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# On mostly expanding diffeomorphisms

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*Abstract*. In this work, we study the class of mostly expanding partially hyperbolic diffeomorphisms. We prove that such a class is $C^r$-open, $r > 1$, among the partially hyperbolic diffeomorphisms and we prove that the mostly expanding condition guarantees the existence of physical measures and provides more information about the statistics of the system. Mañé's classical derived-from-Anosov diffeomorphism on $\mathbb{T}^3$ belongs to this set.

## 1. *Introduction*

Physical measures may be thought of as capturing the asymptotic statistical behavior of large sets of orbits under a dynamical system. There is a strong and well-known connection between the existence of physical measures and abundance, in some proper sense, of non-zero Lyapunov exponents. This is particularly true in the setting of dissipative partially hyperbolic diffeomorphisms, where non-zero central Lyapunov exponents are a crucial ingredient in nearly all of the known results (see [**24**] and [**8**] for two exceptions). During the last 15 years, considerable effort has been made towards an understanding of the statistical behavior of partially hyperbolic diffeomorphisms with non-vanishing central Lyapunov exponents.

In the case studied by Dolgopyat [**11**] and Bonatti and Viana [**7**], they assumed abundance of negative Lyapunov exponents along the center direction (mostly contracting condition). The mostly contracting condition was later shown to be $C^2$ robust, with most of its members satisfying a strong kind of statistical stability: all physical measures persist and vary continuously with small deterministic perturbations of the dynamics [**3**]. Recently, Dolgopyat, Viana and Yang [**12**, **28**] gave a detailed explanation of how bifurcations occur and they gave an exhaustive set of examples. They have also proved a form of continuity of the basins of physical measures.

The present work deals with the analogous but considerably harder case of diffeomorphisms whose central direction exhibits an abundance of positive Lyapunov exponents. We introduce a new notion of *mostly expanding* diffeomorphisms (different from the one introduced in [**1**]) and we prove that they constitute a $C^2$ open set. Moreover, we show that mostly expanding diffeomorphisms exhibit a finite number of physical measures and provide more information about the statistics of the system. In particular, we study a notion of ergodic stability in a non-conservative setting for such diffeomorphisms.

1.1. *Mostly expanding diffeomorphisms.* Let $M$ be a closed Riemannian manifold. We denote the norm obtained from the Riemannian structure by $\|\cdot\|$ and the Lebesgue measure on $M$ by Leb . If $V$, $W$ are normed linear spaces and $A : V \to W$ is a linear map, we define

$$\|A\| = \sup\{\|Av\|/\|v\|,\ v \in V \setminus \{0\}\}$$

and

$$\mathrm{m}(A) = \inf\{\|Av\|/\|v\|,\ v \in V \setminus \{0\}\}.$$

A diffeomorphism $f : M \to M$ is *partially hyperbolic* if there exists a continuous $Df$-invariant splitting of $TM$,

$$TM = E^s \oplus E^c \oplus E^u,$$

and if there exist constants $C \geq 0$ and

$$0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3$$

with $\mu_1 < 1 < \lambda_3$ such that, for all $x \in M$ and every $n \geq 1$,

$$C^{-1}\lambda_1^n \leq \mathrm{m}\,(Df^n(x)|E^s(x)) \leq \|Df^n(x)|E^s(x)\| \leq C\mu_1^n, \tag{1.1}$$

$$C^{-1}\lambda_2^n \leq \mathrm{m}\,(Df^n(x)|E^c(x)) \leq \|Df^n(x)|E^c(x)\| \leq C\mu_2^n, \tag{1.2}$$

$$C^{-1}\lambda_3^n \leq \mathrm{m}\,(Df^n(x)|E^u(x)) \leq \|Df^n(x)|E^u(x)\| \leq C\mu_3^n. \tag{1.3}$$

We always assume that dim $E^\sigma \geq 1$, $\sigma = s, c, u$ unless stated otherwise. We also point out that the set of $C^r$-partially hyperbolic diffeomorphisms, $r \geq 1$, is $C^r$-open [**13**, Corollary 2.17]. For partially hyperbolic diffeomorphisms, it is a well-known fact that there are foliations $\mathcal{F}^\sigma$ tangential to the distributions $E^\sigma$ for $\sigma = s, u$ [**14**]. The leaf of $\mathcal{F}^\sigma$ containing $x$ will be called $W^\sigma(x)$ for $\sigma = s, u$.

An $f$-invariant probability measure $\mu$ is a *Gibbs u-state* or *u-measure* if the conditional measures of $\mu$ with respect to the partition into local strong unstable manifolds are absolutely continuous with respect to Lebesgue measure along the corresponding local strong unstable manifold. Section 2 will be devoted to providing more properties of Gibbs $u$-states.

*Definition 1.1.* A partially hyperbolic diffeomorphism $f : M \to M$ $f$ with $Df$-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ is *mostly expanding along the central direction* if $f$ has positive central Lyapunov exponents almost everywhere with respect to every Gibbs $u$-state for $f$.

There are several notions of asymptotic expansion along the center direction in the literature and we will explain briefly the relationship between mostly expanding and the other similar conditions introduced.

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*Definition 1.2.* A partially hyperbolic diffeomorphism $f: M \to M$ is *strongly mostly expanding along the central direction* if

$$\lambda^c(x) = \liminf_{n\to+\infty} \frac{1}{n} \log \mathrm{m}(Df^n|E^c_x) > 0 \tag{1.4}$$

for a positive Lebesgue measure set of points $x$ in every disk $D^u$ contained in a strong unstable local manifold.

The notion above is a mimic of the mostly contracting notion introduced in [**7**] and it is not the same as in [**1**], where the term mostly expanding was coined.

We prove that if $f$ is mostly expanding in a strong sense, then it is mostly expanding (see Proposition 3.1). We do not know if the reciprocal is true.

As mentioned above, Alves, Bonatti and Viana [**1**] use a different definition of mostly expanding. Their definition is more general than ours. In particular, they allow the strong unstable direction $E^u$ to be trivial, working with a splitting of type $E^s \oplus E^{cu}$. In our setting, we write $E^{cu} = E^c \oplus E^u$ and state their notion of mostly expanding with a different name.

*Definition 1.3.* We say that $f$ is *non-uniformly expanding* along the center-unstable direction (for short $f$ satisfies the *NUE-condition*) if there exists $c_0 > 0$ and $H \subset M$ of positive Lebesgue measure such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \le -c_0 < 0 \tag{1.5}$$

holds for every $x \in H$.

Recently, condition (1.5) was weakened by Alves, Dias, Luzzatto and Pinheiro in [**2**].

*Definition 1.4.* A partially hyperbolic diffeomorphism $f$, as above, is *weakly non-uniformly expanding* along the center-unstable direction (or it satisfies the *wNUE-condition* for short) if there exists $c_0 > 0$ and $H \subset M$ of positive Lebesgue measure such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \le -c_0 < 0. \tag{1.6}$$

We remark that the lim inf in (1.6) implies that the growth only needs to be verified on a subsequence of iterates, in contrast to the lim sup in (1.5), where the condition needs to be verified for all sufficiently large times.

Our first result reveals the motivation of this work and the reason for introducing a new notion of mostly expanding: properties (1.5) and (1.6) are not robust.

THEOREM A. *Satisfying the NUE-condition (or wNUE-condition) on a set $H \subseteq \mathbb{T}^3$ with full Lebesgue measure is not a robust property among the set of partially hyperbolic $C^r$-diffeomorphisms on $\mathbb{T}^3$, $r > 1$.*

The statement above holds also if we replace 'full measure' by 'positive measure'. In fact, in §5, we exhibit examples where each property in the statement above is not

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robust. The examples are inspired by a construction due to Dolgopyat, Hu and Pesin [**4**, Appendix B]. They provide an example of a non-uniformly hyperbolic volume-preserving diffeomorphism on $\mathbb{T}^3$ with countably many ergodic components.

Even though our notion is more restrictive than the NUE-condition or the wNUE-condition (see §4 for details), to be mostly expanding is a robust property, so mostly expanding diffeomorphisms are a good setting in which to looking for robust statistical properties.

THEOREM B. *The class of mostly expanding partially hyperbolic diffeomorphisms constitutes a $C^r$-open subset of* $\mathrm{Diff}^r(M)$, $r > 1$.

