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A NOTE ON THE GENERALISED HYPERSTABILITY OF THE GENERAL LINEAR EQUATION

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Abstract

Let X and Y be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively. We prove new generalised hyperstability results for the general linear equation of the form $g(ax + by) = Ag(x) + Bg(y)$, where $g : X \to Y$ is a mapping and $a, b \in \mathbb{F}$, $A, B \in \mathbb{K}\backslash\{0\}$, using a modification of the method of Brzdęk ['Stability of additivity and fixed point methods', *Fixed Point Theory Appl.* **2013** (2013), Art. ID 285, 9 pages]. The hyperstability results of Piszczek ['Hyperstability of the general linear functional equation', *Bull. Korean Math. Soc.* **52** (2015), 1827–1838] can be derived from our main result.

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1. Introduction and preliminaries

In the sequel, X, Y denote normed spaces over fields $\mathbb{F}$, $\mathbb{K}$, respectively. Also, B^A denotes the set of all functions from a set $A \neq \emptyset$ to a set $B \neq \emptyset$.

First, we recall several results concerning the Hyers–Ulam stability of the Cauchy additive equation

$$g(x + y) = g(x) + g(y) \quad \text{for } x, y \in X,$$

where g maps X into Y. In 1941, Hyers [11] gave a partial answer to a question of Ulam and established the following classical stability result.

THEOREM 1.1 [11]. *Let X, Y be two Banach spaces and $g : X \to Y$ be a function satisfying*

$$\|g(x + y) - g(x) - g(y)\| \leq \delta$$

for some $\delta > 0$ and for all $x, y \in X$. Then the limit

$$A(x) := \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$$

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exists for each $x \in X$ and $A : X \to Y$ is the unique additive Cauchy function such that

$$\|g(x) - A(x)\| \leq \delta$$

for all $x \in X$. Moreover, if $g(tx)$ is continuous in t for each fixed $x \in X$, then the function A is linear.

In 1950, Aoki [2] (see also [18]) proved the following stability result for functions that do not have bounded Cauchy difference.

Theorem 1.2 [2]. *Let X be a normed space, Y a Banach space and $g : X \to Y$ a function satisfying*

$$\|g(x + y) - g(x) - g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for some $\theta \geq 0$, some real number p with $0 \leq p < 1$ and all $x, y \in X$. Then there is a unique additive Cauchy function $A : X \to Y$ such that

$$\|g(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for each $x \in X$.

Remark 1.3. Theorem 1.2 reduces to Theorem 1.1 if $p = 0$.

Later, Gajda [9] obtained the result in Theorem 1.2 for $p > 1$ and gave an example to show that Theorem 1.2 fails when $p = 1$. Rassias [19] proved Theorem 1.2 for the case $p < 0$ and it is now well known that for $p < 0$, if g satisfies (1.1) for all $x, y \in X\backslash\{0\}$, then g must be additive for all $x, y \in X\backslash\{0\}$ (see [4]).

The hyperstability of the general linear equation of the form

$$g(ax + by) = Ag(x) + Bg(y), \tag{1.2}$$

where g maps X into Y, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}$, has been studied by many authors (see [8, 10, 12, 13, 15, 17]). This functional equation includes the Cauchy additive equation ($a = b = A = B = 1$), Jensen's equation ($a = b = A = B = 1/2$) and the linear equation ($a = A$ and $b = B$). The first hyperstability result appears to be due to Bourgin [3]. However, the term hyperstability was used for the first time in [14]. Recently, Piszczek [16] proved the following hyperstability result for the general linear equation.

Theorem 1.4 [16]. *Let X and Y be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p, q \in \mathbb{R}$ with $p + q < 0$ and suppose $g : X \to Y$ satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p\|y\|^q) \quad \text{for } x, y \in X\backslash\{0\}.$$

Then g satisfies the equation

$$g(ax + by) = Ag(x) + Bg(y) \quad \text{for } x, y \in X.$$

The purpose of this work is to prove new generalised hyperstability results for the general linear equation by using a modification of Brzdęk's technique in [5] and the method applied by the authors in [1]. Also, we will show that Piszczek's hyperstability result (Theorem 1.4) and many other hyperstability results can be obtained from our main results.

