ON THE LOWER DERIVATE OF A SET FUNCTION

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Introduction. In (**5**), the following theorem was proved in a very general setting:

(1) *An additive set function is non-negative whenever its lower derivative is non-negative.*

For a continuous additive function of intervals, theorem (1) can be improved as follows:

(2) *A continuous additive set function is non-negative whenever its lower derivative is non-negative except, perhaps, on a countable set.*

Our aim in this paper is to show that theorem (2), the importance of which for Perron integration is well known (see, e.g., **3**, Chapter V), also holds under rather general assumptions. A method very similar to that used in (**5**) will be applied here. In § 1, semihereditary and stable systems of sets are introduced. Their definitions, (1.1 and 1.2), are motivated later by Propositions 4.2 and 4.4. The main results are proved in §§ 2 and 3. They are formulated in topological terms using cluster points of families of sets. In § 4, an easy application of these results gives an analogue of theorem (2).

In order to include such important special cases as additive functions of intervals and additive functions of convex linear cells, the basic system of sets is not assumed to be closed with respect to the formation of set differences. For this reason also the definition of a semihereditary system of sets used here is slightly different from that in (**5**, § 3.2).

It is, of course, well known that the classical form of theorem (2) admits a series of important generalizations (see, e.g., **2**, § 13). However, most of them strongly utilize characteristic properties of real functions, so that there is hardly any hope that they could be transferred into a more general situation.

1. Preliminaries. Throughout, P is a topological space and $P^- = P \cup (\infty)$ is a one-point compactification of P (see **4**, p. 150). For $x \in P^-$, Γ_x denotes a local base for x in P^- (see **4**, p. 50). Let[1] $\sigma \subset \exp P$ be a non-empty system which satisfies the following conditions:

(i) For every $A, B \in \sigma$, $A \cap B \in \sigma$ and $A - B = \bigcup_{i=1}^{n} C_i$, where $C_1, \ldots, C_n$ are disjoint sets from[2] σ;

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[1]exp P is the collection of all subsets of P.

[2]In (**1**), a non-empty system σ satisfying condition (i) is called a *pre-ring*.

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(ii) For every $x \in P$, $\Gamma_x \subset \sigma$.

For $A \subset P^-$, A^- and A° denote the closure and the interior of A in P^-, respectively. If $\delta \subset \exp P$ and $A \subset P^-$, we let $\delta_A = \{B \in \delta : B \subset A\}$.

Definition 1.1. A system $\delta \subset \sigma$ is said to be *semihereditary* if and only if $\sigma_0 \cap \delta \neq \emptyset$ for every finite disjoint collection $\sigma_0 \subset \sigma$ whose union belongs to δ.

Notice that the condition required by this definition is weaker than that required by the corresponding definition in (**5**, § 3.2). These definitions coincide whenever σ is closed with respect to the formation of set differences.

Definition 1.2. A system $\delta \subset \exp P$ is said to be *stable* if and only if $\emptyset \notin \delta$ and for every $A \in \delta$ and every $x \in P^-$ there is $U \in \Gamma_x$ such that $\delta_{A-U} \neq \emptyset$.

We note that no stable system contains a finite set. If δ is a semihereditary or stable system, then so is δ_A for every set $A \subset P^-$.

Example 1.3. Let P be Hausdorff and locally compact, $\sigma = \exp P$, and let $Q \subset P$ be a non-empty set without isolated points. Then the system δ of all sets $A \in \sigma$ for which $A \cap Q$ contains a non-empty subset without isolated points is non-empty, semihereditary, and stable.

Example 1.4. For an ordinal α, $P(\alpha)$ is the set of all ordinals less than α. The set $P(\alpha)$ with the order topology is a locally compact Hausdorff space. We shall prove that there is no non-empty stable system in $P(\alpha)$. This is true for $\alpha = 1$. Assume that it is true for all ordinals β less than α and suppose that there is a non-empty stable system $\delta \subset \exp P(\alpha)$. Since all one-point compactifications of $P(\alpha)$ are homeomorphic to $P(\alpha + 1)$, there is a neighbourhood U of $\alpha + 1$ such that $\delta_{P(\alpha)-U}$ is also a non-empty stable system. Since $P(\alpha) - U \subset P(\beta)$ for some ordinal β less than α, we have obtained a contradiction with the induction hypothesis.