1.2. *Mostly expanding condition and existence of physical measures.* Physical measures may be thought of as capturing the asymptotic statistical behavior of large sets of orbits under a dynamical system. Recall that if $\mu$ is an $f$-invariant measure, then the *basin* of $\mu$ is the set

$$\mathcal{B}(\mu) = \left\{ z \in M : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z)) = \int_M \varphi \, d\mu, \ \text{for all } \varphi \in C^0(M, \mathbb{R}) \right\}.$$

It is well known that if $\mu$ is ergodic, then $\mathcal{B}(\mu)$ has full $\mu$-measure. The measure $\mu$ is a *physical* or *Sinai–Ruelle–Bowen measure* if Leb $(\mathcal{B}(\mu)) > 0$.

There is a strong and well-known connection between the existence of physical measures and abundance, in some proper sense, of non-zero Lyapunov exponents. This is particularly true in the setting of dissipative partially hyperbolic diffeomorphisms, where the asymptotic expansion (or contraction) on the central subbundle is a crucial ingredient in nearly all of the known results. For instance, Alves, Bonatti and Viana [**1**] as well as Alves, Dias, Luzzatto and Pinheiro [**2**] showed that if $f$ is a $C^r$- partially hyperbolic diffeomorphism satisfying the NUE-condition (respectively, wNUE-condition) for $H = M$, then it exhibits finitely many physical measures and the union of their basins covers a full Lebesgue measure subset of $M$. Nevertheless, the techniques and methods that were used by Alves, Dias, Luzzatto and Pinheiro in [**2**] to deal with these weaker assumptions are completely different from those used in Alves Bonatti and Viana in [**1**].

In [**1**], Alves, Bonatti and Viana ask if is it possible to conclude the existence of physical measures if the non-uniform expansion condition (1.5) is replaced by condition (1.4). Our next theorem gives essentially a positive answer to such a question, although instead of requiring that the condition hold on a positive Lebesgue measure set, we require it to hold on a positive leaf volume subset of every unstable disk.

THEOREM C. *If $f$ is a mostly expanding partially hyperbolic diffeomorphism, then $f$ has a finite number of physical measures whose basins together cover Lebesgue almost every point in $M$.*

We point out, however, that there is no possibility of having the phenomenon of intermingled basins of attraction in the mostly expanding case (see Lemma 4.5 and also [**25**]).

1.3. *Mostly expanding and stable ergodicity.* We now consider the question of uniqueness of physical measures, not just for $f$, but also for its small perturbations. This is related to the stable ergodicity problem in a dissipative setting studied in [**3**, **9**, **10**] for mostly contracting diffeomorphisms and by [**27**] in the case of mostly expanding diffeomorphisms. Our examples in §5 show that, in contrast to the mostly contracting case, where it was shown in [**3**] that uniqueness of the physical measure implies robust uniqueness of the physical measure, having the NUE- or wNUE-condition satisfied on a set of full Lebesgue measure does not imply that uniqueness of the physical measure is a robust property. We refer the reader to [**18**, **19**] and the references therein for more exhaustive information.

Recall that a foliation is minimal if its leaves are dense. The strongly stable foliation $\mathcal{F}^s(f)$ of a partially hyperbolic diffeomorphism $f : M \to M$ is $C^r$-robustly minimal if there exists a $C^r$-neighborhood $\mathcal{U}$ of $f$ such that $\mathcal{F}^s(g)$ is minimal for every $g \in \mathcal{U}$.

THEOREM D. *Assume that $f$ is a mostly expanding partially hyperbolic $C^r$-diffeomorphism, $r > 1$. Suppose that the strongly stable (respectively, unstable) foliation $\mathcal{F}^s(f)$ $f$ is $C^r$-robustly minimal. Then any $C^r$ diffeomorphism $g$ close enough to $f$ in the $C^r$ topology has a unique physical measure $\mu_g$ whose basin $\mathcal{B}(\mu_g)$ has full volume in whole manifold $M$.*

The above result holds if we replace the hypothesis of strongly stable (respectively, unstable) foliation by robust transitivity. Conditions under which one of the strong foliations of a partially hyperbolic diffeomorphism is robustly minimal were provided by Pujals and Sambarino [**22**], Bonatti, Diaz and Ures [**5**] and Nobili [**17**].

The work is organized as follows. Section 2 is devoted to recalling known results which will be used later. In §3, we prove Theorem B. Theorem C and Theorem D are proved in §4. As we mentioned previously, in §5, we show examples where the NUE-condition and the wNUE-condition fail to be robust, proving the statement of Theorem A. Finally, in §6, and following the ideas developed in [**7**], we show that the classical example of a non-hyperbolic robustly transitive partial hyperbolic diffeomorphism due to Mañe [**15**] is mostly expanding along the central direction. In particular, the previous result can be applied to such a class of examples.

## 2. *Preliminaries*

We will now summarize the main results related to the existence of a physical measure in the setting of partially hyperbolic diffeomorphisms. The key ingredients are the Gibbs $u$-states.

In this section, $f$ is a partial hyperbolic diffeomorphism. We denote by $\mathcal{M}(M)$ the set of probability measures defined on $M$ provided with the weak* topology and denote by $\mathcal{M}(f)$ the set of invariant probability measures by $f$. It is well known that $\mathcal{M}(f)$ is a convex compact subset of $\mathcal{M}(M)$ and, moreover, that every invariant measure has a decomposition into ergodic measures (cf. [**16**]).

An $f$-invariant probability measure $\mu$ is a *Gibbs $u$-state* or *$u$-measure* if the conditional measures of $\mu$ with respect to the partition into local strong unstable manifolds are absolutely continuous with respect to Lebesgue measure along the corresponding local strong unstable manifold. More precisely, given a point $z \in M$, we define a *foliated box* of $z$ in the following way. Pick a strong unstable disk $D$ with center at $z$ and take a cross section $\Sigma$ to the strong unstable foliation through the center point $z$. Then there exists $\phi : D \times \Sigma \to M$, which is a homeomorphism onto its image such that $\phi$ maps each horizontal $D \times \{y\}$ diffeomorphically to an unstable domain through $y$. We may choose $\phi$ such that $\phi(z, y) = y$ for all $y \in \Sigma$ and $\phi(x, z) = x$ for all $x \in D$. In what follows, we identify $D \times \Sigma$ and each $D \times \{y\}$ with their images under this chart $\phi$. Given any measure $\xi$ on $D \times \Sigma$, we denote by $\hat{\xi}$ the measure on $\Sigma$ defined by

$$\hat{\xi}(B) = \xi(D \times B). \tag{2.1}$$

An $f$-invariant probability measure $\mu$ is a *Gibbs $u$-state* or *$u$-measure* if, for every foliated neighborhood $D \times \Sigma$ such that $\mu(D \times \Sigma) > 0$, the conditional measures of $\mu|(D \times \Sigma)$ with respect to the partition into strong unstable plaques $\{D \times \{y\} : y \in \Sigma\}$ are absolutely continuous with respect to Lebesgue measure along the corresponding plaque.

Gibbs $u$-states play a key role in the theory. If $\mu$ is a physical measure for a partially hyperbolic diffeomorphism, then $\mu$ must be a Gibbs $u$-state [**6**, §11.2.3].

In the early nineteen-eighties, Pesin and Sinai [**20**] proved that the set of the Gibbs $u$-states of $f$ is non-empty for $C^r$- partially hyperbolic diffeomorphisms, $r > 1$. More precisely, denote by $u$ the dimension of the bundle $E^u$. If $D^u$ is a $u$-dimensional disk inside a strong unstable leaf, and $\mathrm{Leb}_{D^u}$ denotes the volume measure induced on $D^u$ by the restriction of the Riemannian metric to $D^u$, then every accumulation point of the sequence of probability measures

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j\left(\frac{\mathrm{Leb}_{D^u}}{\mathrm{Leb}_{D^u}(D^u)}\right)$$

is a Gibbs $u$-state with densities with respect to Lebesgue measure along the strong unstable plaques uniformly bounded away from zero and infinity. It is possible to extend the result obtained by Pesin and Sinai [**6**, Theorem 11.16].

PROPOSITION 2.1. *There exists $E \subseteq M$ intersecting every unstable disk on a full Lebesgue measure subset such that, for any $x \in E$, every accumulation point $\nu$ of*

$$\nu_{n,x} = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \tag{2.2}$$

*is a Gibbs $u$-state.*

The support of any Gibbs $u$-state consists of entire strong unstable leaves [**6**, Corollary 11.14]. Moreover, a convex combination of Gibbs $u$-states is a Gibbs $u$-state. Conversely, if $\mu$ is a Gibbs $u$-state, its ergodic components are Gibbs $u$-states whose densities are uniformly bounded away from zero and infinity [**6**, Lemma 11.13].

