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ON PROBLEMS OF $\mathcal{CF}$-CONNECTED GRAPHS FOR $K_{m,n}$

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Abstract

A connected graph G is $\mathcal{CF}$-connected if there is a path between every pair of vertices with no crossing on its edges for each optimal drawing of G. We conjecture that a complete bipartite graph $K_{m,n}$ is $\mathcal{CF}$-connected if and only if it does not contain a subgraph of $K_{3,6}$ or $K_{4,4}$. We establish the validity of this conjecture for all complete bipartite graphs $K_{m,n}$ for any m, n with $\min\{m, n\} \leq 6$, and conditionally for $m, n \geq 7$ on the assumption of Zarankiewicz's conjecture that $\mathrm{cr}(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

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1. Introduction

Our initial intention was to describe a family of graphs for which it is possible to find an optimal drawing so that after removing the crossed edges we get a disconnected subgraph. For this reason we will only deal with finite connected graphs. In the search for this family of graphs, the importance of various structural properties of the complete bipartite graphs $K_{3,6}$ and $K_{4,4}$ gradually became apparent: $K_{3,6}$ and $K_{4,4}$ are the first of the complete bipartite graphs for which it is possible to achieve a disconnected subgraph induced on the uncrossed edges of some of the optimal drawings. This led us to a new conjecture that a complete bipartite graph $K_{m,n}$ is $\mathcal{CF}$-connected if and only if it does not contain a subgraph of $K_{3,6}$ or $K_{4,4}$. A connected graph is $\mathcal{CF}$-connected if there is a path between every pair of vertices with no crossing on its edges for each optimal drawing of the graph. The problem of reducing the number of crossings on the edges in the drawings of graphs has many applications, the most prominent being VLSI technology. This explains why we study $\mathcal{CF}$-connectedness of graphs only for drawings with the smallest number of crossings, that is, optimal drawings.

Determining the crossing number of the complete bipartite graph $K_{m,n}$ is one of the oldest open crossing number problems. The crossing number, $\mathrm{cr}(K_{m,n})$, of $K_{m,n}$ is bounded above by $Z(m, n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. Zarankiewicz [9] conjectured that $\mathrm{cr}(K_{m,n}) = Z(m,n)$. Recently, this conjecture was proved for all positive integers m, n with $\min\{m,n\} \leq 6$ by Kleitman [5], and Norin and Zwols [7] showed that the conjecture is 'asymptotically at least 90.5% true'. Christian *et al.* [3] showed that, for each fixed integer $m \geq 3$, it is a finite problem to decide whether or not Zarankiewicz's conjecture holds for $K_{m,n}$ for every $n \geq m$. We show that our conjecture follows from Zarankiewicz's conjecture. It would certainly be interesting to know whether these two conjectures are equivalent.

In Section 3, Theorem 3.2 offers quite surprising conclusions about the behaviour of optimal drawings of the graphs obtained by removing one vertex from the complete bipartite graph $K_{m,n}$ (the vertex is removed from the partition with an even number of vertices). The new concept of a crossing sequence will be strongly used in the proofs. Ideas connected with finite integer sequences on a bipartite graph have also been used by Cairns *et al.* [2].

2. Definitions and preliminary results

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings in a drawing of G in the plane. (For the definition of a *drawing*, see [6].) It is easy to see that a drawing with the minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once and no two edges incident with the same vertex cross.

Let D be an optimal drawing of the graph $G = (V, E)$ with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $\mathrm{cr}_D(v_i)$, $i = 1, \ldots, n$, denote the number of crossings on the edges which are incident with the fixed vertex v_i. Since every optimal drawing is a good drawing, each crossing in D is counted on two edges with four vertices at their ends. This means that

$$\sum_{i=1}^{n} \mathrm{cr}_D(v_i) = 4\,\mathrm{cr}_D(G), \tag{2.1}$$

where $\mathrm{cr}_D(G)$ denotes the number of crossings in D. The *crossing sequence* $d_{D(G)}$ of the graph G in the drawing D is the nonincreasing sequence of its vertex crossings $\mathrm{cr}_D(v_i)$. The crossing sequence is an invariant of the drawing of the graph, that is, two isomorphic drawings of a graph have the same crossing sequence. However, the crossing sequence does not, in general, uniquely identify a drawing of a graph. In some cases, nonisomorphic drawings of the same graph have the same crossing sequence. For example, there are two nonisomorphic optimal drawings of the complete bipartite graph $K_{3,4}$. In the proof of Theorem 3.2, we show that just two different crossing sequences can be obtained for all optimal drawings of the graph $K_{3,4}$.

