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# On quasi-conformal self-mappings of the unit disc and elliptic PDEs in the plane

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We prove the following theorem: if $w$ is a quasi-conformal mapping of the unit disc onto itself satisfying elliptic partial differential inequality $|L[w]| \leqslant \mathcal{B}|\nabla w|^2 + \Gamma$, then $w$ is Lipschitz continuous. This result extends some recent results where, instead of an elliptic differential operator, only the Laplace operator is considered.

## 1. Introduction and notation

### 1.1. Quasi-conformal mappings

Let

$$A = \begin{pmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{pmatrix}.$$

We will consider the matrix norm

$$|A| = \max\{|Az| \colon z \in \mathbb{R}^2,\ |z| = 1\}$$

and the matrix function

$$l(A) = \min\{|Az| \colon z \in \mathbb{R}^2,\ |z| = 1\}.$$

Let $D$ and $\Omega$ be subdomains of the complex plane $\boldsymbol{C}$ and let $w = u + \mathrm{i}v \colon D \to \Omega$ be a function that has both partial derivatives at a point $z \in D$. By $\nabla w(z)$, we denote the matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

For the matrix $\nabla w$, we have that

$$|\nabla w| = |\partial w| + |\bar{\partial} w| \tag{1.1}$$

and

$$l(\nabla w) = ||\partial w| - |\bar{\partial} w||, \tag{1.2}$$

where

$$\partial w = w_z := \frac{1}{2}\left(w_x + \frac{1}{\mathrm{i}} w_y\right) \quad \text{and} \quad \bar{\partial} w = w_{\bar{z}} := \frac{1}{2}\left(w_x - \frac{1}{\mathrm{i}} w_y\right).$$





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We say that a function $u\colon D \to \mathbb{R}$ is absolutely continuous on lines (ACL) in the region $D$ if, for every closed rectangle $R \subset D$ with sides parallel to the $x$- and $y$-axes, $u$ is absolutely continuous on almost every (a.e.) horizontal and a.e. vertical line in $R$. Such a function has, of course, partial derivatives $u_x$, $u_y$ a.e. in $D$.

A sense-preserving homeomorphism $w\colon D \to \Omega$, where $D$ and $\Omega$ are subdomains of the complex plane $\boldsymbol{C}$, is said to be $K$-quasi-conformal ($K$-q.c.), with $K \geqslant 1$, if $w$ is ACL in $D$ in the sense that the real and imaginary parts are ACL in D and

$$|\nabla w| \leqslant K l(\nabla w) \quad \text{a.e. on } D \tag{1.3}$$

(see [1, pp. 23–24]). Note that (1.3) can be written as

$$|w_{\bar z}| \leqslant k|w_z| \text{ a.e. on } D, \quad \text{where } k = \frac{K-1}{K+1}, \text{ i.e. } K = \frac{1+k}{1-k}.$$

If, in the previous definition, we replace the condition '$w$ is a sense-preserving homeomorphism' by the condition '$w$ is continuous', then we obtain the definition of a quasi-regular mapping.

### 1.2. Elliptic operator

Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^{2}$ be a symmetric matrix function defined in a domain $D \subset \boldsymbol{C}$ ($a^{ij} = a^{ji}$). Assume that

$$\varLambda^{-1} \leqslant \langle A(z)h, h\rangle \leqslant \varLambda \quad \text{for } |h| = 1, \tag{1.4}$$

where $\varLambda$ is a constant $\geqslant 1$ or, written in coordinates,

$$\varLambda^{-1} \leqslant \sum_{i,j=1}^{2} a^{ij}(z)h_i h_j \leqslant \varLambda \quad \text{for } \sum_{i=1}^{2} h_i^2 = 1. \tag{1.5}$$

In addition, for a certain $\mathfrak{L} \geqslant 0$, we suppose that

$$|A(z) - A(\zeta)| \leqslant \mathfrak{L}|\zeta - z| \quad \text{for any } z, \zeta \in D. \tag{1.6}$$

For

$$L[u] := \sum_{i,j=1}^{2} a^{ij}(z) D_{ij} u(z), \tag{1.7}$$

subjected to (1.5) and (1.6), we consider the differential inequality

$$|L[u]| \leqslant \mathcal{B}|\nabla u|^2 + \varGamma, \tag{1.8}$$

with given $\mathcal{B}, \varGamma \geqslant 0$, or, by using the Einstein convention,

$$|a^{ij}(z) D_{ij} u| \leqslant \mathcal{B}|\nabla u|^2 + \varGamma, \tag{1.9}$$

and call it the *elliptic partial differential inequality*. Observe that, if $A$ is the identity matrix, then $L$ is the Laplace operator $\Delta$. A $C^2$ solution $u\colon D \to \mathbb{R}(\boldsymbol{C})$ of the equation $\Delta u = 0$ is called a harmonic function (mapping) and the corresponding inequality (1.7) is called the *Poisson differential inequality*. This class of harmonic quasi-conformal mappings (HQC) has been subject to recent investigation by several authors; see the subsection below. For the connection between quasi-conformal mappings and PDEs, refer to [2]. See also [8, ch. 12] and [5, 32, 38].

### 1.3. Background and statement of the main result

Let $\gamma$ be a Jordan curve. By the Riemann mapping theorem, there exists a Riemann conformal mapping of the unit disc onto a Jordan domain $\Omega = \operatorname{int}\gamma$. By Carathéodory's theorem, it has a continuous extension to the boundary. Moreover, if $\gamma \in C^{1,\alpha}$, $0 < \alpha < 1$, then the Riemann conformal mapping has $C^{1,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem). We refer the reader to [9] for the proof of the previous result and [22, 25, 35–37] for related results. In particular, a conformal mapping $w$ of the unit disc onto a Jordan domain $\Omega$ with $C^{1,\alpha}$ boundary is Lipschitz continuous, i.e. it satisfies the inequality $|w(z) - w(z')| \leqslant C|z - z'|$, $z, z' \in \mathbb{U} := \{z \in \boldsymbol{C} \colon |z| < 1\}$.

On the other hand, $K$ quasi-conformal mappings between smooth domains are Hölder continuous and the best Hölder constant is $1/K$. So, they are not in general Lipschitz mappings, except if $K = 1$. In this paper we are concerned with an additional condition of a quasi-conformal mapping in order to guarantee its global Lipschitz character.

One 'additional condition' is to assume harmonicity of the mapping. This condition is natural since conformal mappings are quasi-conformal and harmonic. Hence, harmonic quasi-conformal mappings are natural generalizations of conformal mappings. Martio [28] was the first to consider harmonic quasi-conformal mappings on the complex plane.

Recently, there have been a number of authors working on this topic. We list some of the related results below.

(1) If $w$ is a harmonic quasi-conformal mapping of the unit disc onto itself, then $w$ is Lipschitz (Pavlovic theorem, proved in [34]). See also some refinements of Partyka and Sakan [33].