Denote by $\mathcal{G}^u(f) \subseteq \mathcal{M}(f)$ the set of Gibbs $u$-states of $f$. The assertion above implies that $\mathcal{G}^u(f)$ is convex. Furthermore, the set $\mathcal{G}^u(f)$ provided with the weak* topology is closed in $\mathcal{M}(f)$ and so compact [**10**, Theorem 5]. Moreover, given any sufficiently small $C^r$-neighborhood $\mathcal{U}$ of $f$, $r > 1$, the set

$$\mathcal{G}^u(\mathcal{U}, M) = \{(g, \mu) : g \in \mathcal{U} \text{ and } \mu \text{ a Gibbs } u\text{-state of } g\} \tag{2.3}$$

is closed in $\mathcal{U} \times \mathcal{M}(M)$ [**6**, Remark 11.15] when we consider the product topology.

In the partial hyperbolic setting, notions like physical measures, Gibbs $u$-states, non-zero Lyapunov exponents and stable ergodicity are closely related. We will explain some known relationships useful for our purposes. We refer the reader to [**6**, **18**, **19**, **29**], and the references therein, for a complete discussion about the relationships to be discussed now.

As mentioned above, physical measures are Gibbs $u$-states in the setting of partially hyperbolic diffeomorphisms, but the converse is not true even in the uniformly hyperbolic setting, as the reader will notice from the example at the end of this section. It is well known that if $\mu$ is an ergodic Gibbs $u$-state with negative central Lyapunov exponents, that is, if

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n|E^c_x\| < 0$$

for $\mu$-almost every point $x \in M$, then $\mu$ is a physical measure (see [**29**, Theorem 3] and the references therein). The statement follows from classical arguments [**21**].

This is a good motivation for introducing the notion of mostly contracting. A partially hyperbolic diffeomorphism $f$ is *mostly contracting* along the center subbundle if

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n|E^c_x\| < 0 \tag{2.4}$$

for a positive Lebesgue measure set of points $x$ in every disk $D^u$ contained in a strong unstable local manifold. In such cases, it was proved in [**7**] that if $f$ is partially hyperbolic and strongly mostly contracting along the center subbundle, then $f$ admits finitely many ergodic physical measures, and the union of their basins covers a full Lebesgue measure subset of the basin of $M$. A related notion of mostly contracting was studied by Dolgopyat in [**11**]. Ten years later, in [**3**], it was proved that the mostly contracting property (2.4) is equivalent to every (ergodic) Gibbs $u$-state having negative central Lyapunov exponents.

In the setting of partially hyperbolic diffeomorphisms with the absence of strong unstable direction, that is, when $f : M \to M$ is a $C^r$-diffeomorphism, $r > 1$, with decomposition of the tangent bundle $TM = E^s \oplus E^c$, it makes no sense to speak of Gibbs $u$-states. For this setting, Alves, Bonatti and Viana [**1**] introduced the notion of the Gibss $cu$-state using the fact that, in the presence of positive Lyapunov exponents, there are Pesin invariant unstable manifolds. Thus Gibbs $cu$-states correspond to non-uniform versions of Gibbs $u$-states. An invariant probability measure $\mu$ is a *Gibbs $cu$-state* if the $m$ largest Lyapunov exponents are positive $\mu$-almost everywhere, where $m = \dim E^c$, and the conditional measures of $\mu$ along the corresponding local unstable Pesin manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds.

Alves, Bonatti and Viana showed that a $C^r$-non uniformly expanding partially hyperbolic diffeomorphism (recall Definition 1.3), $r > 1$, exhibits (ergodic) Gibbs $cu$-states which are physical measures [**1**]. Moreover, if $H = M$, then $f$ admits finitely many (ergodic) physical measures, and the union of their basins covers a full Lebesgue measure subset of $M$. The same conclusion was reached by Alves, Dias, Luzzatto and Pinheiro in [**2**] under the wNUE-condition (recall Definition 1.4). We record the precise statement obtained by the authors above for future reference.

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PROPOSITION 2.2. [**2**, Theorem A] *Let $f : M \to M$ be a $C^r$, $r > 1$, partially hyperbolic diffeomorphism with decomposition $TM = E^s \oplus E^{cu}$. Assume that there exists a subset $H \subseteq M$ of positive Lebesgue measure on which $f$ is weakly non-uniformly expanding along $E^{cu}$. Then:*

(i) *there exist closed invariant transitive sets $\Omega_1, \ldots, \Omega_\ell$ such that, for Lebesgue almost every $x \in H$, $\omega(x) = \Omega_j$ for some $1 \le j \le \ell$; and*

(ii) *there exist (ergodic) Gibbs cu-states $\mu_1, \ldots, \mu_\ell$ supported on the sets $\Omega_1, \ldots, \Omega_\ell$, whose basins are open up to a zero Lebesgue measure set, such that, for Lebesgue almost every $x \in H$, $x \in \mathcal{B}(\mu_j)$ for some $1 \le j \le \ell$.*

In [**26**], the author established several properties of Gibbs $cu$-states which are similar to the ones for Gibbs $u$-states: that is, [**26**, Theorem 2.1] the ergodic components of a Gibbs $cu$-state $\mu$ are Gibbs $cu$-states whose densities are uniformly bounded away from zero and infinity. Conversely, a convex combination of Gibbs $cu$-states is a Gibbs $cu$-state. The support of any Gibbs $cu$-state consists of entire center-unstable leaves. In the setting of partially hyperbolic diffeomorphisms with non-uniform expansion, every ergodic physical measure is a Gibbs $cu$-state.

In the case where $f$ is a non-uniformly expanding partially hyperbolic diffeomorphism with decomposition of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$, every Gibbs $cu$-state is, in fact, a Gibbs $u$-state with positive central Lyapunov exponents. The converse is not true, even in the case of Anosov diffeomorphisms.

*Example.* Consider two linear Anosov diffeomorphisms $A_1, A_2$ over the torus $\mathbb{T}^2$ with splittings $E_1^u \oplus E_1^s$ and $E_2^u \oplus E_2^s$, respectively, and with (unstable) eigenvalues $\lambda_1 > \lambda_2 > 1$, respectively. Now consider $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$, defined by $f = A_1 \times A_2$. If we consider the decomposition $E^u = E_1^u$, $E^c = E_2^u$ and $E^s = E_1^s \oplus E_2^s$ of the tangent bundle of $\mathbb{T}^2 \times \mathbb{T}^2$, then $f$ is a partially hyperbolic diffeomorphism with positive central Lyapunov exponent at every point. Consider the measure $\mu = \mu_1 \times \mu_2$, where $\mu_1$ is an Gibbs $u$-state for $A_1$ and $\mu_2$ is the Dirac measure supported on a periodic orbit of $A_2$. Then $\mu$ is a Gibbs $u$-state for $f$ whose central Lyapunov exponents are positive, but it is not a Gibbs $cu$-state.

## 3. *Proof of Theorem B*

For every $f \in \mathrm{Diff}^r(M)$ partially hyperbolic and $x \in M$, denote

$$\lambda^c(x, f) := \liminf_{n \to +\infty} \frac{1}{n} \log \mathrm{m}(Df^n | E_x^c). \tag{3.1}$$

The diffeomorphism $f$ is *strongly mostly expanding* (cf. Definition 1.2) if $\lambda^c(x, f) > 0$ for a positive Lebesgue measure set of points $x$ in every disk $D^u$ contained in a strong unstable local manifold. If $x \in M$ is a regular point, the number above is the minimum of the Lyapunov exponents of $x$ whose Osedelec splitting is contained in the central direction. According to Osedelec's theorem (see [**16**, Theorem 10.1]), the set of regular points is a Borel set with total measure. In particular, the set of regular points has full measure with respect to every Gibbs $u$-state. Note that the function defined by (3.1) is $f$ invariant. Then $\lambda^c(x, f) =: \lambda^c(\mu, f)$ is constant for $\mu$-almost every $x$ when $\mu$ is ergodic.