Definition 1.5. Let $\delta \subset \exp P$ and let $x \in P^-$. The point x is said to be a *weak cluster point* of δ if and only if $\delta_U \neq \emptyset$ for every $U \in \Gamma_x$. The point x is said to be a *cluster point*, or a *strong cluster point* of δ if and only if for every $U \in \Gamma_x$ there is an $A \in \delta_U$ such that $x \in A^-$, or $x \in A$, respectively.

The set of all weak cluster points, or cluster points, or strong cluster points of δ is denoted by δ^w, or δ^c, or δ^s, respectively. When no confusion can arise, δ^* stands for δ^w or δ^c or δ^s.

We note that if $\emptyset \in \delta$, then $\delta^w = P^-$. It is easy to see that for δ from Example 1.3, $\delta^w = \delta^c = Q^-$ and $\delta^s = Q^- \cap P$. In particular, $\delta^w = \delta^c = \delta^s$ whenever the space P in that example is compact. Generally, $\delta^w \subset \delta^c \subset \delta^s$ and, as Example 2.2 will show, the inclusions can be proper even for a compact space P.

The following simple properties of the system σ will be useful.

(1.1) *To every collection of sets* $\{A_i\}_{i=1}^n \subset \sigma$ *there is a disjoint collection of sets* $\{B_j\}_{j=1}^m \subset \sigma$ *such that every* B_j *is contained in some* A_i *and* $\bigcup_{j=1}^m B_j = \bigcup_{i=1}^n A_i$.

Proof. Let $A_0 = \emptyset$. For all integers i, j, $0 \leqq j < i \leqq n$, there are disjoint sets $C(i,j,k) \in \sigma$, $k = 1, 2, \ldots, n(i,j)$, such that $A_i - A_j = \bigcup_{k=1}^{n(i,j)} C(i,j,k)$. Given integers i, k_j, $1 \leqq i \leqq n$, $1 \leqq k_j \leqq n(i,j)$, $j = 0, 1, \ldots, i-1$, we let

$$B(i, k_0, \ldots, k_{i-1}) = \bigcap_{j=0}^{i-1} C(i,j,k_j).$$

These sets are clearly disjoint, belong to σ, and $B(i, k_0, \ldots, k_{i-1}) \subset A_i$ for each $i = 1, 2, \ldots, n$ and each combination of $k_0, \ldots, k_{i-1}$. Furthermore,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n}\left(A_i - \bigcup_{j=0}^{i-1} A_j\right) = \bigcup_{i=1}^{n} \bigcap_{j=0}^{i-1} (A_i - A_j) =$$
$$\bigcup_{i=1}^{n} \bigcap_{j=0}^{i-1} \bigcup_{k=1}^{n(i,j)} C(i,j,k) = \bigcup\{B(i, k_0, \ldots, k_{i-1}) : 1 \leqq i \leqq n,$$
$$1 \leqq k_0 \leqq n(i,0), \ldots, 1 \leqq k_{i-1} \leqq n(i, i-1)\}.$$

(1.2) *Let $A \in \sigma$ and let $A_1, \ldots, A_n$ be sets from σ_A. Then there are disjoint sets $B_1, \ldots, B_m$ from σ_A such that $\bigcup_{j=1}^{m} B_j = A - \bigcup_{i=1}^{n} A_i$.*

Proof. For $i = 1, 2, \ldots, n$, there are disjoint sets $C(i, j) \in \sigma$, $j = 1, 2, \ldots, k_i$, such that $A - A_i = \bigcup_{j=1}^{k_i} C(i, j)$. Given integers j_i, $1 \leqq j_i \leqq k_i$, $i = 1, 2, \ldots, n$, we let $B(j_1, \ldots, j_n) = \bigcap_{i=1}^{n} C(i, j_i)$. These sets are disjoint, belong to σ_A, and

$$A - \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (A - A_i) = \bigcap_{i=1}^{n} \bigcup_{j=1}^{k_i} C(i,j) =$$
$$\bigcup\{B(j_1, \ldots, j_n) : 1 \leqq j_i \leqq k_i, i = 1, 2, \ldots, n\}.$$