It will be useful to introduce crossing subsequences as well. Let $K_{m,n}$ be the complete bipartite graph with partitions of sizes $|V_1| = m$ and $|V_2| = n$. In the rest of the paper, the vertices of V_1 and of V_2 will be denoted by u_i and v_j for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, respectively. The *first crossing subsequence* $d_{D(K_m)}$ of the graph $K_{m,n}$ in

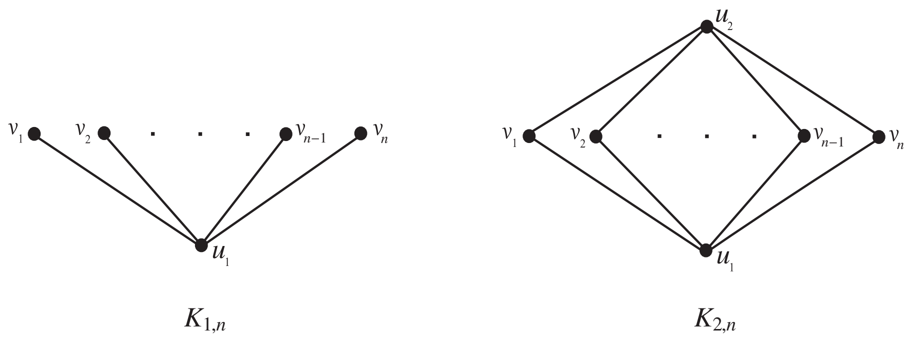

FIGURE 1. Planar drawings of the complete bipartite graphs $K_{1,n}$ and $K_{2,n}$.

the drawing D is the nonincreasing sequence of its vertex crossings $\mathrm{cr}_D(u_i)$ for $u_i \in V_1$. The *second crossing subsequence* $d_{D(K_n)}$ of $K_{m,n}$ is defined by the vertex crossings $\mathrm{cr}_D(v_j)$. Since each edge of the complete bipartite graph $K_{m,n}$ is given by $e = u_i v_j$ for some $u_i \in V_1$ and $v_j \in V_2$, the property (2.1) can be easily adjusted to

$$\sum_{i=1}^{m} \mathrm{cr}_D(u_i) = 2\,\mathrm{cr}_D(K_{m,n}) = \sum_{j=1}^{n} \mathrm{cr}_D(v_j). \tag{2.2}$$

For any optimal drawing D of $G = (V, E)$, let us denote by $CF_{D(G)}$ the subgraph of G with the vertex set $V(G)$ and the edge set $\{e \in E(G) : \mathrm{cr}_D(G) = \mathrm{cr}_D(G \setminus e)\}$. A connected graph G is said to be *$\mathcal{CF}$-connected* if the subgraph $CF_{D(G)}$ is connected for each optimal drawing D of G. Equivalently, a connected graph G is $\mathcal{CF}$-connected if there is a path between every pair of vertices with no crossing on its edges for each optimal drawing D of G. The complete bipartite graphs $K_{1,n}$ and $K_{2,n}$ are $\mathcal{CF}$-connected because they have planar drawings for each $n \geq 1$ (see Figure 1).

3. The complete bipartite graphs $K_{3,n}$ for $3 \leq n \leq 5$

The graph $K_{3,3}$ is nonplanar and it has a unique optimal drawing up to homeomorphism.

THEOREM 3.1 (Bokal and Leaños [1]). *There is only one optimal drawing of $K_{3,3}$.*

THEOREM 3.2. *Let D be any optimal drawing of the complete bipartite graph $K_{m,n}$ for n even and with $\min\{m, n\} \leq 6$. For any vertex v_j with $j = 1, \ldots, n$, the subdrawing induced by D of the subgraph $K_{m,n} \setminus v_j$ obtained by removing v_j from $K_{m,n}$ is also an optimal drawing of $K_{m,n-1}$.*

PROOF. Let D be any optimal drawing of the graph $K_{3,4}$, that is, $\mathrm{cr}_D(K_{3,4}) = 2$. Since $\mathrm{cr}(K_{3,3}) = 1$, the crossing subsequence $d_{D(K_4)}$ consists of values at most one. Otherwise, by deleting the vertex v_j of the partition on four vertices with $\mathrm{cr}_D(v_j) > 1$, a drawing of the graph homeomorphic to $K_{3,3}$ with no crossing is obtained. Further,