PROPOSITION 3.1. *Let $f : M \to M$ be a strongly mostly expanding partially hyperbolic diffeomorphism. Then $f$ is mostly expanding.*

*Proof.* Assume that $f$ is strongly mostly expanding. It is enough to prove the assertion for ergodic Gibbs $u$-states and, in such cases, to prove that all central Lyapunov exponents are positive on a set of positive $\mu$-measure. Pick a foliated neighborhood $D \times \Sigma$ such that $\mu(D \times \Sigma) > 0$. By the definition of Gibbs $u$-states, the conditional measures of $\mu|(D \times \Sigma)$ with respect to the partition into strong unstable disks $\{D \times \{y\} : y \in \Sigma\}$ are absolutely continuous with respect to Lebesgue measure along the corresponding unstable disks. From such disintegration and absolutely continuity, we conclude that, for $\hat{\xi}$-almost every point $y \in \Sigma$ (recall that $\hat{\xi}$ was defined in (2.1)), the central Lyapunov exponents are well defined in a set of Lebesgue full measure in each of the strong unstable disks $D \times \{y\}$. Since $f$ is strongly mostly expanding, such central Lyapunov exponents must be positive. □

Recall that $\mathcal{M}(M)$ denotes the set of probability measures on $M$ provided with the weak* topology and $\mathcal{G}^u(f) \subseteq \mathcal{M}(M)$ denotes the set of Gibbs $u$-states of $f$. Let $\mathcal{S} \subseteq \mathrm{Diff}^r(M) \times \mathcal{M}(M)$ be the set of pairs $(f, \mu)$, where $f$ is a $C^r$ partially hyperbolic diffeomorphism and $\mu$ is a Gibbs $u$-state for $f$. We consider $\mathcal{S}$ endorsed with the product topology induced from $\mathrm{Diff}^r(M) \times \mathcal{M}(M)$.

Let us consider the function $\Lambda^c : \mathcal{S} \to \mathbb{R}$ defined as

$$\Lambda^c(f, \mu) := \int_M \lambda^c(x, f)\, d\mu(x).$$

Now let $\mu$ be any Gibbs $u$-state. By convexity of the set of Gibbs $u$-states of $f$, $\mu$ is a convex combination of ergodic Gibbs $u$-states $(\mu_x)_{x \in M}$. Therefore, the ergodic decomposition theorem implies that

$$\Lambda^c(f, \mu) := \int \lambda^c(x, f)\, d\mu(x) = \iint \lambda^c(x, f)\, d\mu_x\, d\mu(x) = \int \lambda^c(\mu_x)\, d\mu(x). \quad (3.2)$$

The next lemma will be useful in the proof of Proposition 3.5.

LEMMA 3.2. *A partially hyperbolic diffeomorphism $f$ is mostly expanding if and only if $\Lambda^c(f, \mu) > 0$ for every $\mu \in \mathcal{G}^u(f)$.*

*Proof.* Assume that $f$ is mostly expanding. If $\nu$ is an ergodic Gibbs $u$-state,

$$\lambda^c(x, f) := \lambda^c(\nu, f) > 0$$

for $\nu$-almost every point $x \in M$. Then

$$\Lambda^c(f, \nu) = \int_M \lambda^c(x, f)\, d\nu(x) = \lambda^c(\nu, f) > 0.$$

If $\nu$ is not ergodic, then, from (3.2),

$$\Lambda^c(f, \nu) := \int \lambda^c(x, f)\, d\nu(x) = \iint \lambda^c(x, f)\, d\nu_x\, d\nu(x) = \int \lambda^c(\nu_x, f)\, d\nu(x) > 0,$$

since every $\nu_x$ is an ergodic Gibbs $u$-state and then $\lambda^c(\nu_x, f) > 0$.

Now we assume that $\Lambda^c(f, \mu) > 0$ for every $\mu \in \mathcal{G}^u(f)$. If $\mu$ is ergodic, then it follows from $\Lambda^c(f, \mu) = \lambda^c(\mu, f)$ that $\mu$ has positive central Lyapunov exponents. If $\mu$ is non-ergodic and $\lambda^c(x, f) \leq 0$ for every $x \in A$, where $A$ is a set of positive $\mu$-measure, then, by the ergodic decomposition theorem, there exists an ergodic component $\mu_x$ of $\mu$ such that $x \in A$, $\mu_x(A) > 0$ and $\lambda^c(x, f) = \lambda^c(\mu_x, f) \leq 0$, which is a contradiction. □

For every $n \geq 1$, define $L_n : \mathcal{S} \to \mathbb{R}$ by

$$L_n(f, \mu) := \int \log \mathrm{m}(Df^n|E^c_x)\, d\mu(x). \tag{3.3}$$

Recall that the set $\mathcal{S}$ is endowed with the product topology induced from $\mathrm{Diff}^r(M) \times \mathcal{M}(M)$. Therefore $L_n : \mathcal{S} \to \mathbb{R}$ is continuous for every $n \geq 1$. Moreover, for a fixed $(f, \mu) \in \mathcal{S}$, the sequence $(L_n(f, \mu))_{n\geq 1}$ is super additive, that is, for every integer $n$, $m \geq 1$,

$$L_{n+m}(f, \mu) \geq L_n(f, \mu) + L_m(f, \mu). \tag{3.4}$$

This implies that the following limit exists (or is equal to $+\infty$).

$$\alpha = \alpha(f, \mu) = \lim_{n\to+\infty} \frac{1}{n} L_n(f, \mu) = \sup\left\{\frac{1}{n} L_n(f, \mu) : n \in \mathbb{N}\right\}.$$

LEMMA 3.3. *For every $(f, \mu) \in \mathcal{S}$,*

$$\Lambda^c(f, \mu) = \lim_{n\to+\infty} \frac{1}{n} L_n(f, \mu). \tag{3.5}$$

*Proof.* This lemma is a direct consequence of the dominated convergence theorem. In fact, fix $(f, \mu) \in \mathcal{S}$ and let us consider the sequence of $\mu$-integrable functions on $M$ defined by $\Psi_n(x) = (1/n) \log \mathrm{m}(Df^n|E^c_x)$, $x \in M$, $n \geq 1$. Then $\Psi_n$ converges $\mu$-almost every point to $\lambda^c(\cdot, f)$ and, by partial hyperbolicity, $\Psi_n(x) \leq (1/n) \log C^{-1} + \log \lambda_3$, where $C \geq 0$ and $\lambda_3 > 1$ are the constants in (1.3). Hence,

$$\begin{aligned}\Lambda^c(f, \mu) &= \int \lambda^c(x, f)\, d\mu(x) = \lim_{n\to+\infty} \int \Psi_n(x) \\ &= \lim_{n\to\infty} \frac{1}{n} \int \log \mathrm{m}(Df^n|E^c_x)\, d\mu = \lim_{n\to+\infty} \frac{1}{n} L_n(f, \mu).\end{aligned}$$ □

PROPOSITION 3.4. *The function $\Lambda^c : \mathcal{S} \to \mathbb{R}$ is lower semi-continuous.*

*Proof.* Let $(f, \mu) \in \mathcal{S}$ and fix $\epsilon > 0$ arbitrarily. From Lemma 3.3, we can take $n_0 \geq 1$ large so that

$$\frac{1}{n_0} L_{n_0}(f, \mu) > \Lambda^c(f, \mu) - \epsilon.$$

Continuity of $L_{n_0}$ allows us choose, thereafter, a neighborhood $\mathcal{N}$ of $(f, \mu)$ in $\mathcal{S}$ small enough for

$$\frac{1}{n_0} L_{n_0}(g, \mu_g) > \Lambda^c(f, \mu) - \epsilon$$

to hold for any pair $(g, \mu_g) \in \mathcal{N}$. Again from Lemma 3.3,

$$\Lambda^c(g, \mu_g) = \lim \frac{1}{n} L_n(g, \mu_g) = \sup_n \frac{1}{n} L_n(g, \mu_g) \geq \frac{1}{n_0} L_{n_0}(g, \mu_g) \geq \Lambda^c(f, \mu) - \epsilon,$$

which proves the proposition. □

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The next proposition corresponds to the statement in Theorem B.

PROPOSITION 3.5. *The class of mostly expanding partially hyperbolic diffeomorphisms constitutes a $C^r$-open subset of* $\mathrm{Diff}^r(M)$, $r > 1$.