(1.3) *Let $A \in \sigma$ and let $\{U\}$ be an open cover of A^-. Then there is a finite disjoint cover $\{B\} \subset \sigma_A$ of A which refines $\{U\}$.*

Proof. Obviously, the cover $\{U\}$ has an open refinement $\{V_x^\circ\}_{x \in A^-}$ such that $V_x \in \Gamma_x$ for every $x \in A^-$. Since A^- is compact, the cover $\{V_x^\circ\}_{x \in A^-}$ has a finite subcover $\{V_{x_i}^\circ\}_{i=1}^{n}$. Without loss of generality, we may assume that $\{V_{x_i}\}_{i=1}^{n-1} \subset \sigma$. It suffices now to let $A_i = A \cap V_{x_i}$ for $i = 1, 2, \ldots, n-1$ and apply (1.1) and (1.2).

2. Weak cluster points. We start with an elementary observation.

PROPOSITION 2.1. *For any $\delta \subset \exp P$, δ^w is closed in P^-.*

Proof. Let $x \in (\delta^w)^-$ and $U \in \Gamma_x$. Since $U^\circ \cap \delta^w \neq \emptyset$, there is $y \in \delta^w$ and $V \in \Gamma_y$ such that $V \subset U$. It follows that $\delta_U \neq \emptyset$, and thus $x \in \delta^w$.

The following example shows that Proposition 2.1 does not hold if we replace δ^w by δ^c or δ^s.

Example 2.2. Let $P = [0, 1] \times [0, 1]$, $Q = [0, 1) \times [0, 1)$, $R = (0, 1) \times (0, 1)$ and let σ be the family of all Lebesgue measurable subsets of P.

Denote by μ the Lebesgue measure in P and let

$$\delta = \{A \in \sigma_R: \mu(A) > 0 \text{ and } A^- \subset Q\}.$$

Then δ is a non-empty semihereditary, stable system for which $\delta^w = P$, $\delta^c = Q$, and $\delta^s = R$.

PROPOSITION 2.3. *Let $\delta \subset \sigma$ be a non-empty semihereditary system. Then δ^w is non-empty.*

Proof. Assume that the proposition is not true. Then for every $x \in P^-$ we can find an open neighbourhood U_x which does not contain any set $B \in \delta$. Let $A \in \delta$. Then, according to (1.3), there are disjoint sets $B_1, \ldots, B_n$ from $\sigma - \delta$ such that $A = \bigcup_{i=1}^n B_i$. This contradicts the semihereditariness of δ.

THEOREM 2.4. *Let $\delta \subset$ exp P be a non-empty stable system and let $(\delta_A)^* \neq \emptyset$ for every $A \in \delta$. Then δ^* is uncountable whenever it is compact.*

Proof. Suppose that $\delta^* = \{x_1, x_2, \ldots\}$ is countable and choose $A \in \delta$. Since δ is stable, there is $U_1 \in \Gamma_{x_1}$ for which $\delta_{A-U_1} \neq \emptyset$. Thus, we can choose $A_1 \in \delta$ such that $A_1 \subset A - U_1$. Assume that we have already chosen $U_n \in \Gamma_{x_n}$ and $A_n \in \delta$ such that $A_n \subset A - \bigcup_{i=1}^n U_i$. Then, according to the stability of δ, we can find $U_{n+1} \in \Gamma_{x_{n+1}}$ for which $\delta_{A_n - U_{n+1}} \neq \emptyset$. Hence, there is $A_{n+1} \in \delta$ such that $A_{n+1} \subset A_n - U_{n+1} \subset A - \bigcup_{i=1}^{n+1} U_i$. The family $\{U_1^\circ, U_2^\circ, \ldots\}$ is an open cover of δ^*. If δ^* is compact, then $\{U_1^\circ, \ldots, U_N^\circ\}$ covers δ^* for any sufficiently large N. Take such N and let $\gamma = \delta_{A_N}$. Clearly, $\gamma^* \subset \delta^*$. On the other hand, since $\emptyset \notin \gamma$, we have that $\gamma^* \cap \delta^* = \emptyset$. Therefore, $\gamma^* = \emptyset$, which contradicts the assumption.

