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A NOTE ON THE SEQUENT CALCULI G3[mic]$^=$

FRANCO PARLAMENTO
Department of Mathematics, Computer Science and Physics, University of Udine
and
FLAVIO PREVIALE
Department of Mathematics, University of Turin

Abstract. We show that the replacement rule of the sequent calculi **G3[mic]**$^=$ in [8] can be replaced by the simpler rule in which one of the principal formulae is not repeated in the premiss.

§1. Introduction. Extensions of Gentzen's sequent calculus to first order logic with equality by means of rules, for which full cut elimination holds, have been studied in [5], leaving a treatment of such extensions for the sequent calculi free of structural rules to a further work. This note is a specific contribution in that direction as it concerns the sequent calculi **G3[mic]**$^=$ in the second edition [8] of [7] that extends to logic with equality the **G3[mic]** calculi, for minimal, intuitionistic and classical logic introduced in [7]. The latter are among the sequent calculi, free of structural rules, for logic without equality that have evolved from the work of Gentzen through Ketonen, Kleene, Dragalin and Troelstra, in particular by showing that the repetition of the principal formula in the premis(ses) of the logical rules, proposed by Kleene in [2], could actually be dispensed with in most of the cases. **G3[mic]**$^=$ has the following additional rules to deal with equality:

$$\frac{t = t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad\qquad \frac{s = r, P[v/s], P[v/r], \Gamma \Rightarrow \Delta}{s = r, P[v/s], \Gamma \Rightarrow \Delta}\ \text{Rep},$$

where Γ and Δ are finite multisets of formulae, P is an atomic formula and $P[v/r]$ and $P[v/s]$ denote the result of the substitution of the variable v by the terms r and s respectively, and $|\Delta| = 1$ in the intuitionistic case. Ref and Rep are extensions to languages with function symbols of the rules Ref and Repl proposed in [3] (see also [4]), in application of a general method of converting axioms into rules, while preserving the admissibility of the structural rules of the $G3$ systems.

The main purpose of this note is to show, quite in line with the evolution from Keene's original $G3$ systems to the modern ones, that the repetition of the principal formula $P[v/s]$ in the premiss of the rule Rep can be avoided, namely that if the rule

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Rep is replaced by the apparently weaker rule

$$\frac{s = r, P[v/r], \Gamma \Rightarrow \Delta}{s = r, P[v/s], \Gamma \Rightarrow \Delta} \quad \text{Rep}^{-}$$

we obtain calculi **G3[mic]**$^{=-}$ equivalent to **G3[mic]**$^{=}$.

Furthermore we observe that the presence of function symbols causes the failure of the height-preserving admissibility of the left-contraction rule. That is due to the fact, unnoticed in [8], that there are instances of Rep that produce a duplication of the atom $s = r$ in the conclusion, hence also in the premiss, of the rule, such that the result of the contraction of such atoms does not result into an application of a rule of the system. Nevertheless we will show that the left-contraction rule remains admissible. Actually we will provide both a direct and an indirect proof of the admissibility of left-contraction. The former uses a principal induction on the degree of the formula to be contracted, and a secondary induction on the height of the derivation of the premiss. The latter, following the indications in [3], consists in adding suitable rules to **G3[mic]**$^{=}$ so as to obtain a system for which the left-contraction rule is height-preserving admissible. The conclusion follows since the added rules are derivable in **G3[mic]**$^{=}$. Having established that the contraction rules and, therefore, also the cut rule, are admissible in **G3[mic]**$^{=}$, by the equivalence between **G3[mic]**$^{=}$ and **G3[mic]**$^{=-}$, we can conclude that all the structural rules are admissible in **G3[mic]**$^{=-}$ as well. All the above holds if **G3i** is replaced by the Dragalin's multisuccedent sequent calculus for intuitionistic logic in [1].

§2. Preliminaries.

We will refer to [8] for the necessary background on the sequent calculus and the operation of substitution of variables by terms, except that we consider languages with distinct free and bound variables, to be denoted by $u, v, \dots$ and $x, y, \dots$ respectively, so that terms, atomic formulae, and formulae are defined as in [6]. For the rest we will adopt the notations in [8], except that we will denote by $h(T)$ the height of a finite (labeled) tree T, such as a formula or a derivation.

The sequent calculi **G3[mic]**$^{=}$ are obtained by adding the rules Ref and Rep in the Introduction to the **G3[mic]** calculi in [8].

We recall that **G3c** has the following axioms and rules

$$P, \Gamma \Rightarrow \Delta, P \quad Ax \qquad\qquad \bot, \Gamma \Rightarrow \Delta \quad L\bot$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \; L\wedge \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \; R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \; L\vee \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B} \; R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \; L\to \qquad\qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \; R\to$$

$$\frac{A[x/t], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta} \; L\forall \qquad\qquad \frac{\Gamma \Rightarrow \Delta, A[x/v]}{\Gamma \Rightarrow \Delta, \forall x A} \; R\forall$$

$$\frac{A[x/v], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta} \; L\exists \qquad\qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A} \; R\exists$$

G3i has the following axioms and rules:

$$P, \Gamma \Rightarrow P \quad Ax \qquad\qquad \bot, \Gamma \Rightarrow A \quad L\bot$$

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ L\wedge \qquad\qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \wedge B}\ R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\ L\vee \qquad\qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ R\vee_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ R\vee_2$$

$$\frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C}\ L\to \qquad\qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}\ R\to$$

$$\frac{A[x/t], \forall x A, \Gamma \Rightarrow C}{\forall x A, \Gamma \Rightarrow C}\ L\forall \qquad\qquad \frac{\Gamma \Rightarrow A[x/v]}{\Gamma \Rightarrow \forall x A}\ R\forall$$

$$\frac{A[x/v], \Gamma \Rightarrow C}{\exists x A, \Gamma \Rightarrow C}\ L\exists \qquad\qquad \frac{\Gamma \Rightarrow A[x/t]}{\Gamma \Rightarrow \exists x A}\ R\exists$$

G3m is obtained from **G3i** by replacing $L\bot$ by the axiom $\bot, \Gamma \Rightarrow \bot$. **G3[mic]** denotes any of the systems **G3m**, **G3i** or **G3c**.

