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Solutions of the Riemann–Hilbert–Poincaré problem and the Robin problem for the inhomogeneous Cauchy–Riemann equation

Alip Mohammed and **M. W. Wong**
Department of Mathematics and Statistics, York University,
4700 Keele Street, Toronto, Ontario M3J 1P3, Canada
(mwwong@mathstat.yorku.ca)

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The Riemann–Hilbert–Poincaré problem with general coefficient for the inhomogeneous Cauchy–Riemann equation on the unit disc is studied using Fourier analysis. It is shown that the problem is well posed only if the coefficient is holomorphic. If the coefficient has a pole, then the problem is transformed into a system of linear equations and a finite number of boundary conditions are imposed in order to find a unique and explicit solution. In the case when the coefficient has an essential singularity, it is shown that the problem is well posed only for the Robin boundary condition.

1. Introduction

The $\bar{\partial}$-equation has been studied intensively and extensively not only for special domains such as the polydisc and the unit ball [5, 6, 17], but also for general domains [12, 13]. As for the Neumann problem, there are some results for the polydisc [15] and the unit ball [3]. Notwithstanding these results, very little is known about the Riemann–Hilbert–Poincaré (RHP) problem and the Robin problem for either the polydisc or the unit ball. Despite some results in [16] for the polydisc and in [2] for the unit disc in $\mathbb{C}$, the research on these problems is far from being complete and a complete understanding of the Riemann–Hilbert–Poincaré problem for the unit disc in $\mathbb{C}$ is necessary before tackling the problem for higher dimensions.

The Riemann–Hilbert–Poincaré problem for holomorphic functions has some exciting features [2,7,8], which are in sharp contrast with the known results hitherto described [4, 20]. Impressive results are obtained in [2] by reducing the Riemann–Hilbert–Poincaré problem for holomorphic functions to a Fuchsian-type differential equation. However, the analysis in [2] does not cover the Riemann–Hilbert–Poincaré problem with general coefficient.

Let $\mathbb{D}$ be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$ and let $\partial\mathbb{D}$ be its boundary. We are interested in finding a function v in $C^1(\mathbb{D})$ such that

$$\left.\begin{aligned} (\bar{\partial}v)(z) &= f(z), \quad z \in \mathbb{D}, \\ \operatorname{Re}\left[\frac{\partial v}{\partial \nu_\zeta} + \alpha(\zeta)v(\zeta)\right] &= \gamma_0(\zeta), \quad \zeta \in \partial\mathbb{D}, \end{aligned}\right\} \tag{1.1}$$

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where $\bar{\partial} = \frac{1}{2}(\partial/\partial x + \mathrm{i}\partial/\partial y)$ and $\partial v/\partial \nu_\zeta$ denotes the outward normal derivative of v at the point ζ on $\partial\mathbb{D}$, $f \in C(\bar{\mathbb{D}})$, α and γ_0 are Hölder continuous functions on $\partial\mathbb{D}$. The problem (1.1) is known as the Riemann–Hilbert–Poincaré problem for the inhomogeneous Cauchy–Riemann equation on the unit disc $\mathbb{D}$.

Since α is Hölder continuous on $\partial\mathbb{D}$, we can write

$$\alpha(z) = \alpha^+(z) + \alpha^-(z), \quad z \in \partial\mathbb{D}, \tag{1.2}$$

where

$$\alpha^+(z) = \sum_{k=1}^{\infty} \alpha_{-k} z^k \quad \text{and} \quad \alpha^-(z) = \sum_{k=0}^{m} \alpha_k z^{-k}.$$

It is understood that $\alpha_0, \alpha_{\pm 1}, \alpha_{\pm 2}, \dots$ are the Fourier coefficients of α and $m \leqslant \infty$. The choice of '+' and '−' for the subscripts of the Fourier coefficients of α^- and α^+, respectively, is to simplify the notation in the computations involving α^- later. It is well known that the smoothness of α on the boundary $\partial\mathbb{D}$ is related to the decay of the Fourier coefficients α_{-k} when k is large. The continuation of α to a holomorphic function inside the disc is the same as the vanishing of the Fourier coefficients with positive subscripts, and in this case we say that α is holomorphic on $\mathbb{D}$. We say that α has a pole at the origin in $\mathbb{D}$ if $1 \leqslant m < \infty$ and α has an essential singularity at the origin in $\mathbb{D}$ if $m = \infty$.

In [2], problem (1.1) is solved for the case when $f(z) = 0$, $z \in \mathbb{D}$ and α is holomorphic, and also for the case when $f(z) = 0$, $z \in \mathbb{D}$ and

$$\alpha(z) = \alpha_m \bar{z}^m + \alpha_0, \quad z \in \partial\mathbb{D}, \tag{1.3}$$

where α_m and α_0 are complex constants. In this paper we study problem (1.1) first for

$$\alpha(z) = \alpha^-(z) = \sum_{j=0}^{m} \alpha_j \bar{z}^j, \quad z \in \partial\mathbb{D}, \tag{1.4}$$

where $\alpha_0, \alpha_1, \dots, \alpha_m$ are complex constants and $m < \infty$. Then we consider problem (1.1) for $m = \infty$. We show that problem (1.1) with α given by (1.2) can be reduced to one with α given by (1.4).

The case when α is a holomorphic function is satisfactorily solved not only for the unit disc in $\mathbb{C}$ [2] but also for the polydisc in $\mathbb{C}^n$ [16]. Problem (1.1) for the case (1.4) when α is an 'anti-polynomial' is much more difficult. The difficulty can already be seen even for the very special case (1.3) studied in [2]. The level of difficulty increases dramatically from the special two-term case (1.3) to the general anti-polynomial case (1.4) in this paper. This is due to the fact that we have to come to grips with full matrices in the latter instead of diagonal matrices in the former. Full matrices present more theoretical and numerical challenges. Moreover, the general anti-polynomial case is more natural and should find more applications than the two-term situation.

There are some results [4, 20] on problem (1.1) based on the assumption that the norm of α is sufficiently small.

In this paper we show that the number of solutions and the solvability conditions depend not on the smoothness of the function α on the boundary $\partial\mathbb{D}$, but on its interior continuation property. In order to obtain a unique solution, different kinds of boundary conditions need to be imposed even when the norm of α is small.

We show that, for the case when α is holomorphic on $\mathbb{D}$, the RHP problem is well posed. As for the case when α has a pole or an essential singularity at the origin in $\mathbb{D}$, we show that the RHP problem is not well posed and multiple solutions occur. The technique is that instead of treating the equation with a pole or an essential singularity in $\mathbb{D}$, we transform the equation into a regular equation on $\partial\mathbb{D}$. In doing so, the difficulty in tackling with a pole or an essential singularity is avoided. It is also shown that, in the case of an essential singularity, only the Robin problem turns out to be well posed.

The results in this paper play an important role in the study of the third boundary-value problem for pluriharmonic systems on polydiscs (see also [18] for the connections with Hele-Shaw moving boundary-value problems). The closely related Riemann–Hilbert problem with applications to orthogonal polynomials can be found in [9, 10].

2. Homogenization

By [1, theorem 28, p. 84], we know that the general solution to the Cauchy–Riemann equation (1.1) is given by the Pompeiu formula to the effect that

$$v(z) = u(z) + v_0(z), \quad z \in \mathbb{D}, \tag{2.1}$$

where u is an arbitrary holomorphic function on $\mathbb{D}$ and

$$v_0(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta, \quad z \in \mathbb{C}. \tag{2.2}$$

See [1, 19] in this connection. For a holomorphic function u, it is clear that, for all ζ in $\partial\mathbb{D}$,

$$\frac{\partial u}{\partial \nu_\zeta} = \zeta\frac{\mathrm{d}u}{\mathrm{d}\zeta}.$$

Thus, by (2.1), the second equation in (1.1) becomes

$$\mathrm{Re}\left[\zeta\frac{\mathrm{d}u}{\mathrm{d}\zeta} + \alpha(\zeta)u\right] = \gamma(\zeta), \quad \zeta \in \partial\mathbb{D}, \tag{2.3}$$

where

$$\gamma(\zeta) = \gamma_0(\zeta) - \mathrm{Re}\left[\frac{\partial v_0}{\partial \nu_\zeta} + \alpha(\zeta)v_0\right], \quad \zeta \in \partial\mathbb{D}.$$

Thus, to find v, it is enough to find the solution u to the RHP problem for $f(z) = 0$, $z \in \mathbb{D}$, the given function α and the function γ just introduced.

