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CATEGORICITY IN QUASIMINIMAL PREGEOMETRY CLASSES

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Abstract. Quasiminimal pregeometry classes were introduced by [6] to isolate the model theoretical core of several interesting examples. He proves that a quasiminimal pregeometry class satisfying an additional axiom, called excellence, is categorical in all uncountable cardinalities. Recently, [2] showed that the excellence axiom follows from the rest of the axioms. In this paper we present a direct proof of the categoricity result without using excellence.

§1. Introduction. A quasiminimal pregeometry class is a nonelementary class of structures satisfying certain axioms. The notion was introduced by [6] to give canonical axiomatisations of pseudo-exponential fields (in [7]) and other related analytic structures. The original definition of [6] had an additional axiom called excellence, which played a central role in establishing categoricity in cardinalities above $\aleph_1$. (And hence the original terminology of a quasiminimal excellent class.) The notion has evolved through the works of [1] and [4]. In practice checking that the excellence holds has been the most technically difficult part in applications of the categoricity theorem. Some of the original proofs of excellence had gaps, which have only recently been fixed.

Later [2] showed that the excellence axiom is redundant in that it follows from the rest of the axioms. In this paper we present a direct proof of the categoricity result that bypasses excellence altogether. The main new idea is to look at (partial) embeddings that preserve all $\mathcal{L}_{\omega_1,\omega}$ formulas possibly using infinitely many parameters. We call them σ-embeddings. Constructing σ-embeddings by a transfinite recursion presents additional challenges. Given an increasing chain $\langle f_\beta : \beta < \alpha \rangle$ of σ-embeddings, their union $f = \bigcup_{\beta<\alpha} f_\beta$ need not be a σ-embedding. The problem is that f needs to preserve formulas over infinitely (countably) many parameters and if $\mathrm{cf}(\alpha) = \omega$ we cannot guarantee that these parameters all occur at an earlier stage. In case of quasiminimal pregeometry classes the axiom of $\aleph_0$-homogeneity over countable closed models provides a way around this problem in certain situations.

The rest of the paper is organised as follows. The next section fixes the notation and gives the basic definitions. Section 3 establishes infinitary analogues of some well-known elementary properties in first order model theory. Then we prove the categoricity theorem and finish with some concluding remarks.

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§**2. Background.** Let $\mathcal{L}$ be a finitary language, that is, $\mathcal{L}$ has constant, functional, and relation symbols of finite arity. For an infinite cardinal κ the formulas of $\mathcal{L}_{\kappa,\omega}$ are inductively defined as follows.

- Atomic $\mathcal{L}$-formulas are $\mathcal{L}_{\kappa,\omega}$-formulas.
- If Φ is a set of $\mathcal{L}_{\kappa,\omega}$-formulas and $|\Phi| < \kappa$, then $\bigwedge_{\phi\in\Phi}\phi$ and $\bigvee_{\phi\in\Phi}\phi$ are $\mathcal{L}_{\kappa,\omega}$-formulas.
- If ϕ is an $\mathcal{L}_{\kappa,\omega}$-formula and v is a variable, then $\neg\phi$, $\forall v\phi$ and $\exists v\phi$ are $\mathcal{L}_{\kappa,\omega}$-formulas.

In this notation the ordinary first-order language coincides with $\mathcal{L}_{\omega,\omega}$. Note that we do not require $\mathcal{L}_{\kappa,\omega}$-formulas to have finitely many free variables. However, every subformula of an $\mathcal{L}_{\kappa,\omega}$-sentence has finitely many free variables. If M is an $\mathcal{L}$-structure, ϕ is an $\mathcal{L}_{\kappa,\omega}$-formula and θ is a variable assignment, then $M \models_\theta \phi$ is defined as usual. In particular

$$M \models_\theta \bigwedge_{\phi\in\Phi} \phi \text{ if and only if } M \models_\theta \phi \text{ for all } \phi \in \Phi$$

and

$$M \models_\theta \bigvee_{\phi\in\Phi} \phi \text{ if and only if } M \models_\theta \phi \text{ for some } \phi \in \Phi.$$

Two structures M and N are called equivalent in $\mathcal{L}_{\kappa,\omega}$ (in symbols $M \equiv_{\kappa,\omega} N$) if they satisfy the same $\mathcal{L}_{\kappa,\omega}$-sentences.

A formula of $\mathcal{L}_{\infty,\omega}$ is a formula of $\mathcal{L}_{\kappa,\omega}$ for some κ. The notation $M \equiv_{\infty,\omega} N$ means that M and N satisfy the same $\mathcal{L}_{\infty,\omega}$-sentences.