In all the above systems P is an atomic formula ($\bot$ is not regarded as an atomic formula), A, B and C stand for any formula in a first order language (function symbols included), Γ and Δ are finite multisets of formulae and the free variable v does not occur in the conclusion of $L\exists$ and $R\forall$.

In addition to the structural rules of weakening (LW and RW), contraction (LC and RC) and cut (Cut) of the sequent calculus, we will consider also the left-contraction rule restricted to equalities and the left-symmetry rule:

$$\frac{s = r,\ s = r,\ \Gamma \Rightarrow \Delta}{s = r,\ \Gamma \Rightarrow \Delta}\ LC^{=} \qquad\qquad \frac{s = r, \Gamma \Rightarrow \Delta}{r = s, \Gamma \Rightarrow \Delta}\ \text{LS.}$$

In the rules Rep and Rep^{-}, it is not restrictive to assume that the variable v does not occur in r nor in s, since in the representation of a formula as the result of a substitution, as in $P[v/r]$, it can be assumed that v lies outside any given finite set of variables V, in particular that it does not occur in r nor in s. In fact given P and V if we let v' be a variable not occurring in P nor in V and P' be $P[v/v']$, then $P[v/r]$ coincides with $P'[v'/r]$ and v' does not occur in V. We will make use also of the simultaneous replacement of several variables $v_1, \ldots, v_n$ in a formula A with corresponding terms $r_1, \ldots, r_n$, to be denoted with $A[v_1/r_1, \ldots, v_n/r_n]$. As for $P[v/r]$ we may assume that $v_1, \ldots, v_n$ lie outside any given finite set of variables, in particular that $v_1, \ldots, v_n$ do not occur in $r_1, \ldots, r_n$. Assuming the latter condition, we have that $A[v_1/r_1, \ldots, v_n/r_n]$ coincides with any of the possible iterated substitutions that can be performed on A by replacing some of the v_i's by the corresponding r_i's. For example $A[v_1/r_1, v_2/r_2]$ coincides with $(A[v_1/r_1])[v_2/r_2]$ (since v_2 does not occur in r_1) as well as with $(A[v_2/r_2])[v_1/r_1]$ (since v_1 does not occur in r_2). To illustrate the importance

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of the above proviso, we note that assuming that u does not occur in r we have that

$$\frac{q = p, A[u/p, v/r], \Gamma \Rightarrow \Delta}{q = p, A[u/q, v/r], \Gamma \Rightarrow \Delta}$$

is a correct application of Rep$^-$, since $A[u/p, v/r]$ coincides with $(A[v/r])[u/p]$ and $A[u/q, v/r]$ coincides with $(A[v/r])[u/q]$. We will leave to the reader the easy task of specifying the requirement to be made on the variables used in the representation of formulae as the result of substitutions, in order to ensure the correctness of the various derivations that will be displayed in the following.

§3. Admissibility of Rep in G3[mic]$^{=-}$. Let Rep$_1^-$ and Rep$_1$ be the rules Rep$^-$ and Rep as represented in the Introduction, in which it is required that there is at most one occurrence of v in P, i.e. at most one occurrence of r is replaced by s.

LEMMA 3.1. *Rep$^-$ is derivable from Rep$_1^-$, namely the conclusion of a Rep$^-$-inference can be derived from its premiss by using only the rule Rep$_1^-$.*

Proof. By induction on the number of occurrences of v in P. If such a number is $n + 1$, with $n \geq 1$, then let P' be obtained by replacing one occurrence of v in P by a new variable v'. $P[v/r]$ *coincides* with $(P'[v/r])[v'/r]$. Thus from

$$s = r, P[v/r], \Gamma \Rightarrow \Delta$$

namely

$$s = r, (P'[v/r])[v'/r], \Gamma \Rightarrow \Delta$$

by Rep$_1^-$, we obtain

$$s = r, (P'[v/r])[v'/s], \Gamma \Rightarrow \Delta$$

that coincides with

$$s = r, (P'[v'/s])[v/r], \Gamma \Rightarrow \Delta.$$

Since there are n occurrences of v in $P'[v'/s]$, by the induction hypothesis, from the latter sequent, using Rep$_1^-$ we can derive

$$s = r, (P'[v'/s])[v/s], \Gamma \Rightarrow \Delta$$

that coincides with

$$s = r, P[v/s], \Gamma \Rightarrow \Delta.$$

□

DEFINITION 3.1. **G3[mic]**$_1^{=-}$ *is obtained by replacing Rep$^-$ by Rep$_1^-$ in* **G3[mic]**$^{=-}$.

Notation. For the sake of brevity, **G3[mic]**$^{=-}$ and **G3[mic]**$_1^{=-}$ will be denoted also by S and S_1 respectively.

As for **G3[mic]**$^{=}$ we have the following:

LEMMA 3.2. *The weakening rules are height-preserving admissible in S and S_1, i.e. if $\Gamma \Rightarrow \Delta$ has a derivation in S (S_1) of height bounded by n, then also $A, \Gamma \Rightarrow \Delta$ and, in the classical case, $\Gamma \Rightarrow \Delta, A$ have a derivation in S (S_1) of height bounded by n.*

LEMMA 3.3. *Rep is derivable from* Rep_1 *and LW. Therefore Rep is admissible in the system* **G3[mic]**$_1^{=}$.

Proof. As in the proof of Lemma 3.1 let P' be obtained by replacing one of the $n+1$ occurrences of v in P by a new variable v'. The premiss

$$s = r, P[v/s], P[v/r], \Gamma \Rightarrow \Delta$$

of Rep coincides with

$$s = r, (P'[v/s])[v'/s], (P'[v/r])[v'/r], \Gamma \Rightarrow \Delta$$

from which by the left-weakening rule we obtain

$$s = r, (P'[v/s])[v'/s], (P'[v/r])[v'/s], (P'[v/r])[v'/r], \Gamma \Rightarrow \Delta.$$

Then an application of Rep_1 yields

$$s = r, (P'[v/s])[v'/s], (P'[v/r])[v'/s], \Gamma \Rightarrow \Delta$$

that coincides with

$$s = r, (P'[v'/s])[v/s], (P'[v'/s])[v/r], \Gamma \Rightarrow \Delta$$

from which, by the induction hypothesis, we can derive

$$s = r, (P'[v'/s])[v/s], \Gamma \Rightarrow \Delta$$

i.e.

$$s = r, P[v/s], \Gamma \Rightarrow \Delta.$$

□

LEMMA 3.4. a) *Derivability in* S_1 *of the left-contraction rule for equalities* $LC^{=}$: $s = r, \Gamma \Rightarrow \Delta$ *is derivable in* S_1 *from* $s = r, s = r, \Gamma \Rightarrow \Delta$.
b) *Admissibility in* S_1 *and* S *of the left-symmetry rule* LS. *More specifically*: $r = s, \Gamma \Rightarrow \Delta$ *is derivable in* S_1 *and* LW, *from* $s = r, \Gamma \Rightarrow \Delta$.