COROLLARY 2.5. *Let $\delta \subset \sigma$ be a non-empty semihereditary, stable system. Then δ^w is uncountable.*

The corollary follows from Propositions 2.1 and 2.3.

COROLLARY 2.6. *Let P be Hausdorff and locally compact and let σ be closed with respect to the formation of set differences. If $\delta \subset \sigma$ is a non-empty semihereditary, stable system, then δ^c is infinite.*

The corollary follows from (**5**, §§ 4.1 and 4.3).

PROPOSITION 2.7. *Assume that P is regular. Let $\delta \subset$ exp P be a stable system and let $(\delta_A)^* \neq \emptyset$ for every $A \in \delta$. Then $\delta^* \cap P$ has no isolated points.*

Proof. Given $x \in \delta^* \cap P$ and $U \in \Gamma_x$, we can find $A \in \delta$ for which $A^- \cap P \subset U$. Since δ is stable, there are $V \in \Gamma_x$ and $W \in \Gamma_\infty$ such that $\delta_{A-V\cup W} \neq \emptyset$. Select $B \in \delta_{A-V\cup W}$ and $y \in (\delta_B)^*$. Since $\emptyset \notin \delta_B$, $y \in \delta^* \cap U$ and $y \neq x$.

Notice that if P is regular and locally compact, then under the assumptions of Proposition 2.7, not even ∞ can be an isolated point of δ^*; for in this case, P^- is regular.

COROLLARY 2.8. *Let P be Hausdorff and locally compact, and let $\delta \subset \sigma$ be a non-empty semihereditary, stable system. Then the cardinality of δ^w is not less than the continuum.*

By Propositions 2.1 and 2.3 and the previous remark, δ^* is a non-empty perfect subset of a compact Hausdorff space P^-; the corollary follows.

COROLLARY 2.9. *Let P be Hausdorff and locally compact and let σ be closed with respect to the formation of set differences. If $\delta \subset \sigma$ is a semihereditary, stable system, then δ^c has no isolated points.*

The corollary follows from (**5**, §§ 4.1 and 4.3).

3. Cluster points. In a general topological space P we are unable to decide whether δ^c is uncountable for every non-empty semihereditary, stable system $\delta \subset \sigma$. If, e.g., the space P is not Hausdorff or not locally compact, or if the system σ is not closed with respect to the formation of set differences, (**5**, Theorem 4.3) cannot be applied, and hence in this case we do not even know whether δ^c is non-empty. In this section it will be shown that theorems similar to Proposition 2.3 and Theorem 2.4 also hold for δ^c, provided the space P is locally pseudo-metrizable.

PROPOSITION 3.1. *Let P be regular and locally pseudo-metrizable and let $\delta \subset \sigma$ be a semihereditary system containing a non-empty set. Then δ^c is non-empty.*

Proof. Assume first that δ contains a non-empty set A for which $A^- \subset P$. Being compact and locally pseudo-metrizable, A^- is pseudo-metrizable (see **7**, Theorem 2). Using (1.3) we can find sets $C_i \in \sigma$ such that[3] $d(C_i) < 1$, $i = 1, 2, \ldots, n$, and $A = \bigcup_{i=1}^{n} C_i$. From the semihereditariness of δ, it follows that at least one of the sets $C_1, \ldots, C_n$ belongs to δ. We denote it by A_1. It is a matter of an easy induction to construct a decreasing sequence $\{A_k\}_{k=1}^{\infty} \subset \delta_A$ of non-empty sets for which $d(A_k) < 1/k$, $k = 1, 2, \ldots$. Obviously, $\emptyset \neq \bigcap_{k=1}^{\infty} A_k^- \subset \delta^c$.

