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A SOLUTION METHOD FOR LINEAR RATIONAL EXPECTATION MODELS UNDER IMPERFECT INFORMATION

KATSUYUKI SHIBAYAMA
University of Kent at Canterbury

This article presents a solution algorithm for linear rational expectation models under imperfect information, in which some decisions are made based on smaller information sets than others. In our solution representation, imperfect information does not affect the coefficients on crawling variables, which implies that, if a perfect-information model exhibits saddle-path stability, for example, the corresponding imperfect-information models also exhibit saddle-path stability. However, imperfect information can significantly alter the quantitative properties of a model. Indeed, this article demonstrates that, with a predetermined wage contract, the standard RBC model remarkably improves the correlation between labor productivity and output.

Keywords: Linear Rational Expectation Models, Imperfect Information

1. INTRODUCTION

This article presents a solution algorithm for linear rational expectation models under imperfect information. "Imperfect information" in this article signifies that some decisions are made before observing some shocks, whereas others are made after observing them. For example, we can consider a variant of the standard RBC model, in which households predetermine wage (and commit themselves to accommodating any labor demand) before observing today's productivity shock. In this variant, the equations that define the equilibrium are the same as in the standard RBC model, except for the information structure; i.e., the first-order condition (FOC) on labor supply has an expectation operator.

Imperfect information is an interesting consideration for several reasons. First, imperfect information plays an important role in many important classes of models, such as the sticky information model of Mankiw and Reis (2001). Second, researchers often do not know a priori what information is available when each

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decision is made; hence, they may want to estimate the information structure by parameterizing it, or they may want to experiment on a model under several patterns of information structure. It is easy to implement such exercises with our algorithm; once structural equations under the corresponding perfect information are obtained, then the additional input to the algorithm is only the information structure in a model. Third, the obtained numerical result may not be robust for a small change in information structure. Indeed, we present a variant of the RBC model with a predetermined wage contract to demonstrate that changing information structure remarkably improves the model performance in terms of the correlation between labor productivity and output.

This article offers a set of easy-to-use Matlab codes to solve a general class of linear models under imperfect information.[1] The solution method is an extension of the QZ method of Sims (2002). The algorithm solves the system of linear difference equations in the form

$$0 = \tilde{E}_t[Ay_{t+1} + By_t] + C\xi_t, \tag{1}$$

where A, B, and C are proper coefficient matrices, and y_t and ξ_t are the vectors of endogenous and exogenous variables, respectively. The expectation operator $\tilde{E}_t[\,]$ is nonstandard because the information set in each equation can differ from each other (imperfect information).

The algorithm provides the solution of a model in the form of

$$\begin{aligned} \kappa_{t+1} &= H\kappa_t + J\xi^{t,S}, \\ \phi_t &= F\kappa_t + G\xi^{t,S}, \\ \xi^{t,S} &:= \left(\xi_t^T \quad \cdots \quad \xi_{t-S}^T\right),^T \end{aligned}$$

where κ_t and ϕ_t are the vectors of crawling and jump variables, respectively,[2] and ξ_{t-s} is the vector of innovations at time $t - s$, for $s = 0, \ldots, S$, where S is such that the minimum information set in the model includes all information up to time $t - S - 1$. The superscript T indicates transposition, and hence $\xi^{\tau,S}$ is the vertical concatenation of $\{\xi_{\tau-s}\}_{s=0}^{S}$. H, J, F and G are the solution matrices provided by the algorithm.

It is important to note that the state variables in this solution are κ_t and $\xi^{t,S}$. Imperfect information requires expansion of the state space, but this can be done either by expanding the innovation vector or by expanding the set of crawling variables; i.e., representation of the state space is not necessarily unique. Our choice of state variables works, intuitively, because, if past innovations are recorded, we can recover the past crawling variables and hence recover the information available in past periods.

By keeping the number of crawling variables unchanged, it can be shown that the dynamic parts of the solution (i.e., the H and F matrices) are the same as in the corresponding perfect information model.[3] Thus, it is clear that if the corresponding perfect model is saddle-path stable (sunspot, explosive), then an imperfect information model is also saddle-path stable (sunspot, explosive, respectively). That is, the information structure does not alter the dynamic stability property. In this sense, we can say that *qualitatively* an imperfect-information model inherits key properties of the corresponding perfect-information model. However, *quantitatively* imperfect information can have significant effects, as shown in Section 5.

Moreover, invariant H and F matrices imply that the direct effects of imperfect information on impulse response functions (IRFs) last for only S periods after an impulse. In subsequent periods, IRFs follow essentially the same process as in the perfect information counterpart. More specifically, suppose that an endogenous variable a_t is determined S periods in advance (observing κ_{t-S} and ξ_{t-S}). In this case, the IRFs are directly affected by the information imperfection from time t to $t+S-1$. At $t+S$, however, the IRFs show sudden jumps because a_{t+S} starts reacting to innovations at t. Let κ_{t+S} be the values of the crawling variables at the beginning of $t+S$. Then the following IRFs follow exactly the same time path as those under perfect information that start with κ_{t+S} (without innovations). One such example can be found in Dupor and Tsuruga (2005), who argue that the hump-shaped IRFs found in Mankiw and Reis (2001) critically hinge on the assumption of Calvo-style information updating, in which some agents, though their population decreases over time, cannot renew their information forever. By instead constructing Taylor-style staggered information renewal, Dupor and Tsuruga (2005) show that IRFs jump to zero right after the last cohort renews its information set. We show, however, that such sudden jumps in IRFs are rather common observations in imperfect information models.

There are, at least allegedly, three existing treatments of imperfect information.[4] The first remedy for imperfect information is to define *dummy* variables.[5] For example, consider a variant of the standard RBC model, in which labor supply L_t is determined without observing today's innovations. Then the optimal labor supply is determined by

$$0 = E_{t-1}\left[\eta L_t + \sigma C_t - W_t\right], \tag{2}$$

where C_t and W_t are consumption and wage at time t, η and σ are parameters provided by the theory, and $E_{t-1}[\,]$ is the expectation operator with all information up to time $t-1$. Define a dummy variable L_t^* such that

$$\begin{aligned} 0 &= E_t\left[\eta L_{t+1}^* + \sigma C_{t+1} - W_{t+1}\right], \\ L_{t+1} &= L_t^*. \end{aligned}$$

In this method, having the additional crawling variable L_t, the set of crawling variables is expanded. The problem with this method is that it cannot solve the model if some endogenous variables are determined before some (not all) of today's innovations are observed but after the others are observed.

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The second method developed by Wang and Wen (2006) is more closely related to our method, in the sense that they also chose to expand expectation error instead of crawling variables. Apart from the difference in the bases of the algorithm (they employ the method of undetermined coefficients, while we use QZ method), however, there are three major differences. First, our algorithm allows more flexible specification; with our method, an endogenous variable can be determined observing some innovations but not observing the others at t, whereas their method deals with lagged expectations such as the dummy variable method mentioned above. Second, our method only requires two indicator matrices (see the next paragraph), which specify whether each variable is decided with or without observation of each innovation, whereas they require researchers to solve for their Λ_i and Γ_i matrices ($i = 1, \ldots, S$). Third and most importantly, our method reveals sharper analytical results (see note 3, for example).

The other possibility is a modification of the method of undetermined coefficients. According to Christiano (1998), his version of method of undetermined coefficients, like ours, can deal with models in which some endogenous variables are determined before some (not all) of today's innovations are observed but after the others are observed. The most salient difference between his method and ours is in the specification of information structure; Christiano (1998) requires a user to provide only one matrix R that specifies which innovations are to be included in the information set of each expectation operator. Roughly speaking, his R relates equations to observable innovations. In contrast, in the algorithm developed in this paper, a researcher must specify two indicator matrices; one relates innovations to equations [like Christiano (1998)], and the other relates innovations to variables. To understand why the latter matrix is necessary, consider the above example (2). Certainly, it is clear that a researcher must specify the information set of the expectation operator in (2). However, in a given information set, there are generically three possibilities, namely that (a) the representative household fixes labor supply before observing some of today's innovations, (b) it determines wage before innovations (sticky wage), or (c) it decides consumption before innovations. Hence, one more matrix is necessary in our algorithm to specify which of C_t, W_t, or H_t is chosen while not having full information. In general, the quantitative behavior of a model is completely different, depending on which variables are assumed to be decided before observing some information. Indeed, Section 5 shows that the difference between (a) and (b) is very crucial.

The plan of this article is as follows. In Section 2, we define the problem and derive the solution, and show two key observations: (i) if the kth time t variable $y_{k,t}$ is determined without observing the ith time $t-s$ innovations $\xi_{i,t-s}$, then $y_{k,t}$ cannot respond to $\xi_{i,t-s}$; and (ii) if the expectation operator in the jth equation has an information set that includes $\xi_{i,t-s}$, then $\xi_{i,t-s}$ cannot be the source of the expectation error in the jth equation. It turns out that these two restrictions are enough to determine the unique solution coefficients. In Section 3, we discuss the assumptions that are necessary for guaranteeing the existence of a solution. Each of them has some economic meaning, and the existence condition is slightly tighter under imperfect information than under perfect information. In Section 4, the main features of the solution of imperfect information models are briefly discussed. Most of them are direct consequences of the invariant H and F matrices. In Section 5, we demonstrate the effects of imperfect information on the standard RBC model as an example. The final section concludes the discussion.