Let M, N be $\mathcal{L}$-structures and f be a (partial) function from M to N. (We use the notation $f : M \rightharpoonup N$ for partial functions). Then f is called a *(partial) embedding* if it preserves quantifier-free formulas. Note that in particular f preserves the formula $x = x$ and hence f is injective. A bijective embedding is an isomorphism between M and N. The function f is called a *(partial) elementary embedding* if it preserves first-order formulas and a *(partial) σ-embedding* if it preserves all $\mathcal{L}_{\omega_1,\omega}$-formulas (in general using infinitely many parameters from the $\mathrm{dom}(f)$). In other words f is a σ-embedding if $\langle M, m\rangle_{m\in\mathrm{dom}(f)} \equiv_{\omega_1,\omega} \langle N, f(m)\rangle_{m\in\mathrm{dom}(f)}$.

A *back-and-forth system* between $\mathcal{L}$-structures M and N is a nonempty collection F of partial embeddings such that the following two conditions are satisfied:

- for all $f \in F$ and $a \in M$ there is $g \in F$ such that $f \subseteq g$ and $a \in \mathrm{dom}(g)$;
- for all $f \in F$ and $b \in N$ there is $g \in F$ such that $f \subseteq g$ and $b \in \mathrm{img}(g)$.

The following characterisation of $\mathcal{L}_{\infty,\omega}$-equivalence is due to [3].

Theorem 2.1. *Let M and N be $\mathcal{L}$-structures. Then $M \equiv_{\infty,\omega} N$ if and only if there is a back and forth system between them.*

Now we introduce the notion of a quasiminimal pregeometry class from [6]. Our definition follows closely that of [4] . For the definition of pregeometry the reader can consult [5].

Definition 2.2. Let $\mathcal{L}$ be a language. A *quasiminimal pregeometry* class is a class $\mathcal{C}$ of pairs $\langle H, \mathrm{cl}_H\rangle$ where H is an $\mathcal{L}$-structure and $\mathrm{cl}_H : \mathcal{P}(H) \to \mathcal{P}(H)$ is a function satisfying the following conditions.

- *Closure under isomorphisms*
  If $\langle H, \mathrm{cl}_H \rangle \in \mathcal{C}$, H' is an $\mathcal{L}$-structure and $f : H \to H'$ is an isomorphism, then $\langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{C}$, where $\mathrm{cl}_{H'}$ is defined as $\mathrm{cl}_{H'}(X') = f(\mathrm{cl}_H(f^{-1}(X')))$ for $X' \subseteq H'$.
- *Quantifier free theory*
  If $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}'_H \rangle \in \mathcal{C}$, then H and H' satisfy the same quantifier free sentences. In other words the empty function is a partial embedding between any two structures.
- *Pregeometry*
  - For each $\langle H, \mathrm{cl}_H \rangle \in \mathcal{C}$ the function cl_H is a pregeometry on H and the closure of any finite set is countable.
  - If $\langle H, \mathrm{cl}_H \rangle \in \mathcal{C}$ and $X \subseteq H$, then $\mathrm{cl}_H(X)$ is a substructure of H and together with the restriction of cl_H it is in $\mathcal{C}$.
  - If $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{C}$, $X \subseteq H$, $y \in H$ and $f : H \rightharpoonup H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in \mathrm{cl}_H(X)$ if and only if $f(y) \in \mathrm{cl}_{H'}(f(X))$.
- *Uniqueness of the generic type over countable closed models*
  Let $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty and $g : G \to G'$ be an isomorphism. If $x \in H, x' \in H'$ are independent from G and G', respectively, then $g \cup \{\langle x, x' \rangle\}$ is a partial embedding.
- $\aleph_0$-*homogeneity over countable closed models*
  Let $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty and $g : G \to G'$ be an isomorphism. If $g \cup f : H \rightharpoonup H'$ is a partial embedding, $X = \mathrm{dom}(f)$ is finite and $y \in \mathrm{cl}_H(X \cup G)$, then there is $y' \in H'$ such that $g \cup f \cup \{\langle y, y' \rangle\}$ is a partial embedding.

To illustrate the definition we give some examples of quasiminimal pregeometry classes.

Example 2.3. The class of models of a strongly minimal first order theory together with algebraic closure is a quasiminimal pregeometry class, provided that the closure of the empty set is infinite. The latter condition is needed to ensure that the closure of any subset of a given model is a model of the theory itself. This can be readily checked by Tarski-Vaught test, using the strong minimality of the theory.

Let the language $\mathcal{L}$ contain just one binary relation E. Consider the class of $\mathcal{L}$-structures, where E is an equivalence relation and each equivalence class is countable. Define the closure of X to be the set of elements equivalent to some $x \in X$. This class is a quasiminimal pregeometry class. It can be realised as the class of models of an $\mathcal{L}(Q)$-sentence.