(2) If $w$ is a harmonic quasi-conformal mapping between two $C^{1,\alpha}$ Jordan domains, then $w$ is Lipschitz (a result proved in [13]).

(3) If $w$ is a quasi-conformal mapping between two $C^{2,\alpha}$ Jordan domains satisfying the partial differential inequality $|\Delta w| \leqslant C|f_z f_{\bar{z}}|$, then $w$ is Lipschitz (a result proved in [17]).

(4) If $w$ is a quasi-conformal mapping of the unit disc onto itself satisfying the PDE $\Delta w = g$, then this mapping is Lipschitz (a result proved in [21]).

(5) If $w$ is a quasi-conformal mapping between two $C^{2,\alpha}$ Jordan domains satisfying the partial differential inequality $|\Delta w| \leqslant \mathcal{B}|\nabla w|^2 + \Gamma$, then $w$ is Lipschitz (a result proved in [19]).

Note that the proofs of (3)–(5) depend on a Heinz theorem; see [10].

Concerning the bi-Lipschitz character of the HQC class, we refer the reader to [3, 14, 16, 20, 23, 26, 30]. Also, see [18, 29] for some results concerning higher dimensional cases.

For related results about quasi-conformal harmonic mappings with respect to the hyperbolic metric refer to Wan [39] and Marković [27].

More recently, Iwaniec *et al*. [11] have shown that the class of quasi-conformal harmonic mappings is also of interest when considering the modulus of annuli in the complex plane.

In this paper, we study Lipschitz continuity of the class of $K$-q.c. self-mappings of the unit disc satisfying the elliptic differential inequality $|Lw| \leqslant \mathcal{B}|\nabla w|^2 + \Gamma$. This class contains conformal mappings and quasi-conformal harmonic mappings.

The main result of this paper is the following theorem, which is an extension of results (1)–(5) mentioned above.

THEOREM 1.1. *If $a \in \mathbb{U}$ and $w\colon \mathbb{U} \to \mathbb{U}$, $w(a) = 0$ and $w(\mathbb{U}) = \boldsymbol{U}$ is a $K$-q.c. solution of the elliptic partial differential inequality*

$$|L[w]| \leqslant \mathcal{B}|\nabla w|^2 + \Gamma, \tag{1.10}$$

*then $\nabla w$ is bounded by a constant $C(K, \mathcal{B}, \Gamma, \Lambda, \mathfrak{L}, a)$ and $w$ is Lipschitz continuous.*

REMARK 1.2. In [7, pp. 179–180], (1.10) is referred to as the *natural growth condition*. The result is new even for $\mathcal{B} = \Gamma = 0$, i.e. for quasi-conformal (q.c.) solutions to elliptic PDEs with Lipschitz coefficients.

The proof of theorem 1.1 is given in § 3. The methods of the proof differ from the methods of the proof of corresponding results for the HQC class. In § 2, we make some estimates concerning the Green function of the disc, and some estimates concerning the gradient of a solution to the elliptic partial differential inequality, satisfying certain boundary conditions similar to those of Nagumo [31]. We first prove interior estimates for the gradient of a solution $u$ of the elliptic PDE in terms of constants of the elliptic operator and the modulus of continuity of $u$ (theorem 2.5). Then, we recall a theorem of Nagumo [31], which shows that if $u$ is a solution of the elliptic PDE, with vanishing boundary conditions defined in a domain $D$ whose boundary has a bounded curvature from above by a constant $\kappa$, then $|\nabla u(z)| \leqslant \gamma, z \in D$, where $\gamma$ is a constant not depending on $u$, provided that $64\mathcal{B}\Gamma\|u\|_\infty < \pi$ (theorem 2.8). In order to prove theorem 1.1, we first show that the function $u = |w|$ satisfies a certain elliptic differential inequality near the boundary of the unit disc. In order to show an *a priori* bound, we make use of Mori's theorem, which implies that the modulus of continuity of a $K$-q.c. self-mapping of the unit disc depends only on $K$. Using theorem 2.5, we show that the gradient is *a priori* bounded on compacts of the unit disc, while theorem 2.8 serves to demonstrate the *a priori* bound of the gradient of $u$ in some 'neighbourhood' of the boundary of the unit disc. By using the quasi-conformality, we prove that $\nabla w$ is *a priori* bounded as well.

## 2. Auxiliary results

### 2.1. Green's function

If $h(z, w)$ is a real function, then we denote the gradient $(h_x, h_y)$ by $\nabla_z h$.

LEMMA 2.1. *If*

$$h(z, w) = \log\left(\frac{|1 - z\bar{w}|}{|z - w|}\right),$$

*then*

$$\nabla_z h(z, w) = \frac{1 - |w|^2}{(\bar{z} - \bar{w})(w\bar{z} - 1)} \tag{2.1}$$

*and*

$$\partial_w \nabla_z h(z,w) = -\frac{1}{(1-w\bar{z})^2}, \qquad \partial_{\bar{w}} \nabla_z h(z,w) = -\frac{1}{(\bar{w}-\bar{z})^2}. \tag{2.2}$$

*Proof.* First of all, let

$$\nabla_z h = (h_x, h_y) = h_x + \mathrm{i} h_y.$$

Since

$$h_{\bar{z}} = \tfrac{1}{2}(h_x + \mathrm{i} h_y),$$

it follows that

$$\nabla_z h = 2h_{\bar{z}}.$$

Since

$$2h(z) = \log\left(\frac{1-z\bar{w}}{z-w}\frac{1-\bar{z}w}{\bar{z}-\bar{w}}\right),$$

by differentiating we obtain

$$2h_{\bar{z}}(z) = \log\left(\frac{1-\bar{z}w}{\bar{z}-\bar{w}}\right)_{\bar{z}} = \frac{|w|^2-1}{(\bar{z}-\bar{w})^2}\frac{\bar{z}-\bar{w}}{1-\bar{z}w}.$$

This implies (2.1). Then, (2.2) follows from

$$\frac{1-|w|^2}{(\bar{z}-\bar{w})(w\bar{z}-1)} = \frac{w}{w\bar{z}-1} + \frac{1}{\bar{w}-\bar{z}}.$$

□

COROLLARY 2.2. *Let $G(\zeta,\omega)$ be the Green function of the disc $\{\zeta\colon |\zeta-\zeta_0| \leqslant R\}$, defined by*

$$G(\zeta,\omega) := \log\left(\frac{|\varphi(\zeta)-\varphi(\omega)|}{|1-\varphi(\zeta)\overline{\varphi(\omega)}|}\right),$$

*where*

$$\varphi(\zeta) = \frac{1}{R}(\zeta-\zeta_0).$$

*Then,*

$$|\nabla_\zeta G(\zeta,\omega)| \leqslant \frac{2}{|\zeta-\omega|} \tag{2.3}$$

*and*

$$|\partial_{\omega_j}\nabla_\zeta G(\zeta,\omega)| \leqslant \frac{2}{|\zeta-\omega|^2}, \quad j=1,2, \tag{2.4}$$

