J. Inst. Math. Jussieu (2020) **19**(2), 647–661 647
doi:10.1017/S1474748018000166

GENERAL HYPERPLANE SECTIONS OF THREEFOLDS IN POSITIVE CHARACTERISTIC

KENTA SATO[1] AND SHUNSUKE TAKAGI[2]

[1] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan* (ktsato@ms.u-tokyo.ac.jp)
[2] *Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan* (stakagi@ms.u-tokyo.ac.jp)

(Received 9 March 2017; revised 14 March 2018; accepted 16 March 2018; first published online 12 April 2018)

Dedicated to Professor Lawrence Ein on the occasion of his sixty birthday

Abstract In this paper, we study the singularities of a general hyperplane section H of a three-dimensional quasi-projective variety X over an algebraically closed field of characteristic $p > 0$. We prove that if X has only canonical singularities, then H has only rational double points. We also prove, under the assumption that $p > 5$, that if X has only klt singularities, then so does H.

Keywords: Bertini theorem; canonical singularities; klt singularities; MJ-canonical singularities; strongly F-regular singularities

2010 *Mathematics subject classification*: 14B05; 14J17; 13A35

1. Introduction

Reid proved in his historic paper [29] that if a quasi-projective variety over an algebraically closed field of characteristic zero has only canonical singularities, then its general hyperplane section has only canonical singularities, too. This can be deduced from the Bertini theorem for base point free linear systems, and the same argument works for other classes of singularities in the minimal model program such as terminal, klt and log canonical singularities. Then what if the variety is defined over an algebraically closed field of positive characteristic? As a test case, we consider the following question:

Question 1.1. Let X be a three-dimensional quasi-projective variety over an algebraically closed field k of positive characteristic and H be a general hyperplane section of X. If X has only terminal singularities, is H smooth? If X has only canonical (resp. klt, log canonical) singularities, does H have only rational double points (resp. klt singularities, log canonical singularities)?

Since the Bertini theorem for base point free linear systems fails in positive characteristic, Reid's argument does not work in this setting. Thanks to the recent developments in birational geometry, we are able to find a resolution of singularities of X [6, 7] and, if the characteristic of k is larger than 5, run the minimal model program [3, 4, 14, 22], but we do not know in general how to overcome the difficulty arising from the lack of the Bertini theorem.

The terminal case in Question 1.1 is proved affirmatively by the fact that three-dimensional terminal singularities are isolated singularities. The other cases are much more subtle. For example, Reid proved, as a corollary of the result mentioned above, that all closed points of a complex canonical threefold, except finitely many of them, have an analytic neighborhood which is nonsingular or isomorphic to the product of a rational double point and $\mathbb{A}^1_{\mathbb{C}}$. This does not hold in positive characteristic: Hirokado and Ito and Saito [21] and Hirokado [20] gave counterexamples in characteristic two and three, respectively. However, the canonical case in Question 1.1 has remained open.

In this paper, we give an affirmative answer to the canonical and klt cases in Question 1.1, using jet schemes and F-singularities, with the proviso that the characteristic of k is larger than 5 in the klt case. In both cases, we look at every codimension two point x of X. The normal surface singularity $\operatorname{Spec} \mathcal{O}_{X,x}$ is canonical or klt but is not defined over an algebraically closed field. In the canonical case, we observe, as in the case where the base field is algebraically closed, that $\operatorname{Spec} \mathcal{O}_{X,x}$ is a hypersurface singularity. In the klt case, we deduce from the following theorem, which is a generalization of a result of Hara [15], that $\operatorname{Spec} \mathcal{O}_{X,x}$ is strongly F-regular when the characteristic of k is larger than 5.

Theorem 1.2 (Proposition 2.8, Theorem 5.7). *Let $(s \in S)$ be an F-finite normal surface singularity (not necessarily defined over an algebraically closed field) of characteristic $p > 5$ and B be an effective $\mathbb{Q}$-divisor on S whose coefficients belong to the standard set $\{1 - 1/m \mid m \in \mathbb{Z}_{\geqslant 1}\}$. Then $(s \in S, B)$ is klt if and only if $(s \in S, B)$ is strongly F-regular.*

In the canonical case, we see from the above observation that X is MJ-canonical except at finitely many closed points. MJ-canonical singularities, also called Mather–Jacobian canonical singularities, are defined in terms of Mather–Jacobian discrepancies instead of the usual discrepancies and can be viewed as a jet scheme theoretic counterpart of canonical singularities (see [8, 10] for further details). Similarly, in the klt case, we see that if the characteristic of k is larger than 5, then X is strongly F-regular except at finitely many closed points. Strongly F-regular singularities are defined in terms of Frobenius splitting and can be viewed as an F-singularity theoretic counterpart of klt singularities (see, for example, [34]). Applying the Bertini theorem for MJ-canonical singularities due to Ishii and Reguera [24] to the canonical case and the Bertini theorem for strongly F-regular singularities due to Schwede and Zhang [32] to the klt case, we obtain the assertion. To be precise, we can prove a more general result involving a boundary divisor, which is stated as follows.

Main Theorem (Proposition 4.1, Theorems 4.2 and 5.9). *Let X be a three-dimensional normal quasi-projective variety over an algebraically closed field k of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let H be a general hyperplane section of X.*

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

(1) *If* (X, Δ) *is terminal, then* $(H, \Delta|_H)$ *is terminal.*
(2) *If* (X, Δ) *is canonical, then* $(H, \Delta|_H)$ *is canonical.*
(3) *Suppose in addition that* $p > 5$ *and the coefficients of* Δ *belong to the standard set* $\{1 - 1/m \mid m \in \mathbb{Z}_{\geqslant 1}\}$. *If* (X, Δ) *is klt, then* $(H, \Delta|_H)$ *is klt.*

Notation. Throughout this paper, all rings are assumed to be commutative and with a unit element and all schemes are assumed to be noetherian and separated. A variety over a field k means an integral scheme of finite type over k.

2. Preliminaries

2.1. F-singularities

In this subsection, we recall the basic notions of F-singularities, which we will need in § 4.