*Proof.* Let $\mathrm{Diff}^r(M)$, $r > 1$, be a mostly expanding partially hyperbolic diffeomorphism. Since partial hyperbolicity is a $C^1$-open property (and hence it is $C^r$-open for every $r \geq 1$), we need to prove the existence of a small neighborhood where every partially hyperbolic diffeomorphism is mostly expanding. We argue by contradiction. Assume that there is a sequence $(f_n)_{n\geq 1}$ of $C^r$-diffeomorphisms converging to $f$ in the $C^r$-topology, $r > 1$, such that, for every $f_n$, $n \geq 1$, there is a Gibbs $u$-state $\mu_n$ such that $\Lambda^c(f_n, \mu_n) \leq 0$.

Taking a subsequence, if necessary, we may assume that $\mu_n$ converges to $\mu$. As already pointed out (see [**6**, Remark 11.15]), $\mu$ is a Gibbs $u$-state for $f$. Since $f$ is mostly expanding, $\Lambda^c(f, \mu) > 0$. By the lower semi-continuity of $\Lambda^c$,

$$0 \geq \liminf_{n\to+\infty} \Lambda^c(f_n, \mu_n) \geq \Lambda^c(f, \mu) > 0,$$

which is a contradiction. □

## 4. *Proof of Theorem C and Theorem D*

We prove Theorem C by showing that if $f$ is mostly expanding, then it has an iterate that satisfies the weakly expanding condition (1.6) along the center direction. We do this in several steps. Thereafter, we use the weakly expanding condition to show that $f$ has a finite number of physical measures whose basins cover Lebesgue almost every point in the ambient manifold.

LEMMA 4.1. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, be partially hyperbolic and mostly expanding. Then there exists an integer $n_0 \geq 1$ such that*

$$\int \log \mathrm{m}(Df^{n_0}|E^c_x)\, d\mu(x) > 0 \tag{4.1}$$

*for every Gibbs $u$-state $\mu$ of $f$.*

*Proof.* For any $n \geq 1$, consider the set

$$G_n = \{\mu \in \mathcal{G}^u(f) : L_n(f, \mu) > 0\}. \tag{4.2}$$

Because each $L_n$ is continuous, $n \geq 1$, $G_n$ is open in $\mathcal{G}^u(f)$. Since $f$ is mostly expanding, it follows that, given any $\mu \in \mathcal{G}^u(f)$, there exists $n \geq 0$ such that $\mu \in G_n$ (see Lemmas 3.2 and 3.3). Hence the family $\{G_n\}_{n\in\mathbb{N}}$ is an open covering of $\mathcal{G}^u(f)$ and, by compactness, there exist integers $n_1, \ldots, n_k$ such that $\mathcal{G}^u(f) = \bigcup_{j=1}^k G_{n_j}$.

Let $n_0 := n_1 \cdots n_k$. Since the sequence $(L_n(f, \mu))_n$ is super additive,

$$L_{rs}(f, \mu) \geq r L_s(f, \mu) \quad \text{for every integers } r, s \geq 1. \tag{4.3}$$

If $\mu \in \mathcal{G}^u(f)$, then $\mu \in G_{n_j}$ for some $1 \leq j \leq k$. Taking $s = n_j$ and $r = n_0/n_j$ in (4.3) proves that $L_{n_0}(\mu) > 0$ for every $\mu \in \mathcal{G}^u(f)$. □

Note that, for $n_0 \geq 1$, we always have $\mathcal{G}^u(f) \subseteq \mathcal{G}^u(f^{n_0})$, and it may be that $\mathcal{G}^u(f)$ is a proper subset of $\mathcal{G}^u(f^{n_0})$. Lemma 4.1 states that (4.1) holds for every $\mu \in \mathcal{G}^u(f)$. We do not know whether it holds for every $\mu \in \mathcal{G}^u(f^{n_0})$.

LEMMA 4.2. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, be mostly expanding. Then there exists an integer $n_0 \geq 1$ such that $f^{n_0}$ satisfies the wNUE-condition on a set $H \subseteq M$.*

*Moreover, $H \cup f^{-1}(H) \cup \cdots \cup f^{-(n_0-1)}(H)$ has full Lebesgue measure in $M$.*

The proof of Lemma 4.2 makes use of an auxiliary result regarding the lim sup of general sequences.

LEMMA 4.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and let $N \geq 1$ be some integer. Then*

$$\limsup_{n\to\infty} \frac{1}{nN} \sum_{k=0}^{nN-1} a_k \leq \max_{0\leq \ell \leq N-1} \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{kN+\ell}. \tag{4.4}$$

*Proof.* Let $A = \max_{0\leq\ell\leq N-1} \limsup_{n\to\infty} (1/n) \sum_{k=0}^{n-1} a_{kN+\ell}$. Assume that $A < +\infty$; otherwise, there is nothing to prove. For each $\ell = 0, \ldots, N-1$, let

$$A_\ell = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{kN+\ell}.$$

Fix $\epsilon > 0$. For every $\ell = 0, \ldots, N-1$, there exist $m_\ell \geq 1$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} a_{kN+\ell} < A_\ell + \epsilon \quad \text{for every } n \geq m_\ell.$$

Let $m := \max\{m_0, \ldots, m_{N-1}\}$. Then

$$\frac{1}{n} \sum_{k=0}^{n-1} a_{kN+\ell} < A_\ell + \epsilon \quad \text{for every } n \geq m, \text{ and } \ell = 0, \ldots, N-1.$$

Hence,

$$\frac{1}{n} \sum_{\ell=0}^{N-1} \sum_{k=0}^{n-1} a_{kN+\ell} < \sum_{\ell=0}^{N-1} A_\ell + \epsilon \leq N(A+\epsilon)$$

and

$$\frac{1}{nN} \sum_{k=0}^{nN-1} a_k \leq A + \epsilon \quad \text{for every } n \geq m,$$

which proves the lemma. □

We need to remark that it is not possible to change lim sup to lim inf in Lemma 4.3. It is due to this limitation that we are only able to prove *a priori* that an iterate of $f$, $g = f^{n_0}$ satisfies the wNUE-condition on the set of positive Lebesgue measure $H$. Of course, as a consequence of the existence of a physical measure for $g = f^{n_0}$, *a posteriori* $g$ satisfies also the NUE-condition on $H$. In fact, it follows from Proposition 2.2 that $H$ contains the basin of the physical measures of $g$, so we can change the lim sup to lim inf on points belonging to one of such basins.

*Proof of Lemma 4.2.* Recall that if $f$ is partially hyperbolic, we denote $E^{cu} = E^c \oplus E^u$. Let $n_0$ be as in Lemma 4.1 and write $g = f^{n_0}$. Let

$$c_0 := \inf_{\mu\in\mathcal{G}^u(f)} \int \log \mathrm{m}(Dg|E^{cu})\, d\mu. \tag{4.5}$$

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Then $c_0 > 0$ according to Lemma 4.1. Let $E(g)$ be the set of points $x \in M$ such that every accumulation point of the measure

$$\nu_{x,n} = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{g^k(x)} \tag{4.6}$$

belongs to $\mathcal{G}^u(g)$. Then $E(g)$ has full Lebesgue measure in $M$ according to Proposition 2.1. Note that if $x \in E(g)$, then $\nu_{x,n}$ accumulates on $\mu \in \mathcal{G}^u(g)$ if and only if $\nu_{f(x),n}$ accumulates on $f_*\mu \in \mathcal{G}^u(g)$. So $E(g)$ is $f$-invariant. Note also that if $\nu \in \mathcal{G}^u(g)$, then

$$\frac{1}{n_0}(\nu + f_*\nu + \cdots + f_*^{n_0-1}\nu) \in \mathcal{G}^u(f).$$

In particular, every accumulation point $\tilde{\mu}$ of

$$\mu_{x,n} = \frac{1}{n_0}\sum_{\ell=0}^{n_0-1}\nu_{f^\ell(x),n} \tag{4.7}$$

is a Gibbs $u$-state for $f$.