3. The Fuchsian-type equation

Let γ be a given function on $\partial\mathbb{D}$. The problem of finding a holomorphic function G on $\mathbb{D}$ such that

$$\mathrm{Re}[G(z)] = \gamma(z), \quad z \in \partial\mathbb{D}, \tag{3.1}$$

is known as the Schwarz problem for the unit disc $\mathbb{D}$. A solution G to the Schwarz problem (3.1) is given by

$$G(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial\mathbb{D}} \gamma(\zeta) \left[\frac{2}{1 - z/\zeta} - 1 \right] \frac{\mathrm{d}\zeta}{\zeta} + \mathrm{i}\gamma_0,$$

where γ_0 is an arbitrary real constant. Let us write

$$G(z) = \sum_{k=0}^{\infty} g_k z^k, \quad z \in \mathbb{D},$$

and assume in this section that (1.4) is satisfied.

3.1. Pole

Suppose that $m < \infty$. Comparing (3.1) with (2.3), we have

$$\operatorname{Re}\left[zu'(z) + \left(\alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m} \right) u(z) \right] = \operatorname{Re}[G(z)], \quad z \in \partial\mathbb{D}. \tag{3.2}$$

Since the function u is holomorphic on $\mathbb{D}$, the function $zu'(z) + \alpha(z)u(z)$ is holomorphic on $\mathbb{D} \setminus \{0\}$ and has a pole of order at most m at 0. Therefore, by (3.2) and [1, 11, 14], there exist constants $a_1, a_2, \ldots, a_m$ such that

$$zu'(z) + \left(\alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m} \right) u(z) = \sum_{k=1}^{m} \left(\frac{a_k}{z^k} - \bar{a}_k z^k \right) + \mathrm{i}c_0 + G(z)$$

for all z in $\mathbb{D} \setminus \{0\}$. So, we can write

$$zu'(z) + P_m\left(\frac{1}{z}\right) u(z) = \overline{P_m^*\left(\frac{1}{\bar{z}}\right)} - P_m^*(z) + G(z), \quad z \in \mathbb{D} \setminus \{0\}, \tag{3.3}$$

where

$$P_m\left(\frac{1}{z}\right) = \left(\alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m} \right), \quad z \in \mathbb{D} \setminus \{0\},$$

$$P_m^*(z) = \sum_{k=0}^{m} \bar{a}_k z^k, \quad z \in \mathbb{D},$$

and the constant $\mathrm{i}c_0$ is absorbed into P_m^* by just noting that

$$\mathrm{i}c_0 = a_0 - \bar{a}_0.$$

The problem is then reduced to the determination of the holomorphic function u and the coefficients $a_1, a_2, \ldots, a_m$.

Since (3.3) remains the same if $G(z) = \sum_{k=0}^{\infty} g_k z^k$ is replaced by $G_{m-1}(z) = \sum_{k=0}^{m-1} g_k z^k$ as in [2], in this case we need to solve only the equation

$$zu'(z) + \left(\alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m} \right) u(z) = \sum_{k=1}^{m} \left(\frac{a_k}{z^k} - \bar{a}_k z^k \right) + \mathrm{i}c_0 + \sum_{k=0}^{m-1} g_k z^k \tag{3.4}$$

for all z in $\mathbb{D}$.

3.2. Essential singularity

Suppose that $m = \infty$. This means that

$$\alpha^-(z) = \sum_{k=0}^{\infty} \alpha_k z^{-k} = \lim_{m\to\infty} P_m\left(\frac{1}{z}\right), \quad z \in \partial\mathbb{D},$$

and so in this case (3.3) becomes

$$zu'(z) + \alpha^-(z)u(z) = \overline{\alpha^*\left(\frac{1}{\bar{z}}\right)} - \alpha^*(z) + G(z), \quad z \in \mathbb{D} \setminus \{0\}, \tag{3.5}$$

and we are confronted with the problem of finding the two holomorphic functions u and α^* from (3.5).

4. Reduction

In this section, we show that the RHP problem for the case of the general coefficient α given by (1.2) can be reduced to the case when $\alpha = \alpha^-$.

Theorem 4.1. *The problem (2.3) for the case of the general coefficient α given by (1.2) can be reduced to the case when α is given by (1.4).*

Proof. We first assume that

$$\alpha^-(z) = P_m\left(\frac{1}{z}\right) = \left(\alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m}\right), \quad z \in \partial\mathbb{D}.$$

By (2.3) and (3.1), we have

$$\operatorname{Re}\left[zu'(z) + \left(\alpha^+(z) + \alpha_0 + \frac{\alpha_1}{z^1} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m}\right)u(z)\right] = \operatorname{Re}[G(z)] \tag{4.1}$$

for all z in $\partial\mathbb{D}$. Since the function u is holomorphic on $\mathbb{D}$, the function

$$zu'(z) + \left(\alpha^+(z) + P_m\left(\frac{1}{z}\right)\right)u(z)$$

is holomorphic on $\mathbb{D}\setminus\{0\}$ and has a pole of order at most m at 0. Therefore, by (4.1), we have [1, 11, 14]

$$zu'(z) + \left(\alpha^+(z) + \sum_{j=0}^{m} \frac{\alpha_j}{z^j}\right)u(z) = \sum_{k=1}^{m}\left(\frac{a_k}{z^k} - \bar{a}_k z^k\right) + \mathrm{i}c_0 + G(z)$$

for all z in $\mathbb{D} \setminus \{0\}$. Thus,

$$zu'(z) + \left(\alpha^+(z) + P_m\left(\frac{1}{z}\right)\right)u(z) = \overline{P_m^*\left(\frac{1}{\bar{z}}\right)} - P_m^*(z) + G(z) \tag{4.2}$$

for all z in $\mathbb{D}\setminus\{0\}$, where we recall that $P_m^*(z) = \sum_{k=0}^{m} \bar{a}_k z^k$, a_k, $k = 0, 1, 2, \ldots, m$, are arbitrary complex constants and

$$\mathrm{i}c_0 = a_0 - \bar{a}_0.$$

Using the change of variables from u to w given by

$$w(z) = u(z) \exp\left\{ \int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta \right\}, \quad z \in \mathbb{C},$$

(4.2) becomes

$$zw'(z) + P_m\left(\frac{1}{z}\right) w(z) = \left\{ \overline{P_m^*\left(\frac{1}{\bar{z}}\right)} - P_m^*(z) + G(z) \right\} \exp\left\{ \int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta \right\} \tag{4.3}$$

for all z in $\mathbb{D} \setminus \{0\}$. We let E be the function on $\mathbb{D}$ defined by

$$E(z) \exp\left\{ \int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta \right\}, \quad z \in \mathbb{D},$$

and we write E in terms of its power series as

$$E(z) = \sum_{k=0}^{\infty} e_k z^k = E_m(z) + E_r(z), \quad z \in \mathbb{D},$$

where

$$E_m(z) = \sum_{k=0}^{m} e_k z^k, \qquad z \in \mathbb{D},$$

and

$$E_r(z) = z^m \sum_{k=1}^{\infty} e_{k+m} z^k, \quad z \in \mathbb{D}.$$

Then, for all z in $\mathbb{D} \setminus \{0\}$,

$$\begin{aligned}
&\left\{ \overline{P_m^*\left(\frac{1}{\bar{z}}\right)} - P_m^*(z) + G(z) \right\} \exp\left\{ \int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta \right\} \\
&= \overline{P_m^*\left(\frac{1}{\bar{z}}\right)} E_m(z) + \overline{P_m^*\left(\frac{1}{\bar{z}}\right)} E_r(z) + E(z)\{G(z) - P_m^*(z)\} \\
&= \sum_{t=0}^{m} \left\{ \sum_{k=0}^{m-t} a_{k+t} e_k \right\} z^{-t} + \sum_{t=1}^{m-1} \left\{ \sum_{k=0}^{m-t} a_k e_{k+t} \right\} z^t \\
&\quad + \sum_{t=0}^{m-1} \left\{ \sum_{k=1}^{m-t} a_{t+k} e_{k+m} \right\} z^{m-t} + \sum_{t=0}^{m-1} \left\{ \sum_{k=0}^{t} e_k (g_{t-k} - \bar{a}_{t-k}) \right\} z^t + O(z^m).
\end{aligned}$$

If we let

$$c_t = \sum_{k=0}^{m-t} a_{k+t} e_k, \quad t = 0, 1, 2, \dots, m,$$

then we can find constants $f_0, f_1, \dots$ such that, for all $z \in \mathbb{D} \setminus \{0\}$,

$$\left\{ \overline{P_m^*\left(\frac{1}{z}\right)} - P_m^*(z) + G(z) \right\} \exp\left\{ \int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta \right\} = \sum_{t=0}^{m} (c_t z^{-t} - \overline{c_t} z^t) + \sum_{k=0}^{\infty} f_k z^k.$$

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Then, as in [2], equation (4.3) is equivalent to

$$zw'(z) + P_m\left(\frac{1}{z}\right)w(z) = \sum_{t=0}^{m}\left(\frac{c_t}{z^t} - \bar{c}_t z^t\right) + \sum_{k=0}^{m-1} f_k z^k, \quad z \in \mathbb{D} \setminus \{0\}. \tag{4.4}$$