Let X be a scheme of prime characteristic $p > 0$. We say that X is *F-finite* if the Frobenius morphism $F : X \longrightarrow X$ is a finite morphism. When $X = \operatorname{Spec} R$ is an affine scheme, R is said to be F-finite if X is F-finite. For example, a field k of positive characteristic $p > 0$ is F-finite if and only if $[k : k^p] < \infty$. It is known by [26] that every F-finite scheme is locally excellent.

Suppose in addition that X is a normal integral scheme. For an integer $e \geqslant 1$ and a Weil divisor D on X, we consider the following composite map:

$$\mathcal{O}_X \xrightarrow{(F^e)^\#} F^e_* \mathcal{O}_X \xrightarrow{F^e_* \iota} F^e_* \mathcal{O}_X(D),$$

where $F^e : X \longrightarrow X$ is the e-th iteration of Frobenius and $\iota : \mathcal{O}_X \longrightarrow \mathcal{O}_X(D)$ is the natural inclusion.

Strong F-regularity is one of the most basic classes of F-singularities.

Definition 2.1. Let $(R, \mathfrak{m})$ be an F-finite normal local ring of positive characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$. Then the pair (R, Δ) is said to be *strongly F-regular* if for every nonzero element $c \in R$, there exists an integer $e \geqslant 1$ such that the map

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + \operatorname{div}_X(c))$$

splits as an $\mathcal{O}_X$-module homomorphism. When y is a point in an F-finite normal integral scheme Y and B is an effective $\mathbb{Q}$-divisor on Y, we say that $(y \in Y, B)$ is strongly F-regular if $(\mathcal{O}_{Y,y}, B_y)$ is strongly F-regular. We also say that $(y \in Y)$ is strongly F-regular if $(\mathcal{O}_{Y,y}, 0)$ is strongly F-regular.

Remark 2.2. We have the following hierarchy of properties of normal singularities:

$$\text{regular} \Longrightarrow \text{strongly } F\text{-regular} \Longrightarrow \text{Cohen–Macaulay}.$$

Global F-regularity is defined as a global version of strong F-regularity.

Definition 2.3. Let X be an F-finite normal integral scheme and Δ be an effective $\mathbb{Q}$-divisor on X. Then (X, Δ) is said to be *globally F-regular* if for every Cartier divisor

D on X, there exists an integer $e \geqslant 1$ such that the map

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits as an $\mathcal{O}_X$-module homomorphism.

Remark 2.4. If X is an affine scheme, then (X, Δ) is globally F-regular if and only if $(x \in X, \Delta)$ is strongly F-regular for all $x \in X$. However, in general, the former is much stronger than the latter. For example, a smooth projective variety of general type over an algebraically closed field of positive characteristic is not globally F-regular but strongly F-regular at all points.

2.2. Singularities in the minimal model program

In this subsection, we recall the definition of singularities in the minimal model program. Note that we do not assume that the singularities are defined over an algebraically closed field. We start with the definition of canonical divisors on normal schemes.

Suppose that $\pi : Y \longrightarrow X$ is a separated morphism of finite type between schemes. If X has a dualizing complex (see [18, Chapter V, § 2] for the definition of dualizing complexes), then $\omega_Y^\bullet := \pi^! \omega_X^\bullet$ is a dualizing complex on Y, where $\pi^!$ is the twisted inverse image functor associated to π obtained in [18, Chapter VII, Corollary 3.4(a)]. Since any F-finite affine scheme has a dualizing complex by [12], any scheme of finite type over an F-finite local ring has a dualizing complex.

Let X be an excellent integral scheme with a dualizing complex $\omega_X^\bullet$. The *canonical sheaf* ω_X associated to $\omega_X^\bullet$ is the coherent $\mathcal{O}_X$-module defined as the first nonzero cohomology module of $\omega_X^\bullet$. It is well known that ω_X satisfies Serre's second condition (S_2) (see for example [1, (1.10)]). If X is a variety over an algebraically closed field k with structure morphism $f : X \longrightarrow \operatorname{Spec} k$, then the canonical sheaf associated to $f^! k$ coincides with the classical definition of the canonical sheaf of X. When X is a normal scheme, a *canonical divisor* K_X on X associated to $\omega_X^\bullet$ is any Weil divisor K_X on X such that $\mathcal{O}_X(K_X) \cong \omega_X$.

Suppose that X is a normal scheme with a dualizing complex $\omega_X^\bullet$ and fix a canonical divisor K_X on X associated to $\omega_X^\bullet$. Given a proper birational morphism $\pi : Y \longrightarrow X$ from a normal scheme Y, we always choose a canonical divisor K_Y on Y which is associated to $\pi^! \omega_X^\bullet$ and coincides with K_X outside the exceptional locus of f.

Definition 2.5. (i) We say that $(x \in X)$ is a *normal singularity* if $X = \operatorname{Spec} R$ is an affine scheme with a dualizing complex where R is an excellent normal local ring and if x is the unique closed point of X.

(ii) A proper birational morphism $f : Y \longrightarrow X$ between schemes is said to be a *resolution of singularities* of X, or a *resolution* of X for short, if Y is regular. Suppose that X is a normal integral scheme and Δ is a $\mathbb{Q}$-divisor on X. A resolution $f : Y \longrightarrow X$ is said to be a *log resolution* if the union of $f^{-1}(\operatorname{Supp}(\Delta))$ and the exceptional locus of f is a simple normal crossing divisor.

We are now ready to state the definition of singularities in the minimal model program.

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

Definition 2.6. (i) Suppose that $(x \in X)$ is a normal singularity with a dualizing complex $\omega_X^\bullet$ and Δ is an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Given a proper birational morphism $\pi : Y \longrightarrow X$ from a normal scheme Y, we choose a canonical divisor K_Y on Y associated to $\pi^! \omega_X^\bullet$ such that

$$K_Y + \pi_*^{-1}\Delta = \pi^*(K_X + \Delta) + \sum_i a_i E_i,$$

where $\pi_*^{-1}\Delta$ is the strict transform of Δ by π, the a_i are rational numbers and the E_i are π-exceptional prime divisors on Y. We say that $(x \in X, \Delta)$ is *terminal* (respectively *canonical*, *klt*, *plt*) if all the $a_i > 0$ (respectively all the $a_i \geqslant 0$, all the $a_i > -1$ and $\lfloor \Delta \rfloor = 0$, all the $a_i > -1$) for every proper birational morphism $\pi : Y \longrightarrow X$ from a normal scheme Y. We say that $(x \in X)$ is *terminal* (resp. *canonical*, *klt*, *plt*) if so is $(x \in X, 0)$.