Now, suppose that $\infty \in B^-$ for every non-empty set $B \in \delta$ and that $\infty \notin \delta^c$; for, if $\infty \in \delta^c$, there is nothing to prove. Then there exists an open neighbourhood U_∞ of ∞ which does not contain any non-empty set $B \in \delta$. For every $x \in P$, let U_x be a speudo-metrizable neighbourhood of x which is closed in P. Choose a non-empty set $A \in \delta$. By (1.3), there is a finite disjoint cover $\{B\} \subset \delta_A$ of A which refines $\{U_x^\circ\}_{x \in A^-}$. From the semihereditariness of δ, it follows that at least one non-empty set from $\{B\}$, say B, belongs to δ. Clearly, $B^- \cap P \subset U_x$ for some $x \in P$. Using (1.3) again, we can find disjoint non-empty sets $C_1, \ldots, C_n$ from σ such that $d(C_i) < 1$ or $C_i \subset U_\infty$ for $i = 1, 2,$

[3]If Q is a pseudo-metrizable subspace of P and $C \subset Q$, then $d(C)$ denotes a diameter of C with respect to a pseudo-metric coherent with the topology in Q.

$\ldots, n$ and $B = \bigcup_{i=1}^{n} C_i$. At least one of the sets $C_1, \ldots, C_n$ belongs to δ; we denote it by B_1. Since $B_1 \not\subset U_\infty$, we have that $d(B_1) < 1$. We construct inductively a decreasing sequence $\{B_k\}_{k=1}^{\infty} \subset \delta_B$ of non-empty sets for which $d(B_k) < 1/k$, $k = 1, 2, \ldots$. Since $B_k \not\subset U_\infty$, we obtain

$$\emptyset \neq \bigcap_{k=1}^{\infty} (B_k^- - U_\infty) \subset \bigcap_{k=1}^{\infty} (B_k^- \cap P) \subset \delta^c,$$

and the proof is completed.

THEOREM 3.2. *Let P be locally pseudo-metrizable and let $\delta \subset \sigma$ be a non-empty semihereditary, stable system. Then the cardinality of δ^c is not less than the continuum.*

Proof. Since δ is non-empty and stable, we can find a set $A \in \delta$ for which $A^- \subset P$. Being compact, A^- is pseudo-metrizable (see **7**, Theorem 2). By Proposition 3.1, there is $x_0 \in (\delta_A)^c$; for $A \neq \emptyset$. Again, from Proposition 3.1 and the stability of δ follows the existence of $U \in \Gamma_{x_0}$ and $x_1 \in (\delta_{A-U})^c$. According to the definition of a cluster point and the regularity of A^-, there are sets $A_i \in \delta_A$ for which[3] $d(A_i) < 1$, $i = 0, 1$, and $A_0^- \cap A_1^- = \emptyset$. By a simple induction, we can define sets $A_{i_1 \ldots i_n} \in \delta_A$, $n = 1, 2, \ldots$, for which

$$A_{i_1 \ldots i_{n-1} i_n} \subset A_{i_1 \ldots i_{n-1}}, \qquad A^-_{i_1 \ldots i_{n-1} 0} \cap A^-_{i_1 \ldots i_{n-1} 1} = \emptyset,$$

and $d(A_{i_1 \ldots i_n}) < 1/n$, $i_j = 0, 1$, $j = 1, 2, \ldots, n$. Using the axiom of choice we associate with every sequence $\{i_j\}_{j=1}^{\infty}$ of zeros and ones a point[4] $x_{i_1 i_2 \ldots} \in \bigcap_{n=1}^{\infty} A^-_{i_1 \ldots i_n}$; this is possible, since all sets from δ are non-empty. Clearly, $x_{i_1 i_2 \ldots} \in \delta^c$ and since different points are associated with different sequences, the theorem follows.

We shall close this section with a few comments concerning the properties of strong cluster points. The question of the existence of a strong cluster point was answered in a fairly satisfactory way in (**5**, § 4.3, the second part of the theorem and § 2, example). Concerning the cardinality of the set δ^s we know very little. Fortunately, an estimation of the cardinality of δ^s is not essential for the applications treated in the next section (see Theorem 4.9). The following proposition summarizes our knowledge about strong cluster points.