Let $m \to \infty$. Then

$$\alpha^-(z) = \lim_{m\to\infty} P_m\left(\frac{1}{z}\right) = \sum_{k=0}^{\infty} \alpha_k z^{-k}, \quad z \in \partial\mathbb{D},$$

and (4.3) turns out to be

$$zw'(z) + \alpha^-(z)w(z) = \left\{\overline{\alpha^*\left(\frac{1}{\bar{z}}\right)} - \alpha^*(z) + G(z)\right\} \exp\left\{\int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta\right\} \tag{4.5}$$

for all z in $\mathbb{D} \setminus \{0\}$, where

$$\alpha^*(z) = \sum_{k=0}^{\infty} \bar{a}_k z^k, \quad z \in \mathbb{D}.$$

Taking into account the fact that

$$\overline{\alpha^*\left(\frac{1}{\bar{z}}\right)} E(z) = \sum_{t=0}^{\infty}\left\{\sum_{k=0}^{\infty} a_{t+k}e_k\right\} z^{-t} + \sum_{t=1}^{\infty}\left\{\sum_{k=0}^{\infty} a_k e_{k+t}\right\} z^t, \quad z \in \mathbb{D} \setminus \{0\},$$

denoting $\sum_{k=0}^{\infty} a_{t+k}e_k$ by c_t and writing

$$\{G(z) - \alpha^*(z)\} \exp\left\{\int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta\right\} + \sum_{t=1}^{\infty}\left\{\sum_{k=0}^{\infty} a_k e_{k+t}\right\} z^t + \sum_{t=0}^{\infty} \bar{c}_t z^t = \sum_{k=0}^{\infty} f_k z^k$$

for all z in $\mathbb{D}$, we can rewrite the equation (4.5) as

$$zw'(z) + \alpha^-(z)w(z) = \sum_{t=0}^{\infty}\{c_t z^{-t} - \bar{c}_t z^t\} + \sum_{k=0}^{\infty} f_k z^k, \quad z \in \mathbb{D} \setminus \{0\}. \tag{4.6}$$

This completes the proof. □

5. Free parameters

If $\alpha_k = 0$, $k = 1, 2, \ldots$, then the problem is a Schwarz problem for a holomorphic function and therefore the boundary condition (2.3) is enough to determine the holomorphic function u essentially uniquely (see [2]). Otherwise, we have two essentially different cases: (3.3) with poles and (3.5) with essential singularities.

Since equation (3.3) can be transformed into a system of linear equations, it may have a unique solution or several solutions. However, for equation (3.5), there are no known methods available in the literature.

If we look at equations (3.3), (3.5), (4.3) and (4.5) on the boundary, then we have

$$\zeta u'(\zeta) + P_m(\bar{\zeta})u(\zeta) = G(\zeta) - 2\mathrm{i}\,\mathrm{Im}[P_m^*(\zeta)], \quad \zeta \in \partial\mathbb{D}, \tag{5.1}$$

$$\zeta u'(\zeta) + \alpha^-(\zeta)u(\zeta) = G(\zeta) - 2\mathrm{i}\,\mathrm{Im}[\alpha^*(\zeta)], \quad \zeta \in \partial\mathbb{D}, \tag{5.2}$$

$$\zeta w'(\zeta) + P_m(\bar{\zeta})w(\zeta) = \{G(\zeta) - 2\mathrm{i}\,\mathrm{Im}[P_m^*(\zeta)]\}\exp\biggl\{\int_0^\zeta \frac{\alpha^+(\eta)}{\eta}\,\mathrm{d}\eta\biggr\}, \quad \zeta \in \partial\mathbb{D}, \tag{5.3}$$

and

$$\zeta w'(\zeta) + \alpha^-(\zeta)w(\zeta) = \{G(\zeta) - 2\mathrm{i}\,\mathrm{Im}[\alpha^*(\zeta)]\}\exp\biggl\{\int_0^\zeta \frac{\alpha^+(\eta)}{\eta}\,\mathrm{d}\eta\biggr\}, \quad \zeta \in \partial\mathbb{D}. \tag{5.4}$$

Now we have to fix two holomorphic functions from one equation. Usually a half-boundary condition has to be appended to fix one holomorphic function. So, by the Riemann–Hilbert–Poincaré condition (2.3) on the boundary, only one holomorphic function can be fixed and the other remains free. This means that problem (2.3) is not well posed, and therefore the existence of arbitrarily many solutions of the problem is obvious in this case. It is well posed only in the case when $\alpha(\zeta)$ is holomorphic. Thus, we see that properties often assumed for the coefficient α, such as smoothness or positivity, are irrelevant for the solvability of the problem.

Thus, besides the boundary condition (1.1), we impose another boundary condition on v, i.e.

$$\mathrm{Im}\biggl[\zeta\frac{\mathrm{d}v}{\mathrm{d}\zeta} + \alpha(\zeta)v\biggr] = \gamma^0(\zeta), \quad \zeta \in \partial\mathbb{D}. \tag{5.5}$$

So, taking (2.1) into account, the above condition (5.5) becomes

$$\mathrm{Im}[\zeta u'(\zeta) + \alpha(\zeta)u(\zeta)] = \gamma_*(\zeta), \quad \zeta \in \partial\mathbb{D}, \tag{5.6}$$

where

$$\gamma_*(\zeta) = \gamma^0(\zeta) - \mathrm{Im}\biggl[\frac{\partial v_0}{\partial \nu_\zeta} + \alpha(\zeta)v_0(\zeta)\biggr], \quad \zeta \in \partial\mathbb{D}.$$

However, condition (5.6) means that, for w in (5.4),

$$\mathrm{Im}\biggl\{[\zeta w'(\zeta) + \alpha^-(\zeta)w(\zeta)]\exp\biggl\{-\int_0^\zeta \frac{\alpha^+(\eta)}{\eta}\,\mathrm{d}\eta\biggr\}\biggr\} = \gamma_*(\zeta), \quad \zeta \in \partial\mathbb{D}. \tag{5.7}$$

Theorem 5.1. *If the condition (5.5) is fulfilled, then the function α^* can be determined uniquely from equations (5.2) and (5.4) up to one constant term.*

Proof. Taking the imaginary parts on both sides of (5.2) and (5.4), and using (5.6) and (5.7), we have

$$\mathrm{Im}[\alpha^*(\zeta)] = \gamma^*(\zeta), \quad \zeta \in \partial\mathbb{D}, \tag{5.8}$$

where

$$\gamma^*(\zeta) = \tfrac{1}{2}[\gamma_*(\zeta) - \mathrm{Im}\,G(\zeta)], \quad \zeta \in \partial\mathbb{D}.$$

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Since this is a Schwarz problem for the holomorphic function α^*, we have

$$\alpha^*(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \gamma^*(\zeta) \left[\frac{2}{1 - z/\zeta} - 1 \right] \frac{\mathrm{d}\zeta}{\zeta} + \mathrm{i}c_0^*, \quad z \in \mathbb{D}, \tag{5.9}$$

where c_0^* is an arbitrary real constant. Write

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \gamma^*(\zeta) \left[\frac{2}{1 - z/\zeta} - 1 \right] \frac{\mathrm{d}\zeta}{\zeta} + \mathrm{i}c_0^* = \sum_{k=0}^{\infty} \gamma_k^* z^k, \quad z \in \mathbb{D}.$$

Since

$$\alpha^*(z) = \sum_{k=0}^{\infty} \bar{a}_k z^k, \quad z \in \mathbb{D},$$

it follows that

$$a_k = \overline{\gamma_k^*}, \quad k = 1, 2, \dots .$$

Thus, the complex constants a_k, $k = 1, 2, \dots$, are uniquely determined and it remains to determine $\operatorname{Im} a_0 = c_0^*$. This completes the proof. □

This theorem implies that the boundary conditions (5.5) and (1.1) are sufficient to determine the holomorphic function α^* uniquely up to the real part of its zero-order term. In fact, the combination of these two boundary conditions is exactly the Robin boundary condition. This means that the Cauchy–Riemann equation with the Robin boundary condition, in the case when the coefficient has an essential singularity, is well posed provided that the holomorphic function u can be determined uniquely. It will be shown in the following that this is indeed the case.