*where $\omega = \omega_1 + \mathrm{i}\omega_2$, $\omega_1,\omega_2 \in \boldsymbol{R}$.*

*Proof.* Let

$$\varphi(\zeta) = \frac{1}{R}(\zeta - z_0).$$

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Then

$$\varphi'(\zeta) = \frac{1}{R}.$$

Take $z = \varphi(\zeta)$ and $w = \varphi(\omega)$ and define $h(z,w) = G(\zeta,\omega)$. It follows that

$$\nabla_\zeta G(\zeta,\omega) = \nabla_z h(z,w) \cdot \varphi'(\zeta) = \frac{1}{R}\nabla_z h(z,w). \tag{2.5}$$

Thus,

$$|\nabla_\zeta G(\zeta,\omega)| = \frac{1}{R}|\nabla_z h(z,w)|. \tag{2.6}$$

Furthermore,

$$\frac{1-|w|^2}{|1-\bar{z}w|} \leqslant \frac{1-|w|^2}{1-|w|} \leqslant 2. \tag{2.7}$$

Combining (2.7), (2.6) with (2.1), we obtain (2.3). To get (2.4), first observe that for $\omega = \omega_1 + \mathrm{i}\omega_2$

$$\partial_{\omega_1} = \partial_\omega + \partial_{\bar{\omega}} \tag{2.8}$$

and

$$\partial_{\omega_2} = \mathrm{i}(\partial_\omega - \partial_{\bar{\omega}}). \tag{2.9}$$

On the other hand, for $|z| \leqslant 1$ and $|w| \leqslant 1$ we have that

$$\left|\frac{1}{(1-w\bar{z})^2}\right| \leqslant \left|\frac{1}{(w-z)^2}\right|.$$

From (2.8), (2.9), (2.2), (2.5) we deduce (2.4). □

### 2.2. Interior estimates of gradient

LEMMA 2.3. *Let $u\colon \bar{\mathbb{U}} \to \mathbb{C}$ be a continuous mapping. Then, there exists a positive function $\varpi = \varpi_u(t)$, $t \in (0,2)$, such that $\lim_{t\to 0} \varpi_u(t) = 0$ and*

$$|u(z) - u(w)| \leqslant \varpi(|z-w|), \quad z, w \in \mathbb{U}.$$

*The function $\varpi$ is called the modulus of continuity of $u$.*

LEMMA 2.4. *Let $Y: D \to \mathbb{U}$ be a $C^2$ mapping of a domain $D \subset \mathbb{U}$. Define*

$$\mathbb{U}(z_0,\rho) := \{z \in \boldsymbol{C} : |z - z_0| < \rho\}$$

*and assume that the closure of $\mathbb{U}(z_0,\rho)$ is contained in $D$ and let $Z \in \boldsymbol{C}$ be any complex number. Then, we have the estimate*

$$|\nabla h(z_0)| \leqslant \frac{2}{\rho^2}\int_{|y-z_0|=\rho} |Y(y) - Z| \, \mathrm{d}\mathcal{H}^1(y), \tag{2.10}$$

*where $h(z)$, $z \in \overline{\mathbb{U}(z_0,\rho)}$, is the Poisson integral of $Y|_{z_0+\rho\boldsymbol{T}}$ and $\boldsymbol{T}$ is the unit circle. Moreover, $\mathrm{d}\mathcal{H}^1$ is the Hausdorff probability measure (i.e. normalized arc length measure).*

*Proof.* Assume that $v \in C^2(\bar{\mathrm{U}})$ and define

$$H(z) = \int_{\boldsymbol{T}} P(z,\eta) v(\eta)\,\mathrm{d}\mathcal{H}^1(\eta), \tag{2.11}$$

where

$$P(z,\eta) = \frac{1-|z|^2}{|z-\eta|^2}, \quad |\eta| = 1,\ |z| < 1. \tag{2.12}$$

Then, $H$ is a harmonic function. It follows that

$$\langle \nabla H(z), e\rangle = \int_{\boldsymbol{T}} \langle \nabla_z P(z,\eta), e\rangle v(\eta)\,\mathrm{d}\mathcal{H}^1(\eta), \quad e \in \mathbb{R}^2. \tag{2.13}$$

By differentiating (2.12), we obtain

$$\nabla_z P(z,\eta) = \frac{-2z}{|z-\eta|^2} - \frac{2(1-|z|^2)(z-\eta)}{|z-\eta|^{2+2}}.$$

Hence,

$$\nabla_z P(0,\eta) = \frac{2\eta}{|\eta|^4} = 2\eta.$$

Therefore,

$$|\langle \nabla_z P(0,\eta), e\rangle| \leqslant |\nabla_z P(0,\eta)||e| = 2|e|. \tag{2.14}$$

Using (2.13), (2.14) we obtain

$$|\langle \nabla H(0), e\rangle| \leqslant \int_{\boldsymbol{T}} |\nabla_z P(0,\eta)||e||v(\eta)|\,\mathrm{d}\mathcal{H}^1(\eta) = 2|e| \int_{\boldsymbol{T}} |v(\eta)|\,\mathrm{d}\mathcal{H}^1(\eta).$$

Hence, we have that

$$|\nabla H(0)| \leqslant 2 \int_{\boldsymbol{T}} |v(\eta)|\,\mathrm{d}\mathcal{H}^1(\eta). \tag{2.15}$$

Let $v(z) = Y(z_0 + \rho z) - Z$ and let $H(z) = P[v|_{\boldsymbol{T}}](z)$. Then, $H(z) = h(z_0 + \rho z) - Z$ and $\nabla H(0) = \rho \nabla h(z_0)$. Inserting this into (2.15), we obtain

$$\rho|\nabla h(z_0)| = |\nabla H(0)| \leqslant 2 \int_{\boldsymbol{T}} |Y(z_0 + \rho\eta) - Z|\,\mathrm{d}\mathcal{H}^1(\eta). \tag{2.16}$$

Introducing the change of variables $\zeta = z_0 + \rho\eta$ to (2.16), we obtain

$$|\nabla h(z_0)| \leqslant \frac{2}{\rho^2} \int_{|\zeta - z_0| = \rho} |Y(\zeta) - Z|\,\mathrm{d}\mathcal{H}^1(\zeta), \tag{2.17}$$

which is identical to (2.10). □

THEOREM 2.5. *Let $D$ be a bounded domain, whose diameter is $d$. Let $A(z) = \{a^{ij}(z)\}_{i,j=1}^2$ be a symmetric matrix function defined in a domain $\Omega \subset \boldsymbol{C}$ ($a^{ij} = a^{ji}$) satisfying (1.5) and (1.6). Let $u(z)$ be any $C^2$ solution of (1.8) such that*

$$|u(z)| \leqslant M \quad \text{in } D. \tag{2.18}$$

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*Then, there exist constants $C^{(0)}$ and $C^{(1)}$, depending on the modulus of continuity of $u$, $\Lambda$, $\mathfrak{L}$, $B$, $\Gamma$, $M$ and $d$, such that*

$$|\nabla u(z)| < C^{(0)} \rho(z)^{-1} \max_{|\zeta - z| \leqslant \rho(z)} \{|u(\zeta)|\} + C^{(1)}, \tag{2.19}$$

*where* $\rho(z) = \operatorname{dist}(z, \partial D)$.