Finally, we mention mathematically interesting nonelementary classes of pseudo-exponential fields of [7] and group covers of [8].

A class satisfying the above conditions and an additional condition referred to as excellence, is called a *quasiminimal excellent* class in [1,4,6]. It is shown in these works that any two structures in a quasiminimal excellent class of the same uncountable cardinality are isomorphic. As mentioned above [2] showed that excellence follows from other conditions. The terminology of a quasiminimal pregeometry class comes from [2], where the countable dimensional structure is called a quasiminimal pregeometry structure. Thus combining the results of [6] and [2] we get that two

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structures in a quasiminimal pregeometry class of the same uncountable cardinality are isomorphic. In this paper we present a direct proof of this result.

§3. **Properties of Structures in Quasiminimal Pregeometry Classes.** Fix a quasiminimal pregeometry class $\mathcal{C}$. The closure operator is often understood, so we will simply refer to $\mathcal{C}$ as a class of structures instead of a class of pairs. Given a structure $H \in \mathcal{C}$ and a substructure $G \subseteq H$ also in $\mathcal{C}$, we denote $G \preceq H$ the fact that G is closed in H.

PROPOSITION 3.1. *Let $H, H' \in \mathcal{C}$, $H \preceq H'$ and $X \subseteq H$. Then* $\mathrm{cl}_H(X) = \mathrm{cl}_{H'}(X)$.

PROOF. Consider the identity embedding from H to H'. For $y \in H$ we have $y \in \mathrm{cl}_H(X)$ if and only if $y \in \mathrm{cl}_{H'}(X)$. Hence $\mathrm{cl}_H(X) = \mathrm{cl}_{H'}(X) \cap H$. But $\mathrm{cl}_{H'}(X) \subseteq \mathrm{cl}_{H'}(H) = H$. Hence $\mathrm{cl}_H(X) = \mathrm{cl}_{H'}(X)$. ⊣

In view of this, we will drop the subscript from the closure operator whenever no confusion arises. Let us prove some direct consequences of $\aleph_0$-homogeneity and the uniqueness of the generic type.

PROPOSITION 3.2. *Let $H, H' \in \mathcal{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty, $g : G \to G'$ be an isomorphism and $g \cup f : H \rightharpoonup H'$ be a partial embedding with $X = \mathrm{dom}(f)$, $X' = \mathrm{img}(f)$ finite.*

- *The mapping $g \cup f$ extends to an isomorphism $\hat{g} : \mathrm{cl}(X \cup G) \to \mathrm{cl}(X' \cup G')$.*
- *If $y \in H \setminus \mathrm{cl}(X \cup G)$ and $y' \in H' \setminus \mathrm{cl}(X' \cup G')$, then $g \cup f \cup \{\langle y, y'\rangle\}$ is a partial embedding.*

PROOF. For the first assertion note that by the countable closure property both $\mathrm{cl}(X \cup G)$ and $\mathrm{cl}(X' \cup G')$ are countable. Let $\langle a_n : n < \omega\rangle$ and $\langle b_n : n < \omega\rangle$ enumerate $\mathrm{cl}(X \cup G)$ and $\mathrm{cl}(X' \cup G')$, respectively. Construct an increasing family $\langle f_n : n < \omega\rangle$ of finite mappings such that $g \cup f_n$ is a partial embedding as follows. Let $f_0 = f$. For an odd n pick the least m such that a_m is not in the domain of f_{n-1}. By $\aleph_0$-homogeneity there is $b \in \mathrm{cl}(X' \cup G')$ such that $g \cup f_{n-1} \cup \{\langle a_m, b\rangle\}$ is a partial embedding. Put $f_n = f_{n-1} \cup \{\langle a_m, b\rangle\}$. For an even n do the other way around. Then $\hat{g} = \bigcup_{n<\omega} f_n$ is an isomorphism between $\mathrm{cl}(X \cup G)$ and $\mathrm{cl}(X' \cup G')$.

For the second assertion extend $g \cup f$ to an isomorphism $\hat{g} : \mathrm{cl}(X \cup G) \to \mathrm{cl}(X' \cup G')$. Now $\hat{g} \cup \{\langle y, y'\rangle\}$ is a partial embedding by the uniqueness of the generic type. Hence $g \cup f \cup \{\langle y, y'\rangle\}$ is also a partial embedding. ⊣

Next we introduce σ-types and prove σ-saturation of uncountable structures in $\mathcal{C}$.

DEFINITION 3.3. Let $H \in \mathcal{C}$, $A \subseteq H$ and $\bar{v}$ be a finite tuple of variables. A σ-type p (in H) over A in variables $\bar{v}$ is a set of $\mathcal{L}_{\omega_1,\omega}$-formulas with parameters from A and free variables among $\bar{v}$ such that every countable subset is consistent with H. That is for every countable $\Phi \subseteq p$ we have $H \models \exists \bar{v} \bigwedge_{\phi \in \Phi} \phi(\bar{v})$.