In the next section, we give the solution for the case when

$$\deg \alpha^- = m < \infty.$$

6. Solution for a pole

We look for a solution u on $\mathbb{D}$ given by

$$u(z) = \sum_{k=0}^{\infty} u_k z^k, \quad z \in \mathbb{D}.$$

Then, by (3.4), we have

$$\sum_{k=1}^{\infty} k u_k z^k + \left(\alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots + \frac{\alpha_m}{z^m} \right) \sum_{k=0}^{\infty} u_k z^k \\ = \sum_{k=1}^{m} \left(\frac{a_k}{z^k} - \bar{a}_k z^k \right) + \mathrm{i}c_0 + \sum_{k=0}^{m-1} g_k z^k, \quad z \in \mathbb{D}$$

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or, equivalently,

$$\sum_{k=1}^{\infty} k u_k z^{k+m} + (\alpha_0 z^m + \alpha_1 z^{m-1} + \cdots + \alpha_{m-1} z + \alpha_m) \sum_{k=0}^{\infty} u_k z^k = \sum_{k=1}^{m} (a_k z^{m-k} - \bar{a}_k z^{k+m}) + \mathrm{i} c_0 z^m + \sum_{k=0}^{m-1} g_k z^{k+m}, \quad z \in \mathbb{D}. \tag{6.1}$$

Thus, by comparing the coefficients of z^0, z^{m-k}, $k = 1, 2, \dots, m-1$, z^m, z^k, $k = m+1, \dots, 2m-1$, z^{2m} and z^k, $k \geqslant 2m+1$, on both sides of (6.1), we get a system of linear equations given, respectively, by

$$\alpha_m u_0 = a_m, \tag{6.2}$$

$$\sum_{t=k+1}^{m} \alpha_t u_{t-k} = a_k - \alpha_k u_0, \quad k = m-1, \dots, 2, 1, \tag{6.3}$$

$$\alpha_1 u_1 + \cdots + \alpha_{m-1} u_{m-1} + \alpha_m u_m = \mathrm{i} c_0 + g_0 - \alpha_0 u_0, \tag{6.4}$$

$$(k - m + \alpha_0) u_{k-m} + \sum_{j=1}^{m} \alpha_j u_{k-m+j} = g_{k-m} - \bar{a}_{k-m}, \quad k = m+1, \dots, 2m-1, \tag{6.5}$$

$$(\alpha_0 + m) u_m + \alpha_1 u_{m+1} + \cdots + \alpha_{m-1} u_{2m-1} + \alpha_m u_{2m} = -\bar{a}_m, \tag{6.6}$$

$$(k - m + \alpha_0) u_k + \sum_{j=1}^{m} \alpha_j u_{k-m+j} = 0, \quad k \geqslant 2m+1. \tag{6.7}$$

Denoting $\mathrm{i} c_0 + g_0$ by a_0 and using matrices, (6.2)–(6.4) can be written succinctly as

$$\begin{pmatrix} \alpha_m & 0 & \cdots & 0 \\ \alpha_{m-1} & \alpha_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} a_{m-1} \\ a_{m-2} \\ \vdots \\ a_0 \end{pmatrix} - u_0 \begin{pmatrix} \alpha_{m-1} \\ \alpha_{m-2} \\ \vdots \\ \alpha_0 \end{pmatrix}. \tag{6.8}$$

If we denote the $m \times m$ matrix on the left-hand side by $A(m, 1)$, then its determinant $|A(m, 1)|$ is given by

$$|A(m, 1)| = \alpha_m^m.$$

Since $\alpha_m \neq 0$, it follows that $A(m, 1)^{-1}$ exists. Suppose that

$$A(m, 1)^{-1} = \begin{pmatrix} \beta_{11} & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mm} \end{pmatrix}.$$

Then

$$\beta_{jj} = \alpha_m^{-1}, \qquad j = 1, 2, \dots, m,$$
$$\beta_{(j+1)j} = -\alpha_{m-1} \alpha_m^{-2}, \quad j = 1, 2, \dots, m-1,$$

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and, for $k = 2, 3, \ldots, m-1$, $j = 1, 2, \ldots, m-k$,

$$\beta_{(i+k)i} = (-1)^k \alpha_m^{-k-1} \begin{vmatrix} \alpha_{m-1} & \alpha_m & 0 & \cdots & 0 & 0 \\ \alpha_{m-2} & \alpha_{m-1} & \alpha_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m-k+1} & \alpha_{m-k+2} & \alpha_{m-k+3} & \cdots & \alpha_{m-1} & \alpha_m \\ \alpha_{m-k} & \alpha_{m-k+1} & \alpha_{m-k+2} & \cdots & \alpha_{m-2} & \alpha_{m-1} \end{vmatrix}.$$

Since $\alpha_m \neq 0$, for equation (6.8) we obtain

$$U(1, m) = A(m, 1)^{-1} \gamma(m-1, 0), \tag{6.9}$$

where

$$U(1, m) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \quad \text{and} \quad \gamma(m-1, 0) = \begin{pmatrix} a_{m-1} \\ a_{m-2} \\ \vdots \\ a_0 \end{pmatrix} - \frac{a_m}{\alpha_m} \begin{pmatrix} \alpha_{m-1} \\ \alpha_{m-2} \\ \vdots \\ \alpha_0 \end{pmatrix}.$$

For (6.5), (6.6), we have

$$g(1, m) - \bar{a}(1, m) - M(1, m)U(1, m) = A(m, 1)U(m+1, 2m), \tag{6.10}$$

where

$$g(1, m) = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_{m-1} \\ 0 \end{pmatrix}, \qquad \bar{a}(1, m) = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_{m-1} \\ \bar{a}_m \end{pmatrix},$$

$$M(1, m) = \begin{pmatrix} \alpha_0 + 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-2} & \alpha_{m-1} \\ 0 & \alpha_0 + 2 & \alpha_1 & \cdots & \alpha_{m-3} & \alpha_{m-2} \\ 0 & 0 & \alpha_0 + 3 & \cdots & \alpha_{m-4} & \alpha_{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_0 + m - 1 & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha_0 + m \end{pmatrix}$$

and

$$U(m+1, 2m) = \begin{pmatrix} u_{m+1} \\ u_{m+2} \\ u_{m+3} \\ \vdots \\ u_{2m-1} \\ u_{2m} \end{pmatrix}.$$

Then (6.7) is equivalent to the matrix equation

$$A(m, 1)U((\ell+1)m+1, (\ell+2)m) = -M(\ell m+1, (\ell+1)m)U(\ell m+1, (\ell+1)m), \tag{6.11}$$

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where $\ell \geqslant 1$. For $\ell \geqslant 1$, let $C^\ell = (C^\ell_{ij})$ be the matrix defined by

$$C^\ell = A^{-1}(m,1)M(\ell m+1,(\ell+1)m) = (C^\ell_{ij}).$$

Then, by (6.11), for $\ell \geqslant 1$, we obtain

$$\begin{aligned} U((\ell+1)m+1,(\ell+2)m) &= -C^\ell U(\ell m+1,(\ell+1)m) \\ &= \left[(-1)^\ell \prod_{h=1}^{\ell} C^{\ell-h+1}\right] U(m+1,2m). \end{aligned} \tag{6.12}$$

Now, we look at the elements of $C^\ell = (C^\ell_{ij})$ in detail. It is easy to see that, for $j \leqslant i$,

$$\begin{aligned} C^\ell_{ij} &= \beta_{i1}\alpha_{j-1} + \beta_{i2}\alpha_{j-2} + \cdots + \beta_{i(j-1)}\alpha_1 + \beta_{ij}(\alpha_0 + \ell m + j) \\ &= \sum_{t=1}^{j-1} \beta_{it}\alpha_{j-t} + \beta_{ij}(\alpha_0 + \ell m + j) \end{aligned}$$

and, for $j > i$,

$$C^\ell_{ij} = \beta_{i1}\alpha_{j-1} + \beta_{i2}\alpha_{j-2} + \cdots + \beta_{ii}\alpha_{j-i} = \sum_{t=1}^{i} \beta_{it}\alpha_{j-t}.$$

One can see that above the diagonal of the matrix C^ℓ none of the elements C^ℓ_{ij} include the $\alpha_0 + \ell m$ terms and that below and on the diagonal every element C^ℓ_{ij} includes only one $\alpha_0 + \ell m + j$ term.

Theorem 6.1. *Equation (3.4) has a unique polynomial solution which is given in terms of the given function α and the polynomial P^*_m. The polynomial P^*_m has to be determined from a system of linear equations which may have a unique solution, infinitely many solutions or no solutions.*

From (6.9)–(6.12) we can see that the function u can be determined uniquely in terms of α, a and g. Now, $\bar{a}(0,m)$, the coefficient of $P^*_m(z)$, has to be determined and the properties of the solution u have to be studied. We show that the higher order terms of the solution u have zero coefficient and thus $a(0,m)$ can be uniquely determined from the available equations and the finite number of additional pointwise boundary conditions where necessary. These issues are addressed by the lemmas in the following two separate cases.