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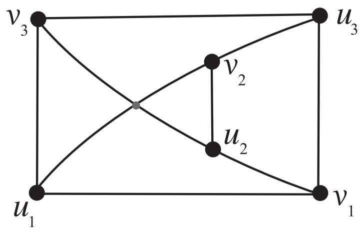

FIGURE 2. Optimal drawing of the complete bipartite graph $K_{3,3}$.

by (2.2) in the form $\sum_{j=1}^{4} \mathrm{cr}_D(v_j) = 4$, none of the values in the subsequence $d_{D(K_4)}$ can be less than one. So, $d_{D(K_4)} = (1, 1, 1, 1)$ and it follows that there is exactly one crossing on the edges incident with each vertex v_j of the partition on four vertices in the drawing D. The proof proceeds in a similar way for the remaining cases and shows that all members of the second crossing subsequence $d_{D(K_n)}$ are equal to $\frac{n-2}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$. □

Theorem 3.2 does not apply for n odd because of the optimal drawing of the graph $K_{3,3}$ in Figure 2 and since $\mathrm{cr}(K_{3,3}) - \mathrm{cr}(K_{3,2}) = 1$. The same holds for the graph $K_{4,3}$ using the drawings of $K_{4,3}$ in Figure 3 and $\mathrm{cr}(K_{4,3}) - \mathrm{cr}(K_{4,2}) = 2$.

COROLLARY 3.3. *There are two nonisomorphic optimal drawings of $K_{3,4}$.*

PROOF. By Theorem 3.2, all optimal drawings of $K_{3,4}$ can be achieved by adding a new vertex v_j with three corresponding edges in some region of the optimal drawing of $K_{3,3}$ as shown in Figure 2 with two vertices u_i and u_k of the subgraph $K_{3,3}$ on its boundary. It follows that no edge of the graph $K_{3,4}$ is crossed by the edges u_iv_j and u_kv_j. Based on the symmetry of the regions, there are only $2 \times 3 = 6$ possible cases, of which only two are nonisomorphic to each other. These two nonisomorphic drawings are shown in Figure 3, along with their crossing sequences. □

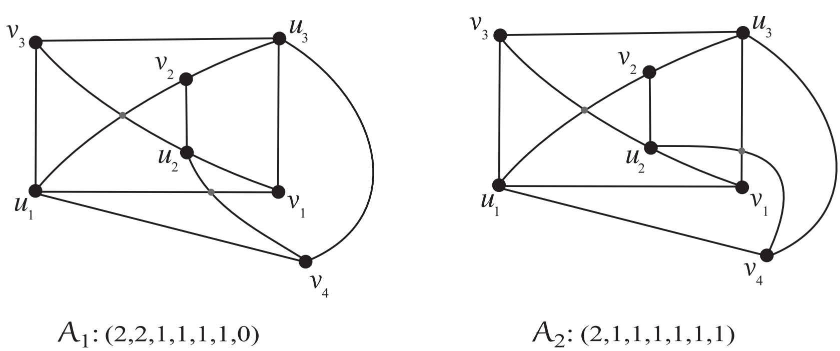

FIGURE 3. Two nonisomorphic optimal drawings of the complete bipartite graph $K_{3,4}$.

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For all optimal drawings of the graphs $K_{3,n}$, $n = 3, 4$, presented in Figures 2 and 3, it is not difficult to verify that the corresponding subgraph $CF_{D(K_{3,n})}$ is connected. So, the complete bipartite graphs $K_{3,3}$ and $K_{3,4}$ are $\mathcal{CF}$-connected.

THEOREM 3.4. *The graph $K_{3,5}$ is $\mathcal{CF}$-connected.*

PROOF. Let D be an optimal drawing of the graph $K_{3,5}$ for which the subgraph $CF_{D(K_{3,5})}$ is disconnected. Since each vertex u_i is adjacent to five different vertices and $\mathrm{cr}_D(K_{3,5}) = 4$, there is no isolated vertex u_i in such a subgraph $CF_{D(K_{3,5})}$. As $\mathrm{cr}(K_{3,4}) = 2$, there is also no isolated vertex v_j in $CF_{D(K_{3,5})}$. Otherwise, by deleting the vertex v_j of the partition on five vertices with $\mathrm{cr}_D(v_j) \geq 3$, a drawing of the graph homeomorphic to $K_{3,4}$ with at most one crossing is obtained. It follows that the second crossing subsequence $d_{D(K_5)}$ consists of values at most two, but not less than one, provided by a certain disconnection of $CF_{D(K_{3,5})}$. Consequently, by (2.2) in the form $\sum_{j=1}^{5} \mathrm{cr}_D(v_j) = 8$, this crossing subsequence must be uniquely determined by $d_{D(K_5)} = (2, 2, 2, 1, 1)$.