2. DERIVATION OF THE SOLUTION

Essentially, our algorithm is an extension of the QZ method used in Sims (2002). Our objective is to obtain the state space representation of a solution that satisfies two key zero restrictions. For the details of the matrix notation, see the Appendix.

2.1. Definition of the Problem

This section defines the inputs and outputs of the algorithm.

Given models. Instead of using expectation operators such as (1), following Sims (2002), we formulate the linear rational models with expectation errors as follows:

$$0 = Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S}, \tag{3}$$

where

$$y_t = \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}, \quad \xi^{t,S} = \begin{pmatrix} \xi_t \\ \vdots \\ \xi_{t-S} \end{pmatrix},$$

$$\begin{aligned} E &:= [\, E_0 \quad E_1 \quad \cdots \quad E_s \quad \cdots \quad E_S \,] \\ &:= \begin{bmatrix} E_{0,11} & \cdots & E_{0,1N} & & E_{s,11} & \cdots & E_{s,1N} & & E_{S-1,11} & \cdots & E_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ E_{0,M1} & \cdots & E_{0,MN} & & E_{s,M1} & \cdots & E_{s,MN} & & E_{S-1,M1} & \cdots & E_{S,MN} \end{bmatrix}. \end{aligned}$$

y_t is the vector of all endogenous variables, in which κ_t is the vector of crawling variables and ϕ_t is that of jump variables. Stock variables are all recorded at the beginning of each period. M is the number of equations, which is equal to the number of endogenous variables; N is the number of innovations; and S is such that the minimum information set includes ξ_{t-S-1}.

ξ_{t-s} is a column vector of i.i.d. innovations at time $t-s$. Limiting ξ_t to be i.i.d. is not restrictive, because we can add the law of motions of serially correlated shocks to the system of equations and treat the shocks themselves as crawling variables.[6]

Only two sets of inputs are required: (i) coefficient matrices A, B, and C, which are typically the same as in perfect-information models; and (ii) indicator matrices IndE and IndV (their elements are either zero or one).[7] The size of IndE is the

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same as that of E in (3), and, if the (i, j)th element of E is zero, then the (i, j)th element of IndE is also zero. Essentially, IndE specifies the information set in each equation in (3). The size of IndV is the same as that of the vertical concatenation $[J^T\ G^T]^T$ (see the next section), and its zero elements represent variables that do not observe each innovation. The values of the nonzero elements in J, G, and E are computed by the algorithm, whereas (the positions of) their zero elements are provided by a user.

Goal of the algorithm. Our objective is to obtain the state space representation of (3),

$$\kappa_{t+1} = H\kappa_t + J\xi^{t,S}, \quad \textbf{(4a)}$$

$$\phi_t = F\kappa_t + G\xi^{t,S}, \quad \textbf{(4b)}$$

where

$$J := \begin{bmatrix} J_0 & J_1 & \cdots & J_s & \cdots & J_S \end{bmatrix}$$
$$:= \begin{bmatrix} J_{0,11} & \cdots & J_{0,1N} & & J_{s,11} & \cdots & J_{s,1N} & & J_{S,11} & \cdots & J_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ J_{0,M_\kappa 1} & \cdots & J_{0,M_\kappa N} & & J_{s,M_\kappa 1} & \cdots & J_{s,M_\kappa N} & & J_{S,M_\kappa 1} & \cdots & J_{S,M_\kappa N} \end{bmatrix}$$
$$G := \begin{bmatrix} G_0 & G_1 & \cdots & G_s & \cdots & G_S \end{bmatrix}$$
$$:= \begin{bmatrix} G_{0,11} & \cdots & G_{0,1N} & & G_{s,11} & \cdots & G_{s,1N} & & G_{S,11} & \cdots & G_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ G_{0,M_\phi 1} & \cdots & G_{0,M_\phi N} & & G_{s,M_\phi 1} & \cdots & G_{s,M_\phi N} & & G_{S,M_\phi 1} & \cdots & G_{S,M_\phi N} \end{bmatrix}.$$

2.2. Two Key Observations

This section shows two zero restrictions. The algorithm seeks the solution that satisfies them.

Repeated substitutions. To obtain the representation of κ_{t+1} and ϕ_t as functions of κ_{t-S} and $\xi_{t-\tau}$ for $\tau = 0, \ldots, 2S-1$, repeat the substitution of the vertically concatenated guess of solution (4) into itself. Defining $\check{H} := [H^T\ F^T]^T$,

$$\begin{pmatrix} \kappa_{t+1} \\ \phi_t \end{pmatrix} = \check{H}\kappa_t + \tilde{\Gamma}\xi^{t,S} = \check{H}\left(H^S\kappa_{t-S} + \sum_{k=1}^{S} H^{k-1}J\xi^{t-k,S}\right) + \tilde{\Gamma}\xi^{t,S} \quad \textbf{(5)}$$

$$= \check{H}H^S\kappa_{t-S} + (\Gamma_0\xi_{t-0} + \Gamma_1\xi_{t-1} + \cdots + \Gamma_S\xi_{t-S})$$
$$+ \check{H}\begin{pmatrix} H^0\left(J_0\xi_{t-1} + J_1\xi_{t-2} + \cdots + J_S\xi_{t-1-S}\right) \\ +H^1\left(J_0\xi_{t-2} + J_1\xi_{t-3} + \cdots + J_S\xi_{t-2-S}\right) + \cdots \\ +H^{S-1}\left(J_0\xi_{t-S} + J_1\xi_{t-S-1} + \cdots + J_S\xi_{t-S-S}\right) \end{pmatrix}$$

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$$= \check{H} H^S \kappa_{t-S} + \Pi_0 \xi_t + \Pi_1 \xi_{t-1} + \cdots + \Pi_s \xi_{t-s} + \cdots + \Pi_S \xi_{t-S}$$
$$+ \text{terms with } \xi_{t-\tau} \text{ for } \tau \geq S+1,$$

where $\tilde{\Gamma} := [\,\Gamma_0 \quad \cdots \quad \Gamma_s \quad \cdots \quad \Gamma_S\,]$ with $\Gamma_s := [\,J_s^T \quad G_s^T\,]^T$, and

$$\Pi_0 := \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix},$$

$$\Pi_1 := \Gamma_1 + \begin{bmatrix} H \\ F \end{bmatrix} J_0 = \begin{bmatrix} J_1 + H J_0 \\ G_1 + F J_0 \end{bmatrix},$$

$$\Pi_2 := \Gamma_2 + \begin{bmatrix} H \\ F \end{bmatrix} (J_1 + H J_0) = \begin{bmatrix} J_2 + H\,(J_1 + H J_0) \\ G_2 + F\,(J_1 + H J_0) \end{bmatrix}, \ldots,$$

$$\Pi_s := \Gamma_s + \begin{bmatrix} H \\ F \end{bmatrix} \left(\sum_{k=0}^{s-1} H^{s-1-k} J_k \right) = \begin{bmatrix} J_s + H \sum_{k=0}^{s-1} H^{s-1-k} J_k \\ G_s + F \sum_{k=0}^{s-1} H^{s-1-k} J_k \end{bmatrix}, \ldots,$$

$$\Pi_S := \Gamma_S + \begin{bmatrix} H \\ F \end{bmatrix} \left(\sum_{k=0}^{S-1} H^{S-1-k} J_k \right) = \begin{bmatrix} J_S + H \sum_{k=0}^{S-1} H^{S-1-k} J_k \\ G_S + F \sum_{k=0}^{S-1} H^{S-1-k} J_k \end{bmatrix}.$$

In the recursive representation,

$$\Pi_0 = \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix}, \tag{6}$$
$$\Pi_s = \Gamma_s + \tilde{H} \Pi_{s-1} \text{ for } s = 1, \ldots, S,$$

where

$$\tilde{H} := \begin{bmatrix} H & 0 \\ F & 0 \end{bmatrix}.$$

Intuitively, equation (6) shows that the (j, k)th element of Π_s is the effect of $\xi_{k,t-s}$ (the kth innovation at time $t-s$) on $y_{j,t}$ (the jth endogenous variable at time t). Thus, given κ_{t-S}, $\Pi_{s,jk}$, which is defined as the (j, k)th element of Π_s, is zero if $y_{j,t}$ is determined without observing $\xi_{k,s}$.

In the matrix representation, (7) becomes

$$\Gamma = M_{\Gamma\Pi} \Pi, \tag{7}$$

where

$$\Gamma := \left[\,\Gamma_0^T \quad \cdots \quad \Gamma_s^T \quad \cdots \quad \Gamma_S^T\,\right]^T, \tag{8a}$$

$$\Pi := \left[\,\Pi_0^T \quad \cdots \quad \Pi_s^T \quad \cdots \quad \Pi_S^T\,\right]^T, \tag{8b}$$

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$$M_{\Gamma\Pi} := \begin{bmatrix} I & & & 0 \\ -\tilde{H} & I & & 0 \\ & \ddots & \ddots & \\ 0 & & -\tilde{H} & I \end{bmatrix}. \tag{8c}$$

$M_{\Gamma\Pi}$ is clearly invertible, and plays a key role in the following.