Proof. a) Since $s = r$ coincides with $P[v/r]$, where P is $s = v$, the following is a derivation in S_1 of $s = r, \Gamma \Rightarrow \Delta$ from $s = r, s = r, \Gamma \Rightarrow \Delta$

$$\cfrac{\cfrac{\cfrac{\mathcal{D}}{s = r, s = r, \Gamma \Rightarrow \Delta}\ \text{Rep}_1^-}{s = r, s = s, \Gamma \Rightarrow \Delta}\ \text{Ref}}{s = r, \Gamma \Rightarrow \Delta}$$

b) The following is a derivation in S_1 and LW of $r = s, \Gamma \Rightarrow \Delta$ from $s = r, \Gamma \Rightarrow \Delta$:

$$\cfrac{\cfrac{\cfrac{\cfrac{s = r, \Gamma \Rightarrow \Delta}{s = r, r = r, \Gamma \Rightarrow \Delta}\ \text{LW}}{s = r, r = s, \Gamma \Rightarrow \Delta}\ \text{Rep}_1^-}{r = r, r = s, \Gamma \Rightarrow \Delta}\ \text{Rep}_1^-}{r = s, \Gamma \Rightarrow \Delta}\ \text{Ref}$$

□

PROPOSITION 3.1. *The rule Rep is admissible in* **G3[mic]**$^{=-}$.

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Proof. We show that the applications of the rule Rep can be eliminated from the derivations in $S + \text{Rep}$. By Lemma 3.1, 3.3 and 3.2, a derivation in $S + \text{Rep}$ can be transformed into a derivation in $S_1 + \text{Rep}_1$ of its endsequent. Thus it suffices to show that if the premiss of a Rep_1- inference, say $s = r, P[v/s], P[v/r], \Gamma \Rightarrow \Delta$, with v that does not occur in s, r and has a single occurrence in P, has a derivation $\mathcal{D}$ in S_1, then also its conclusion $s = r, P[v/s], \Gamma \Rightarrow \Delta$ has a derivation in S_1. The proof is by induction on the height of derivations, but for the induction argument to go through we have to generalize the statement to be proved. In fact, assume that $P[v/s]$ has the form $P^\circ[u/q, v/s]$ and the given derivation of $s = r, P[v/s], P[v/r], \Gamma \Rightarrow \Delta$ has the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ q = p, s = r, P^\circ[u/q, v/s], P^\circ[u/p, v/r], \Gamma \Rightarrow \Delta\end{array}}{q = p, s = r, P^\circ[u/q, v/s], P^\circ[u/q, v/r], \Gamma \Rightarrow \Delta}.$$

Then since $P^\circ[u/q, v/s]$ and $P^\circ[u/p, v/r]$ do not have the form $B[v/s]$ and $B[v/r]$ we could not apply the induction hypothesis to $\mathcal{D}_0$.

To overcome that problem, we generalize the statement to be proved as follows. Let $\vec{q}$ and $\vec{p}$ be the sequences of terms $q_1, \dots q_n$ and $p_1, \dots p_n$ and, similarly, let $\vec{u}$ stand for the sequence of variables $u_1, \dots u_n$ assumed to be distinct from one another and from v and not occurring in $\vec{q}, \vec{p}, s, r$. Let $\vec{q} = \vec{p}$ stand for the sequence of equalities $q_1 = p_1, \dots, q_n = p_n$ and $[\vec{u}/\vec{q}]$ for the substitution $[u_1/q_1, \dots, u_n/q_n]$ and similarly for $[\vec{u}/\vec{p}]$. We proceed by induction on the height $h(\mathcal{D})$ of $\mathcal{D}$ to show that if $\mathcal{D}$ is a derivation in S_1 of

$$\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{q}, v/s], P[\vec{u}/\vec{p}, v/r], \Gamma \Rightarrow \Delta,$$

where each one of the variables in $\vec{u}$ and v has at most occurrence in P, then $\mathcal{D}$ can be transformed into a derivation $\mathcal{D}'$ in S_1 of

$$\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta.$$

The statement we are actually interested in, that yields the admissibility of Rep_1, therefore of Rep, in S, follows by letting $n = 0$.

If $h(\mathcal{D}) = 0$, then $\mathcal{D}$ reduces to a logical axiom. If $\Gamma \cap \Delta \neq \emptyset$ or one of $\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{q}, v/s]$ belongs to Δ, then also $\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{q}, v/s]], \Gamma \Rightarrow \Delta$ is an axiom and we are done. Otherwise $P[\vec{u}/\vec{p}, v/r] \in \Delta$. But then also $\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{p}, v/r], \Gamma \Rightarrow \Delta$ is an axiom and as $\mathcal{D}'$ we can take:

$$\begin{array}{cl}\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{p}, v/r], \Gamma \Rightarrow \Delta & \\ \vdots & \text{Rep}_1^- \ n+1 - \textit{times} \\ \vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta & \end{array}$$

Note that if none of the variable in $\vec{u}, v$ occurs in P, then

$$\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta$$

coincides with $\vec{q} = \vec{p}, s = r, P[\vec{u}/\vec{p}, v/r], \Gamma \Rightarrow \Delta$ and is directly a logical axiom.

If $h(\mathcal{D}) > 0$ and $\mathcal{D}$ ends with a logical inference, since P is atomic, neither $P[\vec{u}/\vec{q}, v/s]$ nor $P[\vec{u}/\vec{p}, v/r]$ nor any of $\vec{q} = \vec{p}$ and $s = r$ can be the principal formula of such an inference and the conclusion follows immediately by the induction hypothesis. The same applies if $\mathcal{D}$ ends with a Ref-inference. If $\mathcal{D}$ ends with a Rep_1^--inference we distinguish the following cases.