If the length of the tuple $\bar{v}$ is n, then we call p an n-type. We can think semantically of a σ-type as a family of $\mathcal{L}_{\omega_1,\omega}$-definable subsets such that each countable subfamily has a nonempty intersection. A σ-type p is *complete* if for every $\mathcal{L}_{\omega_1,\omega}$-formula $\phi(\bar{v})$ we have either $\phi \in p$ of $\neg\phi \in p$. This corresponds to a σ-complete ultrafilter on the σ-algebra of $\mathcal{L}_{\omega_1,\omega}$-definable subsets. A σ-type p is called *isolated* if there is a consistent $\mathcal{L}_{\omega_1,\omega}$-formula $\psi(\bar{x})$ such that $H \models \forall \bar{x}(\psi(\bar{x}) \to \phi(\bar{x}))$ for all $\phi \in p$. A σ-type p is *realised* in H if $\bigcap_{\phi \in p} \phi(H^n) \neq \emptyset$. Clearly, each isolated σ-type is realised. Since $\mathcal{L}_{\omega_1,\omega}$-definable sets are closed under countable intersections, we have the following

Proposition 3.4. *If a σ-type contains a formula defining a countable set, then it is isolated and hence realised.*

Now let us study σ-types in quasiminimal pregeometry structures.

Proposition 3.5. *Let $H \in \mathcal{C}$ and $X \subseteq H$ be countable. Let $a, b \in H \setminus \mathrm{cl}(X)$. Then a and b realise the same σ-type over X.*

Proof. Let $G = \mathrm{cl}(X)$ and $g_0 = \mathrm{id}_G \cup \{\langle a, b\rangle\}$. By the uniqueness of the generic type, g_0 is an embedding. Now consider the collection F of finite extensions of g_0 that are embeddings. It is not empty as $g_0 \in F$. We claim that F is a back-and-forth system. Indeed if $g \in F$ and $y \in H$, then $g = \mathrm{id}_G \cup f$, where f has finite domain. If $y \in \mathrm{cl}(\mathrm{dom}(g))$, then use $\aleph_0$-homogeneity to extend g to y. Otherwise there is $y' \in H \setminus \mathrm{cl}(\mathrm{img}(g))$. By Proposition 3.2 the map $g \cup \{\langle y, y'\rangle\}$ is an embedding. Similarly, we can extend g to an embedding with y in the image.

Now expand the language by adding constant symbols c_g for each $g \in G$ and an additional constant c. Let H_1 be an expansion of H by interpreting c_g by g and c by a. Similarly, let H_2 be an expansion of H by interpreting c_g by g and c by b. By the above there is a back-and-forth system between H_1 and H_2. Hence $H_1 \equiv_{\infty,\omega} H_2$. In particular $H_1 \equiv_{\omega_1,\omega} H_2$. It follows that every formula using parameters from G that is true on a is also true on b. Hence a and b realise the same σ-type over G. Since $X \subseteq G$, elements a and b also realise the same σ-type over X. ⊣

The next corollary establishes the analogy between quasiminimal pregeometry structures and minimality in first-order context. It is also the motivation behind the term quasiminimality.

Corollary 3.6. *Let $H \in \mathcal{C}$ and $\phi(v)$ be an $\mathcal{L}_{\omega_1,\omega}$ formula (possibly using parameters). Then $\phi(H)$ is either countable or cocountable.*

Proof. Suppose otherwise. Let $\bar{c}$ be the parameters used in ϕ. Then $\bar{c}$ is countable. Since both $\phi(H)$ and $\neg\phi(H)$ are uncountable, there are $a \in \phi(H) \setminus \mathrm{cl}(\bar{c})$ and $b \in \neg\phi(H) \setminus \mathrm{cl}(\bar{c})$. This contradicts the fact that a and b realise the same σ-type over $\bar{c}$. ⊣

And now we establish the analogy between the closure in quasiminimal pregeometry classes and the algebraic closure.

Corollary 3.7. *Let $H \in \mathcal{C}$ be uncountable, and $X \subseteq H$ be a countable subset. Then $y \in \mathrm{cl}(X)$ if and only if it satisfies an $\mathcal{L}_{\omega_1,\omega}$-formula that has countably many solutions.*

Proof. Assume that y satisfies $\phi(v)$ that has countably many solutions. Since $\mathrm{cl}(X)$ is countable, there is $y' \in \neg\phi(H) \setminus \mathrm{cl}(X)$. Now y and y' do not satisfy the same σ-type over X. Hence $y \in \mathrm{cl}(X)$.