Zero restrictions. Throughout this paper, we exploit the following two observations.

(1) If the kth set of variables $y_{k,t}$ does not observe the ith set of time $t-s$ innovations $\xi_{i,t-s}$, then $\partial y_{k,t}/\partial \xi_{t-s} = \Pi_{s,ki} = 0$, given κ_{t-S} and $\xi_{t-\tau}$ for $\tau = s+1, \ldots$. Simply put, no decision can respond to unobserved innovations.
(2) If the information set of the expectation operator in the jth equation includes the ith time $t-s$ innovation $\xi_{i,t-s}$, then the realization of the jth equation must hold for any realisation of the ith innovation. The expectation error in each expectation operator occurs only due to innovations that are not included in its information sets. Thus, $E_{s,ji} = 0$.

For example, suppose that labor supply L_t (kth variable, $y_{k,t}$) is decided on before observing today's technology shock (ith shock, $\xi_{i,t}$), but after today's preference shock (lth shock, $\xi_{l,t}$), both of which are i.i.d. If the FOC with respect to L_t is the jth equation, then

$$\begin{aligned} \Pi_{0,ki} &= 0(\xi_{i,t-0} \text{ does not affect } y_{k,t}) \\ E_{0,jl} &= 0(\xi_{l,t-0} \text{ does not cause expectation error in } j\text{th equation}). \end{aligned}$$

Roughly speaking, $E_{0,jl} = 0$ means that if the expectation operator of the jth equation were eliminated from the jth equation, it would still hold in terms of $\xi_{0,l}$. It is the duty of a user to specify the positions of these zero elements in Π and E (by providing IndV and IndE).

2.3. Sketch of Derivation and Key Equations for Computation

The fully detailed derivation is provided in the Appendix. This section briefly describes the skeleton of the derivation and lists the minimum results necessary for computation.

QZ Decomposition. To introduce notations, this section briefly reviews the QZ decomposition (or generalized Schur decomposition). For matrices A and B ($\in \mathbf{C}^{n\times n}$), there exist unitary matrices Q and Z such that

$$\begin{aligned} Q^H A Z &= \Omega_A, \\ Q^H B Z &= \Omega_B, \end{aligned}$$

where Ω_A and Ω_B are both upper triangular matrices, and superscript H indicates a conjugate transpose. Any unitary matrix U satisfies $U^H U = UU^H = I$. Let a_{kk} and b_{kk} be the kth diagonal elements in Ω_A and Ω_B, respectively. Assuming that a_{kk} and b_{kk} are not zero at the same time, then $\lambda_k := b_{kk}/a_{kk}$ for $k = 1, \ldots, n$ are the generalized eigenvalues of the matrix pencil $B - \lambda_k A$.[8]

The basic idea is that, by applying the QZ decomposition to (3), the algorithm separates unstable roots u_t from stable roots s_t, as in Sims (2002):

$$\begin{aligned}
0 &= Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \\
&= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\
&= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\
&\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S},
\end{aligned}$$

where

$$\begin{pmatrix} s_t \\ u_t \end{pmatrix} := Z^H \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}.$$

By using TVCs, the expected values of all unstable roots u_{t+1} are set equal to zero.[9]

Notations for the outputs of QZ decomposition. For later use, we define submatrices as follows:

$$Z^H := \begin{bmatrix} Z^H_{s.} \\ Z^H_{u.} \end{bmatrix} := \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix}, \quad Z := \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix}, \quad Q^H := \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix}, \tag{9a}$$

$$\Omega^A := \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix}, \quad \Omega^B := \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix}, \tag{9b}$$

where subscripts u and s imply unstable and stable roots, respectively. Note that Ω^A_{ss} and Ω^B_{uu} are both invertible by construction.

Additionally, we define four matrices as

$$\Lambda^A_{s\kappa} := \Omega^A_{ss} Z^H_{s\kappa} + \Omega^A_{su} Z^H_{u\kappa}, \tag{10a}$$

$$\Lambda^A_{s\phi} := \Omega^A_{ss} Z^H_{s\phi} + \Omega^A_{su} Z^H_{u\phi}, \tag{10b}$$

$$\Lambda^B_{s\kappa} := \Omega^B_{ss} Z^H_{s\kappa} + \Omega^B_{su} Z^H_{u\kappa}, \tag{10c}$$

$$\Lambda^B_{s\phi} := \Omega^B_{ss} Z^H_{s\phi} + \Omega^B_{su} Z^H_{u\phi}. \tag{10d}$$

Note that all the matrices defined by (10a) are obtained from the outputs of the QZ decomposition.

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Matrix subscripts. We introduce the following notation rule for subscripts on matrices. For a matrix A,

$A_{.x}$ is columns x of a matrix A,
$A_{x.}$ is rows x of a matrix A,
$A_{.\neg x}$ is the columns remaining after the elimination of columns x, and
$A_{\neg x.}$ is the rows remaining after the elimination of rows x,

where x is the name of a set of columns or rows. This notation makes certain matrix operations extremely simple. See the Appendix for further details.

Zero restrictions. As a result of manipulating the matrix equations, it is shown that

$$0 = \Pi + M_{\Pi E}(E + \mathbf{C}), \tag{11}$$

$$M_{\Pi E} := (M_{y\Gamma} M_{\Gamma\Pi}) \backslash \mathbf{Q}, \tag{12}$$

where

$$\Gamma := \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_{S-1} \end{pmatrix}, \quad E := \begin{pmatrix} E_0 \\ \vdots \\ E_{S-1} \end{pmatrix}, \quad \mathbf{C} := \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \quad \mathbf{Q} := \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix}, \tag{13a}$$

$$M_{y\Gamma} := \begin{bmatrix} \Phi & \Lambda^{0A} & & 0 \\ & \ddots & & \\ & & \Phi & \Lambda^{0A} \\ 0 & & & \Phi \end{bmatrix}, \quad \Phi := \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \quad \Lambda^{0A} := \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix}, \tag{13b}$$

and $X \backslash Y = X^{-1}Y$. Our immediate objective is to find E and Π. Bear in mind that, although $M_{y\Gamma}$ is computable solely from the outputs of the QZ decomposition, we can obtain $M_{\Gamma\Pi}$ only after finding H and F (see equation (8c)).

Given $M_{\Gamma\Pi}$, E and Π are computed column by column (i.e., innovation by innovation) in (11). Because some elements in Π and E are zero due to the two zero restrictions, for the ith column (or equivalently for the ith innovation) of (11),

$$0 = \begin{pmatrix} \Pi_{1,i} \\ \vdots \\ \Pi_{k,i}\ (= 0) \\ \vdots \\ \Pi_{M(S+1),i} \end{pmatrix} + M_{\Pi E} \left(\begin{pmatrix} 0 \\ \vdots \\ E_{ji} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} C_{.i} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right), \tag{14}$$

where M in subscripts is the number of equations and hence $M\,(S+1)$ is the number of rows in Π.

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From the kth set of equations in (14),

$$0 = [\, M_{\Pi E}\,]_{kj} E_{ji} + [\, M_{\Pi E}\,]_{kj} \mathbf{C}_{ji} + [\, M_{\Pi E}\,]_{k\neg j} \mathbf{C}_{\neg ji}, \tag{15}$$

which gives the values of the nonzero elements of E. From the remaining equations in (14),

$$\begin{aligned} 0 = {} & \Pi_{\neg ki} + [\, M_{\Pi E}\,]_{\neg kj} \mathbf{C}_{ji} + [\, M_{\Pi E}\,]_{\neg k\neg j} \mathbf{C}_{\neg ji} \\ & -[\, M_{\Pi E}\,]_{\neg kj} ([\, M_{\Pi E}\,]_{kj} \backslash [\, M_{\Pi E}\,]_{k\neg j} \mathbf{C}_{\neg ji} + \mathbf{C}_{ji}), \end{aligned} \tag{16}$$

which gives the nonzero elements of Π.

Here we assume that $[M_{\Pi E}]_{kj}$ is invertible, which, however, is not necessarily true in general. The economic meaning of its invertibility is discussed in Section 3.

Solution. The solution algorithm computes key matrices sequentially. The basic structure is as follows:

(1) Obtain submatrices from the outputs of the QZ decomposition (9a) and (10a).
(2) Obtain H and F from (17a).
(3) Obtain $M_{y\Gamma}$, $M_{\Gamma\Pi}$, and $M_{\Pi E}$ from (13b), (8c), and (12), respectively.
(4) Obtain E and Π from (18) and (19).
(5) Obtain G and J from (20).