Case 1. The shown occurrences of $\vec{q}$, $\vec{p}$, s and r are passive in the last inference of $\mathcal{D}$ and Γ is $q = p, \Gamma'$.

Case 1.1. P is of the form $P^\circ[u/q]$ and $\mathcal{D}$ can be represented as:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/q, \vec{u}/\vec{q}, v/s], P^\circ[u/p, \vec{u}/\vec{p}, v/r], \Gamma' \Rightarrow \Delta\end{array}}{q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/q, \vec{u}/\vec{q}, v/s], P^\circ[u/q, \vec{u}/\vec{p}, v/r], \Gamma' \Rightarrow \Delta}.$$

where u has at most one occurrence in P° and does not occur in $\vec{p}, r$.

By the induction hypothesis applied to $\mathcal{D}_0$ and Γ', $q = p, \vec{q} = \vec{p}$ and P° in place of Γ, $\vec{q} = \vec{p}$ and P respectively, we have a derivation $\mathcal{D}_0'$ in S_1 of

$$q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/q, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta$$

that can be taken as $\mathcal{D}'$.

Case 1.2. P is of the form $P^\circ[u/q]$ and $\mathcal{D}$ can be represented as:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], P^\circ[u/q, \vec{u}/\vec{p}, v/r], \Gamma' \Rightarrow \Delta\end{array}}{q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/q, \vec{u}/\vec{q}, v/s], P^\circ[u/q, \vec{u}/\vec{p}, v/r], \Gamma' \Rightarrow \Delta},$$

where u has at most one occurrence in P and does not occur in $\vec{q}, r, s$. By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ of the same height as $\mathcal{D}_0$ of

$$p = q, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], P^\circ[u/q, \vec{u}/\vec{p}, v/r], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$p = q, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from the following derivation in $S_1 + \mathrm{LC}^= + \mathrm{LS}$, thanks to the derivability in S_1 of $\mathrm{LC}^=$ and the admissibility in S_1 of LS:

$$\dfrac{\dfrac{\dfrac{p = q, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}{p = q, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \mathrm{Rep}_1^-}{q = p, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \mathrm{LS}}{q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \mathrm{LC}^=$$

where the displayed Rep_1^--inference is correct since u has at most one occurrence in P° and does not occur in $\vec{q}, s$.

Note that the application of LS followed by $\mathrm{LC}^=$ can be replaced by

$$\dfrac{\dfrac{p = q, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}{q = p, q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \mathrm{Rep}_1^-}{q = p, \vec{q} = \vec{p}, s = r, P^\circ[u/p, \vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \mathrm{Ref}$$

Case 1.3. The principal formula of the last Rep_1^- inference of $\mathcal{D}$ belongs to Γ'. Then the conclusion follows by applying the induction hypothesis and then the same Rep_1^--inference.

Case 2. One of the shown occurrences of $\vec{q}$, $\vec{p}$, s or r is active in the last inference of $\mathcal{D}$. Without loss of generality we may assume that is either r or s.

Case 2.1. One of the shown occurrences of r is active in the last inference of $\mathcal{D}$.

Case 2.1.1. P has the form $P^\circ[u/q]$ and $\mathcal{D}$ has the form:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ q[v/r]=p,\ \vec{q}=\vec{p},\ s=r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], P^\circ[\vec{u}/\vec{p},\ u/p], \Gamma' \Rightarrow \Delta\end{array}}{q[v/r]=p,\ \vec{q}=\vec{p},\ s=r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], P^\circ[\vec{u}/\vec{p},\ u/q[v/r]], \Gamma' \Rightarrow \Delta},$$

where v has at most one occurrence in q and we may assume it does not occur in P° nor in r, s, p. By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ of the same height as $\mathcal{D}_0$ of

$$q[v/r]=p,\ q[v/s]=p,\ \vec{q}=\vec{p},\ s=r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], P^\circ[\vec{u}/\vec{p},\ u/p], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$q[v/r]=p,\ q[v/s]=p, \vec{q}=\vec{p},\ s=r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from:

$$\begin{array}{cl}
\mathcal{D}_0^{w'} & \\
q[v/r]=p, q[v/s]=p, \vec{q}=\vec{p}, s=r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], \Gamma' \Rightarrow \Delta & \\
\hline
q[v/r]=p, q[v/s]=p, \vec{q}=\vec{p}, r=s, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], \Gamma' \Rightarrow \Delta & \text{LS}\\
\hline
q[v/r]=p, q[v/r]=p, \vec{q}=\vec{p}, r=s, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], \Gamma' \Rightarrow \Delta & \text{Rep}_1^-\\
\hline
q[v/r]=p, \vec{q}=\vec{p}, r=s, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], \Gamma' \Rightarrow \Delta & \text{LC}^=\\
\hline
q[v/r]=p, \vec{q}=\vec{p}, s=r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], \Gamma' \Rightarrow \Delta & \text{LS}
\end{array}$$

where the displayed Rep_1^--inference is correct since, given that v does not occur in r, s, p, we have that $q[v/s]=p$ and $q[v/r]=p$ coincide with $(q=p)[v/s]$ and $(q=p)[v/r]$ respectively and v has at most one occurrence in $q=p$.

Case 2.1.2. r is of the form $r^\circ[u/q]$ and $\mathcal{D}$ can be represented as:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ q=p,\ \vec{q}=\vec{p},\ s=r^\circ[u/q], P[\vec{u}/\vec{q},\ v/s], P[\vec{u}/\vec{p}, v/r^\circ[u/p]], \Gamma' \Rightarrow \Delta\end{array}}{q=p,\ \vec{q}=\vec{p},\ s=r^\circ[u/q], P[\vec{u}/\vec{q},\ v/s], P[\vec{u}/\vec{p},\ v/r^\circ[u/q]], \Gamma' \Rightarrow \Delta},$$

where u has at most one occurrence in r° and it does not occur in P° nor in $\vec{p}, s$. By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ of the same height as $\mathcal{D}_0$ of

$$q=p,\ \vec{q}=\vec{p},\ s=r^\circ[u/q],\ s=r^\circ[u/p], P[\vec{u}/\vec{q},\ v/s], P[\vec{u}/\vec{p},\ v/r^\circ[u/p]], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$q=p,\ \vec{q}=\vec{p},\ s=r^\circ[u/q],\ s=r^\circ[u/p], P[\vec{u}/\vec{q},\ v/s], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from:

