Canad. Math. Bull. Vol. **47** (1), 2004 pp. 12–16

On Newton's Method and Rational Approximations to Quadratic Irrationals

Dedicated to the memory of H. William Oliver, Thomas T. Read Professor of Mathematics at Williams College

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Abstract. In 1988 Rieger exhibited a differentiable function having a zero at the golden ratio $(-1+\sqrt{5})/2$ for which when Newton's method for approximating roots is applied with an initial value $x_0 = 0$, all approximates are so-called "best rational approximates"—in this case, of the form F_{2n}/F_{2n+1}, where F_n denotes the n-th Fibonacci number. Recently this observation was extended by Komatsu to the class of all quadratic irrationals whose continued fraction expansions have period length 2. Here we generalize these observations by producing an analogous result for all quadratic irrationals and thus provide an explanation for these phenomena.

1 Introduction

Let $\varphi = \frac{-1+\sqrt{5}}{2}$ denote the golden ratio and write $\varphi = [0, 1, 1, 1, \dots] = [0, \bar{1}]$ for its continued fraction expansion. The best rational approximations (known as the *convergents*) of φ are defined by the truncated continued fractions $\frac{p_n}{q_n} = [0, 1, 1, \dots, 1]$, where the string of 1's has length n. It follows that $\frac{p_n}{q_n} = \frac{F_n}{F_{n+1}}$, where F_n denotes the n-th Fibonacci number. In 1999, Rieger [5] (see also [4]) produced a differentiable function f_φ with $f_\varphi(\varphi) = 0$, such that if Newton's method for approximating zeros is applied by declaring $x_0 = 0$ and for $n \geq 0$,

$$x_{n+1} = x_n - \frac{f_\varphi(x_n)}{f'_\varphi(x_n)},$$

then $x_n = \frac{p_{2n}}{q_{2n}} = \frac{F_{2n}}{F_{2n+1}}$. That is, the approximations generated by Newton's method with initial value $x_0 = 0$ are precisely the even-indexed convergents of φ.

Very recently, Komatsu [3] (see also his related work [2]) extended Rieger's construction to quadratic irrationals whose continued fraction expansions have period length 2. Specifically, for positive integers a and b, let $\alpha(a, b)$ be the quadratic irrational having continued fraction expansion $[0, a, b, a, b, \dots] = [0, \overline{a, b}]$ and let p_n/q_n denote its n-th convergent. Then Komatsu constructed a function $f_{\alpha(a,b)}$ such

Received by the editors April 16, 2002.
AMS subject classification: 11A55, 11B37.

that $f_{\alpha(a,b)}\big(\alpha(a,b)\big) = 0$ and if Newton's method is applied with $x_0 = 0$ and for $n \geq 0$,

$$x_{n+1} = x_n - \frac{f_{\alpha(a,b)}(x_n)}{f'_{\alpha(a,b)}(x_n)},$$

then $x_n = \frac{p_{2n}}{q_{2n}}$. Komatsu closed his paper with the remark: "Further generalization seems nearly impossible. For example, if $\theta = [0, \overline{a, b, c, d}]$, p_{4n+4}/q_{4n+4} cannot be expressed by the linear relation of p_{4n} and q_{4n}."

The purpose of this note is to introduce a construction that generalizes the previous results to all quadratic irrationals whose reciprocals have purely periodic continued fraction expansions. That is, we show that the observed phenomenon holds for the reciprocal of any reduced quadratic irrational.

Let α be a quadratic irrational having continued fraction expansion $[0, \overline{a_1, a_2, \ldots, a_L}]$ and let p_n/q_n denote the n-th convergent associated with α. By repeating the minimal period once, if necessary, we can assume without loss of generality that L is even. In order to produce the differentiable function having the diophantine structure we seek, we require an auxiliary number. We write $\tilde{\alpha}$ for the quadratic irrational given by $\tilde{\alpha} = [0, \overline{a_L, a_{L-1}, \ldots, a_1}]$ and write P_n/Q_n for the n-th convergent of $\tilde{\alpha}$. We note that $\tilde{\alpha} = -1/\bar{\alpha}$, where $\bar{\alpha}$ denotes the conjugate of α (see, for example, [1]). Our result can now be stated as the following theorem.