H and F. As in Sims (2002), it turns out that the H and F matrices are derived independently from the G and J matrices, based on the coefficient on κ_{t-S} in (6) (see the Appendix for details). Therefore, they are exactly the same as in perfect-information models:

$$F = -Z^H_{u\phi} \backslash Z^H_{u\kappa} = Z_{\phi s} / Z_{\kappa s}, \tag{17a}$$

$$H = -Z_{\kappa s} \left(\Omega^A_{ss} \backslash \Omega^B_{ss} \right) / Z_{\kappa s}. \tag{17b}$$

E and $\mathbf{\Pi}$**.** From (15) and (16), the nonzero elements of E and Π are

$$E_{ji} = -[\, M_{\Pi E}\,]_{kj} \backslash [\, M_{\Pi E}\,]_{k\neg j} \mathbf{C}_{\neg ji} - \mathbf{C}_{ji}, \tag{18}$$

$$\Pi_{\neg ki} = -\left[\, M^{-1}_{\Pi E}\,\right]_{\neg j\neg k} \backslash \mathbf{C}_{\neg ji}, \tag{19}$$

where $M_{\Pi E}$ can be obtained from (8c) and (13a) with the solution of H and F. Note that H and F can be computed without referring to E, Π, or $M_{\Pi E}$. Because $[M_{\Pi E}]_{kj}$ is assumed to be invertible, $\left[M^{-1}_{\Pi E}\right]_{\neg j\neg k}$ is also invertible.

J and G. From the definition of Γ (8a),

$$\Gamma := \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi}\Pi. \qquad \textbf{(20)}$$

Note that, with H and F matrices, $M_{\Gamma\Pi}$ are recovered from (8c).

D. From a given economic model (3) it is obvious that

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix}. \qquad \textbf{(21)}$$

3. ASSUMPTIONS

In this section, we discuss three assumptions. Assumptions 1 and 2 in the following are the same as in the solution method for perfect-information models, whereas Assumption 3 is specific to imperfect-information models. This section omits discussion of the Blanchard–Kahn condition, which is automatically satisfied by Assumption 1.

3.1. Assumption 1: $Z_{u\phi}^H$ Is Invertible

Klein (2000) shows that this assumption is a generalization of the condition derived in Blanchard and Kahn (1980). Boyd and Dotsey (1990) make it clear that the Blanchard–Kahn condition, which counts and compares the numbers of unstable roots and jump variables, is a necessary but not sufficient condition for the existence of a unique solution; they provide a counterexample that satisfies the Blanchard–Kahn counting condition but does not have a stable solution. Intuitively, an invertible $Z_{u\phi}^H$ means that we can always find the values of jump variables such that the expectation of u_{t+1} is a zero vector in any states (TVCs). Heuristically, $Z_{u\phi}^H$ maps jump variables ϕ_t to unstable roots u_t, and its inverse maps u_t to ϕ_t. See King and Watson (1998) for an intuitive exposition.

The existence of the right inverse of $Z_{u\phi}^H$ entails the existence of jump variables, whereas the nonexistence of its left inverse implies the nonuniqueness of jump variables.[10] Note that typically nonuniqueness causes sunspot equilibria.

3.2. Assumption 2: a_{kk} and b_{kk} Are Not Zero at the Same Time

If a_{kk} and b_{kk} are zero at the same time, there exist row vectors X such that $0 = X\xi$; indeed, X is (a scalar multiple of) the kth row of Q [see also Sims (2002)]. The existence of such row vectors generically implies one of the following:

(a) If $X\xi$ is indeed zero, then some equations are not linearly independent of the others. Essentially, there are fewer equations than endogenous variables. At least one equation can be expressed as a linear combination of others, and such a linear combination is X.

(b) If $X\xi$ is nonzero, clearly there is an internal contradiction. One such example is a two-equation, two-variable nondynamic model with no state variables:

$$\phi_{1,t} = \alpha\phi_{2,t} + \xi_t$$
$$\phi_{1,t} = \alpha\phi_{2,t} + \xi_t + \eta_t.$$

Obviously, both do not hold at the same time for nonzero η_t. Because the QZ decomposition is merely a linear transformation, there is an internal inconsistency in the original system of equations (3).

3.3. Assumption 3: $[M_{\Pi E}]_{kj}$ Is Invertible

This condition is specific to imperfect information models, though it is analogous to equation (40) in Sims (2002).[11] Intuitively, if it is not invertible, then the information structure is not consistent. Note that the inverse of $[M_{\Pi E}]_{kj}$, if it exists, maps the jth set of expectation errors to the kth set of innovations to which some endogenous variables cannot respond. Hence, if the inverse of $[M_{\Pi E}]_{kj}$ exists, then expectation errors can equate both sides of the equations for any realization of innovations.

A noninvertible $[M_{\Pi E}]_{kj}$ appears in the following example. Suppose that all production factors and all demand components are decided before today's technology shock is observed. In this case, output varies depending on the realization of technology, whereas demand cannot respond to it. Thus, the goods market does not clear at any price. One important lesson from this is that a researcher must construct consistent models; an arbitrarily specified information structure may have internal inconsistencies.

4. PROPERTIES OF THE SOLUTION

By construction, of course, any solution generated by the algorithm satisfies the following two solution principles (two zero restrictions): (i) if the kth time-t variable $y_{k,t}$ is determined without observing the ith time-$(t-s)$ innovations $\xi_{i,t-s}$, then $y_{k,t}$ cannot respond to $\xi_{i,t-s}$ (i.e., $\partial y_{k,t}/\partial \xi_{i,t-s} = 0$ given κ_{t-S}), and (ii) if the expectation operator in the jth equation has an information set that includes $\xi_{i,t-s}$, $\xi_{i,t-s}$ cannot be the source of the expectation error in the jth equation. In addition, as mentioned in the Introduction, invariant dynamic parts, H and F matrices, imply that imperfect information models inherit the qualitative nature of the corresponding perfect-information model: specifically, (a) the dynamic stability property is not affected by information structure, and (b) the direct effect of imperfect information on IRFs lasts for only the first S periods after an impulse, and then IRFs show sudden jumps.

The rest of this section briefly discusses other interesting features.

4.1. Inference

First, the maximum possible information set at time t (perfect information) is $\{\kappa_{t-j}, \xi_{t-j}\}_{j=0}^{\infty}$ (not including $\{\phi_{t-j}\}_{j=0}^{\infty}$). Importantly, the algorithm does not allow inference. If the information set of economic agents in a model includes all current and past jump variables $\{\phi_{t-j}\}_{j=0}^{\infty}$, then the economic agents can infer most hidden information, which reduces an imperfect-information model to the corresponding perfect-information model in most cases. Hence, one natural interpretation of imperfect information is that agents have to make future decisions in the current period, as in sticky-price models.

4.2. Noisy-Information Models

Second, the algorithm can easily deal with noisy-information models. Suppose an AR(1) shock process A_t follows

$$\ln A_{t+1} = \rho \ln A_t + \sqrt{1-\eta}\xi_t^{ob} + \sqrt{\eta}\xi_t^{uo}, \tag{22}$$

where ξ_t^{ob} and ξ_t^{uo} are the observable and unobservable components of innovation, respectively, and $(1-\eta)/\eta$ is the signal-to-noise ratio. This technique allows us to parameterize the extent of imperfect information.

5. AN EXAMPLE STANDARD RBC MODEL

To demonstrate the quantitative effects of imperfect information, we consider the standard RBC model under imperfect information, focusing on impulse response functions (IRFs) and second moments.

The main economic motivation is to address an overly high $Corr\,(Y_t - H_t, Y_t)$ in the standard RBC model. In the plausible parameter range, the standard RBC model predicts an almost perfect correlation between labor productivity $Y_t - H_t$ and output Y_t, but the correlation is only slightly positive in the data.

Hence, we modify the standard RBC model by adding imperfect information related to the labor market. The relevant equations are

$$0 = bH_t - W_t - \lambda_t, \tag{23a}$$

$$0 = Y_t - H_t - W_t, \tag{23b}$$

where Y_t, H_t, W_t, and λ_t are output, working hours, wage, and the marginal utility of consumption, respectively. All endogenous variables are measured as deviations from their steady-state values in percentage terms. b is a constant, which represents (a multiple of) the elasticity of marginal disutility of labor. The first equation is for the representative household (HH)—the FOC with respect to labor supply—whereas the second is for firms—it equates the marginal product

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of labor $Y_t - H_t$ to wage.[12] The set of state variables under perfect information is $\{K_t, A_t, \xi_t\}$, where K_t and A_t are capital and technology at the beginning of time t, respectively, and ξ_t represents the innovation on technology. Note that A_t is regarded as an endogenous crawling variable, and there is only one i.i.d. exogenous variable ξ_t. That is, A_t is treated as a stock variable.

Assuming that today's innovation affects today's output,

$$Y_t = A_{t+1} K_t^{\alpha} H_t^{1-\alpha},$$

$$\ln A_{t+1} = \rho \ln A_t + \xi_t,$$

where ρ is a parameter that governs the persistence of technology shock.