(ii) Let X be an excellent integral normal scheme with a dualizing complex and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Then (X, Δ) is *terminal* (resp. *canonical*, *klt*, *plt*) if so is the normal singularity $(x \in X = \operatorname{Spec} \mathcal{O}_{X,x}, \Delta_x)$ for every $x \in X$.

Remark 2.7. (i) If X is defined over a field of characteristic zero or $\dim X \leqslant 3$, then it is enough to check the condition in Definition 2.6 only for one f, namely, for a log resolution of (X, Δ).

(ii) We have the following hierarchy of properties of $\mathbb{Q}$-Gorenstein normal singularities:

$$\text{regular} \Longrightarrow \text{terminal} \Longrightarrow \text{canonical} \Longrightarrow \text{klt}.$$

Strong F-regularity implies being klt.

Proposition 2.8 [17, Theorem 3.3]. *Let $(x \in X)$ be an F-finite normal singularity and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If $(x \in X, \Delta)$ is strongly F-regular, then it is klt.*

3. Brief review on surface singularities

In this section, we briefly review the theory of surface singularities. All results are well known if the singularity is defined over an algebraically closed field. Some of the results may be known to experts (even if it is not defined over an algebraically closed field), but we include their proofs here, because we have not been able to find any reference to them.

Definition 3.1. We say that a scheme X is a *surface* if X is a two-dimensional excellent separated integral scheme with a dualizing complex $\omega_X^\bullet$ and that a normal singularity $(x \in X)$ is a *surface singularity* if $\dim \mathcal{O}_{X,x} = 2$.

Remark 3.2. From now on, we often use the results in [25] and [33]. We remark that the base scheme is assumed to be regular in these references, but this assumption is

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

unnecessary in our framework, because our base scheme is a normal surface singularity and its dualizing complex is unique up to isomorphism.

First we recall the definition and basic properties of intersection numbers. Let $(x \in X)$ be a normal surface singularity and $f : Y \longrightarrow X$ be a proper birational morphism from a normal surface Y with exceptional curves $E = \bigcup_i E_i$. For a Cartier divisor D on Y and a Weil divisor $Z = \sum_i a_i E_i$ on Y, We define the intersection number $D \cdot Z$ as follows:

$$D \cdot Z = \sum_i a_i \deg_{E_i/k(x)}(\mathcal{O}_Y(D)|_{E_i}),$$

where $\deg_{E_i/k(x)} : \mathrm{Pic}(E_i) \longrightarrow \mathbb{Z}$ is the degree morphism defined as in [11, Definition 1.4].

Proposition 3.3 [11, Proposition 2.5], [25, Theorem 10.1]. *With the notation above, the following hold.*

(1) *The intersection pairing*

$$\mathrm{Div}(X) \times \bigoplus_i \mathbb{Z}E_i \longrightarrow \mathbb{Z}, \quad (D, Z) \longmapsto D \cdot Z$$

is a bilinear map, where $\mathrm{Div}(X)$ *denotes the group of Cartier divisors on* X.

(2) *If* Z *is Cartier and* $\mathrm{Supp}\ D \subseteq E$, *then* $D \cdot Z = Z \cdot D$.

(3) *If* Y *is regular, then the intersection matrix* $(E_i \cdot E_j)_{i,j}$ *is a negative-definite symmetric matrix.*

Definition 3.4. Let $(x \in X)$ be a normal surface singularity with a dualizing complex $\omega_X^\bullet$. A resolution $f : Y \longrightarrow X$ with exceptional curves $E = \bigcup_i E_i$ is said to be *minimal* if a canonical sheaf K_Y on Y associated to the dualizing complex $f^!\omega_X^\bullet$ is f-nef, that is, $K_Y \cdot E_i \geqslant 0$ for every i. A minimal resolution of X always exists by [25, Theorem 2.25].

We also recall the definition of rational singularities. A normal surface singularity $(x \in X)$ is said to be a *rational singularity* if $R^1 f_* \mathcal{O}_Y = 0$ for every resolution $f : Y \longrightarrow X$.

Proposition 3.5. *Let* $(x \in X)$ *be a normal surface singularity.*

(1) *[27, Proposition 1.2]* $(x \in X)$ *is a rational singularity if and only if* $R^1 f_* \mathcal{O}_Y = 0$ *for some resolution* $f : Y \longrightarrow X$.

(2) *[25, Proposition 2.28] If there exists an effective* $\mathbb{Q}$*-divisor* Δ *on* X *such that* $(x \in X, \Delta)$ *is plt, then* $(x \in X)$ *is a rational singularity.*

In the rest of this section, we work in the following setting.

Setting 3.6. Suppose that $(x \in X)$ is a normal surface singularity with a dualizing complex $\omega_X^\bullet$ and $f : Y \longrightarrow X$ is a (nontrivial) resolution with exceptional curves $E = \bigcup_i E_i$. Let K_Y denote a canonical divisor on Y associated to $f^!\omega_X^\bullet$. Let Z_f denote the fundamental cycle of f, that is, a unique minimal nonzero effective divisor Z on Y such that

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

Supp $Z \subseteq E$ and $-Z$ is f-nef (such a divisor always exists by [25, Definition and Claim 10.3.6] and we have Supp $Z_f = E$ by [25, Claim 10.3.5]).

Let D be a divisor on Y and Z be a nonzero effective divisor on Y such that Supp $Z \subseteq E$. The Euler characteristic $\chi(Z, D)$ is defined by

$$\chi(Z, \mathcal{O}_Y(D)|_Z) = \dim_{k(x)} H^0(Z, \mathcal{O}_Y(D)|_Z) - \dim_{k(x)} H^1(Z, \mathcal{O}_Y(D)|_Z).$$

The following is the Riemann–Roch theorem for curves embedded in a regular surface.