6.1. Case A: $\alpha_0 + k \neq 0$ for all $k \geqslant m+1$

6.1.1. *Properties of solutions*

Lemma 6.2. *Suppose that*

$$\alpha_0 + k \neq 0, \quad k \geqslant m+1. \tag{6.13}$$

Then

$$u_k = 0, \quad k \geqslant m+1. \tag{6.14}$$

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Proof. If we denote $C^\ell C^{\ell-1}$ by $C^{(\ell,\ell-1)}$, then

$$
\begin{aligned}
C^{(\ell,\ell-1)}_{11} &= \sum_{k=1}^{m} C^{\ell}_{1k} C^{\ell-1}_{k1} \\
&= \bigg[\beta_{11}(\alpha_0+\ell m+1)+\sum_{k=1}^{m-1}\alpha_k\beta_{(k+1)1}\bigg]\beta_{11}(\alpha_0+(\ell-1)m+1).
\end{aligned}
$$

Among $C^{(\ell,\ell-1)}_{11}, C^{(\ell,\ell-1)}_{12}, \dots, C^{(\ell,\ell-1)}_{1m}$, $C^{(\ell,\ell-1)}_{11}$ is the only element that includes a second-order term of ℓ. The others only include first-order terms of ℓ due to the fact that the upper triangles of the matrices do not have any ℓ terms. So, for $C^{(\ell,\ell-1)}$ the second-order terms of ℓ do not appear in the upper triangle. They occur only in the lower triangle and along the diagonal. The second-order term of $C^{(\ell,\ell-1)}_{11}$ comes from $C^{\ell}_{11}C^{\ell-1}_{11}$. Let $C^{(\ell,\ell-1,\dots,2,1)} = C^\ell C^{\ell-1}\cdots C^2 C^1$. Then it is easy to see that the highest order term of ℓ, i.e.

$$
\begin{aligned}
&C^{\ell}_{11}C^{\ell-1}_{11}\cdots C^{2}_{11}C^{1}_{11} \\
&\qquad = \beta_{11}^{\ell}(\alpha_0+\ell m+1)(\alpha_0+(\ell-1)m+1)\cdots(\alpha_0+2m+1)(\alpha_0+m+1)
\end{aligned}
$$

is in $C^{(\ell,\ell-1,\dots,2,1)}_{11}$ and that if $u_{m+1}\neq 0$, then

$$
\begin{aligned}
u_{\ell m+1} &= C^{(\ell,\ell-1,\dots,2,1)}_{11}u_{m+1} + C^{(\ell,\ell-1,\dots,2,1)}_{12}u_{m+2} + \cdots + C^{(\ell,\ell-1,\dots,2,1)}_{1m}u_{2m} \\
&= C^{(\ell,\ell-1,\dots,2,1)}_{11}u_{m+1} \to \infty
\end{aligned}
$$

as $\ell\to\infty$. This means that $u_{m+1}=0$. Next,

$$
\begin{aligned}
C^{(\ell,\ell-1)}_{21} &= \sum_{k=1}^{m} C^{\ell}_{2k}C^{\ell-1}_{k1} \\
&= \bigg[\beta_{21}(\beta_{11}(\alpha_0+\ell m+1)+\beta_{22}(\alpha_0+\ell m+2)) \\
&\qquad + \beta_{21}^2\alpha_1 + \sum_{k=3}^{m}(\beta_{21}\alpha_{k-1}+\beta_{22}\alpha_{k-2})\beta_{31}\bigg](\alpha_0+(\ell-1)m+1)
\end{aligned}
$$

and

$$
\begin{aligned}
C^{(\ell,\ell-1)}_{22} &= \sum_{k=1}^{m} C^{\ell}_{2k}C^{\ell-1}_{k2} \\
&= \beta_{22}^2(\alpha_0+\ell m+2)(\alpha_0+(\ell-1)m+2) \\
&\qquad + \beta_{21}\alpha_1[\beta_{11}(\alpha_0+\ell m+1) \\
&\qquad\qquad + \beta_{22}(\alpha_0+(\ell-1)m+2)+\beta_{22}(\alpha_0+lm+2)] \\
&\qquad + \beta_{21}^2\alpha_1^2 + \sum_{k=3}^{m}(\beta_{21}\alpha_{k-1}+\beta_{22}\alpha_{k-2})(\beta_{k1}\alpha_1+\beta_{k2}(\alpha_0+(\ell-1)m+2)).
\end{aligned}
$$

In the element

$$
C^{(\ell,\ell-1)}_{23} = \sum_{k=1}^{m} C^{\ell}_{2k}C^{\ell-1}_{k3},
$$

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the order terms of ℓ cannot appear and therefore only first-order terms of ℓ can occur here. This is also true also for $C_{24}^{(\ell,\ell-1)}, \dots, C_{2m}^{(\ell,\ell-1)}$. Thus, again it is not difficult to see that $C_{21}^{(\ell,\ell-1,\dots,2,1)}$ has the same rate of growth ℓ as $C_{11}^{(\ell,\ell-1,\dots,2,1)}$. Now, assuming that $u_{m+2} \neq 0$ and taking into account the fact that $u_{m+1} = 0$, we have

$$\begin{aligned} u_{\ell m+2} &= C_{21}^{(\ell,\ell-1,\dots,2,1)} u_{m+1} + C_{22}^{(\ell,\ell-1,\dots,2,1)} u_{m+2} + \cdots + C_{2m}^{(\ell,\ell-1,\dots,2,1)} u_{2m} \\ &= C_{21}^{(\ell,\ell-1,\dots,2,1)} u_{m+1} + C_{22}^{(\ell,\ell-1,\dots,2,1)} u_{m+2} = C_{22}^{(\ell,\ell-1,\dots,2,1)} u_{m+2} \to \infty \end{aligned}$$

as $\ell \to \infty$. This means that $u_{m+2} = 0$. Similarly, we obtain

$$u_{m+3} = 0, \dots, u_{2m} = 0,$$

and the proof is complete. □

Clearly, $C^{(\ell,\ell-1,\dots,1)}$ has diagonal dominance for sufficiently large ℓ.

6.1.2. *Solvability and boundary conditions*

By (6.2), we have $u_0 = a_m/\alpha_m$. By lemma 6.2 and (6.6) we have

$$u_m = -\frac{\bar{a}_m}{\alpha_0 + m}$$

if $\alpha_0 + m \neq 0$. So, by (6.4), we have

$$\frac{\alpha_0}{\alpha_m} a_m - \frac{\alpha_m}{\alpha_0 + m} \bar{a}_m = (\mathrm{i}c_0 + g_0) - \alpha(1, m-1)^{\mathrm{T}} U(1, m-1), \tag{6.15}$$

where $\alpha(1, m-1)$ is the column matrix $(\alpha_1, \alpha_2, \dots, \alpha_{m-1})^{\mathrm{T}}$ and the superscript 'T' denotes the transpose of a matrix.

Substituting $U(1, m-1)$ from (6.8) into (6.15), we have

$$\frac{\alpha_0 - \alpha_\#}{\alpha_m} a_m - \frac{\alpha_m}{\alpha_0 + m} \bar{a}_m = (\mathrm{i}c_0 + g_0) - \alpha(1, m-1)^{\mathrm{T}} A(m, 2)^{-1} a(m-1, 1), \tag{6.16}$$

where $\alpha_\# = \alpha(1, m-1)^{\mathrm{T}} A(m, 2)^{-1} \alpha(m-1, 1)$, $a(m-1, 1)$ is the column matrix $(a_{m-1}, \dots, a_2, a_1)^{\mathrm{T}}$ and $A(m, 2)$ is the matrix obtained after the mth row and mth column of $A(m, 1)$ are deleted. Now, if

$$|\alpha_0 - \alpha_\#||\alpha_0 + m| \neq |\alpha_m|^2, \tag{6.17}$$

then

$$a_m = \gamma_\# \sigma_0 - \frac{\bar{\alpha}_0 - \bar{\alpha_\#}}{\bar{\alpha}_m} \tilde{\alpha}(1, m-1)^{\mathrm{T}} a(m-1, 1) - \frac{\alpha_m}{\alpha_0 + m} \bar{\tilde{\alpha}}(1, m-1)^{\mathrm{T}} \bar{a}(m-1, 1), \tag{6.18}$$

where

$$\tilde{\alpha}(1, m-1)^{\mathrm{T}} = \alpha(1, m-1)^{\mathrm{T}} A(m, 2)^{-1},$$

$$\sigma_0 = \left[\frac{\bar{\alpha}_0 - \bar{\alpha}_\#}{\bar{\alpha}_m} (\mathrm{i}c_0 + g_0) + \frac{\alpha_m}{\alpha_0 + m} (-\mathrm{i}c_0 + \bar{g}_0) \right]$$

and

$$\gamma_{\#} = \frac{|\alpha_m|^2|\alpha_0 + m|^2}{|\alpha_0 - \alpha_{\#}|^2|\alpha_0 + m|^2 - |\alpha_m|^4}.$$

Since $u_k = 0$, $k \geqslant m+1$, we get

$$g(1, m-1) - \bar{a}(1, m-1) - M(1, m-1)U(1, m-1) - \alpha(m-1, 1)u_m = 0. \quad (6.19)$$