Theorem *Let α be a quadratic irrational having a continued fraction expansion of the form $\alpha = [0, \overline{a_1, a_2, \ldots, a_L}]$ where, without loss of generality, L is even. Given the notation of the previous paragraph, let $A = \frac{\alpha P_L + Q_L}{\alpha - \bar{\alpha}}$ and write $f_\alpha \colon (\bar{\alpha}, \alpha] \to \mathbb{R}$ for the function defined by*

$$f_\alpha(x) = \left(1 - \frac{x}{\alpha}\right)^{A/P_L} \left(1 - \frac{x}{\bar{\alpha}}\right)^{\bar{A}/P_L}.$$

If the sequence x_n is generated by the initial value $x_0 = 0$, and for $n \geq 0$,

$$x_{n+1} = x_n - \frac{f_\alpha(x_n)}{f'_\alpha(x_n)},$$

then $x_n = \frac{p_{nL}}{q_{nL}}$.

The function f_α defined in the theorem agrees with the previously found functions f_φ and $f_{\alpha(a,b)}$ when suitably specialized. In the case of φ, we must consider $\alpha = [0, \overline{1, 1}]$, that is, $L = 2$.

2 Proof of the Theorem

We begin with the following useful albeit elementary lemma.

Lemma *Let α be a quadratic irrational having a continued fraction expansion of the form $\alpha = [0, \overline{a_1, a_2, \ldots, a_L}]$, and let p_n/q_n denote its n-th convergent. Then for $n \geq 0$,*

$$\frac{q_{(n+1)L-1}}{q_{(n+1)L}} = \frac{q_{L-1}q_{nL} + p_{L-1}q_{nL-1}}{q_L q_{nL} + p_L q_{nL-1}}.$$

Proof of the Lemma By a fundamental correspondence between 2×2 matrices and continued fractions (see, for example, [1] or [6]), we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{(n+1)L} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_{(n+1)L} & p_{(n+1)L-1} \\ q_{(n+1)L} & q_{(n+1)L-1} \end{pmatrix}. \tag{2.1}$$

However, given the periodicity of the continued fraction expansion of α, the previous product of matrices can also be expressed as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_L & 1 \\ 1 & 0 \end{pmatrix} \right)^n \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_L & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} p_{nL} & p_{nL-1} \\ q_{nL} & q_{nL-1} \end{pmatrix} \begin{pmatrix} q_L & q_{L-1} \\ p_L & p_{L-1} \end{pmatrix} = \begin{pmatrix} p_{nL}q_L + p_{nL-1}p_L & p_{nL-1}q_{L-1} + p_{nL-1}p_{L-1} \\ q_{nL}q_L + q_{nL-1}p_L & q_{nL}q_{L-1} + q_{nL-1}p_{L-1} \end{pmatrix}.$$

The lemma now follows by examining the corresponding elements of the second rows of the two equivalent matrices.

Proof of the Theorem If we apply the Lemma to $\tilde{\alpha}$, then for all $n \geq 0$,

$$\frac{Q_{(n+1)L-1}}{Q_{(n+1)L}} = \frac{P_{L-1}\left(\frac{Q_{nL-1}}{Q_{nL}}\right) + Q_{L-1}}{P_L\left(\frac{Q_{nL-1}}{Q_{nL}}\right) + Q_L}. \tag{2.2}$$

By a well-known identity (see [1] or [6]), we have that for all $n \geq 1$,

$$\frac{Q_{nL-1}}{Q_{nL}} = [0, (a_1, a_2, \dots, a_L)^n] = \frac{p_{nL}}{q_{nL}}, \tag{2.3}$$

where by $(a_1, a_2, \dots, a_L)^n$ we mean that the string $a_1, a_2, \dots, a_L$ is repeated n times.

