 4.7 that the sum in $P(0,0)$ is also absolutely convergent. We shall take that as well as absolute convergence of all the other $P(n_1, n_2)$ and their sum over all n_1 and n_2 as proved.

We already know by the previous two propositions that $P(n_1, n_2)$ defined by (4.21) satisfy the balance equations (4.1), (4.2), and (4.3) for $n_1 + n_2 > 1$. It remains to consider the balance equations for $(1,0), (0,1)$, and $(0,0)$. For all K, we have seen in the proof of Proposition 4.6 that the sum

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$$(1-\alpha_1)\alpha_1^{n_1}(1-\beta_2)\beta_2^{n_2} + \sum_{k=1}^{K}[-(1-\alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k})\beta_{2k}^{n_2} + (1-\alpha_{2k+1})\alpha_{2k+1}^{n_1}(1-\beta_{2k+2})\beta_{2k+2}^{n_2}] \quad \mathbf{(4.23)}$$

solves (4.2). Also, for all K, we have seen in the proof of Proposition 4.7 that the sum

$$\sum_{k=1}^{K}[(1-\alpha_{2k})\alpha_{2k}^{n_1}(1-\beta_{2k-1})\beta_{2k-1}^{n_2} - (1-\alpha_{2k})\alpha_{2k}^{n_1}(1-\beta_{2k+1})\beta_{2k+1}^{n_2}] \quad \mathbf{(4.24)}$$

solves (4.2). Hence, the sum of (4.23) and (4.24) also solves (4.2). Consider, in particular, the balance equation for $(n_1,n_2)=(1,0)$:

$$(\mu_1+\mu_2 p_2)P(1,0) = \mu_2 p_2 P(0,0) + \mu_1(1-p_1)P(2,0) + \mu_2 p_2 P(0,1) + \mu_2(1-p_2)P(1,1). \quad \mathbf{(4.25)}$$

It is satisfied by the sum of (4.23) and (4.24). We now look at the sum of (4.23) and (4.24) for $n_1=n_2=0$:

$$\begin{aligned}
&(1-\alpha_1)(1-\beta_2)\\
&\quad + \sum_{k=1}^{K}[-(1-\alpha_{2k+1})(1-\beta_{2k}) + (1-\alpha_{2k+1})(1-\beta_{2k+2})]\\
&\quad + \sum_{k=1}^{K}[(1-\alpha_{2k})(1-\beta_{2k-1}) - (1-\alpha_{2k})(1-\beta_{2k+1})]\\
&= \sum_{k=1}^{2K}(-1)^{k+1}[(1-\alpha_k)(1-\beta_{k+1}) + (1-\alpha_{k+1})(1-\beta_k)]\\
&\quad + (1-\alpha_{2K+1})(1-\beta_{2K+2})\\
&= 1-\alpha_1-\beta_1+\sum_{k=1}^{2K}(-1)^{k+1}(\alpha_k\beta_{k+1}+\alpha_{k+1}\beta_k)\\
&\quad - \beta_{2K+2} + \alpha_{2K+1}\beta_{2K+2}.
\end{aligned}$$

As we will see, $\alpha_k, \beta_k \to 0$ as $k\to\infty$. This property and absolute convergence of the sum $\sum_{k=1}^{\infty}(-1)^{k+1}(\alpha_k\beta_{k+1}+\alpha_{k+1}\beta_k)$ shows that (4.25) is satisfied by $P(1,0)$, $P(0,1)$, $P(1,1)$, and $P(2,0)$ as defined in (4.21) and $P(0,0)$ as defined by (4.22). The proof for the balance equation of (0,1) is symmetric.

Finally, by the absolute convergence of the sum over all n_1 and n_2 of (4.21), we get that (4.4) is redundant and is satisfied automatically by the expressions for $P(n_1,n_2)$ in (4.21) and (4.22). ■

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4.6. Normalizing the Sum of Probabilities

We again take absolute convergence as proved. Based on that, we can calculate various quantities. We first obtain marginal probabilities, which are consistent with (1.3).