From (6.2), (6.3), we have

$$A(m, 2)U(1, m-1) = a(m-1, 1) - \frac{a_m}{\alpha_m}\alpha(m-1, 1). \quad (6.20)$$

Substituting first $u_m = -\bar{a}_m/(\alpha_0 + m)$ into (6.19), then $U(1, m-1)$ from (6.20) and a_m from (6.18) into (6.19), we get

$$C_1 a(m-1, 1) + C_2\bar{a}(m-1, 1) = g(1, m-1) + c_*\alpha(m-1, 1), \quad (6.21)$$

where

$$c_* = \frac{\gamma_{\#}\sigma_0}{\alpha_m}C^0(1, m-1) + \frac{\bar{\gamma}_{\#}\bar{\sigma}_0}{\alpha_0 + m},$$

$$C^0(1, m-1) = M(1, m-1)A(m-1, 2)^{-1},$$

$$C_1(1, m-1) = C^0(1, m-1)\left[1 + \frac{\bar{\alpha}_0 - \bar{\alpha}_{\#}}{|\alpha_m|^2}C_0(1, m-1)\right] + \frac{\bar{\alpha}_m}{|\alpha_0 + m|^2}C_0(1, m-1),$$

$$C_2(1, m-1) = I_{\pi/2} + \left[\frac{1}{\alpha_0 + m}C^0(1, m-1) + \frac{\bar{\alpha}_0 - \bar{\alpha}_{\#}}{(\alpha_0 + m)\alpha_m}\right]C_{\#}(1, m-1).$$

In the above formulae,

$$C_0(1, m-1) = \alpha(m-1, 1)\bar{\tilde{\alpha}}(m-1, 1)^{\mathrm{T}},$$

$$C_{\#}(1, m-1) = \alpha(m-1, 1)\tilde{\alpha}(m-1, 1)^{\mathrm{T}}$$

and

$$I_{\pi/2}a(1, m-1) = a(m-1, 1).$$

If (6.17) does not hold, then

$$(\alpha_0 - \alpha_{\#})(\bar{\alpha}_0 + m) = |\alpha_m|^2$$

or

$$(\alpha_0 - \alpha_{\#})(\bar{\alpha}_0 + m) = -|\alpha_m|^2.$$

If $(\alpha_0 - \alpha_{\#})(\bar{\alpha}_0 + m) = |\alpha_m|^2$, then equation (6.16) becomes

$$\mathrm{Im}\left[\frac{\bar{\alpha}_m}{\bar{\alpha}_0 + m}a_m\right] = \frac{1}{2\mathrm{i}}[(\mathrm{i}c_0 + g_0) - \tilde{\alpha}(1, m-1)^{\mathrm{T}}a(m-1, 1)] \quad (6.22)$$

and a compatibility condition

$$\mathrm{Re}[g_0] = \mathrm{Re}[\tilde{\alpha}(1, m-1)^{\mathrm{T}} a(m-1, 1)] \tag{6.23}$$

must be satisfied.

Both $\mathrm{Im}\, a_m$ and $\mathrm{Re}\, a_m$ must be fixed in terms of the right-hand sides. Thus, an additional half-boundary condition on the real part or on the imaginary part has to be imposed. Suppose that

$$\mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \left[\frac{\partial v}{\partial \nu} + \alpha(\zeta) v\right] \zeta^m \frac{\mathrm{d}\zeta}{\zeta}\right) = h_m. \tag{6.24}$$

Then we get $\mathrm{Re}\, a_m = h_m^0$, where $h_m^0 = h_m - h_m^*$ and

$$h_m^* = \mathrm{Re}\left(\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \left[\frac{\partial v_0}{\partial \nu} + \alpha(\zeta) v_0\right] \zeta^m \frac{\mathrm{d}\zeta}{\zeta}\right).$$

Substituting $\mathrm{Re}\, a_m$ into (6.22), we get

$$\begin{aligned}(\mathrm{Im}\, a_m)\left(\mathrm{Re}\left(\frac{\alpha_m}{\alpha_0 + m}\right)\right) \\ = \left\{\frac{1}{2\mathrm{i}}[(\mathrm{i}c_0 + g_0) - \tilde{\alpha}(1, m-1)^{\mathrm{T}} a(m-1, 1)] + \mathrm{Im}\left(\frac{\alpha_m}{\alpha_0 + m}\right) h_m^0\right\}.\end{aligned}$$

Now, a_m is slightly different from the a_m in (6.18), but equations (6.19), (6.20) are unchanged and so we can again obtain an equation for $a(m-1, 1)$ similar to (6.21).

If

$$(\alpha_0 - \alpha_{\#})(\bar{\alpha}_0 + m) = -|\alpha_m|^2,$$

then equation (6.16) becomes

$$\mathrm{Re}\left[\frac{\bar{\alpha}_m}{\bar{\alpha}_0 + m} a_m\right] = -\tfrac{1}{2}[(\mathrm{i}c_0 + g_0) - \tilde{\alpha}(1, m-1)^{\mathrm{T}} a(m-1, 1)] \tag{6.25}$$

and the compatibility condition

$$c_0 + \mathrm{Im}[g_0] = \mathrm{Im}[\tilde{\alpha}(1, m-1)^{\mathrm{T}} a(m-1, 1)] \tag{6.26}$$

must be satisfied.

The treatment of the rest of (6.25) is similar to the above case (6.22) and this time an additional half-boundary condition on the imaginary part can be imposed. This again leads to an equation similar to (6.21).

If $\alpha_0 + m = 0$, then $a_m = 0$ and $u_0 = 0$. As in the case $\alpha_0 + m \neq 0$, we again obtain equation (6.21) with $c_* = -(\mathrm{i}c_0 + g_0)/\alpha_m$,

$$C_1 = M(1, m-1)A(m, 2)^{-1} - \frac{1}{\alpha_m}\alpha(m-1, 1)\alpha(1, m-1)^{\mathrm{T}} A(m, 2)^{-1}$$

and

$$C_2 = I_{\pi/2}.$$

Thus, in all the cases above we can uniquely determine a_m either by means of the available equations or by imposing an additional half-boundary condition. For the latter case some compatibility conditions may apply. The remaining task is to determine $a(1, m-1)$ from (6.21). This means that we have to determine $2(m-1)$ real unknowns from a system of $2(m-1)$ real equations. This procedure is actually similar to the case of a_m. The difference is that if all $2(m-1)$ real unknowns cannot be determined by the available equations, then we need to determine some real parts and some imaginary parts by imposing some half-boundary conditions and the same number of compatibility conditions attached to the problem. To wit, suppose that we need to determine $\operatorname{Re} a_{\sigma_1}, \dots, \operatorname{Re} a_{\sigma_t}$ and $\operatorname{Im} a_{\mu_1}, \dots, \operatorname{Im} a_{\mu_\tau}$, $1 \leqslant \sigma_t, \mu_\tau \leqslant m-1$. Then we need to impose

$$\operatorname{Re}\left(\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \left[\frac{\partial v}{\partial \nu} + \alpha(\zeta) v\right] \zeta^{\sigma_j} \frac{\mathrm{d}\zeta}{\zeta}\right) = h_{\sigma_j}, \quad j = 1, 2, \dots, t, \tag{6.27}$$

and

$$\operatorname{Im}\left(\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \left[\frac{\partial v}{\partial \nu} + \alpha(\zeta) v\right] \zeta^{\mu_l} \frac{\mathrm{d}\zeta}{\zeta}\right) = h_{\mu_l}, \quad \tau = 1, 2, \dots, \tau. \tag{6.28}$$

Together with these $t + \tau$ half-boundary conditions we have the same number of compatibility conditions.