5.1. Case I: HH Decides Labor Supply before Observing Innovations

In this case, (23a) does not hold. Instead, the labor supply decision is governed by[13]

$$0 = E[\,bH_t - W_t - \lambda_t \mid K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}].$$

Because H_t cannot react to past innovations, for $s = 0, 1, \ldots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}.$$

Figure 1 shows the impulse response functions where $S = 5$, which means that the household decides its labor supply five quarters in advance.

There are several points worth noting here:

Labor hours do not move for the first S periods. That is, $\partial H_t/\partial \xi_{t-s} = 0$ for $s = 0, 1, \ldots, S$.

Labor productivity $(Y_t - H_t)$ and investment show unusual movements for the first S periods. However, after $S + 1$ periods, all endogenous variables follow (linear combinations of) AR(1) processes. This is one example of the proposition that *the direct effect of imperfect information lasts for only S periods after an impulse*.

$Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information (around 0.91), but only slightly.

5.2. Case II: Firms Decide Labor Demand before Observing Innovations

In this case, (23b) does not hold. Instead, the labor demand decision is governed by

$$0 = E[\,Y_t - H_t - W_t \mid K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}].$$

Because H_t cannot react to the innovations, for $s = 0, 1, \ldots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}.$$

The results are not very interesting in terms of economics.

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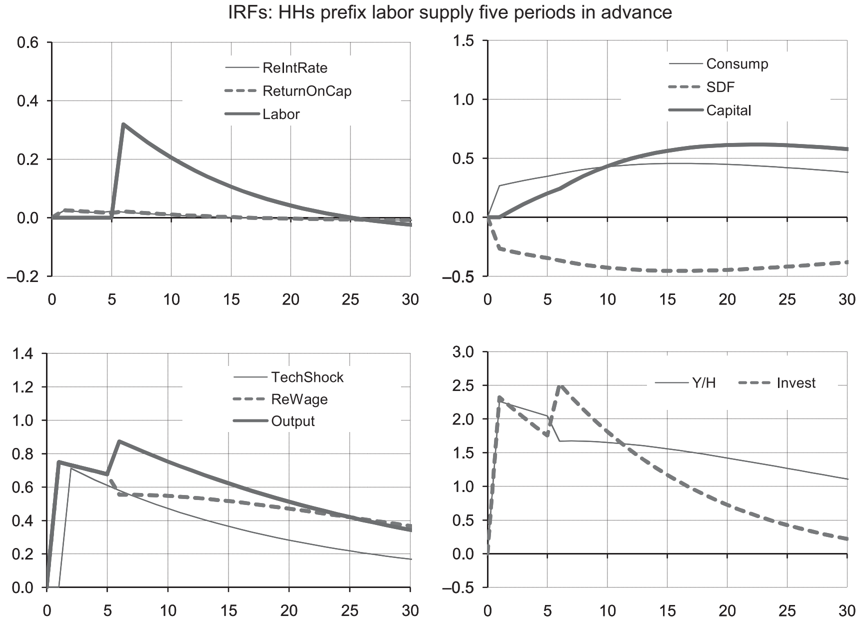

FIGURE 1. Impulse response functions for a positive technology innovation in the standard RBC model, in which labor supply is determined five periods in advance.

The IRFs are almost the same as in Case I, except for wage (hence, the figure is omitted).
Corr $(Y_t - H_t, Y_t)$ is lower than under perfect information, but only slightly.

However, this experiment demonstrates that, to find a solution, it is not enough to specify which endogenous variables are determined with imperfect information; a researcher must also specify which information sets are imperfect. This is evident in that the results of Cases I and II are not the same.

5.3. Case III: HH Decides Wage before Observing Innovations but Accommodates Labor Demand

This case can be regarded as a version of the sticky wage model. The representative household fixes wage before observing innovations, and it commits itself to supplying labor to accommodate labor demand.

In this case, (23a) does not hold. Instead, the labor supply decision is governed by

$$0 = E[bH_t - W_t - \lambda_t \mid K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}].$$

Since W_t cannot react to the innovations, for $s = 0, 1, \ldots, S$,

$$\frac{\partial W_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}.$$

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The results are interesting:

The volatility of labor is much higher, and Corr $(Y_t - H_t, Y_t)$ is much lower than under perfect information.
Given the standard deviation of the innovation, both output and labor are more volatile.
The variance–covariances of most variables other than labor and labor productivity do not change significantly.

The intuition behind these results is quite simple. Without imperfect information, when there is a positive productivity innovation, wage increases, which discourages firms from hiring more labor. As a result, labor does not increase significantly. Indeed, another failure of the standard RBC model is that it predicts too low labor volatility relative to output volatility. During a boom both Y_t and H_t increase, whereas $Y_t - H_t$ increases because the increase in H_t is not large enough. Consequently, both Y_t and $Y_t - H_t$ increase in a boom, which is the (one possible) mechanism behind a high Corr $(Y_t - H_t, Y_t)$ in the standard RBC model.

However, if wage is determined without positive innovation being seen, it does not change quickly; hence, firms are not discouraged from using more labor. Consequently, in a boom both Y_t and H_t increase, whereas $Y_t - H_t$ does not increase very much because the increase in H_t is large enough. Hence, the model predicts a low Corr $(Y_t - H_t, Y_t)$. Indeed, in the otherwise standard RBC model with one-period wage stickiness, the predicted relative volatility of labor almost matches the data. Under the standard parameter set, Corr $(Y_t - H_t, Y_t)$ is negative for $S \geq 2$.

Table 1 shows a summary of the selected second moments for one-period wage stickiness ($S = 1$). One-period wage stickiness significantly improves the labor volatility and the correlation between labor productivity and output, whereas it slightly diminishes the model performance in terms of the relative volatility of investment.

TABLE 1. Comparison between perfect and imperfect information RBC models

	Output	Hours	Consumption	Investment	Corr(Output, Output/Hours)
			Data		
S.d.	1.72	1.59	0.86	8.24	.41
Relative	1.00	0.92	0.50	4.79	
			Standard RBC		
S.d.	1.35	0.47	0.33	5.95	.98
Relative	1.00	0.35	0.24	4.41	
			Imperfect information (RBC with predetermined wage)		
S.d.	2.15	2.10	0.53	7.92	.25
Relative	1.00	0.98	0.25	3.69	

Note. Figures for "data" and "standard RBC" are cited from Cooley and Prescott (1995).

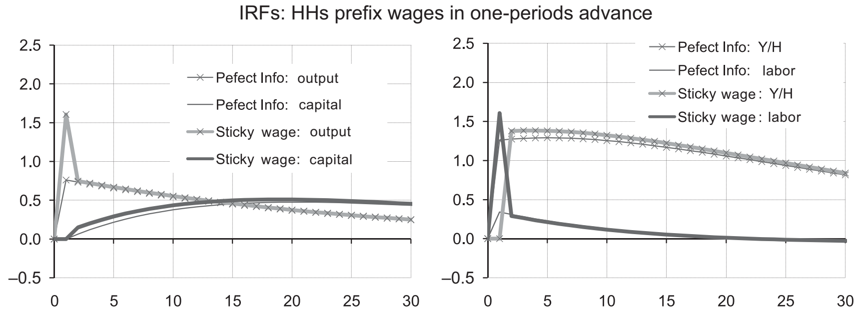

FIGURE 2. Comparison of selected impulse response functions for a positive technology innovation between standard RBC and RBC with wage stickiness.

Figure 2 shows a comparison of selected impulse response functions between perfect- and imperfect-information models. The salient differences appear only in the first period. In the sticky wage model, both labor and output jump in the first period, and the sizes of the jumps are the same; hence, the labor productivity does not change in the first period. Note that the Cobb–Douglas production function implies that the labor productivity is always equal to the wage.

Figure 3 shows the relative volatilities and correlations for different degrees of imperfect information (i.e., for different values of S). As S increases, $\mathrm{Corr}(Y_t - H_t, H_t)$ decreases.

Case III again reveals one computational requirement; simply specifying the information set in each equation is not enough to find a solution. A researcher must also specify which variables are determined without observing perfect information. This is evident in that the results of Cases I and III are not the same.

5.4. Conclusion for RBC under Imperfect Information

Adding one-period wage stickiness is quantitatively enough to overcome the two drawbacks of the standard RBC model—where (a) labor volatility is too small

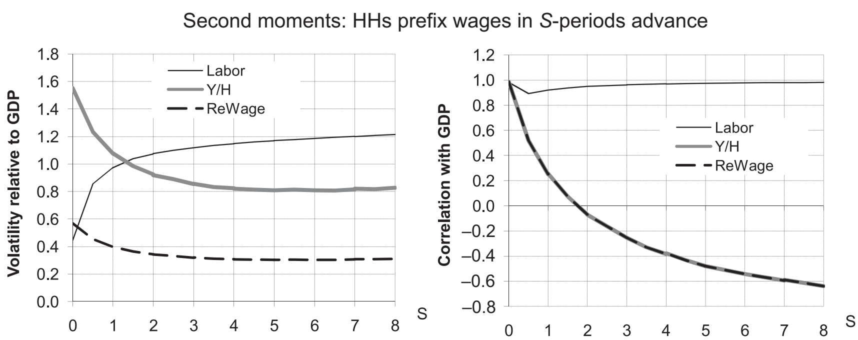

FIGURE 3. Effect of different degrees of imperfect information on selected second moments.