*Proof.* Fix a point $a \in D$ and let $B_p$, $0 < p < 1$, be a closed disc defined by

$$B_p = \{z; |z - a| \leqslant p \operatorname{dist}(a, \partial D)\},$$

with radius

$$R_p = p \operatorname{dist}(a, \partial D).$$

Define the function $\mu_p$ as

$$\mu_p = \max_{z \in B_p} \{|\nabla u| r_p(z)\}, \tag{2.20}$$

where $r_p(z) = \operatorname{dist}(z, \partial B_p) = R_p - |z - a|$. Then, there exists a point $z_p \in B_p$ such that

$$|\nabla u(z_p)| r_p(z_p) = \mu_p, \quad z_p \in B_p. \tag{2.21}$$

We need the following result to proceed.

LEMMA 2.6. *The function $\mu_p$ is continuous on $(0, 1)$ and has a continuous extension at $0$: $\mu_0 = 0$.*

*Proof of lemma 2.6.* Let $p_n$ be a sequence converging to a number $p$, let

$$\mu_{p_n} = |\nabla u(z_n)| r_{p_n}(z_n)$$

and assume it converges to $\mu_p'$. Prove that $\mu_p' = \mu_p$. Passing to a subsequence, we can assume that $z_n \to z_p'$. Then, $z_p' \in B_p$. Thus, $\mu_p' \leqslant \mu_p$. On the other hand, $\mu_{p_n} \geqslant |\nabla u((1 - \varepsilon_n) z_p)| r_{p_n}((1 - \varepsilon_n) z_p)$, where $\varepsilon_n$ is a positive sequence converging to zero. It follows that

$$\mu_p' \geqslant \lim_{n \to \infty} |\nabla u((1 - \varepsilon_n) z_p)| r_{p_n}((1 - \varepsilon_n) z_p) = \mu_p.$$

Furthermore, since $r_p \leqslant R_p = p \operatorname{dist}(a, \partial D)$, we obtain

$$\lim_{p \to 0^+} \mu_p \leqslant |\nabla u(0)| \lim_{p \to 0^+} R_p = 0.$$

□

Now, let $Tz = \zeta$ be a linear transformation of coordinates such that

$$\sum_{i,j=1}^{2} a^{ij}(z_p) D_{ij} u = \Delta v, \tag{2.22}$$

where $v(\zeta) = u(z)$. By [12, lemma 11.2.1], the transformation $T$ can be chosen such that

$$T = \begin{pmatrix} \lambda_1^{-1/2} & 0 \\ 0 & \lambda_2^{-1/2} \end{pmatrix} \cdot R, \tag{2.23}$$

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where $\lambda_1$ and $\lambda_2$ are eigenvalues of the matrix $A(z_p)$ and $R$ is some orthogonal matrix. Then,

$$\frac{1}{\Lambda} \leqslant \lambda_1, \lambda_2 \leqslant \Lambda.$$

Let

$$\nabla^2 u = \begin{pmatrix} D_{11}u & D_{12}u \\ D_{21}u & D_{22}u \end{pmatrix}$$

denote the Hessian matrix of $u$.

Since

$$\nabla^2 u = T^{\mathrm{T}} \nabla^2 v T,$$

we obtain

$$\begin{aligned} \operatorname{tr}(A^{\mathrm{T}} \nabla^2 u) &= \operatorname{tr}(A^{\mathrm{T}} T^{\mathrm{T}} \nabla^2 v T) \\ &= \operatorname{tr}((TA)^{\mathrm{T}} \nabla^2 v T) \\ &= \operatorname{tr}(\nabla^2 v T (TA)^{\mathrm{T}}) \\ &= \operatorname{tr}(\nabla^2 v T A^{\mathrm{T}} T^{\mathrm{T}}) \\ &= \operatorname{tr}(B^{\mathrm{T}} \nabla^2 v), \end{aligned}$$

where

$$B(\zeta) = TA(z)T^{\mathrm{T}}. \tag{2.24}$$

Then,

$$\begin{gathered} B(\zeta_p) = I, \\ b^{ij}(\zeta) D_{ij} v(\zeta) = a^{ij}(z) D_{ij} u(z), \end{gathered} \tag{2.25}$$

where $B(\zeta) = \{b^{ij}\}_{i,j=1}^2$ and

$$\Delta v = (\delta_{ij} - b^{ij}(\zeta)) D_{ij} v + b^{ij}(\zeta) D_{ij} v. \tag{2.26}$$

Furthermore,

$$T(\boldsymbol{U}(z_p, r_p)) \subset T(B_p) \subset T(D) =: D'.$$

From (2.23), we see that $T(D(z_p, r_p))$ is an ellipse with axes equal to $\lambda_1^{-1/2} \cdot r_p$ and $\lambda_2^{-1/2} \cdot r_p$ and with the centre at $\zeta_p = T(z_p)$. Then, $D_\lambda := \{\zeta \colon |\zeta - \zeta_p| \leqslant \lambda r_p\}$ is a closed disc in $T(B_p)$, provided that

$$0 < \lambda < \frac{1}{2\sqrt{\Lambda}}. \tag{2.27}$$

Let $G(\zeta, \omega)$ be the Green function of the disc $D_\lambda$, so that, from (2.26),

$$\begin{aligned} v = -\frac{1}{\pi} \int_{D_\lambda} G(\zeta, \omega)(\delta_{ij} - b^{ij}(\omega)) D_{ij} v(\omega) \, \mathrm{d}\mathcal{L}^2(\omega) \\ - \frac{1}{\pi} \int_{D_\lambda} G(\zeta, \omega) b^{ij}(\omega) D_{ij} v(\omega) \, \mathrm{d}\mathcal{L}^2(\omega) + h(\zeta), \end{aligned}$$