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2. Main results

In this section, we prove some new generalised hyperstability results for the general linear equation (1.2) which extend Theorem 1.4. One of the main tools used in the proof is the following fixed point result that can be derived from [7].

THEOREM 2.1 [7]. *Let U be a nonempty set, Y be a Banach space, $f_1, \ldots, f_k : U \to U$ and $L_1, \ldots, L_k : U \to \mathbb{R}_+$ be given mappings. Suppose that $\mathcal{T} : Y^U \to Y^U$ is an operator satisfying the inequality*

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^{k} L_i(x)\|(\xi - \mu)(f_i(x))\|$$

for all $\xi, \mu \in Y^U$ and for all $x \in U$. Assume that the functions $\varepsilon : U \to \mathbb{R}_+$ and $\varphi : U \to Y$ satisfy the conditions

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x) \quad \text{and} \quad \varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty,$$

for each $x \in U$, where $\Lambda : \mathbb{R}_+^U \to \mathbb{R}_+^U$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)) \tag{2.1}$$

for all $\delta \in \mathbb{R}_+^U$ and $x \in U$. Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x)$$

for all $x \in U$. Moreover, $\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x)$ for all $x \in U$.

Now we prove the first important result in this paper.

THEOREM 2.2. *Let X and Y be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$ and let $u, v : X \to \mathbb{R}_+$ be functions such that*

$$M_0 := \left\{n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a + bn)s_2(a + bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1\right\}$$

is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \leq tu(x) \text{ for all } x \in X\}$$

and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \leq tv(x) \text{ for all } x \in X\}$$

for $n \in \mathbb{F}\backslash\{0\}$ and s_1, s_2 satisfy the two conditions

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;

(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

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Suppose that $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \le u(x)v(y) \quad \textit{for } x, y \in X\backslash\{0\}. \tag{2.2}$$

Then g satisfies the equation

$$g(ax + by) = Ag(x) + Bg(y) \quad \textit{for } x, y \in X\backslash\{0\}. \tag{2.3}$$

Proof. Without loss of generality we may assume that Y is a Banach space, because otherwise we can replace Y by its completion.

From the condition (W_2), we first assume that $\lim_{n\to\infty} s_2(n) = 0$. Replacing y by mx for $m \in \mathbb{N}$ in (2.2),

$$\|g(ax + bmx) - Ag(x) - Bg(mx)\| \le u(x)v(mx) \quad \text{for } x \in X\backslash\{0\}$$

and

$$\left\|\frac{1}{A}g(ax + bmx) - g(x) - \frac{B}{A}g(mx)\right\| \le \left|\frac{1}{A}\right|u(x)v(mx) \quad \text{for } x \in X\backslash\{0\}. \tag{2.4}$$

We now define the operator $\mathcal{T}_m : Y^{X\backslash\{0\}} \to Y^{X\backslash\{0\}}$ by

$$(\mathcal{T}_m)\xi(x) := \frac{1}{A}\xi((a + bm)x) - \frac{B}{A}\xi(mx) \quad \text{for } x \in X\backslash\{0\}, \xi \in Y^{X\backslash\{0\}}$$

and put

$$\varepsilon_m(x) := \left|\frac{1}{A}\right|u(x)v(mx) \le \left|\frac{1}{A}\right|s_2(m)u(x)v(x) \quad \text{for } x \in X\backslash\{0\}.$$

Then the inequality (2.4) takes the form

$$\|\mathcal{T}_m g(x) - g(x)\| \le \varepsilon_m(x) \quad \text{for } x \in X\backslash\{0\}.$$