PROPOSITION 3.3. *Let P be Hausdorff and let σ be closed with respect to the formation of set differences. If $\delta \subset \sigma$ is a semihereditary system which contains a non-empty compact set, then δ^s is non-empty. If, in addition, δ is stable and Γ_x is an open local base for every $x \in P$, then δ^s has no isolated points and is, therefore, infinite.*

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[4]Notice that if the space P is locally metrizable, we need not employ the axiom of choice.

The first part of the proposition follows from (**5**, §§ 4.1 and 4.3). The proof of the second part is similar to the proof of Proposition 2.7.

4. Set functions.

Here we shall apply the results from previous sections to superadditive functions of sets.

Definition 4.1. An extended real-valued function F defined on σ is said to be *superadditive* if and only if

$$F\left(\bigcup_{i=1}^{n} A_i\right) \geqq \sum_{i=1}^{n} F(A_i)$$

for every disjoint collection of sets $A_1, \ldots, A_n$ from σ for which $\bigcup_{i=1}^{n} A_i$ belongs to σ and $\sum_{i=1}^{n} F(A_i)$ has meaning.

The family of all superadditive functions on σ is denoted by $\mathfrak{S}$. For $F \in \mathfrak{S}$, $\sigma(F) = \{A \in \sigma: F(A) < 0\}$. The following proposition corresponds to that in (**5**, § 3.3).

PROPOSITION 4.2. *If $F \in \mathfrak{S}$, then $\sigma(F)$ is a semihereditary system. On the other hand, if $\delta \subset \sigma$ is a semihereditary system, there is an $F \in \mathfrak{S}$ such that $\sigma(F) = \delta$.*

Proof. Let $\sigma_0 = \{A_i\}_{i=1}^{n}$ be a disjoint collection of sets from σ for which $\bigcup_{i=1}^{n} A_i \in \sigma(F)$. Since

$$0 > F\left(\bigcup_{i=1}^{n} A_i\right) \geqq \sum_{i=1}^{n} F(A_i)$$

whenever the sum on the right side has meaning, it follows that $\sigma_0 \cap \sigma(F) \neq \emptyset$. In order to prove the second part of the proposition, it suffices to let $F(A) = -1$ for $A \in \delta$ and $F(A) = 0$ otherwise.

For $x \in P^-$ and $U \in \Gamma_x$ we let

$$\sigma_w(x, U) = \sigma_U, \quad \sigma_c(x, U) = \{A \in \sigma_U: x \in A^-\}, \quad \sigma_s(x, U) = \{A \in \sigma_U: x \in A\}.$$

When no confusion is possible, $\sigma_*(x, U)$ stands for $\sigma_w(x, U)$ or $\sigma_c(x, U)$ or $\sigma_s(x, U)$.

A set $A \subset P$ is called *small* whenever $A \subset U$ for some $U \in \bigcup\{\Gamma_x: x \in P\}$ or $A \subset P - V$ for some $V \in \Gamma_\infty$. Let $G \in \mathfrak{S}$ be a non-negative function such that $G(A) < +\infty$ for every small set $A \in \sigma$. Intuitively, G is some kind of a locally finite "measure" on σ.

Definition 4.3. Let F be an extended real-valued function on σ. For every $A \in \sigma$ put: $(F/G)(A) = F(A)/G(A)$ if the ratio $F(A)/G(A)$ has meaning, and $(F/G)(A) = +\infty$ otherwise.[5] Given $x \in P^-$, we call the number

$$L_*(F, x) = \sup_{U \in \Gamma_x} \left[\inf_{A \in \sigma_*(x,U)} F(A) \right]$$

[5]We let $a/0 = +\infty$ for $a > 0$ and $a/0 = -\infty$ for $a < 0$; the symbols $a/(\pm\infty)$ and $0/0$ are undefined.

the *lower limit* of F at x, and the number

$$D_*(F, x) = L_*(F/G, x)$$

the *lower derivate* of F at x.

Notice that we actually have three definitions of the lower limit and the lower derivate according to whether $*$ stands for w or c or s. For every $x \in P^-$ we have the obvious relations:

(4.1) $L_w(F, x) \leqq L_c(F, x) \leqq L_s(F, x)$,

(4.2) $D_w(F, x) \leqq D_c(F, x) \leqq D_s(F, x)$.