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where $\mathrm{d}\mathcal{L}^2(z) = \mathrm{d}x\,\mathrm{d}y$ is the Lebesgue two-dimensional measure in the complex plane and $h(\zeta)$ is the harmonic function which takes the same values as $v(\zeta)$ for $\zeta \in \partial D_\lambda$. Then,

$$|\nabla v(\zeta_p)| \leqslant \mathcal{P} + \mathcal{Q} + \mathcal{R}, \tag{2.28}$$

where

$$\begin{aligned}\mathcal{P} &= \left|\frac{1}{\pi}\int_{D_\lambda} \nabla_\zeta G(\zeta_p,\omega) b^{ij}(\omega) D_{ij} v(\omega)\,\mathrm{d}\mathcal{L}^2(\omega)\right|,\\ \mathcal{Q} &= \left|\frac{1}{\pi}\int_{D_\lambda} \nabla_\zeta G(\zeta_p,\omega)(\delta_{ij} - b^{ij}(\omega)) D_{ij} v(\omega)\,\mathrm{d}\mathcal{L}^2(\omega)\right|,\\ \mathcal{R} &= |\nabla_\zeta h(\zeta_p)|.\end{aligned}$$

Furthermore, it follows by (1.6) that $A$ is differentiable almost everywhere. From (2.24), we obtain

$$DB(\zeta)\cdot T = T \cdot DA(z) \cdot T^t \quad \text{for a.e. } z.$$

Here $DA(z)$ is the differential operator defined by

$$A(z+h) = A(z) + DA(z)h + o(|h|).$$

Note that $DA(z)h$ is a matrix. Since $\Lambda^{-1/2}|z| \leqslant |Tz| \leqslant \Lambda^{1/2}|z|$, and bearing in mind (1.6), we obtain

$$\|DB(\zeta)| \leqslant |T|^3 \|DA(z)\| \leqslant \Lambda^{3/2}\mathfrak{L}. \tag{2.29}$$

In the previous formula we mean the following norms: the norm of a matrix $L$ is defined by $|L| = \max\{|Lh|\colon |h| = 1\}$, and the norm of an operator $DX(z)$ by $\|DX(z)\| = \max\{|DA(z)h|\colon |h| = 1\}$ $(X = A, B)$. Thus,

$$|B(\zeta) - B(\zeta_p)| = |B(\zeta) - \mathrm{I}| \leqslant \Lambda^{3/2}\mathfrak{L}|\zeta - \zeta_p|. \tag{2.30}$$

As

$$|T(z) - T(z_p)| \leqslant \lambda r_p(z_p),$$

by using the inequalities

$$\begin{aligned} r_p(z_p) &\leqslant d(z, z_p) + r_p(z),\\ d(z, z_p) &\leqslant \Lambda^{1/2}|T(z) - T(z_p)|\end{aligned}$$

and, by (2.20),

$$|\nabla u(z)| r_p(z) \leqslant \mu_p,$$

we obtain

$$|\nabla u(z)| \leqslant (1 - \lambda\Lambda^{1/2})^{-1} r_p(z_p)^{-1}\mu_p \quad \text{for } z \in T^{-1}(D_\lambda) (\subset B_p).$$

From (2.27), we obtain

$$(1 - \lambda\Lambda^{1/2})^{-2} < 4. \tag{2.31}$$

Bearing in mind that $\nabla u(z) = \nabla v(\zeta)\cdot T$, we obtain

$$|\nabla v(\zeta)| \leqslant 2\Lambda^{1/2} r_p(z_p)^{-1}\mu_p \tag{2.32}$$

for $\zeta \in D_\lambda$.

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Since

$$|a^{ij}(z)D_{ij}u| \leqslant \mathcal{B}|\nabla u|^2 + \Gamma,$$
$$|b^{ij}(\zeta)D_{ij}v(\zeta)| = |a^{ij}(z)D_{ij}u(z)|,$$

it follows that

$$|b^{ij}(\zeta)D_{ij}v(\zeta)| \leqslant \mathcal{B}|T|^2|\nabla v|^2 + \Gamma = \mathcal{B}\Lambda|\nabla v|^2 + \Gamma \tag{2.33}$$

and, therefore, from (2.32), we find that

$$|b^{ij}(\zeta)D_{ij}v(\zeta)| \leqslant 4\Lambda^2\mathcal{B}r_p(z_p)^{-2}\mu_p^2 + \Gamma. \tag{2.34}$$

Now, we divide the proof into four steps.

Step 1 (estimation of $\mathcal{P}$). From (2.3) and (2.34), we have that

$$\begin{aligned}
&\left|\frac{1}{\pi}\int_{D_\lambda} \nabla_\zeta G(\zeta_p,\omega)b^{ij}(\omega)D_{ij}v(\omega)\,\mathrm{d}\mathcal{L}^2(\omega)\right| \\
&\qquad \leqslant \frac{2}{\pi}\int_{|\omega-\zeta_p|\leqslant\lambda r_p(z_p)} \frac{1}{|\omega-\zeta_p|}|b^{ij}(\omega)D_{ij}v(\omega)|\,\mathrm{d}\mathcal{L}^2(\omega) \\
&\qquad \leqslant \frac{2}{\pi}\int_{|\omega-\zeta_p|\leqslant\lambda r_p(z_p)} \frac{1}{|\omega-\zeta_p|}(4\Lambda^2\mathcal{B}r_p(z_p)^{-2}\mu_p^2 + \Gamma)\,\mathrm{d}\mathcal{L}^2(\omega).
\end{aligned}$$

Therefore,

$$\mathcal{P} \leqslant \frac{16\Lambda^2\mathcal{B}\lambda\mu_p^2}{r_p} + 4\Gamma r_p\lambda. \tag{2.35}$$

Step 2 (estimation of $\mathcal{Q}$). Let $\boldsymbol{n_\omega} = (\cos\alpha_1, \cos\alpha_2)$ be the unit inner vector of $\partial D_\lambda$ at $\omega$. Then, from Green's formula

$$\int_{\partial D_\lambda} \sum_{i=1}^{2} u_i(\omega)\cos\alpha_i\,\mathrm{d}\mathcal{H}^1(\omega) = \int_{D_\lambda} (\partial_{\omega_1}u_1 + \partial_{\omega_2}u_2)\,\mathrm{d}\mathcal{L}^2(\omega),$$

proceeding as in [31, Theorem 2], we obtain

$$\begin{aligned}
\mathcal{Q} \leqslant{}& \left|\frac{1}{\pi}\int_{|\omega-\zeta_p|=\lambda r_p(z_p)} \nabla_\zeta G(\zeta_p,\omega)(\delta_{ij} - b^{ij}(\omega))\partial_i v(\omega)\cos\alpha_j\,\mathrm{d}\mathcal{H}^1(\omega)\right| \\
&+ \left|\frac{1}{\pi}\int_{|\omega-\zeta_p|\leqslant\lambda r_p(z_p)} \nabla_\zeta G(\zeta_p,\omega)\partial_{\omega_j}b^{ij}(\omega)\partial_i v(\omega)\,\mathrm{d}\mathcal{L}^2(\omega)\right| \\
&+ \left|\frac{1}{\pi}\int_{|\omega-\zeta_p|\leqslant\lambda r_p(z_p)} \partial_{\omega_j}\nabla_\zeta G(\zeta_p,\omega)(\delta_{ij} - b^{ij}(\omega))\partial_i v(\omega)\,\mathrm{d}\mathcal{L}^2(\omega)\right|.
\end{aligned} \tag{2.36}$$