Fix some $x \in E(g)$ and, for any integer $n \geq 1$, set

$$a_n = \log \mathrm{m}(Dg|E^{cu}_{f^n(x)}). \tag{4.8}$$

For every $0 \leq \ell \leq n_0 - 1$ fixed,

$$\begin{aligned}\int \log \mathrm{m}(Dg|E^{cu}_x)\, d\nu_{f^\ell(x),n} &= \frac{1}{n}\sum_{k=0}^{n-1}\log \mathrm{m}(Dg|E^{cu}_{g^k(f^\ell(x))})\\ &= \frac{1}{n}\sum_{k=0}^{n-1}\log \mathrm{m}(Dg|E^{cu}_{f^{kn_0+\ell}(x)})\\ &= \frac{1}{n}\sum_{k=0}^{n-1}a_{kn_0+\ell}.\end{aligned}$$

So

$$\begin{aligned}\int \log \mathrm{m}(Dg|E^{cu}_x)\, d\mu_{x,n} &= \frac{1}{n_0}\sum_{\ell=0}^{n_0-1}\int \log \mathrm{m}(Dg|E^{cu}_x)\, d\nu_{f^\ell(x),n}\\ &= \frac{1}{n_0}\sum_{\ell=0}^{n_0-1}\frac{1}{n}\sum_{k=0}^{n-1}a_{kn_0+\ell}.\end{aligned}$$

For every sufficiently large $n \geq 1$ it must be that

$$\begin{aligned}\frac{1}{nn_0}\sum_{\ell=0}^{n_0-1}\sum_{k=0}^{n-1}a_{kn_0+\ell} &= \int \log \mathrm{m}(Dg|E^{cu}_x)\, d\mu_{x,n}\\ &> \frac{1}{2}\inf_{\mu\in\mathcal{G}^u(f)}\int \log \mathrm{m}(Dg|E^{cu})\, d\mu = c_0/2.\end{aligned}$$

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In particular,

$$\limsup_{n\to\infty} \frac{1}{nn_0} \sum_{k=0}^{n-1} \sum_{\ell=0}^{n_0-1} a_{kn_0+\ell} \geq c_0/2. \tag{4.9}$$

Hence, applying Lemma 4.3 with $N = n_0$, we conclude that there exists some $0 \leq \ell \leq n_0 - 1$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_{kn_0+\ell} \geq c_0/2. \tag{4.10}$$

We have now proved that $f^{\ell}(x) \in H$ for some $0 \leq \ell n_0 - 1$. Since $x \in E$ was chosen arbitrarily, this implies that $E = H \cup f^{-1}(H) \cup \cdots \cup f^{-(n_0-1)}(H)$. To finish the proof, note that $H$ corresponds to the set of points $x \in M$ for which

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \mathrm{m}(Dg|E^{cu}_{g^k(x)}) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \mathrm{m}(Dg|E^{cu}_{f^{n_0 k}(x)}) \geq c_0/2 > 0,$$

which is equivalent to

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \|(Dg|E^{cu}_{g^k(x)})^{-1}\| \leq -c_0/2 < 0, \tag{4.11}$$

so $g = f^{n_0}$ satisfies the wNUE-condition on $H$.

The next lemma allows us to conclude Theorem C.

LEMMA 4.4. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, be partially hyperbolic and mostly expanding. Then there exist finitely many ergodic Gibbs cu-states $\mu_1, \ldots, \mu_\ell$ all of which are physical measures. The union of their basins has full Lebesgue measure in $M$.*

*Proof.* From Lemma 4.2, there exists an integer $n_0 \geq 1$ such that $f^{n_0}$ satisfies the wNUE-condition on a set $H \subseteq M$. So we can apply Proposition 2.2 to $g = f^{n_0}$ and then we conclude that there exist finitely many ergodic Gibbs $cu$-states (of $g$) $\nu_1, \ldots, \nu_\ell$ whose basins have non-empty interior and are such that

$$\mathrm{Leb}\left(H \Big\backslash \bigcup_{j=1}^{\ell} \mathcal{B}(\nu_j)\right) = 0.$$

As we noted above, if $\nu_j$ is a Gibbs $cu$-state for $g = f^{n_0}$, $j \in \{1, \ldots, \ell\}$, then

$$\mu_j := \frac{1}{n_0}(\nu_j + f_*\nu + \cdots + f_*^{n_0-1}\nu_j)$$

is a Gibbs $cu$-state for $f$. Of course, each $\mathcal{B}(\mu_j)$ is contained in $H \cup f^{-1}(H) \cup \cdots \cup f^{-(n_0-1)}(H)$ and

$$\mathrm{Leb}\left(\bigcup_{k=0}^{n_0-1} f^{-k}(H) \Big\backslash \bigcup_{j=1}^{\ell} \mathcal{B}(\mu_j)\right) = 0.$$

Since $\bigcup_{k=0}^{n_0-1} f^{-k}(H)$ has full Lebesgue measure in $M$ (see Lemma 4.2), then, for almost every $x \in M$, $x \in \mathcal{B}(\mu_j)$ for some $1 \leq j \leq \ell$. □

The following remark is a key ingredient in the proof of Theorem D.

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LEMMA 4.5. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, be partially hyperbolic and mostly expanding. Let $\mu$ be a physical measure for $f$. Then its basin is open in $M$ up to a zero Lebesgue measure subset.*

*Proof.* We prove the Lemma for completeness, even though it is contained in the second statement of Proposition 2.2 (see also [**2**]). Since $\mu$ is ergodic, $\mu$-almost every point belongs to $\mathcal{B}(\mu)$. Moreover, since $\mu$ is a Gibbs $cu$-state, $\mu$-almost every point lies in an unstable leaf $F$ on which $\mathrm{Leb}_F$-almost every point also belongs to $\mathcal{B}(\mu)$. More precisely, there exists a set $\mathcal{L}(\mu)$ of unstable leaves such that:
(i) $\mu\big(\bigcup_{F\in\mathcal{L}(\mu)} F\big) = 1$; and
(ii) $\mathcal{B}(\mu)$ has full $\mathrm{Leb}_F$-measure for every $F \in \mathcal{L}(\mu)$.
For each $F \in \mathcal{L}(\mu)$, let $\mathcal{A}(F)$ be the $s$-saturated set consisting of the union of all strong stable manifolds of points in $F$. Then $\mathcal{A}(F)$ is an open set and, by absolute continuity of the stable foliation, $\mathcal{B}(\mu) \cap \mathcal{A}(F)$ has full Lebesgue measure in $\mathcal{A}(F)$. Let $\mathcal{A} = \bigcup_{F\in\mathcal{L}(\mu)} \mathcal{A}(F)$. Since $\mathcal{B}(\mu)$ has full Lebesgue measure in each $\mathcal{A}(F)$, it has full Lebesgue measure in $\mathcal{A}$. Note that $\mathcal{A}$ is a neighborhood of $\operatorname{supp}\mu$. Therefore, given any point $x \in \mathcal{B}(\mu)$, there is an iterate $f^n(x)$ that belongs to $\mathcal{A}$. But $\mathcal{A}$ is invariant, so, in fact, we must have $x \in \mathcal{A}$. Hence $\mathcal{A}$ is an open set containing $\mathcal{B}(\mu)$ and $\mathcal{B}(\mu)$ has full Lebesgue measure in $\mathcal{A}$. □

The next lemma implies Theorem D.

LEMMA 4.6. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, be transitive, partially hyperbolic and mostly expanding. Then $f$ has a unique physical measure whose basin has full Lebesgue measure in $M$.*

*Proof.* Assume that there are two physical measures $\mu$ and $\nu$ for $f$. The basins $\mathcal{B}(\mu)$ and $\mathcal{B}(\nu)$ are open up to a zero Lebesgue measure subset. From topological transitivity, there is a non-negative integer $n$ such that $f^n(\mathcal{B}(\mu)) \cap \mathcal{B}(\nu) \neq \emptyset$. Hence $\mu = \nu$. □

## 5. *Dolgopyat–Hu–Pesin blocks*

In [**4**, Appendix B], Dolgopyat, Hu and Pesin provide an example of a non-uniformly hyperbolic volume-preserving diffeomorphism on $\mathbb{T}^3$ with countably many ergodic components. As a key step in the construction, they consider a linear Anosov diffeomorphism $A$ on $\mathbb{T}^2$ and the map $F : [0, 1] \times \mathbb{T}^2 \to [0, 1] \times \mathbb{T}^2$ defined by $F = I \times A$, where $I$ is the identity map. Then, by a suitable perturbation of $F$, they construct a $C^\infty$ diffeomorphism $g$ on $M = [0, 1] \times \mathbb{T}^2$, with the following properties.

- $g$ is a partially hyperbolic volume-preserving diffeomorphism on $M$.
- The connected components of the boundary, $W_i := \{i\} \times \mathbb{T}^2$, $i = 0, 1$ are left invariant by $g$ and $g|W_i = F|W_i$ is a linear Anosov map.
- $g$ is ergodic with respect to Lebesgue measure and has positive central Lyapunov exponents almost everywhere.
- $g$ can be chosen to be as $C^r$ close to $F$ as desired, for any $r \geq 2$.