Next we define the functions $N(x)$ and $D(x)$ by

$$N(x) = \frac{P_{L-1}x + Q_{L-1}}{P_L x + Q_L},$$

and

$$D(x) = N(x) - x = \frac{-\left(P_L x^2 + (Q_L - P_{L-1})x - Q_{L-1}\right)}{P_L x + Q_L}.$$

We now wish to factor the numerator of $D(x)$. We begin by noting that

$$\frac{1}{\tilde{\alpha}} = \left[a_L, a_{L-1}, \dots, a_1, \frac{1}{\tilde{\alpha}}\right],$$

which in view of the relation $\tilde{\alpha} = -1/\bar{\alpha}$, yields

$$-\bar{\alpha} = [a_L, a_{L-1}, \dots, a_1, -\bar{\alpha}].$$

https://doi.org/10.4153/CMB-2004-002-4 Published online by Cambridge University Press

The previous identity together with the correspondence between continued fractions and matrices given in (2.1) implies that

$$\begin{aligned}\begin{pmatrix} a_L & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{L-1} & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -\bar{\alpha} & 1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} Q_L & Q_{L-1} \\ P_L & P_{L-1} \end{pmatrix}\begin{pmatrix} -\bar{\alpha} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -Q_L\bar{\alpha} + Q_{L-1} & Q_L \\ -P_L\bar{\alpha} + P_{L-1} & P_L \end{pmatrix}.\end{aligned}$$

It thus follows that

$$-\bar{\alpha} = \frac{-Q_L\bar{\alpha} + Q_{L-1}}{-P_L\bar{\alpha} + P_{L-1}},$$

or equivalently,

$$P_L\bar{\alpha}^2 + (Q_L - P_{L-1})\bar{\alpha} - Q_{L-1} = 0.$$

Therefore we conclude that

$$D(x) = -\frac{P_L(x-\alpha)(x-\bar{\alpha})}{P_Lx + Q_L}.$$

It is easy to verify that for $\bar{\alpha} < x < \alpha$, $D(x) > 0$. A direct calculation reveals the identity

$$\frac{1}{D(x)} = \frac{1}{P_L}\left(\frac{A}{\alpha - x} + \frac{\bar{A}}{x - \bar{\alpha}}\right),$$

where $A = \frac{\alpha P_L + Q_L}{\alpha - \bar{\alpha}}$. Thus for $0 \le x < \alpha$, we have

$$\exp\left(-\int_0^x \frac{dt}{D(t)}\right) = \exp\left(-\frac{1}{P_L}\ln\left(\left(1 - \frac{x}{\alpha}\right)^{-A}\left(1 - \frac{x}{\bar{\alpha}}\right)^{-\bar{A}}\right)\right) = f_\alpha(x).$$

Hence for $0 \le x < \alpha$, we see that

$$\frac{f_\alpha'(x)}{f_\alpha(x)} = \frac{-1}{D(x)}$$

and therefore

$$x - \frac{f_\alpha(x)}{f_\alpha'(x)} = x + D(x) = N(x) = \frac{P_{L-1}x + Q_{L-1}}{P_Lx + Q_L}.$$

Thus given how the sequence x_n was defined, we see that $x_0 = 0$ and for $n \ge 0$,

$$x_{n+1} = \frac{P_{L-1}x_n + Q_{L-1}}{P_Lx_n + Q_L}.$$

So $x_1 = Q_{L-1}/Q_L$ and therefore by the recurrence in (2.2), for $n \ge 1$, $x_n = Q_{nL-1}/Q_{nL}$, which by identity (2.3) implies that $x_n = p_{nL}/q_{nL}$. As L is even, we have that $0 \le x_n < \alpha$, which allows us to iterate this process and thus complete the proof.

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