$$\begin{array}{cl}
\mathcal{D}_0^{w'} & \\
q=p, \vec{q}=\vec{p},\ s=r^\circ[u/q],\ s=r^\circ[u/p], P[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta & \\
\hline
q=p, \vec{q}=\vec{p},\ s=r^\circ[u/q],\ s=r^\circ[u/q], P[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta & \text{Rep}_1^-\\
\hline
q=p, \vec{q}=\vec{p},\ s=r^\circ[u/q], P[\vec{u}/\vec{q}, v/s], \Gamma \Rightarrow \Delta & \text{LC}^=
\end{array}$$

where the displayed Rep_1^--inference is correct since, given that u does not occur in s, we have that $s=r^\circ[u/p]$ and $s=r^\circ[u/q]$ coincide with $(s=r^\circ)[u/p]$ and $(s=r^\circ)[u/q]$ respectively and u has at most one occurrence in $s=r^\circ$.

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Case 2.1.3. r is of the form $r^\circ[u/q]$ and $\mathcal{D}$ can be represented as:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ q = p,\ \vec{q} = \vec{p},\ s = r^\circ[u/p], P[\vec{u}/\vec{q},\ v/s], P[\vec{u}/\vec{p},\ v/r^\circ[u/q]], \Gamma' \Rightarrow \Delta\end{matrix}}{q = p,\ \vec{q} = \vec{p},\ s = r^\circ[u/q], P[\vec{u}/\vec{q},\ v/s], P[\vec{u}/\vec{p},\ v/r^\circ[u/q]], \Gamma' \Rightarrow \Delta},$$

where u has at most one occurrence in r° and does not occur in s.

By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ in S_1 of the same height as $\mathcal{D}_0$ of

$$q = p,\ \vec{q} = \vec{p},\ s = r^\circ[u/p],\ s = r^\circ[u/q], P[\vec{u}/\vec{q},\ v/s], P[\vec{u}/\vec{p},\ v/r^\circ[u/q]], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$q = p,\ \vec{q} = \vec{p},\ s = r^\circ[u/p],\ s = r^\circ[u/q], P[\vec{u}/\vec{q},\ v/s], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0^{w'}\\ q = p, \vec{q} = \vec{p},\ s = r^\circ[u/q],\ s = r^\circ[u/p], P[\vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta\end{matrix}}{q = p, \vec{q} = \vec{p},\ s = r^\circ[u/q],\ s = r^\circ[u/q], P[\vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \text{Rep}_1^-}{q = p, \vec{q} = \vec{p},\ s = r^\circ[u/q], P[\vec{u}/\vec{q}, v/s], \Gamma' \Rightarrow \Delta}\ \text{LC}^=$$

where the displayed Rep_1^--inference is correct since, given that u does not occur in s, we have that $s = r^\circ[u/p]$ and $s = r^\circ[u/q]$ coincide with $(s = r^\circ)[u/p]$ and $(s = r^\circ)[u/q]$ respectively and u has at most one occurrence in $s = r^\circ$.

Case 2.2. One of the shown occurrences of s is active in the last inference of $\mathcal{D}$.

Case 2.2.1. P has the form $P^\circ[u/q]$, and $\mathcal{D}$ can be represented as:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ q[v/s] = p,\ \vec{q} = \vec{p},\ s = r, P^\circ[\vec{u}/\vec{q},\ u/p], P^\circ[\vec{u}/\vec{p},\ u/q[v/r]], \Gamma' \Rightarrow \Delta\end{matrix}}{q[v/s] = p,\ \vec{q} = \vec{p},\ s = r, P^\circ[\vec{u}/\vec{q},\ u/q[v/s]], P^\circ[\vec{u}/\vec{p},\ u/q[v/r]], \Gamma' \Rightarrow \Delta},$$

where v has at most one occurrence in q and does not occur in P° nor in p.

By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ of the same height as $\mathcal{D}_0$ of

$$q[v/s] = p,\ p = q[v/r],\ \vec{q} = \vec{p},\ s = r, P^\circ[\vec{u}/\vec{q},\ u/p], P^\circ[\vec{u}/\vec{p},\ u/q[v/r]], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$q[v/s] = p,\ p = q[v/r], \vec{q} = \vec{p},\ s = r, P^\circ[\vec{u}/\vec{q},\ u/p], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0^{w'}\\ q[v/s] = p,\ p = q[v/r], \vec{q} = \vec{p},\ s = r, P^\circ[\vec{u}/\vec{q}, u/p], \Gamma' \Rightarrow \Delta\end{matrix}}{q[v/s] = p,\ p = q[v/r],\ \vec{q} = \vec{p}, s = r, P^\circ[\vec{u}/\vec{q}, u/p[v/s]], \Gamma' \Rightarrow \Delta}\ \text{Rep}_1^-}{q[v/s] = p,\ p = q[v/s],\ \vec{q} = \vec{p}, s = r, P^\circ[\vec{u}/\vec{q}, u/p[v/s]], \Gamma' \Rightarrow \Delta}\ \text{Rep}_1^-}{q[v/s] = p,\ q[v/s] = p, \vec{q} = \vec{p}, r = s, P^\circ[\vec{u}/\vec{q}, u/p[v/s]], \Gamma' \Rightarrow \Delta}\ \text{LS}}{q[v/s] = p, \vec{q} = \vec{p}, r = s, P^\circ[\vec{u}/\vec{q}, u/p[v/s]], \Gamma' \Rightarrow \Delta}\ \text{LC}^=$$

where, since v has at most one occurrence in q, the first displayed Rep_1^--inference is correct, given that v does not occur in P° and the second one is correct, given that v does not occur in p.

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Case 2.2.2. s is of the form $s^\circ[u/q]$ and $\mathcal{D}$ can be represented as:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ q=p,\ \vec{q}=\vec{p},\ s^\circ[u/q]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/p]], P[\vec{u}/\vec{p},\ v/r], \Gamma' \Rightarrow \Delta\end{array}}{q=p,\ \vec{q}=\vec{p},\ s^\circ[u/q]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/q]], P[\vec{u}/\vec{p},\ v/r], \Gamma' \Rightarrow \Delta},$$

where u has at most one occurrence in s° and does not occur in P nor in r.