For a fixed natural number $m \in M_0$, the operator $\Lambda_m : \mathbb{R}_+^{X\backslash\{0\}} \to \mathbb{R}_+^{X\backslash\{0\}}$ defined by

$$\Lambda_m\eta(x) := \left|\frac{1}{A}\right|\eta((a + bm)x) + \left|\frac{B}{A}\right|\eta(mx) \quad \text{for } \eta \in \mathbb{R}_+^{X\backslash\{0\}}, x \in X\backslash\{0\}$$

has the shape given in (2.1) with $k = 2$, $f_1(x) = (a + bm)x$, $f_2(x) = mx$, $L_1(x) = |1/A|$, $L_2(x) = |B/A|$ for all $x \in X$. Furthermore, for each $\xi, \mu \in Y^{X\backslash\{0\}}$ and $x \in X\backslash\{0\}$,

$$\begin{aligned}
\|(\mathcal{T}_m)\xi(x) - (\mathcal{T}_m)\mu(x)\| &= \left\|\frac{1}{A}\xi((a + bm)x) - \frac{B}{A}\xi(mx) - \frac{1}{A}\mu((a + bm)x) + \frac{B}{A}\mu(mx)\right\| \\
&\le \left|\frac{1}{A}\right|\|\xi((a + bm)x) - \mu((a + bm)x)\| + \left|\frac{B}{A}\right|\|\xi(mx) - \mu(mx)\| \\
&= \sum_{i=1}^{2} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\| \\
&= \sum_{i=1}^{2} L_i(x)\|(\xi - \mu)(f_i(x))\|.
\end{aligned}$$

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Next, we will show that for each $n \in \mathbb{N}_0$ and $x \in X\backslash\{0\}$,

$$\Lambda_m^n \varepsilon_m(x) \leq \left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^n s_2(m)u(x)v(x). \tag{2.5}$$

It is easy to see that inequality (2.5) holds for $n = 0$. By the definitions of Λ_m and ε_m,

$$\begin{aligned}
\Lambda_m \varepsilon_m(x) &\leq \left|\frac{1}{A}\right| \cdot \left|\frac{1}{A}\right| s_2(m)u((a+bm)x)v((a+bm)x) + \left|\frac{B}{A}\right| \cdot \left|\frac{1}{A}\right| s_2(m)u(mx)v(mx) \\
&\leq \left|\frac{1}{A}\right| \cdot \left|\frac{1}{A}\right| s_2(m)s_1(a+bm)s_2(a+bm)u(x)v(x) \\
&\quad + \left|\frac{B}{A}\right| \cdot \left|\frac{1}{A}\right| s_2(m)s_1(m)s_2(m)u(x)v(x) \\
&= \left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right] s_2(m)u(x)v(x).
\end{aligned}$$

Using the above relation and iterating the argument,

$$\begin{aligned}
\Lambda_m^2 \varepsilon_m(x) &= \left|\frac{1}{A}\right| \Lambda_m \varepsilon_m((a+bm)x) + \left|\frac{B}{A}\right| \Lambda_m \varepsilon_m(mx) \\
&\leq \left|\frac{1}{A}\right| \left\{\left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]\right. \\
&\qquad \left. \cdot\, s_2(m)u((a+bm)x)v((a+bm)x)\right\} \\
&\qquad + \left|\frac{B}{A}\right| \left\{\left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right] s_2(m)u(mx)v(mx)\right\} \\
&= \left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^2 s_2(m)u(x)v(x).
\end{aligned}$$

In the same way, by induction,

$$\Lambda_m^n \varepsilon_m(x) = \left|\frac{1}{A}\right| \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^n s_2(m)u(x)v(x),$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}_0$. By summing the geometric progression,

$$\begin{aligned}
\varepsilon^*(x) &= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= \left|\frac{1}{A}\right| \left[\frac{s_2(m)u(x)v(x)}{1 - |1/A| s_1(a+bm)s_2(a+bm) - |B/A| s_1(m)s_2(m)}\right]
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and $m \in M_0$. From Theorem 2.1, for each $m \in M_0$ there is a unique solution $G_m : X\backslash\{0\} \to Y$ of the equation