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and (b) the correlation between labor productivity and output is too high—without deteriorating other dimensions of the model performance. This example shows the possibility that the information structure has significant quantitative effects.

6. CONCLUSIONS

This article has developed an algorithm for linear rational models under imperfect information. Imperfect information is important because it includes many interesting classes of models, such as sticky-information and noisy-signal models.

The algorithm exploits two observations: (1) if an endogenous variable $y_{k,t}$ is decided without observing an innovation $\xi_{i,t-s}$, then $y_{k,t}$ is not affected by $\xi_{i,t-s}$ (i.e., $\partial y_{k,t}/\partial \xi_{i,t-s} = 0$ given κ_{t-S}); (2) if the information set in the jth equation includes $\xi_{i,t-s}$, then $\xi_{i,t-s}$ cannot be the source of expectation error in the jth equation ($E_{s,ji} = 0$). The solution is defined by these two zero restrictions, and it turns out that they are enough to determine unique solutions.

The state space representation chosen in this algorithm is the set of crawling variables and current and past innovations. This representation reveals that the dynamic parts of the solution (i.e., the H and F matrices) are the same as under the corresponding perfect-information models. Invariant H and F matrices imply that (a) the dynamic property, such as sunspot or saddle-path stability, is not altered by the information structure, and (b) impulse response functions are not (directly) affected by the information structure after the first S periods, where S is such that the minimum information set in a model has all the information up to time S. These findings show that *qualitatively* imperfect information models inherit the properties of their perfect information counterparts.

However, as the RBC example demonstrates, *quantitatively* imperfect information may be important. Hence, it is desirable to check for robustness in terms of the information structure, and our Matlab program offers an easy way to conduct such experiments. Once structural equations are obtained, the additional inputs to the algorithm are only two zero–one matrices.

NOTES

1. The codes and a manual for them are available at http://www.kent.ac.uk/economics/papers/papers07.html.

2. Crawling and jump variables are essentially the same concepts as predetermined and nonpredetermined variables in the literature. Indeed, they are interchangeable under perfect information, which is a special case of imperfect information. However, the traditional terminologies *predetermined/nonpredetermined* could be misleading, in the sense that typical nonpredetermined variables such as consumption and wage can be *already determined before the current period* under imperfect information.

3. See Wang and Wen (2006). They point out that the dynamic parts under imperfect information have the same roots as those under perfect information, which is a corollary of our result.

4. There are three types of methods for perfect information models.

   1. King and Watson's method (1998 and 2002) [see also Woodford (undated)] implements a two-stage substitution. First, nondynamic jump variables are substituted out, and then, dynamic jump variables are substituted out from the system of equations.

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2. In the QZ method of Sims (2002) [see also Klein (2000)], the QZ decomposition is applied to matrices on endogenous variables. Recognizing that (1) roots that correspond to nondynamic jump variables are infinite, and (2) roots that correspond to dynamic jump variables are larger than one in absolute terms, the transversality conditions (TVCs) eliminate both types of jump variables at once.
3. The method of undetermined coefficients of Uhlig (1999) [see also Christiano (1998)] substitutes a guessed solution into the given system of equations; the resulting matrix polynomial is solved directly. In principle, this method does not require that given equations be first-order difference equations. Higher-order matrix polynomials can be numerically solved (see the Appendix).

5. See Uhlig (1999), for example.

6. See Woodford (undated). This technique simplifies the algebra and computation significantly.

7. See the manual for further details. Note that we do not explicitly mention these two indicator matrices in the rest of this article.

8. The generalized eigenvalues have properties similar to those of forward operators F; $x_{t+1} = F x_t$.

9. Remember that all innovations are assumed to be i.i.d. Note also that, if the expectations of u_{t+1} must be zero under perfect information, they must be also zero under imperfect information. This can be shown by simply applying the iterated linear projection. See the Appendix for more extensive discussion.

10. See Uhlig (1999) for a treatment of nonuniqueness.

11. Note, however, that Sims's condition is related to time-$(t+1)$ expectation errors, whereas our discussion in the following is related to time-τ expectation errors ($\tau < t$).

12. Note that because all endogenous variables are represented as log-deviations from their steady state, $Y_t - H_t$ is the deviation of "output per labor hour" (i.e., labor productivity). The Cobb–Douglas production function implies that the marginal product of labor is $(1-\alpha)$ times labor productivity, which means that the percent change of labor productivity is exactly the same as that of the marginal product of labor. In other words, in the Cobb–Douglas production function, $Y_t - H_t$ represents both the percent deviation of labor productivity and the marginal product of labor.

13. Exactly speaking, the information set is $\{K_{t-j}, A_{t-j}, \xi_{t-j}\}_{j=S+1}^{\infty}$, but only $\{K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}\}$ suffices to determine the state of the economy.

14. There are two comments. First, (A.4) must hold for *any* realization of κ_{t-1} and ξ_{t-s} for $s = 0, 1, \ldots$. Hence, it is *not* possible that TVCs hold under imperfect information but not under perfect information. Second, if an information set does not include, for example, $\xi_{i,t-s}$, then the relevant expected value of u_{t+s} is the RHS with setting $\xi_{i,t-s} = 0$. Hence, if TVCs hold for the full information set, they hold for nonfull information sets as well.

15. Remember that an invertible $Z_{u\phi}^H$ implies an invertible $Z_{s\kappa}^H$.

16. For the F matrix, note that

$$Z^H Z = \begin{bmatrix} Z_{s\kappa}^H & Z_{s\phi}^H \\ Z_{u\kappa}^H & Z_{u\phi}^H \end{bmatrix} \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix} = \begin{bmatrix} Z_{s\kappa}^H Z_{\kappa s} + Z_{s\phi}^H Z_{\phi s} & Z_{s\kappa}^H Z_{\kappa u} + Z_{s\phi}^H Z_{\phi u} \\ Z_{u\kappa}^H Z_{\kappa s} + Z_{u\phi}^H Z_{\phi s} & Z_{u\kappa}^H Z_{\kappa u} + Z_{u\phi}^H Z_{\phi u} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Looking at the lower left element,

$$\begin{aligned} Z_{u\kappa}^H Z_{\kappa s} + Z_{u\phi}^H Z_{\phi s} &= 0 \\ -Z_{u\kappa}^H Z_{\kappa s} &= Z_{u\phi}^H Z_{\phi s} \\ -Z_{u\phi}^H \backslash Z_{u\kappa}^H &= Z_{\phi s} / Z_{\kappa s}. \end{aligned}$$

Also, remember that

$$Z_{\kappa s}^{-1} = Z_{s\kappa}^H - Z_{s\phi}^H \left(Z_{u\phi}^H \backslash Z_{u\kappa}^H \right)$$

and that Ω_{ss}^A is invertible by the reordering of QZ decomposition.

17. Though this process is not necessary, it reduces the computational burden.

REFERENCES

Blanchard, Olivier Jean and Charles M. Kahn (1980) The solution of linear difference models under rational expectations. *Econometrica* 48, 1305–1312.

Boyd, John H. and Michael Dotsey (1990) Interest Rate Rues and Nominal Determinacy. Working paper, Federal Reserve Bank of Richmond.

Christiano, Lawrence J. (1998) Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients. NBER technical working papers 225.

Cooley, Thomas F. and Edward C. Prescott (1995) Economic growth and business cycles. In Thomas F. Cooley (ed.), *Frontiers of Business Cycle Research*, pp. 1–38. Princeton, NJ: Princeton University Press.

Dupor, Bill and Takayuki Tsuruga (2005) Sticky information: The impact of different information updating assumptions. *Journal of Money, Credit and Banking* 37, 1143–1152.

King, Robert G. and Mark W. Watson (1998) The solution of singular linear difference systems under rational expectations. *International Economic Review* 39, 1015–1026.

King, Robert G. and Mark W. Watson (2002) System reduction and solution algorithms for singular linear difference systems under rational expectations. *Computational Economics* 20, 57–86.

Klein, Paul (2000) Using the generalized Schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control* 10, 1405–1423.

Mankiw, Gregory N. and Ricardo Reis (2001) Sticky Information Versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve. NBER working papers 8290.

Sims, Christopher A. (2002) Solving linear rational expectations models. *Computational Economics* 20, 1–20.

Uhlig, Harald (1999) A toolkit for analyzing nonlinear dynamic stochastic models easily. In Ramon Marimon and Andrew Scott (eds.), *Computational Methods for the Study of Dynamic Economics*, pp. 30–61. Oxford, UK: Oxford University Press.

Wang, Pengfei and Yi Wen (2006) Solving Linear Difference Systems with Lagged Expectations by a Method of Undetermined Coefficients. Working paper 2006-003C, Federal Reserve Bank of St. Louis.

Woodford, Michael (undated) Reds-Solds User's Guide. Mimeo, Princeton University.