By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ of the same height as $\mathcal{D}_0$ of

$$q=p,\ \vec{q}=\vec{p},\ s^\circ[u/q]=r,\ s^\circ[u/p]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/p]], P[\vec{u}/\vec{p},\ v/r], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$q=p,\ \vec{q}=\vec{p},\ \ s^\circ[u/q]=r,\ s^\circ[u/p]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/p]], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from:

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_0^{w'}\\ q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, s^\circ[u/p]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/p]], \Gamma' \Rightarrow \Delta\end{array}}{q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, s^\circ[u/p]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/q]], \Gamma' \Rightarrow \Delta}\ \mathrm{Rep}_1^-}{q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, s^\circ[u/q]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/q]], \Gamma' \Rightarrow \Delta}\ \mathrm{Rep}_1^-}{q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/q]], \Gamma \Rightarrow \Delta}\ \mathrm{LC}^=$$

where, since u has at most one occurrence in s°, the first displayed Rep_1^--inference is correct, given that u does not occur in P and the second one is correct, given that u does not occur in r.

Case 2.2.3. s is of the form $s^\circ[u/q]$ and $\mathcal{D}$ can be represented as:

$$\frac{\begin{array}{c}\mathcal{D}_0\\ q=p,\ \vec{q}=\vec{p},\ s^\circ[u/p]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/q]], P[\vec{u}/\vec{p},\ v/r], \Gamma' \Rightarrow \Delta\end{array}}{q=p,\ \vec{q}=\vec{p},\ s^\circ[u/q]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/q]], P[\vec{u}/\vec{p},\ v/r], \Gamma' \Rightarrow \Delta},$$

where u has at most one occurrence in s° and does not occur in r.

By height-preserving weakening we have a derivation $\mathcal{D}_0^w$ of the same height as $\mathcal{D}_0$ of

$$q=p,\ \vec{q}=\vec{p},\ s^\circ[u/p]=r,\ s^\circ[u/q]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/q]], P[\vec{u}/\vec{p},\ v/r], \Gamma' \Rightarrow \Delta.$$

By induction hypothesis there is a derivation $\mathcal{D}_0^{w'}$ in S_1 of

$$q=p,\ \vec{q}=\vec{p},\ s^\circ[u/p]=r,\ s^\circ[u/q]=r, P[\vec{u}/\vec{q},\ v/s^\circ[u/q]], \Gamma' \Rightarrow \Delta.$$

Then $\mathcal{D}'$ can be obtained from:

$$\cfrac{\cfrac{\begin{array}{c}\mathcal{D}_0^{w'}\\ q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, s^\circ[u/p]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/q]], \Gamma' \Rightarrow \Delta\end{array}}{q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, s^\circ[u/q]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/q]], \Gamma' \Rightarrow \Delta}\ \mathrm{Rep}_1^-}{q=p, \vec{q}=\vec{p}, s^\circ[u/q]=r, P[\vec{u}/\vec{q}, v/s^\circ[u/q]], \Gamma \Rightarrow \Delta}\ \mathrm{LC}^=$$

where the displayed Rep_1^--inference is correct since u has at most one occurrence in s° and does not occur in r. □

As an immediate consequence we obtain our main result:

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THEOREM 3.1. *A sequent is derivable in* **G3[mic]**$^{=}$ *if and only if it is derivable in* **G3[mic]**$^{=-}$.

Proof. Let $\mathcal{D}$ be a derivation in **G3[mic]**$^{=}$ of $\Gamma \Rightarrow \Delta$. By induction on the height of $\mathcal{D}$, thanks to Proposition 3.1, it is straightforward that $\mathcal{D}$ can be transformed into a derivation in **G3[mic]**$^{=-}$. Conversely, given a derivation $\mathcal{D}$ in **G3[mic]**$^{=-}$ of $\Gamma \Rightarrow \Delta$, it suffices to apply the admissibility of the left-weakening rule, to show that $\mathcal{D}$ can be transformed into a derivation in **G3[mic]**$^{=}$ of $\Gamma \Rightarrow \Delta$. □

§4. Admissibility of the structural rules in G3[mic]$^{=}$. As shown in [8], for **G3[mic]**$^{=}$ the following hold. If a sequent $\Gamma \Rightarrow \Delta$ has a derivation of height bounded by n and t does not contain any variable used as proper in a $L\exists$ or $R\forall$-inference, then also $\Gamma[v/t] \Rightarrow \Delta[v/t]$ has a derivation of height bounded by n (term substitution lemma). The weakening rules are height preserving admissible and all the rules are height preserving invertible with the exception of $R\vee_1, R\vee_2, R\exists$ and $L\rightarrow$ in the case of **G3[mi]**. Still, if the conclusion of a $L\rightarrow$ has a derivation of height bounded by n in **G3[mi]**, then also its second premiss has a derivation in **G3[mi]** of height bounded by n. The right-contraction rule is height-preserving admissible. Furthermore, if the language does not contain function symbols, also the left-contraction rule is height-preserving admissible in **G3[mic]**$^{=}$. From the admissibility of the weakening and contraction rules it follows that also the cut rule is admissible, hence that all the structural rules are admissible.

We note that, when function symbols are present, the height-preserving admissibility of the left-contraction rule fails. For example $a = f(a), a = f(a) \Rightarrow a = f(f(a))$ has the following derivation of height 1:

$$\frac{a = f(a),\ a = f(a),\ a = f(f(a)) \Rightarrow a = f(f(a)}{a = f(a), a = f(a) \Rightarrow a = f(f(a))}$$

but $a = f(a) \Rightarrow a = f(f(a))$ cannot have a derivation of height less than or equal to 1 in **G3[mic]**$^{=}$.

Yet the admissibility of the left-contraction rule in **G3[mic]**$^{=}$ holds as it can be established, on the ground of the height-preserving admissibility of LW, the invertibility of the rules and the following remark.