PROPOSITION 4.9: *For $n_i > 0$,*

$$\sum_{n_{3-i}=0}^{\infty} P(n_1, n_2) = (1-\rho_i)\rho_i^{n_i}.$$

PROOF: We make heavy use of the absolute convergence to change the order of summations and group sums of positive and negative terms. For $n_1 > 0$, the sum is

$$\begin{aligned}
&\sum_{n_2=0}^{\infty} P(n_1, n_2) \\
&\quad = \sum_{n_2=0}^{\infty}\sum_{k=1}^{\infty} (-1)^{k+1}[(1-\alpha_k)\alpha_k^{n_1}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}(1-\beta_k)\beta_k^{n_2}] \\
&\quad = \sum_{k=1}^{\infty} (-1)^{k+1}\Bigg[(1-\alpha_k)\alpha_k^{n_1}\sum_{n_2=0}^{\infty}(1-\beta_{k+1})\beta_{k+1}^{n_2} \\
&\qquad\qquad + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}\sum_{n_2=0}^{\infty}(1-\beta_k)\beta_k^{n_2}\Bigg] \\
&\quad = \sum_{k=1}^{\infty} (-1)^{k+1}[(1-\alpha_k)\alpha_k^{n_1} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}] \\
&\quad = (1-\alpha_1)\alpha_1^{n_1} - (1-\alpha_2)\alpha_2^{n_1} + (1-\alpha_2)\alpha_2^{n_1} - (1-\alpha_3)\alpha_3^{n_1} + \cdots \\
&\quad = (1-\alpha_1)\alpha_1^{n_1}.
\end{aligned}$$

The case of $i = 2$ is symmetric. ∎

We next calculate $P(n_1 = 0, n_2 > 0)$ and $P(n_1 > 0, n_2 = 0)$.

PROPOSITION 4.10:

$$\sum_{n_2=1}^{\infty} P(0, n_2) = \rho_2 + \sum_{k=1}^{\infty} (-1)^k(\alpha_k\beta_{k+1} + \alpha_{k+1}\beta_k),$$

$$\sum_{n_1=1}^{\infty} P(n_1, 0) = \rho_1 + \sum_{k=1}^{\infty} (-1)^k(\alpha_k\beta_{k+1} + \alpha_{k+1}\beta_k).$$

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PROOF: We again make heavy use of the absolute convergence:

$$\sum_{n_2=1}^{\infty} P(0,n_2)$$

$$= \sum_{n_2=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1}[(1-\alpha_k)(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})(1-\beta_k)\beta_k^{n_2}]$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1}\left[(1-\alpha_k)\sum_{n_2=1}^{\infty}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\sum_{n_2=1}^{\infty}(1-\beta_k)\beta_k^{n_2}\right]$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1}[(1-\alpha_k)\beta_{k+1} + (1-\alpha_{k+1})\beta_k]$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1}(\beta_k + \beta_{k+1}) + \sum_{k=1}^{\infty} (-1)^k(\alpha_k\beta_{k+1} + \alpha_{k+1}\beta_k)$$

$$= \beta_1 + \sum_{k=1}^{\infty} (-1)^k(\alpha_k\beta_{k+1} + \alpha_{k+1}\beta_k).$$

The other case is symmetric. ■

Finally, we have the following proposition.

PROPOSITION 4.11: *The probabilities* $P(n_1,n_2)$ *in (4.21) and (4.22) sum up to* 1.

PROOF: By the previous two propositions and (4.22),

$$\sum_{n_1,n_2} P(n_1,n_2) = \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{\infty} P(n_1,n_2) + \sum_{n_2=1}^{\infty} P(0,n_2) + P(0,0) = 1. \quad ■$$

4.7. Absolute Convergence

In the previous sections, we made heavy use of the absolute convergence of the sums in (4.21) and (4.22). This will be proved next.