$$G_m(x) = \frac{1}{A} G_m((a+bm)x) - \frac{B}{A} G_m(mx)$$

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such that

$$\|g(x) - G_m(x)\| \leq \left|\frac{1}{A}\right| \left[\frac{s_2(m)u(x)v(x)}{1 - |1/A|s_1(a+bm)s_2(a+bm) - |B/A|s_1(m)s_2(m)}\right], \tag{2.6}$$

for all $x \in X\backslash\{0\}$. Now we will show that for each $n \in \mathbb{N}_0$,

$$\begin{aligned} &\|\mathcal{T}_m^n g(ax+by) - A\mathcal{T}_m^n g(x) - B\mathcal{T}_m^n g(y)\| \\ &\quad \leq [|1/A|s_1(a+bm)s_2(a+bm) + |B/A|s_1(m)s_2(m)]^n u(x)v(y) \end{aligned} \tag{2.7}$$

for all $x, y \in X\backslash\{0\}$. If $n = 0$, then (2.7) is simply (2.2). So take $r \in \mathbb{N}_0$ and suppose that (2.7) holds for $n = r$ and $x, y \in X\backslash\{0\}$. Then

$$\begin{aligned} &\|\mathcal{T}_m^{r+1} g(ax+by) - A\mathcal{T}_m^{r+1} g(x) - B\mathcal{T}_m^{r+1} g(y)\| \\ &= \left\| \frac{1}{A}\mathcal{T}_m^r g((a+bm)(ax+by)) - \frac{B}{A}\mathcal{T}_m^r g(m(ax+by)) \right. \\ &\quad \left. - A\Big(\frac{1}{A}\mathcal{T}_m^r g((a+bm)x) + \frac{B}{A}\mathcal{T}_m^r g(mx)\Big) - B\Big(\frac{1}{A}\mathcal{T}_m^r g((a+bm)y) + \frac{B}{A}\mathcal{T}_m^r g(my)\Big) \right\| \\ &\leq \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^r \left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm)u(x)v(y) \\ &\quad + \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^r \left|\frac{B}{A}\right| s_1(mx)s_2(mx)u(x)v(y) \\ &= \left[\left|\frac{1}{A}\right| s_1(a+bm)s_2(a+bm) + \left|\frac{B}{A}\right| s_1(m)s_2(m)\right]^{r+1} u(x)v(y). \end{aligned}$$

By induction, the inequality (2.7) holds for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (2.7),

$$G_m(ax+by) = AG_m(x) + BG_m(y) \quad \text{for } x, y \in X\backslash\{0\}.$$

Thus, $\{G_m\}_{m\in M_0}$ is a sequence of functions satisfying (2.3). From (2.6) and the assumptions (W_1) and $s_2(n) \to 0$, it follows, $G_m(x) \to g(x)$ as $m \to \infty$, so that g is a solution of the general linear equation (2.3).

On the other hand, if $\lim_{n\to\infty} s_1(n) = 0$, we replace x by my for $m \in \mathbb{N}$ in (2.2) and an analogous calculation leads to the same result. This completes the proof. □

Recently, Brzdęk [6] proved that if $g : X \to Y$ satisfies the general linear equation on $X\backslash\{0\}$, then g satisfies the general linear equation on X. This observation and Theorem 2.2 yield the following result.

Theorem 2.3. *Let X and Y be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$ and let $u, v : X \to \mathbb{R}_+$ be functions such that*

$$M_0 := \left\{n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1\right\}$$

is an infinite set, and

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \leq tu(x) \text{ for all } x \in X\}$$

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and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

for $n \in \mathbb{F}\backslash\{0\}$ *satisfy the two conditions*

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;
(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

Suppose that $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \le u(x)v(y) \quad \text{for } x, y \in X\backslash\{0\}.$$

Then g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y) \quad \text{for } x, y \in X.$$

Remark 2.4. From Theorem 2.3, if $A = B = 0$ and g satisfies (2.2), then

$$g(ax + by) = 0$$

for all $x, y \in X$. This implies that $g(x) = 0$ for all $x \in X$.