LEMMA 4.1. a) *$P, \Gamma \Rightarrow \Delta$ is derivable in* **G3[mic]**$^{=}$ *from $P, P, \Gamma \Rightarrow \Delta$ by means of LW, Rep and Ref.*

b) *If $P, P, \Gamma \Rightarrow \Delta$ has a derivation in* **G3[mic]**$^{=}$ *of height bounded by n then $P, \Gamma \Rightarrow \Delta$ has a derivation in* **G3[mic]**$^{=}$ *of height bounded by $n + 2$.*

Proof. a) P can be seen as $P'[v/s]$, where s is any term occurring in P and P' is obtained from P by replacing one occurrence of s by a new variable v or, more simply, as $P'[v/s]$ where P' is P, v does not occur in P and s is any term. In any case $P, P, \Gamma \Rightarrow \Delta$ coincides with $P'[v/s], P'[v/s], \Gamma \Rightarrow \Delta$. From that, by LW and Rep, the sequent $P, \Gamma \Rightarrow \Delta$ can be derived as follows:

$$\dfrac{\dfrac{\dfrac{P, P, \Gamma \Rightarrow \Delta}{s = s, P'[v/s], P'[v/s], \Gamma \Rightarrow \Delta}\ \text{LW}}{s = s, P'[v/s], \Gamma \Rightarrow \Delta}\ \text{Rep}}{P, \Gamma \Rightarrow \Delta}\ \text{Ref}$$

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$b)$ follows immediately from the proof of $a)$ by the height preserving admissibility of LW in $\mathbf{G3[mic]}^{=}$ □

PROPOSITION 4.1. a) *If $A, A, \Gamma \Rightarrow \Delta$ is derivable in* $\mathbf{G3[mic]}^{=}$, *then also $A, \Gamma \Rightarrow \Delta$ is derivable in* $\mathbf{G3[mic]}^{=}$.

b) *If $\Gamma \Rightarrow \Delta, A, A$ is derivable in* $\mathbf{G3c}^{=}$, *then also $\Gamma \Rightarrow \Delta, A$ is derivable in* $\mathbf{G3c}^{=}$.

Proof. We proceed by a principal induction on the height $h(A)$ of A and a secondary induction on the height of the given derivation $\mathcal{D}$ of $A, A, \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, A, A$. For **G3c**, $a)$ and $b)$ are proved simultaneously as follows. If $h(A) = 0$ and A is $\perp$ then $a)$ is immediate since $A, \Gamma \Rightarrow \Delta$ is an instance of $L\perp$. As for $b)$ we note that $\perp$ cannot be principal in the last inference of $\mathcal{D}$ and $b)$ is a straightforward consequence of the secondary induction hypothesis. Otherwise A is an atom P. Then $a)$ follows immediately by the previous Lemma (with an increase of the height of the given derivation). As for $b)$, P can be principal in the last inference of $\mathcal{D}$ only if $\Gamma \Rightarrow \Delta, P, P$ is an axiom, in which case $\Gamma \Rightarrow \Delta, P$ is still an axiom. If P is not principal in the last inference of $\mathcal{D}$, then $b)$ is again a straightforward consequence of the secondary induction hypothesis. If $h(A) > 0$ the proof is essentially the same as the one for **G3[mic]** (see [8] or [4]), except that when A is principal in the last inference of $\mathcal{D}$ and the induction hypothesis on the height of $\mathcal{D}$ is used twice. The first time we may use the secondary induction hypothesis, but the second time we need to use the principal one. For example in case $a)$, if A is $B \wedge C$ so that $\mathcal{D}$ has the form:

$$\cfrac{\begin{array}{c}\mathcal{D}_0\\ B, C, B \wedge C, \Gamma \Rightarrow \Delta\end{array}}{B \wedge C, B \wedge C, \Gamma \Rightarrow \Delta}$$

we proceed as follows. By the invertibility of the $L\wedge$-rule, there is derivation $\mathcal{D}^i$ of the same height as $\mathcal{D}_0$ of $B, C, B, C, \Gamma \Rightarrow \Delta$. By the secondary induction hypothesis applied to $\mathcal{D}^i$ there is a derivation $\mathcal{D}_0'$ of $B, C, C, \Gamma \Rightarrow \Delta$. $h(\mathcal{D}_0')$ can be larger than $h(\mathcal{D}_0^i) = h(\mathcal{D}_0)$, however $h(C) < h(B \wedge C)$, so that we can apply the principal induction hypothesis to $\mathcal{D}_0'$ to obtain a derivation of $B, C, \Gamma \Rightarrow \Delta$, to which it suffices to apply the last $R\wedge$-inference of $\mathcal{D}$ to obtain the desired derivation of $B \wedge C, \Gamma \Rightarrow \Delta$. □

Alternatively, the admissibility of the left-contraction rule for $\mathbf{G3[mic]}^{=}$ can be obtained by following the method in [4] of adding to the system the rules needed to ensure that when an instance of a rule of the system produces a duplication of a principal atom in the conclusion and the premiss, then the result of their contraction is still an instance of a rule of the system (*closure condition* in Definition 6.1.7 of [4]), and then observing that the added rules are derivable in $\mathbf{G3[mic]}^{=}$.

In fact, in the case of $\mathbf{G3[mic]}^{=}$, it is easy to determine which atoms P determine the duplication of the equality $s = r$ in the conclusion, hence in the premiss, of the rule Rep. For $P[v/s]$ to coincide with $s = r$, obviously P must be an equality, say $s^\circ = r^\circ$. If v occurs in s° then s° must coincide with v, for, otherwise, $s^\circ[v/s]$ cannot coincide with s, while, if v does not occur in s°, then s° coincides with s. Furthermore if v occurs in r°, then $r^\circ[v/s]$ must coincide with r, while if v does not occur in r°, then r° coincides with r. Therefore, if v occurs in v but not in r°, P is $v = r$ and Rep takes the form:

$$\frac{s = r, s = r, r = r, \Gamma \Rightarrow \Delta}{s = r, s = r, \Gamma \Rightarrow \Delta}.$$

Contracting the duplications of $s = r$, both in the premiss and the conclusion, yields an instance of the rule Ref, so that no new rule need to be added to the system ([8] p. 134).