Conversely, assume that $y \in \mathrm{cl}(X)$. Pick $y' \in H \setminus \mathrm{cl}(X)$. Since the closure is determined by the language, the map $\mathrm{id}_X \cup \{\langle y, y'\rangle\}$ is not an embedding. Hence there is a quantifier-free formula $\phi(v)$ over X satisfied by y but not y'. Now $\neg\phi(H)$ cannot by countable (as it implies that $y' \in \mathrm{cl}(X)$). Hence $\phi(H)$ is countable. ⊣

Next we introduce the infinitary analogue of saturation and prove this property for uncountable structures in quasiminimal pregeometry classes.

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Definition 3.8. A structure H is called σ-saturated if for every $X \subset H$ with $|X| < |H|$ every σ-type over X is realised in H.

Proposition 3.9. *Let $H \in \mathcal{C}$ be uncountable. Then H is σ-saturated.*

Proof. Let $X \subset H$ be a subset with $|X| < |H|$ and let p be a σ-type over X in n variables. We prove by induction on n that p is realised in H.

Let $n = 1$. Put $G = \mathrm{cl}(X)$. Then $|G| = |X| + \aleph_0 < |H|$. So there is $y \in H \setminus G$. If y realises p, then we are done. So assume the opposite. Then there is a formula $\phi(v) \in p$ such that $H \models \neg\phi(y)$. Now since $y \notin \mathrm{cl}(X)$, we have that $\neg\phi(H)$ is uncountable. Hence $\phi(H)$ is countable. But then p is isolated and hence realised in H.

Assume the hypothesis for n. Let p be an $n+1$-type. As before let $G = \mathrm{cl}(X)$. We claim that there is $x \in H$ such that $q_x = \{\phi(\bar{v}, x) : \phi(\bar{v}, w) \in p\}$ is a σ-type. Assume the opposite. Then for every x there is a countable subset $p_x \subset p$ such that

$$H \models \neg\exists\bar{v} \bigwedge_{\phi\in p_x} \phi(\bar{v}, x).$$

Pick $y \in H \setminus G$ and let Y be the set of parameters used in formulas of p_y. Then $Y \subseteq X$ is countable and $y \notin \mathrm{cl}(Y)$. But since any two elements outside $\mathrm{cl}(Y)$ realise the same type over Y, for every $z \in H \setminus \mathrm{cl}(Y)$ we have

$$H \models \neg\exists\bar{v} \bigwedge_{\phi\in p_y} \phi(\bar{v}, z).$$

Now let $W = p_y \cup \bigcup_{x\in\mathrm{cl}(Y)} p_x$. Then W is countable and we have that

$$H \models \neg\exists w\exists\bar{v} \bigwedge_{\phi\in W} \phi(\bar{v}, w).$$

This contradicts the fact that p is a σ-type. Thus for some $x \in H$ we have that $q_x = \{\phi(\bar{v}, x) : \phi(\bar{v}, w) \in p\}$ is a σ-type. By induction hypothesis q_x is realised in H and hence p is also realised in H. ⊣

§4. The Categoricity Theorem. In this section we prove that any two structures of the same uncountable cardinality in a quasiminimal pregeometry class are isomorphic. Let $H, H' \in \mathcal{C}$ be of the same uncountable cardinality. Since we can construct a back-and-forth system between H and H', we have that $H \equiv_{\omega_1,\omega} H'$. In other words the empty embedding is a σ-embedding. In analogy with first-order case we would like to extend a partial σ-embedding to map H onto H'. By σ-saturation we can extend any σ-embedding to any one element (and recursively to any finite number of elements). At limit stages however, we need to take unions. But the union of σ-embeddings may not be a σ-embedding. However, the union of σ-embeddings is always an embedding and in some cases this is sufficient to get a σ-embedding.

Proposition 4.1. *Let $H, H' \in \mathcal{C}$ be uncountable, subsets $G \subset H, G' \subset H'$ be countable closed and let $g : G \to G'$ be an isomorphism. Then g is a partial σ-embedding between H and H'.*

Proof. By $\aleph_0$-homogeneity and Proposition 3.2 the set of embeddings between H and H' that are finite extensions of g is a back-and-forth system. Hence if we add constant symbols for G in H and for G' in H' the resulting structures will be $\mathcal{L}_{\omega_1,\omega}$-equivalent. Therefore g is a σ-embedding. $\dashv$

We can use this result to extend a σ-embeddings to the closure of its domain provided the latter is countable.