Without lost of generality, let $\mathrm{cr}_D(v_j) = 2$, for $j = 1, 2, 3$, and $\mathrm{cr}_D(v_4) = \mathrm{cr}_D(v_5) = 1$. Due to the disconnection of $CF_{D(K_{3,5})}$, the three vertices u_1, u_2 and u_3 cannot all be in the same component. So, in some component of $CF_{D(K_{3,5})}$, the vertices v_4 and v_5 are adjacent to two of these vertices. Again without loss of generality, let these be the vertices u_1 and u_2. Since $\mathrm{cr}_D(v_j) = 2$ for $j = 1, 2, 3$, the drawing D' induced by D and obtained by removing the vertex v_j from $K_{3,5}$ is an optimal drawing of $K_{3,4}$. Using the idea from the proof of Theorem 3.2, its second crossing subsequence has the form $d_{D'(K_4)} = (1, 1, 1, 1)$. It follows that the edges v_4u_3 and v_5u_3 cross in this good drawing D' of $K_{3,4}$ and thus also in the optimal drawing D of $K_{3,5}$. This contradiction completes the proof. □

4. Non-$\mathcal{CF}$-connected graphs

Figure 4 shows an optimal drawing D of the complete bipartite graph $K_{3,n}$ for $n \geq 6$. This drawing D forces a disconnected subgraph $CF_{D(K_{3,n})}$, which makes the next result obvious.

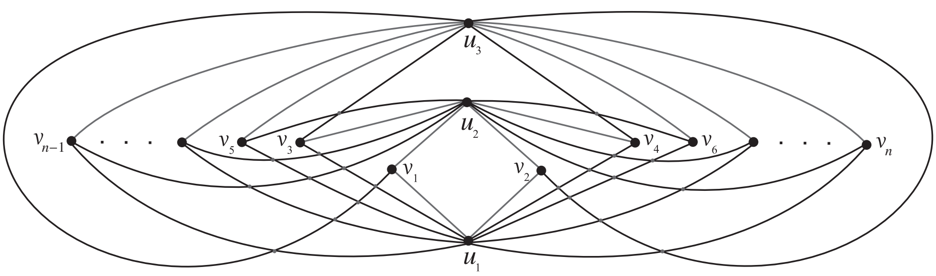

FIGURE 4. Optimal drawing of the graph $K_{3,n}$ for any $n \geq 6$.

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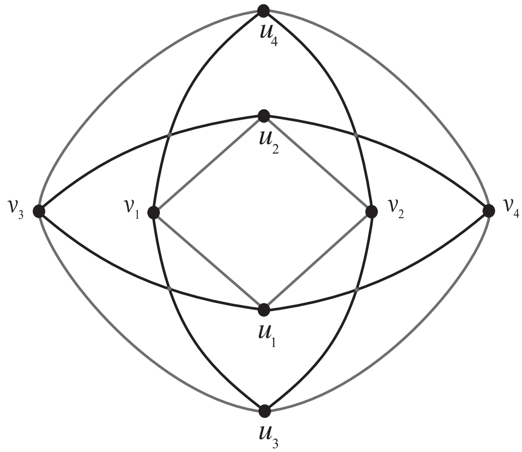

FIGURE 5. Optimal drawing of the graph $K_{4,4}$.

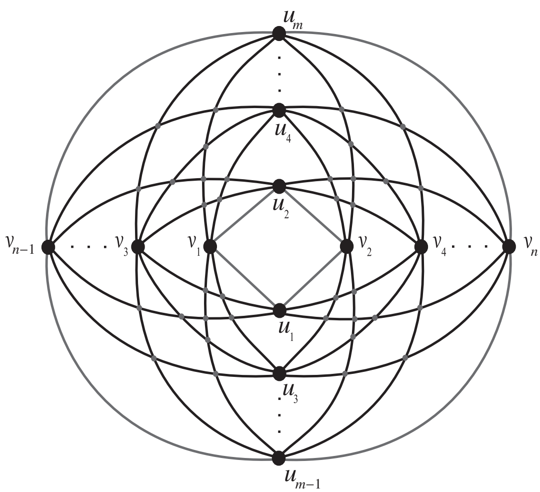

FIGURE 6. Optimal drawing of the graph $K_{m,n}$ for any $m, n \geq 4$ with $\min\{m, n\} \leq 6$.