Proposition 3.7. *For each divisor D on Y and each nonzero effective divisor $Z = \sum_i a_i E_i$ on Y such that* Supp $Z \subseteq E$, *we have*

$$\chi(Z, D) = \frac{(2D - K_Y - Z) \cdot Z}{2} = \chi(Z, 0) + D \cdot Z.$$

Proof. We will prove the assertion by induction on $a := \sum_i a_i$. If $a = 1$, then $Z = E_i$ for some i and the assertion immediately follows from [33, Theorem 2.5(1) and Theorem 2.7] (see also Remark 3.2). Suppose that $a \geqslant 2$. Then there exists i such that $a_i \geqslant 1$, so set $Z' := Z - E_i$ and consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{E_i}(-Z') \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{Z'} \longrightarrow 0.$$

By the additivity of the Euler characteristic and applying the induction hypothesis to Z' and E_i, one has

$$\begin{aligned} \chi(Z, D) &= \chi(Z', D) + \chi(E_i, D - Z') \\ &= \frac{(2D - K_Y - Z') \cdot Z'}{2} + \frac{(2(D - Z') - K_Y - E_i) \cdot E_i}{2} \\ &= \frac{(2D - K_Y - Z) \cdot Z}{2}. \end{aligned}$$ □

Proposition 3.8 [2]. *Suppose that $(x \in X)$ is a rational singularity. Let $\mathfrak{m}_x$ denote the maximal ideal on X corresponding to x and $e(x \in X)$ be the Hilbert–Samuel multiplicity of $\mathcal{O}_{X,x}$. Then the following hold for each integer $n \geqslant 0$.*

(1) $\mathfrak{m}_x^n = H^0(Y, \mathcal{O}_Y(-nZ_f))$.

(2) $\dim_{k(x)} \mathfrak{m}^n/\mathfrak{m}^{n+1} = -nZ_f^2 + 1$.

(3) $e(x \in X) = -Z_f^2$.

Proof. First note that $\mathfrak{m}_x^n \mathcal{O}_Y = \mathcal{O}_Y(-nZ_f)$, which follows from essentially the same argument as for [2, Theorem 4], where X is assumed to be defined over an algebraically closed field. Since in a rational surface singularity, the product of integrally closed ideals is again integrally closed (see [27, Theorem 7.2]), $\mathfrak{m}_x^n$ is integrally closed. Therefore,

$$\mathfrak{m}_x^n = H^0(Y, \mathfrak{m}_x^n \mathcal{O}_Y) = H^0(Y, \mathcal{O}_Y(-nZ_f)).$$

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

For (2), we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-(\ell+1)Z_f) \longrightarrow \mathcal{O}_Y(-\ell Z_f) \longrightarrow \mathcal{O}_{Z_f}(-\ell Z_f) \longrightarrow 0$$

for each integer $\ell \geqslant 0$. Since $H^1(Y, \mathcal{O}_Y(-\ell Z_f)) = H^1(Y, \mathcal{O}_Y(-(\ell+1)Z_f)) = 0$ by [25, Proposition 10.9(1)] and $H^2(Y, \mathcal{O}_Y(-(\ell+1)Z_f)) = 0$ by [19, Chapter III, Corollary 11.2], which is valid for any proper morphism, we have $H^1(Z_f, \mathcal{O}_{Z_f}(-\ell Z_f)) = 0$. Therefore,

$$\begin{aligned}\chi(Z_f, -\ell Z_f) &= \dim_{k(x)} H^0(Z_f, \mathcal{O}_{Z_f}(-\ell Z_f)) \\ &= \dim_{k(x)} H^0(Y, \mathcal{O}_Y(-\ell Z_f))/H^0(Y, \mathcal{O}_Y(-(\ell+1)Z_f)) \\ &= \dim_{k(x)} \mathfrak{m}_x^{\ell}/\mathfrak{m}_x^{\ell+1},\end{aligned}$$

where the last equality follows from (1). In particular, $\chi(Z_f, 0) = 1$. Thus, wee see from Proposition 3.7 that

$$\dim_{k(x)} \mathfrak{m}_x^n/\mathfrak{m}_x^{n+1} = \chi(Z_f, -nZ_f) = 1 - nZ_f^2.$$

Finally, (3) immediately follows from (2), because

$$\dim_{k(x)} \mathcal{O}_{X,x}/\mathfrak{m}_x^n = \sum_{i=0}^{n-1} \dim_{k(x)} \mathfrak{m}_x^i/\mathfrak{m}_x^{i+1} = \frac{-Z_f^2}{2} n^2 + O(n). \qquad \square$$

We will use the following result to prove the canonical case of the main theorem in § 3.

Proposition 3.9. *A normal surface singularity $(x \in X)$ is canonical if and only if $(x \in X)$ is either a regular point or a rational double point. In particular, a canonical surface singularity is a hypersurface singularity.*

Proof. We may assume by Proposition 3.5 that $(x \in X)$ is a rational singularity. Since the assertion is obvious when X is regular, we can also assume that $(x \in X)$ is not regular.

Suppose that $f : Y \longrightarrow X$ is a minimal resolution, and let $\Delta = f^*K_X - K_Y$. Then Δ is an anti-f-nef $\mathbb{Q}$-divisor on Y, so $\Delta \geqslant 0$ by [25, Claim 10.3.5]. On the other hand, it follows from Proposition 3.7 that

$$2 = 2\chi(Z_f, 0) = -(K_Y + Z_f) \cdot Z_f = -K_Y \cdot Z_f - Z_f^2 = \Delta \cdot Z_f - Z_f^2,$$

which implies that $\Delta \cdot Z_f = Z_f^2 + 2$.

If $(x \in X)$ is a canonical singularity, then we have $\Delta = 0$ and hence $Z_f^2 = -2$. We see by Proposition 3.8(3) that $e(x \in X) = 2$, that is, $(x \in X)$ is a rational double point. Using Proposition 3.8(2), (3), we can verify that every rational double point is a hypersurface singularity.

Conversely, if $(x \in X)$ is a rational double point, then $Z_f^2 = -2$ by Proposition 3.8(3) and therefore $\Delta \cdot Z_f = 0$. Since $-\Delta$ is f-nef and $\operatorname{Supp} Z_f = E$, the intersection number $\Delta \cdot E_i$ has to be zero for all i. This means by [25, Claim 10.3.5] that $\Delta = 0$. It then follows from the fact that $K_Y + \Delta = f^*K_X$ and $(Y, \Delta) = (Y, 0)$ is canonical that $(x \in X)$ is a canonical singularity. $\square$

4. Terminal and canonical singularities

The terminal case of the main theorem is an immediate consequence of a Bertini theorem for Hilbert–Samuel multiplicity.