APPENDIX

A.1. EXTENSION OF UHLIG'S THEOREM 3

PROPOSITION 1 (Extension of Uhlig's Theorem 3). *To find a $m \times m$ matrix X that solves the matrix polynomial*

$$\Theta_n X^n - \Theta_{n-1} X^{n-1} - \cdots - \Theta_1 X - \Theta_0 = 0, \tag{A.1}$$

given $m \times m$ coefficient matrices $\{\Theta_{n'}\}_{n'=0}^{n}$, define the $nm \times nm$ matrices Ξ and Δ by

$$\Xi = \begin{bmatrix} \Theta_{n-1} & \cdots & \Theta_1 & \Theta_0 \\ I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Theta_n & 0 & \cdots & 0 \\ 0 & I & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I \end{bmatrix},$$

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and obtain the generalized eigenvalues λ *and the generalized eigenvector* s *such that* $\lambda \Delta s = \Xi s$. *Then* s *can be written as*

$$s = \begin{pmatrix} \lambda^{n-1} x \\ \vdots \\ \lambda x \\ x \end{pmatrix}$$

for some $x \in \mathbf{R}^m$, *and*

$$X = \Omega \Lambda \Omega^{-1},$$

where $\Omega = [x_1, \ldots, x_m]$ *and* $\Lambda = diag(\lambda_1, \ldots, \lambda_m)$.

Proof. Almost identical to Uhlig (1999). ■

A.2. MATRIX OPERATIONS

To pick up and drop out columns and rows from a matrix, as in the main text, we define (i) $[A]_{.x}$ as columns x of a matrix A, (ii) $[A]_{x.}$ as rows x of a matrix A, (iii) $[A]_{.\neg x}$ as the columns remaining after the elimination of columns x, and (iv) $[A]_{\neg x.}$ as the rows remaining after the elimination of rows x, where x is the name of a set of columns or rows. The brackets are used simply because they often clarify notation, and often can be omitted (i.e., $[B]_{.\neg y} = B_{.\neg y}$). The dot . implies all rows or columns (e.g., $B_{..} = B$). It is quite easy to show the following formulae:

$$\begin{aligned} [AB] &= [A]_{.\neg x} [B]_{\neg x.} + [A]_{.x} [B]_{x.} \,, \\ [AB]_{.\neg y} &= [A] [B]_{.\neg y} \,, \\ [AB]_{\neg x.} &= [A]_{\neg x.} [B] \,, \\ [AB]_{\neg x \neg y} &= [A]_{\neg x.} [B]_{.\neg y} \,. \end{aligned}$$

An example for the first formula is

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} [b_{11} \quad b_{12}] + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} [b_{21} \quad b_{22}]$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix},$$

where $x = 2$.

Note that this notation is consistent with other matrix subscripts; for example, the rows of $Z_{s\kappa}$ are related to stable roots s and its columns are related to crawling variables κ.

A.3. INVERTIBLE $Z^H_{u\phi}$ IMPLIES INVERTIBLE $Z^H_{s\kappa}$

PROPOSITION 2. *For an invertible matrix* Z, *which is partitioned as*

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

if Z_{11} *is invertible, then* $[Z^{-1}]_{22}$ *is also invertible.*

Proof. Define

$$Z_L := \begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix},$$
$$Z_R := \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix}.$$

Note that $Z_L Z Z_R$ has full rank because all of Z_L, Z, and Z_R have full rank, and note that

$$\begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \end{bmatrix}.$$

Hence, $G := Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}$ must have full-rank.

For a full-rank matrix with an invertible upper left submatrix, the well-known formula tells us that

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Z_{11}^{-1} + Z_{11}^{-1}Z_{12}G^{-1}Z_{21}Z_{11}^{-1} & -Z_{11}^{-1}Z_{12}G^{-1} \\ -G^{-1}Z_{21}Z_{11}^{-1} & G^{-1} \end{bmatrix}.$$

Note that the RHS exists because we know that both Z_{11} and G are invertible. Thus, $\left[Z^{-1}\right]_{22}$ is invertible. ■

Since Z is unitary, $Z^{-1} = Z^H$, which implies $G^{-1} = [Z^{-1}]_{22} = Z_{22}^H$. Since Z_{22}^H has full rank, its conjugate transpose $Z_{22} (= [Z_{22}^H]^H)$ also has full rank. This proposition is very useful; e.g., some final results in Klein (2000) can be significantly simplified.

A.4. FULL DERIVATION

This section provides the full derivation. For the notation, see the main text.

Applying the QZ decomposition to (3),

$$\begin{aligned} 0 &= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\ &= \begin{bmatrix} \Omega_{ss}^A & \Omega_{su}^A \\ 0 & \Omega_{uu}^A \end{bmatrix}\begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega_{ss}^B & \Omega_{su}^B \\ 0 & \Omega_{uu}^B \end{bmatrix}\begin{pmatrix} s_t \\ u_t \end{pmatrix} \\ &\quad + \begin{bmatrix} Q_{s.}^H \\ Q_{u.}^H \end{bmatrix} C\xi_t + \begin{bmatrix} Q_{s.}^H \\ Q_{u.}^H \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q_{s.}^H \\ Q_{u.}^H \end{bmatrix} E\xi^{t,S}, \end{aligned} \quad \textbf{(A.2)}$$

where s_t and u_t are stable and unstable roots, respectively, such that

$$\begin{pmatrix} s_t \\ u_t \end{pmatrix} := \begin{bmatrix} Z_{s\kappa}^H & Z_{s\phi}^H \\ Z_{u\kappa}^H & Z_{u\phi}^H \end{bmatrix}\begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}.$$

Imperfect information requires a slightly careful treatment of TVCs. Focusing on the lower half of (A.2)

$$0 = \Omega_{uu}^A u_{t+1} + \Omega_{uu}^B u_t + Q_{u.}^H C\xi_t + Q_{u.}^H D\xi_{t+1} + Q_{u.}^H E\xi^{t,S}. \quad \textbf{(A.3)}$$

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Iterating it forward,

$$\lim_{l\to\infty}\left\{\begin{array}{c}\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l}u_{t+l}\\ +\sum_{s=1}^{l-1}\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{s}\left(\Omega_{uu}^{B}\backslash Q_{u.}^{H}\right)\left(C\xi_{t+s}+D\xi_{t+1+s}+E\tilde{\xi}^{t+s,S}\right)\end{array}\right\}$$

$$= -u_t-\left(\Omega_{uu}^{B}\backslash Q_{u.}^{H}\right)C\xi_t-\sum_{l=0}^{S}\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l}\left(\Omega_{uu}^{B}\backslash Q_{u.}^{H}\right)E\hat{\xi}^{t+l,S}, \qquad \textbf{(A.4)}$$

where

$$\xi^{t+l,S}=\begin{pmatrix}\xi_{t+l}\\ \vdots\\ \xi_{t+1}\\ \xi_t\\ \vdots\\ \xi_{t+l-S}\end{pmatrix}=\hat{\xi}^{t+l,S}+\tilde{\xi}^{t+l,S}:=\begin{pmatrix}0\\ \vdots\\ 0\\ \xi_t\\ \vdots\\ \xi_{t+l-S}\end{pmatrix}+\begin{pmatrix}\xi_{t+l}\\ \vdots\\ \xi_{t+1}\\ 0\\ \vdots\\ 0\end{pmatrix},$$

where $A\backslash B=A^{-1}B$ and $A/B=AB^{-1}$.

There are many information sets, under each of which TVCs must be satisfied—that is, TVCs are (seemingly) tighter under imperfect information. However, if the perfect-information counterpart satisfies TVCs, corresponding imperfect-information models also satisfy them automatically due to the law of iterated linear projection.[14] Thus, the same logic holds as in the perfect-information case; because $\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l}\to 0$ as $l\to 0$ by construction, the expected value of u_{t+l} explodes for any nonzero value of the RHS of (A.4), which contradicts the TVCs. Note that the inside the limit operator in the LHS shows the expected value of u_{t+l} (the realisation of u_{t+l} plus expectation errors) times $\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l}$. Hence, the RHS of (A.4) must be zero.