Proposition 4.2. *Let $H, H' \in \mathcal{C}$ be uncountable and let $g : H \rightharpoonup H'$ be a partial σ-embedding with $X = \mathrm{dom}(g)$, $X' = \mathrm{img}(g)$ countable. Then g extends to a σ-embedding $\hat{g} : H \rightharpoonup H'$ with $\mathrm{dom}(\hat{g}) = \mathrm{cl}(X)$ and $\mathrm{img}(\hat{g}) = \mathrm{cl}(X')$.*

Proof. Let $\mathrm{cl}(X) = \{a_n : n < \omega\}$ and $\mathrm{cl}(X') = \{a'_n : n < \omega\}$. Construct an increasing sequence $f_0 \subseteq f_1 \subseteq f_2 \subseteq \cdots$ of σ-embeddings as follows. Let $f_0 = g$. For an odd n, pick the least m not in the domain of f_n. Let p be the σ-type of a_m over $\mathrm{dom}(f_n)$. Consider the σ-type $p' = \{\phi(x, f_n(\bar{b})) : \phi(x, \bar{b}) \in p\}$. The set p' is a σ-type as f_n is a σ-embedding. By σ-saturation of H', the type p' is realised by some $c \in H'$. Let $f_{n+1} = f_n \cup \{\langle a_m, c\rangle\}$. For an even n go the other direction.

Now take $\hat{g} = \bigcup_{n<\omega} f_n$. Then $\hat{g}$ is an embedding between the countable closed sets $\mathrm{cl}(X)$ and $\mathrm{cl}(X')$. By Proposition 4.1 the embedding $\hat{g}$ is a σ-embedding. $\dashv$

In particular every countable embedding that extends to the closure of its domain must be a σ-embedding.

Corollary 4.3. *Let $H, H' \in \mathcal{C}$ be uncountable, $G \subset H$, $G' \subset H'$ be countable closed or empty, $g : G \to G'$ an isomorphism, $a \in H \setminus G$ and $a' \in H' \setminus G'$. Then $g \cup \{\langle a, a'\rangle\}$ is a σ-embedding.*

Let us now prove a technical lemma. The argument we use is a direct generalisation of an argument given in [1], [4].

Lemma 4.4. *Let $H, H' \in \mathcal{C}$ be uncountable, $B \subseteq H$ be countable and independent and $f : \mathrm{cl}(B) \to H'$ be a closed embedding. Let $X \subset H$ be finite independent over B. Assume that for every $Y \subsetneq X$, we have $f_Y : \mathrm{cl}(BY) \to H$ compatible σ-embeddings extending $f = f_\emptyset$ (i.e., if $Y_1 \subseteq Y_2$, then $f_{Y_1} \subseteq f_{Y_2}$). Define $g = \bigcup_{Y \subsetneq X} f_Y$. Then there is a cofinite subset $B' \subseteq B$ such that $g|_{\mathrm{cl}(B'X)}$ is a σ-embedding.*

Proof. If $|X| \leq 1$ the assertion is trivial, so assume that $X = \{x_1, \ldots, x_n\}$ where $n > 1$. Note that if $Y_1, Y_2 \subsetneq X$ and $x \in \mathrm{cl}(BY_1) \cap \mathrm{cl}(BY_2)$, then $x \in \mathrm{cl}(B(Y_1 \cap Y_2))$. Hence by the compatibility condition $g = \bigcup_{Y \subsetneq X} f_Y$ is a well defined function.

Let $Y_i = X \setminus \{x_i\}$ and $h_k = \bigcup_{i=1}^{k} f_{Y_i}$. So that $g = h_n$. Thus h_k is a mapping from $\bigcup_{i=1}^{k} \mathrm{cl}(BY_i)$. We prove by induction on k that there is a cofinite subset B_k of B such that the restriction $h_k|_{\bigcup_{i=1}^{k} \mathrm{cl}(B_k Y_i)}$ is a σ-embedding. For $k = 1$, we have $h_k = f_{Y_1}$ so we can simply take $B_1 = B$.

Assume that we have constructed B_{k-1}. Pick $b \in B_{k-1}$ arbitrary and let $B_k = B_{k-1} \setminus \{b\}$. Let $C_k = \bigcup_{i=1}^{k-1} \mathrm{cl}(B_k Y_i)$. We should show that $h_k|_{C_k \cup \mathrm{cl}(B_k Y_k)}$ is a σ-embedding.