The meaning of the first item above, that is, that $g$ is partially hyperbolic, could potentially be confusing, as we are dealing with a manifold with boundary, and no definition of partial hyperbolicity has been provided in this case. In fact, there is no

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need for that. The diffeomorphism $g$ is constructed so that it becomes a $C^\infty$ partially hyperbolic diffeomorphism on $\mathbb{T}^3$ when identifying $\{0\} \times \mathbb{T}^2$ with $\{1\} \times \mathbb{T}^2$ in the natural manner. This is what enables them to glue two or more copies of $g$ together.

We observe that if $\mu$ is a Gibbs $u$-state for $A$, then $\delta_0 \times \mu$ is a $u$-measure for $g$ with zero central Lyapunov exponent, where $\delta_0$ is the Dirac measure concentrated on zero. Therefore $g$ is not mostly expanding, although it does satisfy the NUE-condition along the center (or center-unstable) direction (1.5).

Although the construction of the diffeomorphism $g$ by Dolgopyat, Hu, and Pesin was done as an intermediate step in providing an example of a diffeomorphism with non-zero Lyapunov exponents almost everywhere and yet having countably many ergodic components, it turns out to be useful when thinking about the NUE-condition of [**1**] and how it compares with our notion of mostly expanding along the central direction.

Gluing copies of $g$ together is an easy matter. Indeed, given $0 < \lambda < 1$ and $0 \le \tau < 1 - \lambda$, we define the squeezing and sliding action

$$\begin{aligned} L_{\lambda,\tau} : [0, 1) \times \mathbb{T}^2 &\to [\tau, \tau + \lambda) \times \mathbb{T}^2 \\ (x, y, z) &\mapsto (\lambda x + \tau, y, z). \end{aligned}$$

Now suppose that we wish to construct an example of a diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ satisfying the NUE-condition and exhibiting precisely $k$ physical measures. All we need to do is identify $\mathbb{T}^3$ with $[0, 1) \times \mathbb{T}^2$ in the obvious way and take $f$ to be the diffeomorphism satisfying

$$f|_{[i/k,(i+1/k))} = L_{1/k,i/k} \circ g \circ L^{-1}_{1/k,i/k}, \quad i = 0, \ldots, k.$$

PROPOSITION 5.1. *Having condition (1.5) or (1.6) satisfied on a set of full Lebesgue measure is not a robust property.*

*Proof.* Pick any $0 < \epsilon < 1$ and define $f_\epsilon$ by

$$\begin{aligned} f_\epsilon(x, y, z) &= L_{1-\epsilon,0} \circ g \circ L^{-1}_{1-\epsilon,0}(x, y, z), \quad x \in [0, 1 - \epsilon) \\ f_\epsilon(x, y, z) &= (x, A(y, z)), \quad x \in [1 - \epsilon, 1). \end{aligned}$$

By construction, $f_\epsilon$ approaches $g$ as $\epsilon \to 0$ in any $C^r$ topology, but none of the $f_\epsilon$ satisfy the NUE-condition. □

PROPOSITION 5.2. *Having condition (1.5) or (1.6) satisfied on a set of positive Lebesgue measure is not a robust property.*

*Proof.* Let $f$ be a diffeomorphism on $\mathbb{T}^3$ obtained by gluing two blocks, as above, with different sign on their central Lyapunov exponents. More precisely, let $f$ be such that

$$f|_{[0,1/3)\times\mathbb{T}^2} = L_{1/3,0} \circ g \circ L^{-1}_{1/3,0}$$

and

$$f|_{[1/3,1)\times\mathbb{T}^2} = L_{2/3,1/3} \circ g^{-1} \circ L^{-1}_{2/3,1/3}.$$

Then both $f$ satisfies (1.5) and (1.6) on a set of positive measure. Note that the integrated central Lyapunov exponent of $f$ is negative. From [**23**], we know that, for every $r > 1$, there is a $C^r$ diffeomorphism $\tilde{f}$ arbitrarily close to $f$ in the $C^r$ topology such that $\tilde{f}$

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is ergodic. In particular, the central Lyapunov exponent of $\tilde{f}$ is constant Lebesgue almost everywhere. Since $\tilde{f}$ is $C^1$ close to $f$, this Lyapunov exponent is negative. In particular, $\tilde{f}$ fails to satisfy conditions (1.5) and (1.6) on a set of positive Lebesgue measure. □

PROPOSITION 5.3. *Having condition (1.5) or (1.6) satisfied on a set of full measure does not imply that the number of physical measures varies upper semi-continuously with the dynamics.*

*Proof.* To prove this claim, we need to show that the number of physical measures can undergo an explosion. All we have to do to accomplish that is to modify $f_\epsilon$ in Claim 1 on $[1-\epsilon, 1) \times \mathbb{T}^2$: that is,

$$f_\epsilon(x, y, z) = L_{1-\epsilon,0} \circ g \circ L^{-1}_{1-\epsilon,0}(x, y, z), \quad x \in [0, 1-\epsilon), \tag{5.1}$$

$$f_\epsilon(x, y, z) = L_{\epsilon,1-\epsilon} \circ g' \circ L^{-1}_{\epsilon,1-\epsilon}(x, y, z), \quad x \in [1-\epsilon, 1), \tag{5.2}$$

where $g' : [0, 1] \times \mathbb{T}^2$ is a diffeomorphism satisfying all the properties of $g$ and is sufficiently close to $F$ in the $C^r$ topology. The resulting $f_\epsilon$ has two physical measures and is arbitrarily close to $g$ (which has one). □

## 6. *Derived-from-Anosov example*

Here we show that a classical example of a non-hyperbolic robustly transitive diffeomorphism due to Mañe [**15**] satisfies our notion of partial hyperbolicity with mostly expanding central direction in the strong sense. We do that following the ideas developed in [**7**].

We start with a linear Anosov diffeomorphism $f_0 : \mathbb{T}^3 \to \mathbb{T}^3$ with three different eigenvalues

$$\lambda_s < 1/3 < 1 < \lambda_c \ll 3 < \lambda_u.$$

We consider the splitting

$$T\mathbb{T}^3 = E_0^s \oplus E_0^c \oplus E_0^u$$

into eigenspaces corresponding to these eigenvalues. Let $p_0 \in \mathbb{T}^3$ be a fixed point of $f_0$ and consider $\delta > 0$ (to be fixed later). We deform $f_0$ along a one-parameter family of diffeomorphisms $f_t$, $t \geq 0$, by isotopy inside $V = B(p_0, \delta)$ to make it go through a pitchfork bifurcation. The expansion in the strong unstable subbundle $E_t^u$ remains large everywhere and the same is true for the contraction in the strong stable subbundle $E_t^s$. More precisely, for every parameter $t \geq 0$,

$$|Df_t|E_t^s| < 1/3 < 3 < |Df_t|E_t^u|.$$

Then the distortion along the strong unstable leaves remains uniformly bounded in the whole family $f_t$.

The center subbundle $E_t^c$ restricted outside $V$ also remains expanding: that is,

$$3 > |Df_t(x)|E_t^c(x)| \geq \eta_c(t) > 1 \quad \text{for every } x \in \mathbb{T}^3 \setminus V, \tag{6.1}$$

where $\eta_c(0) = \lambda_c > 1$. Moreover, for every parameter $0 \leq t < t_0$, the deformation can be done in a such a way that (6.1) can be extended to every $x \in \mathbb{T}^3$ by choosing carefully the

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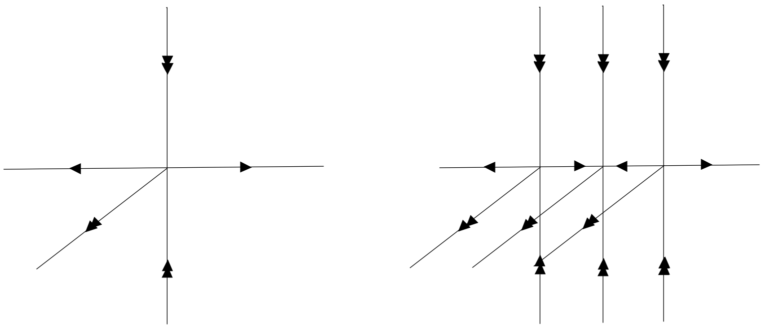

FIGURE 1. Deformation of $f_0$ along the $E_0^c$ direction in the point $p_0$.

constant $\eta_c(t) > 1$ and so $f_t$ is expanding along the central direction $E_t^c$. If we denote by $p_t$, $t \geq 0$, the continuation of the hyperbolic fixed point $p_0$ of $f_0$ inside the neighborhood $V$, the eigenvalue $\lambda_c(p_t) > 1$ if $0 \leq t < t_0$ becomes $\lambda_c(p_{t_0}) = 1$. Then, for parameters $t_0 + \epsilon > t > t_0$ slightly greater than $t_0$, the continuation $p_t$ is a saddle point whose stable index changes from one to two, and two other saddle points $q_t^1, q_t^2 \in V$, of stable index one, are created (see Figure 1).