The last number we have to fix is $\mathrm{i}c_0 = a_0 - \bar{a}_0$. Therefore, we need

$$\operatorname{Im}\left(\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \left[\frac{\partial v}{\partial \nu} + \alpha(\zeta) v\right] \frac{\mathrm{d}\zeta}{\zeta}\right) = h_0. \tag{6.29}$$

Then we obtain $c_0 = h_0^0$, where $h_0^0 = h_0 - h_0^*$ and

$$h_0^* = \operatorname{Im}\left(\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \left[\frac{\partial v_0}{\partial \nu} + \alpha(\zeta) v_0\right] \frac{\mathrm{d}\zeta}{\zeta}\right).$$

6.2. Case B: $\boldsymbol{\alpha_0 + k_0 = 0}$ for some $\boldsymbol{k_0 \geqslant m + 1}$

LEMMA 6.3. *Suppose that*

$$\alpha_0 + k_0 = 0, \quad k_0 \geqslant m + 1. \tag{6.30}$$

Then

$$u_k = 0, \quad k \geqslant m + 1, \quad k \neq m + k_0^*, \tag{6.31}$$

where $k_0^* \neq 0$ *and*

$$k_0^* = k_0 \bmod m.$$

If $k_0^* = 0$*, then* $k \neq m + k_0^*$ *must be replaced by* $k \neq 2m$*.*

Proof. Without loss of generality, we assume that $0 \leqslant k_0^* \leqslant m - 1$. Then there exists a non-negative integer t such that $k_0 = tm + k_0^*$. As in case A, by (6.12) we obtain

$$u_{m+1} = 0, u_{m+2} = 0, \dots, u_{m+k_0^*-1} = 0.$$

Of particular interest is the term $u_{m+k_0^*}$. A distinct feature here (and not in case A) is that the term $C_{k_0^* k_0^*}^{(t,t-1)}$ does not give a second-order term of t. Now, by straightforward but tedious computations, we can write

$$C_{k_0^* k_0^*}^{(t,t-1)} = A(\alpha_0 + (t-1)m + k_0^*) + B$$

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for some constants A and B which do not depend on t. For case B, the highest order term of t in $C^{(\ell,\ell-1,\dots,t,t-1,\dots,2,1)}_{k_0^*k_0^*}$ not only lacks the term $\beta_{k_0^*k_0^*}(\alpha_0+tm+k_0^*)$ as in case A, but it can be zero due to the linear term $C^{(t,t-1)}_{k_0^*k_0^*}$. Therefore, we may not get $u_{m+k_0^*}=0$ and $u_{\ell m+k_0^*}=0$ immediately as in case A. However, looking at the equations for $u_{\ell m+k_0^*+1},\dots,u_{\ell m+m}$, we see that the main term, i.e. the term on the diagonal, has the dominant part, which is the highest order term of ℓ. So, in the same way as in case A, we get

$$u_{m+k_0^*+1}=u_{\ell m+k_0^*+1}=0,\dots,u_{2m}=u_{\ell m+m}=0.$$

Now, at this stage, the equation for $u_{\ell m+k_0^*}$ is simplified to

$$C^{(\ell,\ell,\dots,2,1)}_{k_0^*k_0^*}u_{m+k_0^*}=u_{\ell m+k_0^*}$$

and if $A(\alpha_0+(t-1)m+k_0^*)+B=0$, then we get

$$u_{\ell m+k_0^*}=0.$$

If $A(\alpha_0+(t-1)m+k_0^*)+B\neq 0$, then the highest order term of ℓ in $C^{\ell,\ell-1,\dots,2,1}$ goes to ∞ as $\ell\to\infty$. Thus,

$$u_{\ell m+k_0^*}=0.$$

This completes the proof. □

From lemma 6.3, it is clear that $u_{m+k_0^*}$ is the only term that may not be 0 and the remaining analysis is similar to that in case A.

7. Solution for an essential singularity

We begin by rewriting α^- in the form

$$\alpha^-(\zeta)=\alpha_0+\zeta\alpha_1^-(\zeta),\quad \zeta\in\partial\mathbb{D},$$

where

$$\alpha_1^-(\zeta)=\sum_{k=1}^{\infty}\alpha_k\zeta^{-k-1},\quad \zeta\in\partial\mathbb{D}.$$

By changing the function u to w by means of

$$w(z)=u(z)\exp\biggl\{\int_\infty^z\alpha_1^-(\zeta)\,\mathrm{d}\zeta\biggr\},\quad z\in\mathbb{D}\setminus\{0\},$$

equation (3.5) becomes

$$zw'(z)+\alpha_0w(z)=F(z),\quad z\in\mathbb{D}\setminus\{0\},\tag{7.1}$$

where

$$F(z)=H_*(z)+G_*(z),\quad z\in\mathbb{D}\setminus\{0\},$$
$$H_*(z)=\biggl\{\overline{\alpha^*\biggl(\frac{1}{\bar z}\biggr)}-\alpha^*(z)\biggr\}\exp\biggl\{-\int_\infty^z\alpha_1^-(\zeta)\,\mathrm{d}\zeta\biggr\},\quad z\in\mathbb{D}\setminus\{0\},$$

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and

$$G_*(z) = G(z)\exp\biggl\{-\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta\biggr\}, \quad z \in \mathbb{D}\setminus\{0\}.$$

Write

$$H_*(z) = \sum_{k=-\infty}^{\infty} h_k z^k, \quad z \in \mathbb{D}\setminus\{0\},$$

$$G_*(z) = \sum_{k=-\infty}^{\infty} g_k^* z^k, \quad z \in \mathbb{D}\setminus\{0\},$$

$$F(z) = \sum_{k=-\infty}^{\infty} f_k z^k, \quad z \in \mathbb{D}\setminus\{0\},$$

and from (7.1),

$$w(z) = \sum_{k=-\infty}^{\infty} w_k z^k, \quad z \in \mathbb{D}\setminus\{0\}.$$

Let H_*^+, H_*^-, G_*^+, G_*^-, F^+, F^-, w^+ and w^- be functions defined by

$$H_*^+(z) = \sum_{k=0}^{\infty} h_k z^k, \qquad z \in \mathbb{D},$$

$$H_*^-(z) = \sum_{k=1}^{\infty} h_k z^{-k}, \qquad z \in \mathbb{D}\setminus\{0\},$$

$$G_*^+(z) = \sum_{k=0}^{\infty} g_k^* z^k, \qquad z \in \mathbb{D},$$

$$G_*^-(z) = \sum_{k=1}^{\infty} g_{-k}^* z^{-k}, \quad z \in \mathbb{D}\setminus\{0\},$$

$$F^+(z) = \sum_{k=0}^{\infty} f_k z^k, \qquad z \in \mathbb{D},$$

$$F^-(z) = \sum_{k=1}^{\infty} f_{-k} z^{-k}, \quad z \in \mathbb{D}\setminus\{0\},$$

$$w^+(z) = \sum_{k=0}^{\infty} w_k z^k, \qquad z \in \mathbb{D},$$

and

$$w^-(z) = \sum_{k=1}^{\infty} w_{-k} z^{-k}, \quad z \in \mathbb{D}\setminus\{0\}.$$

Then equation (7.1) becomes

$$zw^{+\prime}(z) + \alpha_0 w^+(z) = F^+(z), \quad z \in \mathbb{D}, \tag{7.2}$$

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and

$$zw^{-\prime}(z) + \alpha_0 w^-(z) = F^-(z), \quad z \in \mathbb{D} \setminus \{0\}, \tag{7.3}$$

where

$$F^+(z) = H_*^+(z) + G_*^+(z), \quad z \in \mathbb{D},$$
$$F^-(z) = H_*^-(z) + G_*^-(z), \quad z \in \mathbb{D} \setminus \{0\}.$$

Since equations (7.2) and (7.3) are different in character, we treat them separately in the following subsections.

7.1. The homogeneous problem

In this subsection, we assume that

$$F(z) = 0, \quad z \in \mathbb{D} \setminus \{0\}.$$

Furthermore, we assume that

$$G(z) = 0, \quad z \in \mathbb{D} \setminus \{0\},$$

in (7.1) and we can solve the corresponding homogeneous equation.

7.1.1. *The singular equation*

Lemma 7.1. *Let w^- be holomorphic on $\mathbb{D} \setminus \{0\}$ and satisfy*

$$zw^{-\prime}(z) + \alpha_0 w^-(z) = 0, \quad z \in \mathbb{D} \setminus \{0\}. \tag{7.4}$$

If w^- is not identically zero, then α_0 must be an integer.

Proof. Let $\partial\mathbb{D}$ be positively oriented. By (7.4), we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \frac{w^{-\prime}(z)}{w^-(z)}\, \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \frac{-\alpha_0}{z}\, \mathrm{d}z.$$

Therefore,

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \frac{w^{-\prime}(z)}{w^-(z)}\, \mathrm{d}z = -\alpha_0.$$

Thus, by the argument principle, α_0 is an integer. □

Lemma 7.2. *If α_0 is a negative integer and satisfies (7.4), then*

$$w^-(z) = 0, \quad z \in \mathbb{D} \setminus \{0\}.$$

Proof. If α_0 is a negative integer, then by (7.4) we have

$$(z^{\alpha_0} w^-(z))' = \alpha_0 z^{\alpha_0 - 1} w^-(z) + z^{\alpha_0} w^{-\prime}(z) = z^{\alpha_0 - 1}(\alpha_0 w^-(z) + zw^{-\prime}(z)) = 0$$

for all z in $\mathbb{D} \setminus \{0\}$. This means that $z^{\alpha_0} w^-(z)$ is equal to a constant for all z in $\mathbb{D} \setminus \{0\}$. Since

$$w^-(z) = \sum_{k=1}^{\infty} w_{-k} z^{-k}, \quad z \in \mathbb{D} \setminus \{0\},$$

the proof is complete. □

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We need three additional lemmas. Since the proofs are straightforward, we omit them.