Proposition 4.1. *Let X be a three-dimensional normal quasi-projective variety over an algebraically closed field of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that the pair (X, Δ) is terminal. Then $(H, \Delta|_H)$ is terminal for a general hyperplane section H of X. In particular, H is smooth.*

Proof. Let U be the locus of the points $x \in X$ such that X is regular at x and $\mathrm{mult}_x(\Delta) < 1$. Since the regular locus of X is open and Hilbert–Samuel multiplicity is upper semicontinuous, U is an open subset of X. Then every codimension two point $x \in X$ lies in U by [25, Theorem 2.29(1)], because $(\mathrm{Spec}\,\mathcal{O}_{X,x}, \Delta_x)$ is a two-dimensional terminal pair. Hence, $X \setminus U$ consists of only finitely many closed points and a general hyperplane section H of X is contained in U. It follows from a Bertini theorem for Hilbert–Samuel multiplicity [9, Proposition 4.5], which holds in arbitrary characteristic, that H is smooth and $\mathrm{mult}_x(\Delta|_H) = \mathrm{mult}_x(\Delta) < 1$ for all $x \in \mathrm{Supp}\,\Delta|_H$. Thus, applying [25, Theorem 2.29(1)] again, we see that $(H, \Delta|_H)$ is terminal. □

The canonical case of the main theorem can be deduced from a Bertini theorem for MJ-canonical singularities. The proof was inspired by a discussion with Shihoko Ishii, whom we thank.

Theorem 4.2. *Let X be a three-dimensional normal quasi-projective variety over an algebraically closed field k of characteristic $p > 0$ and Δ be an effective $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Suppose that the pair (X, Δ) is canonical. Then $(H, \Delta|_H)$ is canonical for a general hyperplane section H of X. In particular, H has only rational double points.*

Proof. First we will show the case where $\Delta = 0$. Since X has only canonical singularities, $\mathrm{Spec}\,\mathcal{O}_{X,x}$ is a two-dimensional scheme with only canonical singularities for every codimension two point $x \in X$. It then follows from Proposition 3.9 that the local ring $\mathcal{O}_{X,x}$ is a hypersurface for every codimension two point $x \in X$. Since the l.c.i. locus of X is open by [13], X is a l.c.i. except at finitely many closed points. By the fact that l.c.i. canonical singularities are MJ-canonical (see for example [24, Remark 2.7(iv)]), X has only MJ-canonical singularities except at finitely many closed points. Thus, applying a Bertini theorem for MJ-canonical singularities [24, Corollary 4.11], we see that H has only rational double points.

Next we consider the case where $\Delta \neq 0$. Let U_1 be the locus of the points $x \in X$ such that X is regular at x and $\mathrm{mult}_x(\Delta) \leqslant 1$, and $U_2 = X \setminus \mathrm{Supp}\,\Delta$. Note that U_1 and U_2 are both open subsets of X (the openness of U_1 follows from the openness of the regular locus of X and the upper semicontinuity of Hilbert–Samuel multiplicity). Then every codimension two point $x \in X$ lies in $U_1 \cup U_2$ by [25, Theorem 2.29(2)], because $(\mathrm{Spec}\,\mathcal{O}_{X,x}, \Delta_x)$ is a two-dimensional canonical pair. Hence, we may assume that $X = U_1 \cup U_2$. We have already seen that the assertion holds when $\Delta = 0$, so $(H \cap U_2, \Delta|_{H \cap U_2}) = (H \cap U_2, 0)$ is canonical for a general hyperplane section H of X. On the other hand, it

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

follows from a Bertini theorem for Hilbert–Samuel multiplicity [9, Proposition 4.5] that $H \cap U_1 \subsetneq U_1$ is smooth and $\mathrm{mult}_x(\Delta|_{H\cap U_1}) = \mathrm{mult}_x(\Delta) \leqslant 1$ for all $x \in \mathrm{Supp}\,\Delta|_{H\cap U_1}$. Thus, applying [25, Theorem 2.29(2)] again, we see that $(H \cap U_1, \Delta|_{H\cap U_1})$ is canonical. □

Corollary 4.3. *Let X be a three-dimensional quasi-projective variety over an algebraically closed field k of characteristic $p \geqslant 5$ with only canonical singularities. Then there exists a zero-dimensional closed subscheme $Z \subset X$ with the following property: for every closed point $x \in X \setminus Z$, the maximal-adic completion $\widehat{\mathcal{O}_{X,x}}$ of the local ring of X at x is either regular or isomorphic to the formal tensor product $k[[u, v, w]]/(f)\widehat{\otimes}_k k[[t]]$ of a rational double point $k[[u, v, w]]/(f)$ and the one-dimensional formal power series ring $k[[t]]$.*

Proof. An immediate application of Theorem 4.2 to [21, Theorem 3] yields the desired result. □

Remark 4.4. When the characteristic p is less than 5, there exist counterexamples to Corollary 4.3 due to Hirokado and Ito and Saito [21] and Hirokado [20]. In particular, in characteristic three, an exhaustive list of examples is given in [21, Theorem 3].

5. Klt singularities

In this section, we will prove the klt case of the main theorem. Since we often use the results of [25] and [33] in this section, the reader is referred to Remark 3.2.

First we prove some lemmas needed for the proof of Theorem 1.2.

Definition 5.1. Let $I \subseteq [0, 1]$ be a subset. We define the subset $D(I) \subseteq [0, 1]$ by

$$D(I) := \left\{ \frac{m - 1 + \sum_{j=1}^{n} i_j}{m} \;\middle|\; m \in \mathbb{Z}_{\geqslant 1}, n \in \mathbb{Z}_{\geqslant 0}, i_j \in I \text{ for } j = 1, \ldots, n \right\} \cap [0, 1].$$

In particular, $D(\emptyset) = \{1 - 1/m \mid m \in \mathbb{Z}_{\geqslant 1}\}$ is the set of standard coefficients.