Let τ be an automorphism of $\mathrm{cl}(B_{k-1}X)$ that fixes $\mathrm{cl}(B_{k-1}Y_k)$ and swaps x_k with b. Let e be a σ-embedding from $\mathrm{cl}(B_{k-1}X)$ into H' extending $h_{k-1}|_{\bigcup_{i=1}^{k-1} \mathrm{cl}(B_{k-1} Y_i)}$ that agrees with f_{Y_k} on Y_k. If $k \geq 3$ then e is simply an extension of $h_{k-1}|_{\bigcup_{i=1}^{k-1} \mathrm{cl}(B_{k-1} Y_i)}$ to the closure of its domain (provided by Proposition 4.2). For $k = 2$ we have $X = Y_1 \cup \{x_1\}$ and e is an extension of $h_1 \cup \{\langle x_1, f_{Y_2}(x_1)\rangle\}$ to the closure of its domain. Finally, consider $\tau = e\theta^{-1}e^{-1}f_{Y_k}\theta$. The automorphism θ

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takes $C_{k-1} \cup \mathrm{cl}(B_k Y_k)$ to $\mathrm{cl}(B_{k-1} Y_k)$ and f_{Y_k} is defined on it. Hence τ is well-defined on $C_{k-1} \cup \mathrm{cl}(B_k Y_k)$. Since all these maps preserve $\mathcal{L}_{\omega_1,\omega}$-formulas, so does τ.

We claim that on $C_{k-1} \cup \mathrm{cl}(B_k Y_k)$ the mappings τ and h_k agree (and therefore h_k preserves $\mathcal{L}_{\omega_1,\omega}$-formulas). Let $a \in C_{k-1}$. Denote $Y_{ik} = Y_i \cap Y_k = X \setminus \{x_i, x_k\}$. We have

$$\theta(a) \in \bigcup_{i=1}^{k-1} \mathrm{cl}(B_{k-1} Y_{ik}).$$

Since f_{Y_k} agrees with f_{Y_i} on $\mathrm{cl}(B_{k-1} Y_{ik})$, it agrees with h_{k-1} on $\bigcup_{i=1}^{k-1} \mathrm{cl}(B_{k-1} Y_{ik})$. Also e agrees with h_{k-1} on $\bigcup_{i=1}^{k-1} \mathrm{cl}(B_{k-1} Y_i)$ and h_k agrees with h_{k-1} on C_{k-1}. Thus

$$\tau(a) = e\theta^{-1}e^{-1} f_{Y_k} \theta(a) = h_{k-1}\theta^{-1} h_{k-1}^{-1} h_{k-1} \theta(a) = h_{k-1}(a) = h_k(a).$$

Now let $a \in \mathrm{cl}(B_k Y_k)$. Then it is fixed by θ. Also e and f_{Y_k} preserve the closure and agree on $B_k Y_k$. Hence $e^{-1} f_{Y_k}(a) \in \mathrm{cl}(B_k Y_k)$ is fixed by θ. Hence

$$\tau(a) = e\theta^{-1}e^{-1} f_{Y_k} \theta(a) = ee^{-1} f_{Y_k}(a) = f_{Y_k}(a) = h_k(a).$$

This completes the proof. ⊣

We can now prove the main result of this paper. The main difference between our approach and the existing literature is the focus on σ-embeddings. The existing proofs of categoricity start with an ordinary embedding (i.e., a function that preserves quantifier free formulas) and extend it to an isomorphism between two structures of the same cardinality. At certain stages of the construction one needs to extend an embedding with domain of a special form to its closure. The condition of excellence is precisely the statement that this is possible. However, if we have a σ-embedding at hand, then we can always extend it to the closure of its domain by Proposition 4.2 (provided the domain is countable). This is where we bypass the need for excellence.

THEOREM 4.5. *Let $H, H' \in \mathcal{C}$ be uncountable, let B, B' be bases of H, H', respectively and let $g : B \to B'$ be a bijection. Then g extends to an isomorphism $\hat{g} : H \to H'$.*

PROOF. Let $B = \{b_\alpha : \alpha < \kappa\}$, $B' = \{b'_\alpha : \alpha < \kappa\}$ and $g(b_\alpha) = b'_\alpha$. For $n < \omega$, let $G_n = \mathrm{cl}(\{b_m : n \le m < \omega\})$ and $G'_n = \mathrm{cl}(\{b'_m : n \le m < \omega\})$. By $\aleph_0$-homogeneity there is an isomorphism $f_0 : G_0 \to G'_0$ extending g. By Proposition 4.1 the embedding f_0 is a σ-embedding.

For each finite subset $X \subset B$ we construct a number n_X and a surjective σ-embedding $f_X : \mathrm{cl}(G_{n_X} X) \to \mathrm{cl}(G'_{n_X} X')$ that extends g and satisfies the following condition: whenever $X \subseteq Y$, we have $n_X \le n_Y$ and $f_X|_{\mathrm{cl}(G_{n_Y} X)} = f_Y|_{\mathrm{cl}(G_{n_Y} X)}$.