From Theorem 2.2, by the same technique as in the proof of Brzdęk [6, Corollary 4.8], we get the following hyperstability results for inhomogeneous functional equations.

Corollary 2.5. *Let* X *and* Y *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $C : X \times X \to Y$ *a given mapping and* $u, v : X \to \mathbb{R}_+$ *functions such that*

$$M_0 := \left\{n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a + bn)s_2(a + bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1\right\}$$

is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

for $n \in \mathbb{F}\backslash\{0\}$ *satisfy the two conditions*

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;
(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

Suppose that $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y) - C(x, y)\| \le u(x)v(y) \quad \text{for } x, y \in X\backslash\{0\}$$

and the functional equation

$$f(ax + by) = Af(x) + Bf(y) + C(x, y) \quad \text{for } x, y \in X \tag{2.8}$$

has a solution $f_0 : X \to Y$. *Then,* g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y) + C(x, y) \quad \text{for } x, y \in X.$$

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COROLLARY 2.6. *Let X and Y be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $C \in Y$ a given mapping and $u, v : X \to \mathbb{R}_+$ functions such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\}$$

is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

for $n \in \mathbb{F}\backslash\{0\}$ satisfy the two conditions

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;
(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

Suppose that $g : X \to Y$ satisfies the inequality

$$\|g(ax+by) - Ag(x) - Bg(y) - C\| \le u(x)v(y) \quad \text{for } x, y \in X\backslash\{0\}.$$

Then g satisfies the equation

$$g(ax+by) = Ag(x) + Bg(y) + C \quad \text{for } x, y \in X.$$

PROOF. Note that the function $f_0 : X \to Y$, defined by

$$f_0(x) = \frac{C}{1-A-B} \quad \text{for } x \in X,$$

satisfies the functional equation (2.8). The result now follows from Corollary 2.5. □

To end this section, we give another simple application of Theorem 2.3.

COROLLARY 2.7. *Let X and Y be two normed spaces over fields $\mathbb{F}$ and $\mathbb{K}$, respectively, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$ and let $u, v : X \to \mathbb{R}_+$ be functions such that*

$$M_0 := \left\{ n \in \mathbb{N} : \left|\frac{1}{A}\right| s_1(a+bn)s_2(a+bn) + \left|\frac{B}{A}\right| s_1(n)s_2(n) < 1 \right\}$$

is an infinite set, where

$$s_1(n) := \inf\{t \in \mathbb{R}_+ : u(nx) \le tu(x) \text{ for all } x \in X\}$$

and

$$s_2(n) := \inf\{t \in \mathbb{R}_+ : v(nx) \le tv(x) \text{ for all } x \in X\}$$

for $n \in \mathbb{F}\backslash\{0\}$ satisfy the two conditions

(W_1) $\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = 0$;
(W_2) $\lim_{n\to\infty} s_1(n) = 0$ *or* $\lim_{n\to\infty} s_2(n) = 0$.

Suppose that $H : X \times X \to Y$ is a mapping with $H(w, z) \neq 0$ for some $w, z \in X$ and that H satisfies the inequality

$$\|H(x, y)\| \le u(x)v(y) \quad \text{for } x, y \in X\backslash\{0\}.$$

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Then the functional equation

$$h(ax + by) = Ah(x) + Bh(y) + H(x, y) \quad \textit{for } x, y \in X, \tag{2.9}$$

has no solutions in the class of functions $h : X \to Y$.

PROOF. Suppose that $h : X \to Y$ is a solution to (2.9). Then (2.2) holds and, according to Theorem 2.3, h satisfies the general linear equation. This implies that $H(w, z) = 0$ for all $w, z \in X$, which is a contradiction. This completes the proof. □

3. Further observations

In this section, we show that the hyperstability result of Piszczek [16, Theorem 2.1] and some hyperstability results for inhomogeneous general linear equations can be derived from our main results.