If $x \in P$, then also the following hold:

(4.3) $D_*(F, x) > -\infty$ implies that $L_*(F, x) > -\infty$,

(4.4) $D_*(F, x) \geqq 0$ implies that $L_*(F, x) \geqq 0$,

(4.5) $D_*(F, x) > -\infty$ and $L_*(-G, x) \geqq 0$ imply that $L_*(F, x) \geqq 0$.

Furthermore, $L_s(F, \infty) = D_s(F, \infty) = +\infty$; for, $\sigma_s(\infty, U) = \emptyset$ for every $U \in \Gamma_\infty$.

From now on we shall assume that, in addition to conditions (i) and (ii) of § 1, the system σ also satisfies the following condition:

(iii) There is a fixed integer $p \geqq 1$ such that for every $U \in \Gamma_\infty$, $U \cap P$ can be written as a disjoint union of p sets from σ.

Notice that if P is an n-dimensional Euclidean space, then the system σ consisting of the empty set and all non-degenerate one-side-open intervals (bounded or unbounded) satisfies conditions (i)–(iii).

PROPOSITION 4.4. *Let $F \in \mathfrak{S}$, $F(\emptyset) \geqq 0$, and let $L_c(F, x) \geqq 0$ for all $x \in P^-$. Then $\sigma(F)$ is a semihereditary, stable system.*

Proof. According to Proposition 4.2, it suffices to show that $\sigma(F)$ is stable. Obviously, $\emptyset \notin \sigma(F)$. Let $A \in \sigma(F)$ and $x \in P^-$. Since $L_c(F, x) \geqq 0$, there is $U \in \Gamma_x$ such that $F(B) > F(A)/p$ for every $B \in \sigma_c(x, U)$. According to (iii), there are disjoint sets $C_1, \ldots, C_p$ from σ whose union is $U \cap P$. Letting $A_i = A \cap C_i$, $i = 1, 2, \ldots, p$, we can find disjoint sets $B_1, \ldots, B_n$ from σ such that $\bigcup_{i=1}^n B_i = A - \bigcup_{i=1}^p A_i = A - U$ (see (1.2)). Since

$$F(A) \geqq \sum_{i=1}^{p} F(A_i) + \sum_{i=1}^{n} F(B_i)$$

whenever the right side has meaning and since $F(A_i) > F(A)/p$ whenever $x \in A_i^-$, it follows that among the sets $A_1, \ldots, A_p, B_1, \ldots, B_n$ there is at least one, say C, for which $x \notin C^-$ and $F(C) < 0$. Hence, $[\sigma(F)]_{A-V}$ is not empty for all sufficiently small $V \in \Gamma_x$.

Notice that if $x \in P^-$ and $L_w(F, x) \geqq 0$, then also $L_c(F, x) \geqq 0$ and $F(\emptyset) \geqq 0$. On the other hand, Example 4.5 shows that even for a compact Hausdorff space P the condition $L_c(F, x) \geqq 0$ cannot be replaced by $L_s(F, x) \geqq 0$ in Proposition 4.4. Example 4.6 shows that Proposition 4.4 is incorrect without assumption (iii).

Example 4.5. Let $P = [-1, 1]$ and let σ be the system of all subintervals of P. For $A \in \sigma$ we let $F(A) = 1$ if $0 \in A - A^{\circ}$, $F(A) = -1$ if $0 \in A^{-} - A$, and $F(A) = 0$ otherwise. Then[6] $F \in \mathfrak{S}$, $L_s(F, x) = 0$ for all $x \in P^{-}$, and $\sigma(F)$ is not stable.

Example 4.6. Let P be the interval $[0, 1]$ with the discrete topology and let σ be the system of all, possibly degenerate, subintervals of P. Obviously, σ satisfies conditions (i) and (ii) and does not satisfy condition (iii). For $A \in \sigma$, let $F(A) = -\mu(A)$, where μ is the Lebesgue measure in P. It is easy to see that[6] $F \in \mathfrak{S}$ and $L_w(F, x) = 0$ for all $x \in P^{-}$. However, since $P \in \sigma(F)$ and $[\sigma(F)]_{P-U} = \emptyset$ for every $U \in \Gamma_{\infty}$, $\sigma(F)$ is not stable.