By using the Cauchy–Schwarz inequality, (2.3), (2.4), (2.29), (2.30), (2.32), we obtain

$$\mathcal{Q} \leqslant 8\Lambda^2\mathfrak{L}\lambda\mu_p + 4\Lambda^2\mathfrak{L}\lambda\mu_p + 4\Lambda^2\mathfrak{L}\lambda\mu_p,$$

i.e.

$$\mathcal{Q} \leqslant 16\Lambda^2\mathfrak{L}\lambda\mu_p. \tag{2.37}$$

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Step 3 (estimation of $\mathcal{R}$). Let $\varpi(t) = \varpi_v(t)$ be the modulus of continuity of $v$ as in lemma 2.3. From (2.10), for $Z = v(\zeta_p)$ $(Z = 0)$, $Y(\zeta) = v(\zeta)$ and $\rho = \lambda r_p(z_p)$, by using lemmata 2.3 and 2.4, we obtain

$$\begin{aligned} \mathcal{R} \leqslant |\nabla h(z_p)| &\leqslant \frac{2}{\lambda^2 r_p(z_p)^2} \int_{|\omega-\zeta_p|=\lambda r_p(z_p)} |v(\omega) - Z| \, \mathrm{d}\mathcal{H}^1(\omega) \\ &\leqslant \frac{2}{\lambda r_p(z_p)} \max\{|v(\zeta) - Z| \colon |\zeta - \zeta_p| = \lambda r_p(z_p)\} \\ &\leqslant \frac{\min\{2\varpi(\lambda r_p(z_p)), 2K\}}{\lambda r_p(z_p)}, \end{aligned} \tag{2.38}$$

where

$$K = \sup_{|z-a|\leqslant\rho(a)} |u(z)|. \tag{2.39}$$

Step 4 (completing the proof). As

$$|\nabla v(\zeta_p)| \geqslant \Lambda^{-1/2} |\nabla u(z_p)| = \Lambda^{-1/2} r_p(z_p)^{-1} \mu_p$$

and $r_p(z_p) < 2\rho(a) \leqslant d$, from (2.28), (2.35), (2.37) and (2.38), we get that

$$A_0 \mu_p^2 + B_0 \mu_p + C_0 \geqslant 0, \tag{2.40}$$

where

$$\begin{aligned} A_0 &= 16\mathcal{B}\Lambda^2\lambda, \\ B_0 &= 16\Lambda^2\mathfrak{L}\lambda r_p(z_p) - \Lambda^{-1/2} \end{aligned}$$

and

$$C_0 = 4\Gamma r_p^2(z_p)\lambda + \frac{2\min\{\varpi(\lambda r_p(z_p)), K\}}{\lambda}.$$

We can take $\lambda > 0$ depending on $\varpi$, $\Lambda$, $\mathfrak{L}$, $B$, $\Gamma$ and $d$ so small that

$$B_0^2 > 4A_0C_0 \tag{2.41}$$

and

$$16\Lambda^2\mathfrak{L}\lambda r_p(z_p)\lambda \leqslant 1/2\Lambda^{-1/2}. \tag{2.42}$$

Let $\mu_1$ and $\mu_2$ $(\mu_1 < \mu_2)$ be the distinct real roots of the equation

$$A_0\mu^2 + B_0\mu + C_0 = 0. \tag{2.43}$$

Then, from (2.40), we have that

$$\mu_p \leqslant \mu_1 \quad \text{or} \quad \mu_p \geqslant \mu_2.$$

Lemma 2.6 asserts that $\mu_p$ depends on $p$ continuously for $0 < p < 1$ and $\lim_{p\to 0} \mu_p = 0$. Then, we have only $\mu_p \leqslant \mu_1$. And, letting $p$ tend to 1, by the definition of $\mu_p$,

$$|\nabla u(a)| \leqslant \mu_1 \rho(a)^{-1}. \tag{2.44}$$

As $\mu_1$ is the smaller root of (2.43),

$$\begin{aligned}\mu_1 &= \frac{-B_0 - \sqrt{B_0^2 - 4A_0C_0}}{2A_0} \\ &= \frac{2C_0}{-B_0 + \sqrt{B_0^2 - 4A_0C_0}} \\ &\leqslant -\frac{2C_0}{B_0}.\end{aligned}$$

From (2.44) and (2.39), we get that

$$|\nabla u(a)| \leqslant C^{(0)}\rho(a)^{-1} \sup_{|z-a|\leqslant\rho(a)} |u(z)| + C^{(1)}, \tag{2.45}$$

where $C^{(0)}$ and $C^{(1)}$ depend on $\Lambda$, $\mathfrak{L}$, $B$, $M$, $\Gamma$, $d$ and on the modulus of continuity of $u$. □

## 2.3. Boundedness of gradient

Definition 2.7. We say that a domain $D$ satisfies the *exterior sphere condition* for some $\kappa > 0$ if to any point $p$ of $\partial D$ there corresponds a ball $B_p \subset \boldsymbol{C}$ with radius $\kappa$ such that $\bar{D} \cap B_p = \{p\}$.

Theorem 2.8 (*A priori* bound, see [31, lemma 2]). *Let $D$ be a complex domain with diameter $d$ satisfying the exterior sphere condition for some $\kappa > 0$. Let $u(z)$ be a twice differentiable mapping satisfying (1.8) in $D$ satisfying the boundary condition $u = 0$ $(z \in G)$. Assume, in addition, that $|u(z)| \leqslant M$, $z \in D$,*

$$\frac{4}{\pi} \cdot 16\mathcal{B}\Gamma M < 1 \tag{2.46}$$

*and $u \in C(\bar{D})$. Then,*

$$|\nabla u| \leqslant \gamma, \quad z \in D, \tag{2.47}$$

*where $\gamma$ is a constant depending only on $\kappa$, $M$, $\mathcal{B}$, $\Gamma$, $\mathfrak{L}$, $\Lambda$ and $d$.*

Remark 2.9. See [8, theorem 15.9] for a related result.

In the statement of [31, lemma 2], instead of (2.46),

$$16\mathcal{B}\Gamma M < 1$$

appears. However, a related proof relies on [31, theorem 2], which, it seems, only works under the condition (2.46). Indeed, the right-hand side of the inequality in the first line of [31, p. 214] should be multiplied by

$$\frac{2\Gamma(1 + m/2)}{\sqrt{\pi}\Gamma((m+1)/2)},$$

where $m$ is the dimension of the space (in our case $m = 2$) and

$$\frac{2\Gamma(1 + 2/2)}{\sqrt{\pi}\Gamma((2+1)/2)} = \frac{4}{\pi}.$$

## 3. Proof of the main theorem

We need the following lemmas.