PROPOSITION 4.12:

$$\sum_{(n_1,n_2)\neq(0,0)} \sum_{k=1}^{\infty} [(1-\alpha_k)\alpha_k^{n_1}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}(1-\beta_k)\beta_k^{n_2}] < \infty.$$

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PROOF:

$$\sum_{(n_1,n_2)\neq(0,0)} \sum_{k=1}^{\infty} [(1-\alpha_k)\alpha_k^{n_1}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}(1-\beta_k)\beta_k^{n_2}]$$

$$= \sum_{k=1}^{\infty}\left[(1-\alpha_k)\sum_{n_2=1}^{\infty}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\sum_{n_2=1}^{\infty}(1-\beta_k)\beta_k^{n_2}\right]$$

$$+ \sum_{k=1}^{\infty}\sum_{n_1=1}^{\infty}\left[(1-\alpha_k)\alpha_k^{n_1}\sum_{n_2=0}^{\infty}(1-\beta_{k+1})\beta_{k+1}^{n_2} \right.$$

$$\left. + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}\sum_{n_2=0}^{\infty}(1-\beta_k)\beta_k^{n_2}\right]$$

$$= \sum_{k=1}^{\infty}((1-\alpha_k)\beta_{k+1} + (1-\alpha_{k+1})\beta_k) + \sum_{k=1}^{\infty}(\alpha_k + \alpha_{k+1})$$

$$< 2\sum_{k=1}^{\infty}(\alpha_k + \beta_k).$$

In the next proposition, we show that the sequences α_k and β_k decrease geometrically, and, hence, the last sum converges. ■

PROPOSITION 4.13: *For all $k \geq 0$,*

$$\alpha_{k+1} \leq \rho_1 \beta_k, \qquad \beta_{k+1} \leq \rho_2 \alpha_k.$$

PROOF: For $k = 0$, we have $\alpha_0 = \beta_0 = 1, \alpha_1 = \rho_1$, and $\beta_1 = \rho_2$ and this is the only case of equality. For $k \geq 1$, by (4.15),

$$\alpha_{k+1}^{-1} = (p_2^{-1} + \rho_1^{-1})\beta_k^{-1} - \alpha_{k-1}^{-1} - \frac{1-p_2}{p_2} > (p_2^{-1} + \rho_1^{-1} - 1)\beta_k^{-1} - p_2^{-1} + 1$$

$$> \rho_1^{-1}\beta_k^{-1},$$

where the first inequality follows from $\beta_k^{-1} > \alpha_{k-1}^{-1}$ and the second from $\beta_k^{-1} > 1$. The proof for β_{k+1} is symmetric. ■

We can also get the asymptotic rate of decay of α_k and β_k:

PROPOSITION 4.14: *As $k \to \infty$,*

$$\frac{\alpha_{k+1}}{\alpha_{k-1}}, \frac{\beta_{k+1}}{\beta_{k-1}} \to \frac{\mu_1 + \mu_2 - \sqrt{(\mu_1+\mu_2)^2 - 4\mu_1\mu_2 p_1 p_2}}{\mu_1 + \mu_2 + \sqrt{(\mu_1+\mu_2)^2 - 4\mu_1\mu_2 p_1 p_2}}.$$

PROOF: The parameters α_{k+1} and α_{k-1} are the roots of (4.5) with $\beta = \beta_k$. Dividing (4.5) by β_k^2, we get that α_{k+1}/β_k and α_{k-1}/β_k are the roots of

$$[\mu_1 p_1 + \mu_1(1-p_1)\beta]\gamma^2 - [(\mu_1+\mu_2) - \mu_2(1-p_2)\beta]\gamma + \mu_2 p_2 = 0, \qquad \textbf{(4.26)}$$

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with $\beta = \beta_k$. As $k \to \infty$, then $\beta_k \to 0$ by Proposition 4.13 and, thus,

$$\frac{\alpha_{k+1}}{\beta_k} \to \gamma_1, \qquad \frac{\alpha_{k-1}}{\beta_k} \to \gamma_2,$$

where $0 < \gamma_1 < 1 < \gamma_2$ are the roots of (4.26) with $\beta = 0$. Hence,

$$\frac{\alpha_{k+1}}{\alpha_{k-1}} \to \frac{\gamma_1}{\gamma_2} = \frac{\mu_1 + \mu_2 - \sqrt{(\mu_1+\mu_2)^2 - 4\mu_1\mu_2 p_1 p_2}}{\mu_1 + \mu_2 + \sqrt{(\mu_1+\mu_2)^2 - 4\mu_1\mu_2 p_1 p_2}}.$$