THEOREM 4.1. *If $n \geq 6$, then the graphs $K_{3,n}$ are not $\mathcal{CF}$-connected.*

Figure 5 shows an optimal drawing of the complete bipartite graph $K_{4,4}$. This drawing can be easily generalised to the optimal drawing D of the graph $K_{m,n}$ in Figure 6 for any $m, n \geq 4$. The optimality of such a drawing is ensured for $\min\{m, n\} \leq 6$ according to Zarankiewicz's conjecture, and a disconnection of the subgraph $CF_{D(K_{m,n})}$ is obtained by the two disjoint cycles on four vertices. This gives the following result.

THEOREM 4.2. *If $n \geq 4$, then the graphs $K_{4,n}$, $K_{5,n}$ and $K_{6,n}$ are not $\mathcal{CF}$-connected.*

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5. Zarankiewicz's conjecture

Zarankiewicz's conjecture about the crossing number of the complete bipartite graph $K_{m,n}$ says that the upper bound $\mathrm{cr}(K_{m,n}) \le Z(m,n)$ holds with equality, where the Zarankiewicz number $Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is defined for all $m,n \ge 1$. Guy [4] proved this conjecture for the graphs $K_{4,n}$ and Kleitman [5] for the graphs $K_{5,n}$. Several exact values for the crossing number of the graphs $K_{m,n}$ are based on the following theorem presented in [4].

THEOREM 5.1. *If Zarankiewicz's conjecture holds for the graph $K_{m,n-1}$, where n is even, then it holds also for the graph $K_{m,n}$.*

For n even, the idea of the second crossing sequence $d_{D(K_n)}$ in the proof of Theorem 3.2 can also be used to give the proof of Theorem 5.1 provided that

$$\frac{n-2}{2}\frac{n-2}{2}\Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor + \frac{n-2}{2}\Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor = \frac{n}{2}\frac{n-2}{2}\Big\lfloor \frac{m}{2} \Big\rfloor \Big\lfloor \frac{m-1}{2} \Big\rfloor.$$

From Theorem 5.1 and Kleitman's result, Zarankiewicz's conjecture is true for all positive integers m,n with $\min\{m,n\} \le 6$. Woodall [8] confirmed the validity of the conjecture for the graphs $K_{7,7}$ and $K_{7,9}$, which means that it also holds for the graphs $K_{7,8}$, $K_{7,10}$, $K_{8,8}$, $K_{8,9}$ and $K_{8,10}$ by Theorem 5.1. Using the flag algebra framework, Norin and Zwols [7] obtained the best known asymptotic lower bound for the crossing number of the complete bipartite graphs $K_{m,n}$ with $m \ge 9$ in the form

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor} \ge \frac{0.905m}{m-1},$$

which implies that the conjecture is 'asymptotically at least 90.5% true'.

It is not difficult to verify that the drawing of $K_{m,n}$ shown in Figure 6 forces exactly $Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ crossings for all positive integers $m,n \ge 7$. So, the next result is obvious.

THEOREM 5.2. *If $m,n \ge 7$ and $\mathrm{cr}(K_{m,n}) = Z(m,n)$, then $K_{m,n}$ is not $\mathcal{CF}$-connected.*

6. Conclusions

We have seen that crossing sequences of optimal drawings can be used to find certain families of $\mathcal{CF}$-connected graphs, but mainly for graphs with already well-known crossing numbers. Theorem 3.2 gives conditions under which removing some vertex from an optimal drawing of the complete bipartite graph $K_{m,n}$ yields an optimal drawing of $K_{m,n-1}$ with a restriction to n even. Assuming the validity of Zarankiewicz's conjecture, the same result holds for all even natural numbers n when $m,n \ge 7$. In this case, all members of the second crossing subsequence $d_{D(K_n)}$ of each optimal drawing D of $K_{m,n}$ are equal to $\frac{n-2}{2}\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$.

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