Lemma 5.2 (cf. [28, Lemma 4.3]). *Let X be a normal surface and Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is plt. Let C be a regular curve which is an irreducible component of $\lfloor \Delta \rfloor$ and $\mathrm{Diff}_C(B)$ be the different of $B := \Delta - C$ on C (see [25, Definition 2.34] for the definition of the $\mathbb{Q}$-divisor $\mathrm{Diff}_C(B)$). If the coefficients of B belong to a subset $I \subset [0, 1] \cap \mathbb{Q}$, then the coefficients of $\mathrm{Diff}_C(B)$ belong to $D(I)$.*

Proof. We may assume that $(x \in X)$ is a surface singularity, that is, $X = \mathrm{Spec}\, R$ for a two-dimensional normal local ring. Let $f : Y \longrightarrow X$ be a minimal log resolution of (X, C) with exceptional curves $E = \bigcup_i E_i$ defined as in [25, Definition 2.25 (b)] and m be the determinant of the negative of the intersection matrix of the E_i. Then by [25, (3.36.1)],

$$\mathrm{Diff}_C(B) = \frac{m - 1}{m}[x] + B|_C.$$

Let $B = \sum_j b_j B_j$ be the irreducible decomposition of B. Since mD is Cartier for every Weil divisor D on X by [25, Proposition 10.9(3)], we see that $\mathrm{ord}_{[x]}(mB_j)|_C$ is an integer.

Thus,

$$\mathrm{ord}_{[x]}\mathrm{Diff}_C(B) = \frac{m-1}{m} + \sum_j \frac{b_j}{m}\mathrm{ord}_{[x]}(mB_j)|_C \in D(I),$$

which completes the proof of the proposition. □

A *log Fano* pair (X, Δ) is a pair of a normal projective variety over a field k and an effective $\mathbb{Q}$-divisor on X such that $-(K_X + \Delta)$ is ample and (X, Δ) is klt.

Proposition 5.3 (cf. [5, Corollary 4.1], [36, Theorem 4.2]). *Let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. There exists a positive constant $p_0(I)$ depending only on I with the following property: if k is an F-finite field of characteristic $p > p_0(I)$ and $(\mathbb{P}^1_k, B = \sum_{i=1}^m a_i P_i)$ is a log Fano pair such that every coefficient a_i belongs to $D(I)$ and every point P_i is a k-rational point of $\mathbb{P}^1_k$, then $(\mathbb{P}^1_k, B)$ is globally F-regular. Moreover, when $I = \emptyset$, we may take the constant $p_0(I)$ to be 5.*

Proof. The proof is essentially the same as in the case where k is algebraically closed. When k is algebraically closed, the first assertion is nothing but [5, Theorem 4.1] and the second one follows from [36, Theorem 4.2]. □

Definition 5.4. A *curve* X over a field k is a one-dimensional integral scheme of finite type over k. The *arithmetic genus* $g(X)$ of a complete curve X over k is defined to be $\dim_k H^1(X, \mathcal{O}_X)/\dim_k H^0(X, \mathcal{O}_X)$. We remark that $g(X)$ is independent of the choice of the base field k.

We extend Proposition 5.3 to the case of an arbitrary curve of genus zero.

Proposition 5.5. *Let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. There exists a positive constant $p_1(I)$ depending only on I with the following property: if k is an F-finite field of characteristic $p > p_1(I)$ and (C, B) is a log Fano pair such that C is a complete curve over k and the coefficients of B belong to $D(I)$, then (C, B) is globally F-regular. Moreover, when $I = \emptyset$, we may take $p_1(I)$ to be 5.*

Proof. Let $p_0(I)$ be the constant as in Proposition 5.3. Set $a(I) = \min\{a \in D(I) \mid a \neq 0\}$ and $p_1(I) = \max\{p_0, \lceil 2/a(I) \rceil\}$. If $I = \emptyset$, then $a(I) = 1/2$, so we can take $p_1(\emptyset) = 5$ by Proposition 5.3. We say that a pair (C, B) satisfies the condition $(\star)$ if C is a complete curve over an F-finite field of characteristic $p > p_1(I)$ and (C, B) is a log Fano pair such that the coefficients of B belong to $D(I)$. Note that if (C, B) satisfies $(\star)$, then C is a complete regular curve with $g(C) = 0$ by [33, Corollary 2.6].

Let (C, B) be a log Fano pair over a field k satisfying the condition $(\star)$. First, we will show that we can reduce to the case where $C \cong \mathbb{P}^1_k$. Replacing k by its finite extension if necessary, we may assume that $k = H^0(C, \mathcal{O}_C)$. Suppose that $C \not\cong \mathbb{P}^1_k$. Then C is isomorphic to a conic in $\mathbb{P}^2_k$ by [25, Lemma 10.6(3)], and hence there exists a closed point $P \in C$ whose residue field $k(P)$ is a quadratic extension of k. Let $l := k(P)$ and consider the pullback $f : C_l := C \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, l \longrightarrow C$ of C. Note that C_l is connected, because $[l : k] = 2$ and l is not contained in the function field $K(C)$ of C, so $K(C) \otimes_k l$ is a domain. Since $p > p_1 > 2 = [l : k]$, the pullback f is a surjective étale morphism. It

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

follows from [25, Lemma 10.6(3)] again, together with the fact that a regular conic with an l-rational point is isomorphic to $\mathbb{P}^1_l$, that $C_l \cong \mathbb{P}^1_l$. By the fact that being klt is preserved under surjective étale morphisms [25, Proposition 2.15], the pair $(C_l, B_l := f^*B)$ satisfies the condition $(\star)$. On the other hand, a pair is globally F-regular if and only if its affine cone is strongly F-regular [30, Proposition 5.3], and strong F-regularity is preserved under finite étale morphisms of degree not divisible by p ([16, Theorem 3.3], [31, Corollary 6.31 and Proposition 7.4]). Therefore, (C, B) is globally F-regular if and only if so is (C_l, B_l). Thus, replacing k by l and C by C_l, we may assume that $C = \mathbb{P}^1_k$.