LEMMA 3.1 (Kalaj [15]). *Every $K$-q.c. mapping $w(z) = \rho(z)S(z)\colon D \to \Omega$, $D, \Omega \subset \mathbb{C}$, $\rho = |w|$, $S(z) = \mathrm{e}^{\mathrm{i}s(z)}$, $s(z) \in [0, 2\pi)$, satisfies the inequalities*

$$\rho|\nabla S| \leqslant K|\nabla\rho| \tag{3.1}$$

*and*

$$|\nabla\rho| \leqslant K\rho|\nabla S| \tag{3.2}$$

*almost everywhere on $D$. Inequalities (3.1) and (3.2) are sharp; the equality*

$$\rho|\nabla S| = |\nabla\rho| \tag{3.3}$$

*holds if $w$ is a 1-quasi-regular mapping. We also have that*

$$K^{-1}|\nabla w| \leqslant |\nabla\rho| \leqslant |\nabla w|. \tag{3.4}$$

LEMMA 3.2. *If $w = \rho S\colon \mathbb{U} \to \mathbb{U}$, $\rho = |w|$, is twice differentiable, then*

$$L[\rho] = \rho(a^{11}|p|^2 + 2a^{12}\langle p, q\rangle + a^{22}|q|^2) + \langle L[w], S\rangle, \tag{3.5}$$

*where $p = D_1 S$ and $q = D_2 S$.*

*If, in addition, $w$ is $K$-q.c. and satisfies*

$$|L[w]| = \left|\sum_{i,j=1}^{2} a^{ij}(z) D_{ij} w\right| \leqslant \mathcal{B}|\nabla w|^2 + \Gamma, \tag{3.6}$$

*then there exists a constant $\Theta$ depending on $K$, $\mathcal{B}$ and $\Gamma$, such that*

$$|L[\rho]| \leqslant \frac{\Theta}{\rho}|\nabla\rho|^2 + \Gamma. \tag{3.7}$$

*Proof.* Let $w = (w_1, w_2)$ (here $w_i$ are real), $S = (S_1, S_2)$ and let $f = (f_1, f_2)$. For real differentiable functions $a$ and $b$, define the bilinear operator by

$$D[a, b] = \sum_{k,l=1}^{2} a^{kl}(z) D_k a(z) D_l b(z).$$

Since $w_i = \rho S_i$, $i \in \{1, 2\}$, and

$$\rho = \sum_{i=1}^{2} S_i w_i,$$

we obtain

$$L[w_i] = S_i L[\rho] + \rho L[S_i] + 2D[\rho, S_i], \quad i \in \{1, 2\}, \tag{3.8}$$

and

$$L[\rho] = \sum_{i=1}^{2} w_i L[S_i] + \sum_{i=1}^{2} S_i L[w_i] + 2\sum_{i=1}^{2} D[S_i, w_i]. \tag{3.9}$$

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From (3.8), we obtain

$$\begin{aligned} L[\rho] &= L[\rho]|S|^2 \\ &= \sum_{i=1}^{2} S_i \cdot S_i L[\rho] \\ &= \sum_{i=1}^{2} S_i L[w_i] - \rho \sum_{i=1}^{2} S_i L[S_i] - 2 \sum_{i=1}^{2} S_i D[\rho, S_i]. \end{aligned} \tag{3.10}$$

By adding (3.9) and (3.10), we obtain

$$L[\rho] = \sum_{i=1}^{2} (D[S_i, w_i] - S_i D[\rho, S_i]) + \langle L[w], S \rangle.$$

On the other hand,

$$\begin{aligned} D[S_i, w_i] - S_i D[S_i, \rho] &= \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l w_i - S_i \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l \rho \\ &= \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i (\rho D_l S_i + S_i D_l \rho) - S_i \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l \rho \\ &= \rho \sum_{k,l=1}^{2} a^{kl}(z) D_k S_i D_l S_i, \quad i = 1, 2. \end{aligned}$$

Thus,

$$\begin{aligned} L[\rho] &= \rho \sum_{i,k,l=1}^{2} a^{kl}(z) D_k S_i D_l S_i + \langle L[w], S \rangle \\ &= \rho (a^{11} |p|^2 + 2a^{12} \langle p, q \rangle + a^{22} |q|^2) + \langle L[w], S \rangle, \end{aligned}$$

where $p = (D_1 S_1, D_1 S_2)$ and $q = (D_2 S_1, D_2 S_2)$. Therefore,

$$\begin{aligned} |L[\rho]| &\leqslant \Lambda \rho (|p|^2 + |q|^2) + (\mathcal{B} |\nabla w|^2 + \Gamma) \\ &= \Lambda \rho \|\nabla S\|^2 + (\mathcal{B} |\nabla w|^2 + \Gamma), \end{aligned}$$

provided (3.6) holds. Here, $\| \cdot \|$ is the Hilbert–Schmidt norm, which satisfies the inequality $\|P\| \leqslant \sqrt{2} |P|$. If $w$ is $K$-q.c., then, according to (3.1) and (3.3), we have that

$$|L[\rho]| \leqslant 2K\Lambda |\nabla \rho|^2 \rho^{-1} + (\mathcal{B} K |\nabla \rho|^2 + \Gamma).$$

Taking $\Theta = 2K\Lambda + \mathcal{B}K$, we obtain (3.7). □

LEMMA 3.3. *If $f = u + \mathrm{i}v$ is a $K$-q.c. mapping satisfying the elliptic differential inequality, then $u$ and $v$ satisfy the elliptic differential inequality.*

*Proof.* Let

$$A := |\nabla u|^2 = 2(|u_z|^2 + |u_{\bar z}|^2) = \tfrac{1}{2}(|f_z + \bar f z|^2 + |f_{\bar z} + \bar f_z|^2)$$

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and

$$B := |\nabla v|^2 = 2(|v_z|^2 + |v_{\bar{z}}|^2) = \tfrac{1}{2}(|f_z - \bar{f}z|^2 + |f_{\bar{z}} - \bar{f}_z|^2).$$

Then,

$$\frac{A}{B} = \frac{|1+\mu|^2}{|1-\mu|^2},$$

where $\mu = \bar{f}z/f_z$. Since $|\mu| \leqslant k = (K-1)/(K+1)$,

$$\frac{(1-k)^2}{(1+k)^2} \leqslant \frac{A}{B} \leqslant \frac{(1+k)^2}{(1-k)^2}. \tag{3.11}$$

As

$$|L[f]| = |L[u] + \mathrm{i}L[v]| \leqslant \mathcal{B}|\nabla f|^2 + \varGamma \leqslant \mathcal{B}(|\nabla u|^2 + |\nabla v|^2) + \varGamma,$$

the relation (3.11) yields

$$|L[u]| \leqslant \mathcal{B}\biggl(1 + \frac{(1+k)^2}{(1-k)^2}\biggr)|\nabla u|^2 + \varGamma$$

and

$$|L[v]| \leqslant \mathcal{B}\biggl(1 + \frac{(1+k)^2}{(1-k)^2}\biggr)|\nabla v|^2 + \varGamma.$$

□

Before proving the main results of this paper, let us recall one of the most fundamental results concerning quasi-conformal mappings.