It is an open question whether for every non-empty semihereditary stable system $\delta \subset \sigma$ there is a function $F \in \mathfrak{S}$ with $\sigma(F) = \delta$ and such that $L_*(F, x) \geqq 0$ for all $x \in P^{-}$.

THEOREM 4.7. *Let $F \in \mathfrak{S}$ and let $Z \subset P$ be a countable set. If $L_w(F, x) \geqq 0$ for all $x \in Z \cup (\infty)$ and $D_w(F, x) \geqq 0$ for all $x \in P - Z$, then $F(A) \geqq 0$ for every set $A \in \sigma$.*

Proof. Suppose that $F(A) < 0$ for some $A \in \sigma$. Since $\sigma(F)$ is stable (see (4.4) and the remark following Proposition 4.4), there is a small set $B \in \sigma$ such that $F(B) < 0$. Therefore, $\epsilon = -F(B)/[G(B) + 1]$ is a well-defined positive number. For $C \in \sigma$, we let $H(C) = F(C) + \epsilon G(C)$ whenever $F(C) + \epsilon G(C)$ has meaning and $H(C) = -\infty$ otherwise. It is rather easy to see that $H \in \mathfrak{S}$, $H(B) = -\epsilon < 0$, and $L_w(H, x) \geqq L_w(F, x)$ for all $x \in P^{-}$. From (4.4), the remark following Proposition 4.4, and Corollary 2.5, it follows that there is $x_0 \in (P - Z) \cap [\sigma(H)]^{w}$. Hence $D_w(H, x_0) \leqq 0$. On the other hand, $D_w(H, x_0) \geqq D_w(F, x_0) + \epsilon > 0$, which is a contradiction.

If the space P is locally compact and Hausdorff, then according to Corollary 2.8, Theorem 4.7 holds for any set $Z \subset P$, which has cardinality less than the continuum.

THEOREM 4.8. *Assume that P is locally pseudo-metrizable. Let $F \in \mathfrak{S}$ and let $Z \subset P$ be a set with cardinality less than the continuum. If $L_c(F, x) \geqq 0$ for all $x \in Z \cup (\infty)$ and $D_c(F, x) \geqq 0$ for all $x \in P - Z$, then $F(A) \geqq 0$ for every non-empty set $A \in \sigma$.*

The theorem follows from Theorem 3.2 and Proposition 4.4 and its proof is a verbatim repetition of the proof of Theorem 4.7.

THEOREM 4.9. *Assume that P is Hausdorff and that σ is closed with respect to the formation of set differences. Let $F \in \mathfrak{S}$ and let $Z \subset P$ be a countable set. If $L_s(F, x) \geqq 0$ and $L_s(-G, x) \geqq 0$ for all $x \in Z$ and if $D_s(F, x) \geqq 0$ for all $x \in P - Z$, then $F(A) \geqq 0$ for every non-empty compact set $A \in \sigma$.*

[6]In fact, the function F is additive.

Proof. Let $Z = \{x_1, x_2, \ldots\}$ and choose $\epsilon > 0$. For $A \in \sigma$ we let

$$H(A) = F(A) + \epsilon\sum_n 2^{-n}\chi_A(x_n),$$

where χ_A is the characteristic function of A in P. Obviously, $H \in \mathfrak{S}$, $D_s(H, x) \geqq D_s(F, x)$ for all $x \in P$, and $D_s(H, x) = +\infty$ for all $x \in Z$. According to (**5**, § 5.2), $H(A) \geqq 0$ for every non-empty compact set $A \in \sigma$. Since $F \geqq H - \epsilon$, the theorem follows from the arbitrariness of ϵ.

Notice that condition (iii) was not needed for the proof of Theorem 4.3.

Theorems 4.7 and 4.8 are of essential importance in the general theory of the Perron integration which will be given in (**6**). Theorem 4.8 is particularly important for the geometric applications of the Perron integral.

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