Assume that we have constructed such embeddings. Define $\hat{g} : H \to H'$ as follows. For every $x \in H$, we have $x \in \mathrm{cl}(X)$ for some finite $X \subset B$. Define $\hat{g}(x) = f_X(x)$. By the assumption on the embeddings, the result does not depend on the choice of X. Now $\hat{g}$ is surjective. Indeed for $x' \in H'$ pick finite $X' \subset B'$ such that $x' \in \mathrm{cl}(X')$. Let $X = g^{-1}(X')$. Then $x' \in \mathrm{img}(f_X)$. Since f_X is an embedding $f_X^{-1}(x') \in \mathrm{cl}(X)$. Hence $\hat{g}(f_X^{-1}(x')) = x'$. Also if $\bar{x} \in H$ is a finite tuple, choose $X \subset B$ a finite set such that $\bar{x} \in \mathrm{cl}(X)$. Then $\hat{g}(\bar{x}) = f_X(\bar{x})$, preserves quantifier free formulas. Thus $\hat{g}$ is an isomorphism.

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We now proceed to the construction of σ-embeddings f_X by a well-founded induction on the partial order of subsets of B. Take $n_\emptyset = 0$ and $f_\emptyset = f_0$. If $X = \{b_\alpha\}$ is a singleton do the following. If $\alpha < \omega$, then take $n_X = \alpha + 1$, otherwise take $n_X = 0$. Then $f_0|_{G_{n_X}} : G_{n_X} \to G'_{n_X}$ is an isomorphism. By Corollary 4.3, the map $f_0 \cup \{\langle b_\alpha, b'_\alpha\rangle\}$ is a σ-embedding. So by Proposition 4.1 it extends to an isomorphism $f_X : \mathrm{cl}(G_{n_X} X) \to \mathrm{cl}(G'_{n_X} X')$.

Assume that $|X| > 1$ and we have already constructed n_Y and f_Y for every $Y \subsetneq X$. Let $n = \max\{n_Y : Y \subsetneq X\}$. Then each $g_Y = f_Y|_{\mathrm{cl}(G_n Y)}$ is a σ-embedding of $\mathrm{cl}(G_n Y)$ onto $\mathrm{cl}(G'_n Y')$, where $Y' = g(Y)$. Now if $x \in \mathrm{dom}(g_{Y_1}) \cap \mathrm{dom}(g_{Y_2})$ for $Y_1, Y_2 \subsetneq X$, then $x \in \mathrm{dom}(g_{Y_1 \cap Y_2})$. Thus if we define $g_X = \bigcup_{Y \subsetneq X} g_Y$, then g_X is a well defined function. By the previous Lemma there is some number $m \geq n$ such that $g_X|_{\bigcup_{Y \subsetneq X} \mathrm{cl}(G_m Y)}$ is a σ-embedding. Let f_X be its extension to the closure of the domain $\mathrm{cl}(\bigcup_{Y \subsetneq X} \mathrm{cl}(G_m Y)) = \mathrm{cl}(G_m X)$ and $n_X = m$. ⊣

§5. Concluding Remarks. Ever since its introduction by Shelah, excellence has been the key notion for categoricity in nonelementary classes. What our methods show is that in some very natural mathematical examples one can use infinitary logic instead. The fact that σ-embeddings and associated infinitary notions occur in natural mathematical contexts is remarkable in itself. This opens up a possibility of a broader use of infinitary logic both in elementary and nonelementary setting.

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REFERENCES

[1] John Baldwin, ***Categoricity***, American Mathematical Society, Providence, RI, 2009.

[2] Martin Bays, Bradd Hart, Tapani Hyttinen, Meeri Kesälä, and Jonathan Kirby, *Quasiminimal structures and excellence*. ***Bulletin of London Mathematical Society***, vol. 46 (2014), pp. 155–163.

[3] Carol Karp, Finite-quantifier equivalence, In ***The Theory of Models. Proceedings of the 1963 International Symposium at Berkeley*** (John Addison, Leon Henkin, and Alfred Tarski, editors), North-Holland, Amsterdam, 1965.

[4] Jonathan Kirby, *On quasiminimal excellent classes*, this Journal, vol. 75 (2010), no. 2, pp. 551–564.

[5] David Marker, ***Model Theory: An Introduction***, Springer-Verlag, New York, 2002.

[6] Boris Zilber, *A categoricity theorem for quasi-minimal excellent classes*, ***Logic and its Applications*** (Andreas Blass and Yi Zhang, editors), American Mathematical Society, Providence, RI, 2005.

[7] ———, *Pseudo-exponentiation on algebraically closed fields of characteristic zero*. ***Annals of Pure and Applied Logic***, vol. 132 (2005), pp. 67–95.

[8] ———, *Covers of the multiplicative group of an algebraically closed field of characteristic zero*. ***Journal of the London Mathematical Society***, vol. 74 (2006), no. 1, pp. 41–58.

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