The proof for β_{k+1}/β_k is similar. ■

From the geometric decay of the sequences α_k and β_k, we can further conclude the following:

Corollary 4.15:

(i) *The sum that defines* $P(0,0)$ *in (4.22) is absolutely convergent.*
(ii) *For all* $m, n \geq 0, m + n > 0$, *the sum defining* $\mathbb{E}\left(\binom{X_1}{m}\binom{X_2}{n}\right)$ *in (2.5) is absolutely convergent.*

4.8. Nonnegativity and Ergodicity

From Propositions 4.8, 4.11, and 4.12 it follows that $P(n_1, n_2)$ given by (4.21) and (4.22) are a nonnull, absolutely convergent solution of the balance equations, which sums up to 1. From Theorem 1 in Foster [4], we can immediately conclude the following:

Corollary 4.16: *The Markov jump process* $X(t) = (X_1(t), X_2(t))$ *is ergodic when* $\rho_1, \rho_2 < 1$, *and its equilibrium probabilities are given by the solution* $P(n_1, n_2)$ *defined by (4.21) and (4.22).*

4.9. Queue Length Correlation

In this subsection, we show that the correlation between X_1 and X_2 is always negative, which is equivalent to

$$\begin{aligned}\mathbb{E}(X_1 X_2) &= \sum_{k=1}^{\infty} (-1)^{k+1} \left[\frac{\alpha_k}{(1-\alpha_k)} \frac{\beta_{k+1}}{(1-\beta_{k+1})} + \frac{\alpha_{k+1}}{(1-\alpha_{k+1})} \frac{\beta_k}{(1-\beta_k)} \right] \\ &< \frac{\rho_1}{1-\rho_1} \frac{\rho_2}{1-\rho_2}. \end{aligned} \tag{4.27}$$

The terms in the infinite sum (4.27) are alternating and decreasing in magnitude, since $\alpha_1 > \beta_2 > \alpha_3 > \cdots$ and $\beta_1 > \alpha_2 > \beta_3 > \cdots$. Hence, it suffices to show that

$$\frac{\alpha_1}{(1-\alpha_1)} \frac{\beta_2}{(1-\beta_2)} + \frac{\alpha_2}{(1-\alpha_2)} \frac{\beta_1}{(1-\beta_1)} < \frac{\rho_1}{1-\rho_1} \frac{\rho_2}{1-\rho_2},$$

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which can be verified by straightforward calculations. This proves the following proposition:

PROPOSITION 4.17: *If* $\rho_1, \rho_2 < 1$, *then* $\mathrm{Corr}(X_1, X_2) < 0$.

Figure 5 displays the correlation for the symmetric system $\mu_1 = \mu_2 = \mu$ and $p_1 = p_2 = p$. To find the limiting value of the correlation as $p \uparrow 1$, note that in the symmetric case,

$$\alpha_k = \beta_k = 1 - \frac{1}{2} k(k+1)(1-p) + O(1-p)^2,$$

which can be derived from the recursive relations for the sequences α_k and β_k. Hence, from (4.27) and using that

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = \frac{p}{1-p}, \qquad V(X_1) = V(X_2) = \frac{p}{(1-p)^2},$$

we obtain

$$\lim_{p\uparrow 1} \mathrm{Corr}(X_1, X_2) = \lim_{p\uparrow 1} \frac{\mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)}{\sqrt{V(X_1)V(X_2)}} = 8 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)^2(k+2)} - 1$$
$$= \frac{2}{3}\pi^2 - 7.$$

Note that exactly the same asymptotic correlation value appears in the calculations of Boxma and van Houtum [3, p. 488], which is curious, since the two models are quite different.

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