Next, we will show that we can reduce to the case where every point in Supp B is a k-rational point. Let $B = \sum_i a_i P_i$ be the irreducible decomposition of B. Since $(C = \mathbb{P}^1_k, B)$ is log Fano, we have $\sum_i a_i[k(P_i) : k] = \deg_{\mathbb{P}^1_k/k}(B) < 2$. In particular, $[k(P_i) : k] < 2/a(I) \leqslant p_1(I) < p$ for every i. Hence, there exists a Galois extension K/k such that $k(P_i) \subseteq K$ for every i and p does not divide $[K : k]$. Since the condition $(\star)$ and global F-regularity are preserved under finite Galois base field extensions of degree not divisible by p as we have seen above, replacing k by K and $(\mathbb{P}^1_k, B)$ by $(\mathbb{P}^1_K, B_K)$ if necessary, we may assume that every point in Supp B is a k-rational point. The assertion is now an immediate consequence of Proposition 5.3. □

Lemma 5.6 (cf. [5, Proposition 2.13]). *Let $(x \in X)$ be a normal surface singularity with a dualizing complex $\omega^\bullet_X$ and B be an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. If $(x \in X, B)$ is klt, then there exists a proper birational morphism $f : Y \longrightarrow X$ from a normal surface Y such that*

(1) *the exceptional locus C of f is a complete regular curve with $g(C) = 0$;*
(2) *$(Y, B_Y + C)$ is plt, where $B_Y = f^{-1}_* B$ is the strict transform of B by f; and*
(3) *$-(K_Y + B_Y + C)$ is f-ample, where K_Y is a canonical divisor on Y associated to the dualizing complex $f^!\omega^\bullet_X$.*

Proof. We can apply essentially the same argument as in the case where X is defined over an algebraically closed field [5, Proposition 2.13], using the minimal model program for excellent surfaces [33, Theorem 1.1] instead of that for surfaces over an algebraically closed field. Then (2) and (3) are easily verified. Hence, it is enough to check (1). Since $(Y, B_Y + C)$ is plt, the complete curve C has to be regular by [25, Theorem 2.31]. The pair $(Y, B_Y + C)$ being plt also implies by Proposition 3.5 that Y has only rational singularities, from which it follows that $g(C) = 0$ (see [25, Lemma 10.8(3)]). □

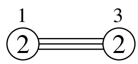
Theorem 1.2 is a consequence of Proposition 2.8 and the following theorem, which gives a generalization of [5, Theorem 1.1] (see also [15]) to the case where the base field is not necessarily algebraically closed.

Theorem 5.7 (cf. [15], [5, Theorem 1.1]). *Let $I \subseteq [0, 1] \cap \mathbb{Q}$ be a finite set. There exists a positive constant $p_1(I)$ depending only on I with the following property: if $(x \in X)$ is an F-finite normal surface singularity of characteristic $p > p_1(I)$ and $(x \in X, B)$ is a klt pair such that the coefficients of B belong to $D(I)$, then $(x \in X, B)$ is strongly F-regular. Moreover, when $I = \emptyset$, we can take $p_1(I)$ to be 5.*

https://doi.org/10.1017/S1474748018000166 Published online by Cambridge University Press

Proof. We employ the same strategy as the proof of [5, Theorem 1.1]. Let $p_1(I)$ be the constant as in Proposition 5.5. Suppose that $(x \in X, B)$ is a klt pair such that $(x \in X)$ is an F-finite normal surface singularity of characteristic $p > p_1(I)$ and the coefficients of B belong to $D(I)$. Take a proper birational morphism $f : Y \longrightarrow X$ with exceptional prime divisor C as in Lemma 5.6, and let $B_Y = f_*^{-1}B$ be the strict transform of B by f and $B_C = \mathrm{Diff}_C(B_Y)$ be the different of B_Y on C. Note that by Lemma 5.2 and [28, Lemma 4.4], the coefficients of B_C belong to $D(D(I)) = D(I) \cup \{1\}$. Since (C, B_C) is log Fano by adjunction [25, Theorem 3.36 and Lemma 4.8], the coefficients of B_C are less than 1, and then (C, B_C) is globally F-regular by Proposition 5.5. Applying the same argument as in the case where X is defined over an algebraically closed field ([5, Proposition 2.11] and [14, Lemma 2.12]), we can conclude that (X, B) is strongly F-regular. □

Example 5.8 (cf. [25, 2.26]). Let k be a non-algebraically closed field of characteristic $p > 0$ with an element $a \in k$ that is not a cubic power in k. Let $(\mathbf{0} \in X)$ be the origin of the hypersurface $(x^2 = y^3 - az^3 + y^4 + z^4) \subset \mathbb{A}^3_k$, which is a klt surface singularity. Its minimal resolution has two exceptional curves C_1, C_2 and their dual graph is



,

where the numbers above the circles are the $\dim_k H^0(C_i, \mathcal{O}_{C_i})$ and the numbers inside the circles are the $-C_i^2/\dim_k H^0(C_i, \mathcal{O}_{C_i})$. Note that this graph does not appear in the list of the dual graphs of exceptional curves for the minimal resolutions of klt surface singularities over an algebraically closed field (see [35]). On the other hand, $(\mathbf{0} \in X)$ is strongly F-regular if and only if $p > 3$.

We are now ready to prove the klt case of the main theorem.

Theorem 5.9. *Let X be a three-dimensional normal quasi-projective variety over an algebraically closed field of characteristic $p > 5$ and Δ be an effective $\mathbb{Q}$-divisor on X whose coefficients belong to the standard set $\{1 - 1/m \mid m \in \mathbb{Z}_{\geqslant 1}\}$. Suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and (X, Δ) is klt. Then $(H, \Delta|_H)$ is klt for a general hyperplane section H of X.*

Proof. Let U be the locus of the points $x \in X$ such that (X, Δ) is strongly F-regular. Note that U is an open subset of X (the $\Delta = 0$ case was proved in [23, Theorem 3.3] and the general case follows from a very similar argument). Every codimension two point $x \in X$ lies in U, because $(\mathrm{Spec}\, \mathcal{O}_{X,x}, \Delta_x)$ is strongly F-regular by Theorem 5.7. Hence, $X \setminus U$ consists of only finitely many closed points and a general hyperplane section H of X is contained in U. It then follows from a Bertini theorem for strongly F-regular pairs [32, Corollary 6.6] that $(H, \Delta|_H)$ is strongly F-regular. Since strongly F-regular pairs are klt by Proposition 2.8, we obtain the assertion. □

Acknowledgements. The authors are indebted to Shihoko Ishii for valuable discussions, especially on the proof of Theorem 4.2. They are also grateful to Mircea Mustaţă and Hiromu Tanaka for helpful conversations. The first author was partially supported by the Program for Leading Graduate Schools, MEXT, Japan and JSPS KAKENHI Grant Number JP17J04317. The second author was partially supported by JSPS KAKENHI Grant Numbers JP26400039, JP15H03611, JP15KK0152, JP16H02141 and JP17H02831. Last, they thank the referee for useful comments.