If v does not occur in s° but it occurs in r°, then P is $s = r^\circ$, and (given that r coincides with $r^\circ[v/s]$) Rep takes the form:

$$\frac{s = r^\circ[v/s], s = r^\circ[v/s], s = r^\circ[v/r^\circ[v/s]], \Gamma \Rightarrow \Delta}{s = r^\circ[v/s], s = r^\circ[v/s], \Gamma \Rightarrow \Delta}.$$

Contracting the duplications of $s = r^\circ[v/s]$ in the premiss and conclusion yields the rule:

$$\frac{s = r^\circ[v/s], s = r^\circ[v/r^\circ[v/s]], \Gamma \Rightarrow \Delta}{s = r^\circ[v/s], \Gamma \Rightarrow \Delta}\ \text{Rep}_1^*$$

If v occurs both in s° and r°, then P is $v = r^\circ$ and Rep takes the form:

$$\frac{s = r^\circ[v/s], s = r^\circ[v/s], r^\circ[v/s] = r^\circ[v/r^\circ[v/s]], \Gamma \Rightarrow \Delta}{s = r^\circ[v/s], s = r^\circ[v/s], \Gamma \Rightarrow \Delta}.$$

Contracting the duplications of $s = r^\circ[v/s]$ in the premiss and conclusion yields the rule:

$$\frac{s = r^\circ[v/s], r^\circ[v/s] = r^\circ[v/r^\circ[v/s]], \Gamma \Rightarrow \Delta}{s = r^\circ[v/s], \Gamma \Rightarrow \Delta}\ \text{Rep}_2^*$$

Finally if v does not occur in s° nor in r°, then P is $s = r$ and Rep takes the form:

$$\frac{s = r, s = r, s = r, \Gamma \Rightarrow \Delta}{s = r, s = r, \Gamma \Rightarrow \Delta}.$$

Contracting the duplication of $s = r$ in the premiss yields the left-contraction rule for equalities LC$^=$.

Since Rep$_1^*$ and Rep$_2^*$ and LC$^=$ do not belong to **G3[mic]**$^=$ they must be added to obtain a system satisfying the closure condition. However, for the purpose of achieving the height-preserving admissibility of the left-contraction rule, the addition of LC$^=$ is not needed.

Let **G3[mic]**$^{=*}$ be **G3[mic]**$^=$ + Rep$_1^*$ + Rep$_2^*$. Proceeding, as indicate in [8] (4.6.3) or, in more detail, in [4] (6.2), by induction on the height of derivations only, we can establish the height-preserving admissibility of the left-contraction rule in **G3[mic]**$^{=*}$. We note that when dealing with the instance of Rep with premiss $s = r, s = r, s = r, \Gamma \Rightarrow \Delta$ it suffices to apply twice the induction hypothesis to obtain a derivation of $s = r, \Gamma \Rightarrow \Delta$, so that the introduction of $LC^=$ into the system is not necessary.

As a straightforward consequence we indirectly obtain the admissibility of the left-contraction rule in **G3[mic]**$^=$.

PROPOSITION 4.2. *If $A, A, \Gamma \Rightarrow \Delta$ has a derivation in* **G3[mic]**$^=$ *of height bounded by n, then $A, \Gamma \Rightarrow \Delta$ has a derivation in* **G3[mic]**$^=$ *of height bounded by $3n$.*

Proof. A derivation $\mathcal{D}$ of $A, A, \Gamma \Rightarrow \Delta$ in **G3[mic]**$^=$ of height bounded by n, being a derivation in **G3[mic]**$^{=*}$, can be transformed into a derivation $\mathcal{D}'$ of $A, \Gamma \Rightarrow \Delta$ in **G3[mic]**$^{=*}$ of the same height as $\mathcal{D}$. Then it suffices to replace the Rep$_1^*$ and Rep$_2^*$-inferences of $\mathcal{D}'$ by the derivation in Ref + Rep+ LW of their conclusion from their premiss obtained as follows. We first weaken the premiss, say $s = r^\circ[v/s], s =$

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$r^\circ[v/r^\circ[v/s]], \Gamma \Rightarrow \Delta$ into $s = r^\circ[v/s], s = r^\circ[v/s], s = r^\circ[v/r^\circ[v/s]], \Gamma \Rightarrow \Delta$, then a Rep-inference yields $s = r^\circ[v/s], s = r^\circ[v/s], \Gamma \Rightarrow \Delta$, from which the conclusion $s = r^\circ[v/s], \Gamma \Rightarrow \Delta$ is obtained as in in the proof of Lemma 4.1 by means of (a single application of) LW, Rep and Ref. Thus any Rep_1^* or Rep_2^*-inference is replaced by two applications of LW and Rep and an application of Ref. Finally it suffices to apply the height-preserving admissibility of LW to obtain a derivation in $\mathbf{G3[mic]}^=$ of $A, \Gamma \Rightarrow \Delta$ of height bounded by $3n$. □

Since also the left-contraction rule is admissible in $\mathbf{G3[mic]}^=$, all the structural rules are admissible in the system. As an immediate consequence, by Theorem 3.1, we have the following:

THEOREM 4.1. *The structural rules are admissible in* $\mathbf{G3[mic]}^{=-}$.

§5. Extension to Dragalin's multisuccedent intuitionistic calculus. All the previous results hold, with no essential changes, if **G3i** is replaced by Dragalin's multisuccedent intuitionistic calculus in [1], denoted with m- **G3i'** in [8], that is obtained from **G3c** by replacing the rules $L\to$, $R\to$ and $R\forall$ by:

$$\frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\ L^{i'}\to \qquad\qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \to B}\ R^{i}\to$$

$$\frac{\Gamma \Rightarrow A[x/v]}{\Gamma \Rightarrow \Delta, \forall x A}\ R^{i}\forall$$

In particular adding to m- **G3i'** the rules Ref, Rep, Rep_1^* and Rep_2^* yields a system, say m- $\mathbf{G3i'}^{=*}$, in which the contraction rules are height-preserving admissible and a system equivalent to m- $\mathbf{G3i'}^{=*}$, is obtained if the rules Rep_1^* and Rep_2^* are omitted and the rule Rep is replaced by the rule Rep^-.

The same can be said for the system, denoted with m- **G3i** in [8], that differs from m- **G3i'** only because the first premiss of the rule $L^{i'}\to$ is replaced by $A \to B, \Gamma \Rightarrow \Delta, A$, as well as for the corresponding systems for minimal logic.

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DEPARTMENT OF MATHEMATICS, COMPUTER SCIENCE AND PHYSICS
UNIVERSITY OF UDINE
VIA DELLE SCIENZE 206, 33100 UDINE, ITALY
E-mail: franco.parlamento@uniud.it

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TURIN
VIA CARLO ALBERTO 10, 10123 TORINO, ITALY
E-mail: flavio.previale@unito.it