LEMMA 7.3. *Let α_0 be a non-negative integer satisfying (7.4). Then*

$$w^-(z) = Cz^{-\alpha_0}, \quad z \in \mathbb{D} \setminus \{0\},$$

where C is an arbitrary complex constant.

LEMMA 7.4. *If $\alpha_0 - k \neq 0, k \in \mathbb{N}$, then*

$$w^-(z) = \sum_{k=1}^{\infty} \frac{h_{-k}}{\alpha_0 - k} z^{-k}, \quad z \in \mathbb{D} \setminus \{0\},$$

is a solution to the differential equation

$$zw^{-\prime}(z) + \alpha_0 w^-(z) = H_*^-(z), \quad z \in \mathbb{D} \setminus \{0\}. \tag{7.5}$$

LEMMA 7.5. *Suppose that there exists a non-negative integer k_0 such that $\alpha_0 - k_0 = 0$ and w^- is holomorphic on $\mathbb{D} \setminus \{0\}$ satisfying*

$$zw^{-\prime}(z) + \alpha_0 w^-(z) = C_0 z^{-k_0}, \quad z \in \mathbb{D}, \tag{7.6}$$

where C_0 is a constant. Then $C_0 = 0$ and hence

$$w^-(z) = Cz^{-k_0}, \quad z \in \mathbb{D} \setminus \{0\},$$

where C is an arbitrary constant.

7.1.2. *The non-singular equation*

We need five preliminary lemmas before the formulation of theorems 7.11 and 7.12. They are analogues of the lemmas from § 7.1.1.

LEMMA 7.6. *Let w^+ be holomorphic on $\mathbb{D}$ and satisfy*

$$zw^{+\prime}(z) + \alpha_0 w^+(z) = 0, \quad z \in \mathbb{D}. \tag{7.7}$$

If w^+ is not identically zero, then α_0 must be an integer.

LEMMA 7.7. *If α_0 is a positive integer satisfying (7.7), then*

$$w^+(z) = 0, \quad z \in \mathbb{D}.$$

LEMMA 7.8. *Let α_0 be a non-positive integer satisfying (7.7). Then*

$$w^+(z) = Cz^{-\alpha_0}, \quad z \in \mathbb{D},$$

where C is an arbitrary complex constant.

LEMMA 7.9. *If $\alpha_0 + k \neq 0$, $k \in \mathbb{N} \cup \{0\}$, then*

$$w^+(z) = \sum_{k=0}^{\infty} \frac{h_k}{\alpha_0 + k} z^k, \quad z \in \mathbb{D},$$

is a solution to the differential equation

$$zw^{+\prime}(z) + \alpha_0 w^+(z) = H_*^+(z), \quad z \in \mathbb{D}. \tag{7.8}$$

LEMMA 7.10. *Suppose that there exists a non-negative integer k_0 such that $\alpha_0+k_0 = 0$ and w^+ is a holomorphic function on $\mathbb{D}$ satisfying*

$$zw^{+\prime}(z) + \alpha_0 w^+(z) = C_0 z^{k_0}, \quad z \in \mathbb{D}, \tag{7.9}$$

where C_0 is a constant. Then $C_0 = 0$ and hence

$$w^+(z) = Cz^{k_0}, \quad z \in \mathbb{D},$$

where C is an arbitrary constant.

From the preceding lemmas, we have the following theorems.

THEOREM 7.11.

(i) *If $\alpha_0 \notin \mathbb{Z}$, then the homogeneous equation (3.5) with $G(z) = 0$ has a non-trivial solution u given by*

$$u(z) = \exp\left\{-\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta\right\}\left[\sum_{k=0}^\infty \frac{h_k}{\alpha_0+k} z^k + \sum_{k=1}^\infty \frac{h_{-k}}{\alpha_0-k} z^{-k}\right], \quad z \in \mathbb{D}\setminus\{0\}.$$

(ii) *If there exists a non-negative integer k_0 such that $\alpha_0 + k_0 = 0$, then we have the following two cases.*

(a) *If $h_{-\alpha_0} \neq 0$, then the homogeneous equation (3.5) with $G(z) = 0$ and $H_*^+(z) = 0$ has a non-trivial solution u given by*

$$u(z) = \exp\left\{-\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta\right\}\left[Cz^{-\alpha_0} + \sum_{k=1}^\infty \frac{h_k}{\alpha_0-k} z^{-k}\right], \quad z \in \mathbb{D}\setminus\{0\},$$

where C is an arbitrary complex constant.

(b) *If $h_{-\alpha_0} = 0$, then equation (3.5) has a non-trivial solution u given by*

$$u(z) = \exp\left\{-\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta\right\} \times \left[Cz^{-\alpha_0} + \sum_{k\geqslant 0, k\neq -\alpha_0} \frac{h_k}{\alpha_0+k} z^k + \sum_{k=1}^\infty \frac{h_{-k}}{\alpha_0-k} z^{-k}\right]$$

for all z in $\mathbb{D}\setminus\{0\}$, where C is an arbitrary constant.

THEOREM 7.12. *If there exists a positive integer k_0 such that $\alpha_0 - k_0 = 0$, then we have the following two cases.*

(i) *If $h_{-\alpha_0} \neq 0$, then equation (3.5) has a non-trivial solution u given by*

$$u(z) = \exp\left\{-\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta\right\}\left[Cz^{-\alpha_0} + \sum_{k=0}^\infty \frac{h_k}{\alpha_0+k} z^k\right], \quad z \in \mathbb{D}\setminus\{0\},$$

where C is an arbitrary complex constant.

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(ii) *If $h_{-\alpha_0} = 0$, then equation (3.5) has a non-trivial solution u given by*

$$u(z) = \exp\biggl\{ -\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta \biggr\} \biggl[Cz^{-\alpha_0} + \sum_{k\geqslant 1, k\neq\alpha_0} \frac{h_{-k}}{\alpha_0 - k} z^{-k} + \sum_{k=0}^{\infty} \frac{h_k}{\alpha_0 + k} z^k \biggr]$$

for all z in $\mathbb{D}\setminus\{0\}$, where C is an arbitrary constant.

Remark 7.13. If

$$\exp\biggl\{ -\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta \biggr\}$$

is replaced by

$$\exp\biggl\{ -\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta \biggr\} - \int_0^z \frac{\alpha^+(\zeta)}{\zeta}\,\mathrm{d}\zeta,$$

then the above two theorems are also true for equation (4.5) with $G(z) = 0$, $z \in \mathbb{D}\setminus\{0\}$.

7.2. The inhomogeneous problem

It is an elementary fact that if we can find a special solution to equation (7.1), then we have the general solution to equation (3.5).

Theorem 7.14. (i) *If $\alpha_0 \notin \mathbb{Z}$, then the inhomogeneous equation (3.5) is solvable and the solution u is given by*

$$u(z) = \exp\biggl\{ -\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta \biggr\} \biggl[\sum_{k=0}^{\infty} \frac{h_k + g_k^*}{\alpha_0 + k} z^k + \sum_{k=1}^{\infty} \frac{h_{-k} + g_{-k}^*}{\alpha_0 - k} z^{-k} \biggr]$$

for all z in $\mathbb{D}\setminus\{0\}$.

(ii) *If there exists a non-negative integer k_0 such that $\alpha_0 + k_0 = 0$, then the inhomogeneous equation (3.5) is solvable if and only if*

$$h_{-\alpha_0} + g_{-\alpha_0}^* = 0.$$

If this condition is satisfied, then the solution u to equation (3.5) is given by

$$u(z) = \exp\biggl\{ -\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta \biggr\} \times \biggl[Cz^{-\alpha_0} + \sum_{k\geqslant 0, k\neq -\alpha_0} \frac{h_k + g_k^*}{\alpha_0 + k} z^k + \sum_{k=1}^{\infty} \frac{h_{-k} + g_{-k}^*}{\alpha_0 - k} z^{-k} \biggr]$$

for all z in $\mathbb{D}\setminus\{0\}$, where C is an arbitrary constant.

(iii) *If there exists a positive integer k_0 such that $\alpha_0 - k_0 = 0$, then the inhomogeneous equation (3.5) is solvable if and only if $h_{\alpha_0} + g_{-\alpha_0}^* = 0$. If this condition is satisfied, then the solution u to the inhomogeneous equation (3.5) is given* *by*

$$u(z) = \exp\left\{ -\int_\infty^z \alpha_1^-(\zeta)\,\mathrm{d}\zeta \right\} \times \left[Cz^{-\alpha_0} + \sum_{k=1,k\neq\alpha_0}^{\infty} \frac{h_{-k}+g^*_{-k}}{\alpha_0 - k} z^{-k} + \sum_{k=0}^{\infty} \frac{h_k + g^*_k}{\alpha_0 + k} z^k \right]$$

for all z in $\mathbb{D}\setminus\{0\}$, where C is an arbitrary constant.

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