References

1. Y. Aoyama, Some basic results on canonical modules, *J. Math. Kyoto. Univ.* **23** (1983), 85–94.
2. M. Artin, On isolated rational singularities of surfaces, *Amer. J. Math.* **88** (1966), 129–136.
3. C. Birkar, Existence of flips and minimal models for 3-folds in char p, *Ann. Sci. Éc. Norm. Supér. (4)* **49** (2016), 169–212.
4. C. Birkar and J. Waldron, Existence of Mori fibre spaces for 3-folds in char p, *Adv. Math.* **313** (2017), 62–101.
5. P. Cascini, Y. Gongyo and K. Schwede, Uniform bounds for strongly F-regular surfaces, *Trans. Amer. Math. Soc.* **368** (2016), 5547–5563.
6. V. Cossart and O. Piltant, Resolution of singularities of threefolds in positive characteristic. I, *J. Algebra* **320** (2008), 1051–1082.
7. V. Cossart and O. Piltant, Resolution of singularities of threefolds in positive characteristic. II, *J. Algebra* **321** (2009), 1836–1976.
8. T. de Fernex and R. Docampo, Jacobian discrepancies and rational singularities, *J. Eur. Math. Soc.* **16** (2014), 165–199.
9. T. de Fernex, L. Ein and M. Mustaţă, Bounds for log canonical thresholds with applications to birational geometry, *Math. Res. Lett.* **10** (2003), 219–236.
10. L. Ein and S. Ishii, Singularities with respect to Mather–Jacobian discrepancies, in *Commutative Algebra and Noncommutative Algebraic Geometry, Vol. II*, Mathematical Sciences Research Institute Publications, Volume 68, pp. 125–168 (Cambridge Univ. Press, New York, 2015).
11. W. Fulton, *Intersection Theory,* second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 2 (Springer, Berlin, 1998).
12. O. Gabber, Notes on some t-structures, in *Geometric Aspects of Dwork Theory, Vol. I, II*, pp. 711–734 (Walter de Gruyter GmbH & Co. KG, Berlin, 2004).
13. S. Greco and M. G. Marinari, Nagata's criterion and openness of loci for Gorenstein and complete intersection, *Math. Z.* **160** (1978), 207–216.
14. C. Hacon and C. Xu, On the three dimensional minimal model program in positive characteristic, *J. Amer. Math. Soc.* **28** (2015), 711–744.
15. N. Hara, Classification of two-dimensional F-regular and F-pure singularities, *Adv. Math.* **133** (1998), 33–53.
16. N. Hara and S. Takagi, On a generalization of test ideals, *Nagoya Math. J.* **175** (2004), 59–74.
17. N. Hara and K.-i. Watanabe, F-regular and F-pure rings vs. log terminal and log canonical singularities, *J. Algebra Geom.* **11** (2002), 363–392.
18. R. Hartshorne, *Residues and Duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, with an appendix by P. Deligne, Lecture Notes in Mathematics, Volume 20 (Springer-Verlag, Berlin–New York, 1966).
19. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Volume 52, (Springer-Verlag, Berlin–New York, 1977).

20. M. Hirokado, Canonical singularities of dimension three in characteristic 2 which do not follow Reid's rules, *Kyoto J. Math.* (to appear) arXiv:1607.08664.
21. M. Hirokado, H. Ito and N. Saito, Three dimensional canonical singularities in codimension two in positive characteristic, *J. Algebra* **373** (2013), 207–222.
22. K. Hashizume, Y. Nakamura and H. Tanaka, Minimal model program for log canonical threefolds in positive characteristic, preprint, arXiv:1711.10706.
23. M. Hochster and C. Huneke, Tight closure and strong F-regularity, *Mém. Soc. Math. France* **38** (1989), 119–133.
24. S. Ishii and J. Reguera, Singularities in arbitrary characteristic via jet schemes, in *Hodge Theory and L^2 Analysis*, Advanced Lectures in Mathematics (ALM), Volume 39, pp. 419–449 (International Press, Somerville, MA, 2017). Higher Education Press, Beijing.
25. J. Kollár, *Singularities of the Minimal Model Program*, Cambridge Tracts in Mathematics, Volume 200 (Cambridge University Press, Cambridge, 2013).
26. E. Kunz, On Noetherian rings of characteristic p, *Amer. J. Math.* **98** (1976), 999–1013.
27. J. Lipman, Rational singularities, with applications to algebraic surfaces and unique factorization, Publications Mathématiques. Institut de Hautes Études Scientifiques, Volume 36, pp. 195–279. (1969).
28. J. McKernan and Y. Prokhorov, Threefold thresholds, *Manuscripta Math.* **114** (2004), 281–304.
29. M. Reid, Canonical 3-folds, in *Journées de géométrie algébrique d'Angers 1979* (ed. A. Beauville), pp. 273–310 (Sijthoff & Noordhoff, Alphen, 1980).
30. K. Schwede and K. E. Smith, Globally F-regular and log Fano varieties, *Adv. Math.* **224** (2010), 863–894.
31. K. Schwede and K. Tucker, On the behavior of test ideals under finite morphisms, *J. Algebraic. Geom.* **23** (2014), 399–443.
32. K. Schwede and W. Zhang, Bertini theorems for F-singularities, *Proc. Lond. Math. Soc.* **107** (2013), 851–874.
33. H. Tanaka, Minimal model program for excellent surfaces, *Ann. Inst. Fourier.* (to appear) arXiv:1608.07676.
34. S. Takagi and K.-i. Watanabe, F-singularities: applications of characteristic p methods to singularity theory, *Sugaku Expositions* **31** (2018), 1–42.
35. K. Watanabe, Plurigenera of normal isolated singularities. I, *Math. Ann.* **250** (1980), 65–94.
36. K.-i. Watanabe, F-regular and F-pure normal graded rings, *J. Pure Appl. Algebra* **71** (1991), 341–350.