Note that, for every $\beta > 0$, we can choose $\epsilon > 0$ such that for every $0 \leq t < t_0 + \epsilon$ and every $x \in V$,

$$|Df_t(x)|E_t^c(x)| > 1 - \beta.$$

Then, for $0 \leq t < t_0 + \epsilon$, the diffeomorphism $f_t$ is partially hyperbolic. Hence there exist unique foliations $\mathcal{F}_t^s$ and $\mathcal{F}_t^u$ tangential to $E_t^s$ and $E_t^u$, respectively. Moreover, it follows from [**14**] that $f_t$ has an invariant central foliation $\mathcal{F}_t^c$ tangential to the central direction $E_t^c$, $0 \leq t < t_0 + \epsilon$. Since it remains normally contracting all the way during the isotopy, the center-unstable foliation $\mathcal{F}_t^{cu}$ is leaf conjugate to the unstable foliation $\mathcal{F}_0^{cu}$, which means that there exists a homeomorphism $h : \mathbb{T}^3 \to \mathbb{T}^3$ which sends leaves of $\mathcal{F}_t^{cu}$ into leaves of $\mathcal{F}_0^{cu}$ and conjugates the dynamics of the leaves. More precisely,

$$h(W_t^{cu}(f_t(x))) = W_0^{cu}(f_0(h(x))).$$

If we fix a diffeomorphism $f_t$ with $t$ slightly greater than $t_0$ in the family above, we obtain a $C^1$-open set $\mathcal{U}$ of diffeomorphisms containing $f_t$ such that every $f \in \mathcal{U}$ satisfies the following.

(i) $f$ is partially hyperbolic and uniquely integrable. We have an $Df$-invariant splitting into three non-trivial subbundles

$$T\mathbb{T}^3 = E^s \oplus E^c \oplus E^u$$

satisfying

$$|Df|E^s| < 1/3 < 3 < |Df|E^u|, \tag{6.2}$$

and, for every $x \in V$,

$$|Df(x)|E^c(x)| > 1 - \beta. \tag{6.3}$$

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(ii) There exist unique foliations $\mathcal{F}^*$ tangential to $E^*$, respectively, where $* = s, c, u$.
(iii) $f$ has three hyperbolic fixed saddles inside $V$ contained in a same central leaf $\mathcal{F}^c(p)$: two saddles with stable index one and one saddle with stable index two.

Recall that for a $u$-segment $\gamma$, by (6.2),

$$\text{length}(f(\gamma)) \geq 3 \cdot \text{length}(\gamma),$$

so we can assume that there are constants $L \geq 0$ such that, given any $u$-segment $\gamma$ with

$$\text{length}(\gamma) \geq L,$$

there is an integer $k = k(\gamma) \geq 1$ such that we may write

$$f(\gamma) = \gamma_1 \cup \cdots \cup \gamma_k$$

as the disjoint union of $u$-segments $\gamma_i$ satisfying

$$2L \geq \text{length}(\gamma_i) \geq L, \quad i = 1, \ldots, k.$$

By redefining $V$, if necessary, we can assume that there is $0 < \tau_0 < 1$ such that, given any $u$-segment $\gamma$ with length$(\gamma) \geq L$, if $I_V = \{i : \gamma_i \cap V \neq \emptyset,\ i = 1, \ldots, k(\gamma)\}$, then

$$\sum_{j \in I_V} \text{length}(\gamma_j) \leq \tau_0 \cdot \text{length}(f(\gamma)).$$

Of course, $L \geq 0$ can be chosen close the size of $V$, so the image of any $u$-segment with length bigger than $L$ spends a positive fraction $1 - \tau_0$ (in length) outside $V$.

For any integer $k \geq 1$ and $0 < \alpha < 1$, we define

$$M(k, \alpha) = \left\{ x \in \gamma : \frac{1}{k} \sum_{j=0}^{k-1} \mathbf{1}_V(f^j(x)) \geq \alpha \right\},$$

where $\mathbf{1}_V$ is the characteristic function of $V$. Note that $x \in M(k, \alpha)$ if its orbit spends a fraction of time bigger than $\alpha > 0$ inside $V$ (until time $k$). Of course, the complement of the set

$$\bigcap_{n \geq 1} \bigcup_{k \geq n} M(k, \alpha)$$

corresponds to the set of points $x \in \gamma$ that spend at least a fraction $1 - \alpha$ outside $V$.

LEMMA 6.1. *There is $0 < \alpha_0 < 1$, depending on $\alpha_0 = \alpha_0(\tau_0, |Df|E^u|)$, such that, for every $u$-segment $\gamma$ and every $\alpha_0 \leq \alpha < 1$, the set*

$$M_\gamma := \bigcap_{n \geq 1} \bigcup_{k \geq n} M(k, \alpha)$$

*has zero Lebesgue measure in $\gamma$.*

The proof of Lemma 6.1 can be found in [**7**, §6.3]. We need to remark that $\alpha_0$ is a constant such that it essentially depends on the distortion bound along the unstable direction. That allows us to fix $\alpha > \alpha_0$ and to choose $\beta$ close to zero such that

$$3^{(1-\alpha)}(1 - \beta)^\alpha = \lambda > 1. \tag{6.4}$$

PROPOSITION 6.2. *Let $f \in \mathcal{U}$ be as above. Then, for any u-segment $\gamma$ and Lebesgue almost every point $x \in \gamma$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log |(Df^n|E^c_x)^{-1}| < 0.$$

*In particular, $f$ is mostly expanding.*

*Proof.* Let $\gamma$ be a $u$-segment and consider $x \in \gamma$. Since $E^c$ is a one-dimensional subbundle,

$$|Df^n(x)|E^c(x)| = \prod_{j=0}^{n-1} |Df(f^j(x))|E^c(f^j(x))|.$$

If we denote by $J_V(x) = \{j : f^j(x) \in V, 0 \le j \le n-1\}$ the iterates of $x$ belonging to $V$, then, by (6.3), the derivative of $f^j(x)$ along the center direction is controlled, if $j \in J_V(x)$, by

$$|Df(f^j(x))|E^c(f^j(x))| > 1 - \beta.$$

For the iterated $j \notin J_V(x)$, by (6.2),

$$|Df(f^j(x))|E^c(f^j(x))| > 3.$$

Then

$$|Df^n(x)|E^c(x)| > 3^{n-|J_V(x)|}(1-\beta)^{|J_V(x)|}.$$

It follows from Lemma 6.1 that for Lebesgue almost every $x \in \gamma \setminus M_\gamma$, that is, there exist $N \ge 1$ such that, for every $n \ge N$,

$$\frac{|J_V(x)|}{n} \le \alpha < 1.$$

Then, taking into account (6.4),

$$\begin{aligned} |Df^n(x)|E^c(x)| &> 3^{n-|J_V(x)|}(1-\beta)^{|J_V(x)|} \\ &> 3^{(1-\alpha)n}(1-\beta)^{\alpha n} \\ &> \lambda^n > 1. \end{aligned}$$

Finally,

$$\frac{1}{n} \log |Df^n(x)|E^c(x)| > \log \lambda > 0.$$

Then we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log |Df^n(x)|E^c(x)| > \log \lambda > 0.$$

□

*Remark.* The previous example can be generalized to $\mathbb{T}^{n+2}$, $n \ge 1$, beginning from a linear Anosov diffeomorphism $f_0$ with decomposition

$$T\mathbb{T}^3 = E^s_0 \oplus E^c_0 \oplus E^u_0,$$

where $\dim E^u_0 = \dim E^c_0 = 1$ and $\dim E^s_0 = n$.

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