PROPOSITION 3.4 (Mori). *If* $w\colon \mathbb{U} \to \mathbb{U}$, $w(0) = 0$, *is a* $K$*-q.c. harmonic mapping of the unit disc onto itself, then*

$$|w(z_1) - w(z_2)| \leqslant 16|z_1 - z_2|^{1/K}, \quad z_1, z_2 \in \mathbb{U}.$$

Mori's theorem for q.c. self-mappings of the unit disc has been generalized in various directions in the plane and in the space. See, for example, [4, 6, 24].

*Proof of theorem 1.1.* The main thrust of the proof is to estimate the gradient of $w$ in some 'neighbourhood' of the boundary together with some interior estimate in the rest of the unit disc. Set $\alpha, \beta \in \boldsymbol{R}$ such that $\frac{1}{2}(1 + |a|) \leqslant \alpha < 1$ and $\beta = \frac{1}{2}(\alpha + 1)$. Define $D_\alpha = \{z\colon |z| \leqslant \beta\}$ and $A_\alpha = \{z\colon \alpha \leqslant |z| < 1\}$.

Let $w = (w_1, w_2)$. According to theorem 2.5 and lemma 3.3, there exists a constant $C_i$ depending only on the modulus of continuity of $w_i$, $\mathcal{B}$, $\varGamma$, $K$, $\varLambda$, $\mathfrak{L}$ and $\alpha$, such that

$$|\nabla w_i(z)| \leqslant C_i, \quad z \in D_\alpha, \quad i = 1, 2. \tag{3.12}$$

By Mori's theorem, the modulus of continuity of $w_i$ depends only on $K$ and $a$. Thus,

$$|\nabla w(z)| \leqslant |\nabla w_1| + |\nabla w_2| \leqslant C_1 + C_2 = C_3(K, \mathcal{B}, \varGamma, \varLambda, \mathfrak{L}, \alpha), \quad z \in D_\alpha. \tag{3.13}$$

As $w$ is a $K$-q.c. self-mapping of the unit disc, by Mori's theorem [40], it satisfies the inequality

$$4^{1-K}\left|\frac{a-z}{1-z\bar{a}}\right|^K \leqslant |w(z)|, \quad |z|<1, \tag{3.14}$$

where $a = w^{-1}(0)$. Let $u = |w|$. From lemma 3.2 and (3.14), we find that

$$|L[u]| \leqslant 2^{3K-2}\left(\frac{1+|a|}{1-|a|}\right)^K \varTheta|\nabla u|^2 + \varGamma, \quad (1+|a|)/2 < |z| < 1. \tag{3.15}$$

Let $g$ be a function

$$g\colon A_\alpha \to \mathbb{R}$$

defined as

$$g(z) = \begin{cases} 1 & \text{if } \beta < |z| \leqslant 1, \\ 1 + (u(z)-1)\dfrac{\exp(1/(|z|^2-\beta^2))}{\exp(1/(\alpha^2-\beta^2))} & \text{if } \alpha \leqslant |z| \leqslant \beta. \end{cases}$$

Define

$$\phi(z) := \frac{\exp(1/(|z|^2-\beta^2))}{\exp(1/(\alpha^2-\beta^2))}.$$

Then,

$$L[g] = \begin{cases} 0 & \text{if } \beta < |z| \leqslant 1, \\ (u(z)-1)L[\phi] + \phi L[u] + D[u,\phi] & \text{if } \alpha \leqslant |z| \leqslant \beta. \end{cases}$$

Therefore,

$$|L[g]| \leqslant \begin{cases} 0 & \text{if } \beta < |z| \leqslant 1, \\ \mathcal{B}_1|\nabla u|^2 + \varGamma_1 & \text{if } \alpha \leqslant |z| \leqslant \beta, \end{cases} \tag{3.16}$$

where

$$\mathcal{B}_1 = 2^{3K-2}\left(\frac{1+|a|}{1-|a|}\right)^K (2K\varLambda + \mathcal{B}K)$$

and $\varGamma_1$ is a constant depending only on $K$, $\mathcal{B}$, $\varGamma$, $\varLambda$, $\mathfrak{L}$ and $\alpha$. By (3.4), (3.13) and (3.16), we have that

$$|L[g]| \leqslant C_4(K, \mathcal{B}, \varGamma, \varLambda, \mathfrak{L}, \alpha), \quad z \in A_\alpha, \tag{3.17}$$

and

$$|\nabla g| \leqslant C_5(K, \mathcal{B}, \varGamma, \varLambda, \mathfrak{L}, \alpha), \quad z \in A_\alpha. \tag{3.18}$$

Furthermore, by (3.15), (3.17), (3.18) and $|a+b|^2 \leqslant 2(|a|^2+|b|^2)$, we have that

$$\begin{aligned} |L[u-g]| &\leqslant |L[u]| + |L[g]| \\ &\leqslant \mathcal{B}_1|\nabla u|^2 + C_7(K, \mathcal{B}, \varGamma, \varLambda, \mathfrak{L}, \alpha) \\ &\leqslant 2\mathcal{B}_1|\nabla u - \nabla g|^2 + C_8(K, \mathcal{B}, \varGamma, \varLambda, \mathfrak{L}, \alpha), \quad z \in A_\alpha. \end{aligned}$$

By Mori's theorem, there exists a constant $\alpha = \alpha(K, a) < 1$, such that

$$M = \max\{|u(z) - g(z)|\colon z \in A_\alpha\}$$

is small enough, satisfying the inequality

$$\frac{64}{\pi} \cdot 2\mathcal{B}_1 M \Lambda < 1. \tag{3.19}$$

Thus, $\tilde{u} = u - g$ satisfies the conditions of theorem 2.8 in the domain $D = A_\alpha$. The conclusion is that $\nabla u$ is bounded in $\beta < |z| < 1$ by a constant depending only on $K$, $\mathcal{B}$, $\Gamma$, $\Lambda$, $\mathfrak{L}$ and $a$ and on the modulus of continuity of $\tilde{u}$. From Mori's theorem, the modulus of continuity of $u$ depends only on $K$ and $a$. Combining (3.18) with (3.4), we obtain

$$|\nabla w| \leqslant C_0(K, \mathcal{B}, \Gamma, \Lambda, \mathfrak{L}, a), \quad \beta < |z| < 1. \tag{3.20}$$

From (3.13) and (3.20), we obtain the desired conclusion. □

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