COROLLARY 3.1 [16]. *Let* X *and* Y *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p, q \in \mathbb{R}$ *with* $p + q < 0$ *and suppose* $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \leq c\|x\|^p\|y\|^q \quad \textit{for } x, y \in X\backslash\{0\}.$$

Then g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y) \quad \textit{for } x, y \in X.$$

PROOF. Define $u, v : X \to \mathbb{R}_+$ by

$$u(x) := s\|x\|^p \quad \text{and} \quad v(x) := r\|x\|^q,$$

where $r, s \in \mathbb{R}_+$ with $sr = c$.

First, assume that $c > 0$ and hence $r, s > 0$. Then

$$s_1(n) = \inf\{t \in \mathbb{R}_+ : u(nx) \leq tu(x) \text{ for all } x \in X\} = |n|^p$$

and

$$s_2(n) = \inf\{t \in \mathbb{R}_+ : v(nx) \leq tv(x) \text{ for all } x \in X\} = |n|^q.$$

Thus,

$$\lim_{n\to\infty} s_1(\pm n)s_2(\pm n) = \lim_{n\to\infty} |n|^{p+q} = 0.$$

Since $p, q \in \mathbb{R}$ with $p + q < 0$, either $p < 0$ or $q < 0$ and, correspondingly, either $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0$.

On the other hand, if $c = 0$, then $r = 0$ or $s = 0$. From the definitions of s_1 and s_2, it follows that $\lim_{n\to\infty} s_1(n) = 0$ or $\lim_{n\to\infty} s_2(n) = 0$.

Finally, it is easy to see that M_0 is an infinite set. Therefore, all the conditions in Theorem 2.3 hold and we obtain the result. □

By using the same technique, we also get the following two results.

COROLLARY 3.2. *Let* X *and* Y *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p, q \in \mathbb{R}$ *with* $p + q < 0$ *and let* $C : X \times X \to Y$ *be a given mapping. Suppose that* $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y) - C(x, y)\| \leq c\|x\|^p\|y\|^q \quad \textit{for } x, y \in X\backslash\{0\},$$

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and the functional equation

$$f(ax + by) = Af(x) + Bf(y) + C(x, y) \quad \text{for } x, y \in X,$$

has a solution $f_0 : X \to Y$. *Then* g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y) + C(x, y) \quad \text{for } x, y \in X.$$

COROLLARY 3.3. *Let* X *and* Y *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p, q \in \mathbb{R}$ *with* $p + q < 0$ *and* $C \in Y$. *Suppose that* $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y) - C\| \leq c\|x\|^p\|y\|^q \quad \text{for } x, y \in X\backslash\{0\}.$$

Then g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y) + C \quad \text{for } x, y \in X.$$

4. Open problems

The following hyperstability results were also obtained by Piszczek in [16].

THEOREM 4.1 [16]. *Let* X *and* Y *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p, q \in \mathbb{R}$ *with* $p + q > 0$ *and suppose* $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \leq c\|x\|^p\|y\|^q \quad \text{for } x, y \in X\backslash\{0\}.$$

If either $q > 0$ *and* $|a|^{p+q} \neq |A|$, *or* $p > 0$ *and* $|b|^{p+q} \neq |B|$, *then* g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y) \quad \text{for } x, y \in X.$$

THEOREM 4.2 [16]. *Let* X *and* Y *be two normed spaces over fields* $\mathbb{F}$ *and* $\mathbb{K}$, *respectively,* $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}\backslash\{0\}$, $c \geq 0$, $p, q > 0$, *and suppose* $g : X \to Y$ *satisfies the inequality*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \leq c\|x\|^p\|y\|^q \quad \text{for } x, y \in X.$$

If $|a|^{p+q} \neq |A|$ *or* $|b|^{p+q} \neq |B|$, *then* g *satisfies the equation*

$$g(ax + by) = Ag(x) + Bg(y), \quad \text{for } x, y \in X.$$

We do not know whether Theorems 4.1 and 4.2 can be extended along the lines of the results given in Section 3.

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