Therefore,

$$\begin{aligned}-\Omega_{uu}^{B}u_t&=-\Omega_{uu}^{B}Z_{u\kappa}^{H}\kappa_t-\Omega_{uu}^{B}Z_{u\phi}^{H}\phi_t\\ &=Q_{u.}^{H}C\xi_t+\Omega_{uu}^{B}\sum_{l=0}^{S}\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l}\left(\Omega_{uu}^{B}\backslash Q_{u.}^{H}\right)E\hat{\xi}^{t+l,S}\\ &=Q_{u.}^{H}C\xi_t+\sum_{l=0}^{S}\left(-\Omega_{uu}^{A}/\Omega_{uu}^{B}\right)^{l}Q_{u.}^{H}E\hat{\xi}^{t+l,S}.\end{aligned} \qquad \textbf{(A.5)}$$

Substituting our "guess solution" (4) into (A.5),

$$\begin{aligned}0&=\left(\Omega_{uu}^{B}Z_{u\kappa}^{H}+\Omega_{uu}^{B}Z_{u\phi}^{H}F\right)\kappa_t+\Omega_{uu}^{B}Z_{u\phi}^{H}G\xi^{t,S}+Q_{u.}^{H}C\xi_t\\ &\quad+\sum_{l=0}^{S}\left(-\Omega_{uu}^{A}/\Omega_{uu}^{B}\right)^{l}Q_{u.}^{H}E\hat{\xi}^{t+l,S}.\end{aligned} \qquad \textbf{(A.6)}$$

Similarly, from the upper half,

$$\begin{aligned}0&=\Omega_{ss}^{A}\left(Z_{s\kappa}^{H}\kappa_{t+1}+Z_{s\phi}^{H}\phi_{t+1}\right)+\Omega_{su}^{A}\left(Z_{u\kappa}^{H}\kappa_{t+1}+Z_{u\phi}^{H}\phi_{t+1}\right)\\ &\quad+\Omega_{ss}^{B}\left(Z_{s\kappa}^{H}\kappa_t+Z_{s\phi}^{H}\phi_t\right)+\Omega_{su}^{B}\left(Z_{u\kappa}^{H}\kappa_t+Z_{u\phi}^{H}\phi_t\right)\\ &\quad+Q_{s.}^{H}C\xi_t+Q_{s.}^{H}D\xi_{t+1}+Q_{s.}^{H}E\xi^{t,S}.\end{aligned} \qquad \textbf{(A.7)}$$

Again, by substituting (A.4) into (A.7), after some manipulation,

$$\begin{aligned}0 &= \left(\Lambda^A_{s\phi}FH + \Lambda^A_{s\kappa}H + \Lambda^B_{s\phi}F + \Lambda^B_{s\kappa}\right)\kappa_t \\ &+ \Lambda^A_{s\phi}G\xi^{t+1,S} + Q^H_{s.}D\xi_{t+1} + Q^H_{s.}C\xi_t \\ &+ \left(\Lambda^A_{s\phi}FJ + \Lambda^A_{s\kappa}J + \Lambda^B_{s\phi}G + Q^H_{s.}E\right)\xi^{t,S}. \end{aligned} \quad \textbf{(A.8)}$$

Though the definitions of $\Lambda^A_{s\kappa}$, $\Lambda^A_{s\phi}$, $\Lambda^B_{s\kappa}$, and $\Lambda^B_{s\phi}$ are (10a) in the main text, the following definition may be more useful:

$$\begin{bmatrix}\Lambda^A_{s\kappa} & \Lambda^A_{s\phi} \\ \Lambda^B_{s\kappa} & \Lambda^B_{s\phi}\end{bmatrix} := \begin{bmatrix}\Omega^A_{ss} & \Omega^A_{su} \\ \Omega^B_{ss} & \Omega^B_{su}\end{bmatrix}\begin{bmatrix}Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi}\end{bmatrix}. \quad \textbf{(A.9)}$$

Expanding $\xi^{t+1,S}$ and $\xi^{t,S}$ in (A.8) and (A.6),

$$\begin{aligned}0 &= \left(\Lambda^A_{s\phi}FH + \Lambda^A_{s\kappa}H + \Lambda^B_{s\phi}F + \Lambda^B_{s\kappa}\right)\kappa_t \\ &+ \left(\Lambda^A_{s\phi}G_0 + Q^H_{s.}D\right)\xi_{t+1} \\ &+ \left(\Lambda^A_{s\phi}G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right)J_0 + \Lambda^B_{s\phi}G_{0.} + Q^H_{s.}E_{0.} + Q^H_{s.}C\right)\xi_t \\ &+ \left(\Lambda^A_{s\phi}G_2 + \left(\Omega^A_{ss}/Z_{\kappa s}\right)J_1 + \Lambda^B_{s\phi}G_{1.} + Q^H_{s.}E_{1.}\right)\xi_{t-1} + \cdots \\ &+ \left(\Lambda^A_{s\phi}G_S + \left(\Omega^A_{ss}/Z_{\kappa s}\right)J_{S-1} + \Lambda^B_{s\phi}G_{S-1.} + Q^H_{s.}E_{S-1.}\right)\xi_{t-(S-1)} \\ &+ \left(\left(\Omega^A_{ss}/Z_{\kappa s}\right)J_S + \Lambda^B_{s\phi}G_{S.} + Q^H_{s.}E_{S.}\right)\xi_{t-S},\end{aligned}$$

$$\begin{aligned}0 &= \left(\Omega^B_{uu}Z^H_{u\kappa} + \Omega^B_{uu}Z^H_{u\phi}F\right)\kappa_t \\ &+ \sum_{s=1}^{S}\left\{\Omega^B_{uu}Z^H_{u\phi}G_s + \left[\sum_{k=0}^{S-s}\left(-\Omega^A_{uu}/\Omega^B_{uu}\right)^k Q^H_{u.}E_{k+s}\right]\right\}\xi_{t-s} \\ &+ \left\{Q^H_{u.}C + \Omega^B_{uu}Z^H_{u\phi}G_0 + \left[\sum_{k=0}^{S}\left(-\Omega^A_{uu}/\Omega^B_{uu}\right)^k Q^H_{u.}E_k\right]\right\}\xi_t.\end{aligned}$$

Because these matrix equations must hold for any realization of κ_t, $\xi_{t-\tau}$ for $\tau = -1, 0, 1, \ldots, S$,

$$0 = \Lambda^A_{s\phi}FH + \Lambda^A_{s\kappa}H + \Lambda^B_{s\phi}F + \Lambda^B_{s\kappa}, \quad \textbf{(A.10a)}$$

$$0 = \Omega^B_{uu}Z^H_{u\kappa} + \Omega^B_{uu}Z^H_{u\phi}F, \quad \textbf{(A.10b)}$$

$$0 = \Lambda^A_{s\phi}G_{0.} + Q^H_{s.}D, \quad \textbf{(A.11a)}$$

$$0 = 0, \quad \textbf{(A.11b)}$$

$$0 = \Lambda^A_{s\phi}G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right)J_0 + \Lambda^B_{s\phi}G_0 + Q^H_{s.}E_{S.} + Q^H_{s.}C, \quad \textbf{(A.12a)}$$

$$0 = \Omega^B_{uu}Z^H_{u\phi}G_0 + \left[\sum_{s=0}^{S}\left(-\Omega^A_{uu}/\Omega^B_{uu}\right)^s Q^H_{u.}E_s\right] + Q^H_{u.}C, \quad \textbf{(A.12b)}$$

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$$0 = \Lambda^A_{s\phi} G_{s+1} + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_s + \Lambda^B_{s\phi} G_s + Q^H_{s.} E_s, \quad \textbf{(A.13a)}$$

$$0 = \Omega^B_{uu} Z^H_{u\phi} G_s + \left[\sum_{k=0}^{S-s} \left(-\Omega^A_{uu}/\Omega^B_{uu}\right)^k Q^H_{u.} E_{k+s}\right], \quad \textbf{(A.13b)}$$

for $s = 1, \ldots, S-1$,

$$0 = \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_S + Q^H_{s.} E_S, \quad \textbf{(A.14a)}$$

$$0 = \Omega^B_{uu} Z^H_{u\phi} G_S + Q^H_{u.} E_S. \quad \textbf{(A.14b)}$$

Since (A.10a) and (A.10b) do not include G, J, D, E, or Π, these two matrix equations can be solved for H and F independently. Thus, assuming $Z^H_{u\phi}$ has a (right) inverse,[15]

$$F = -Z^H_{u\phi} \backslash Z^H_{u\kappa} = Z_{\phi s}/Z_{\kappa s},$$
$$H = -Z_{\kappa s}\left(\Omega^A_{ss} \backslash \Omega^B_{ss}\right) ./Z_{\kappa s}$$

Note that the H and F matrices are the same as in the corresponding perfect-information model.[16]

Vertically concatenating matrix equations (A.12a)–(A.14b) in pairs,

$$0 = \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + \sum_{k=0}^{S} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H \left(E_k + C\right), \quad \textbf{(A.15a)}$$

$$0 = \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + \sum_{k=0}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} \qquad \text{for } s = 1, \ldots, S-1, \quad \textbf{(A.15b)}$$

$$0 = \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S. \quad \textbf{(A.15c)}$$

Note that

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix} \left(\begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+2} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \sum_{k=0}^{S-(s+1)} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s+1} \right)$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \sum_{k=1}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s}. \quad \textbf{(A.16)}$$

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Subtracting (A.16) from each of (A.15a–c),[17]

$$0 = \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + Q^H E_k + Q^H C, \quad \textbf{(A.17a)}$$

$$0 = \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + Q^H E_{k+s}, \quad \textbf{(A.17b)}$$

for $s = 1, \ldots, S-1$,

$$0 = \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S, \quad \textbf{(A.17c)}$$

and again vertically concatenating these equations,

$$0 = M_{y\Gamma} \Gamma + \mathbf{Q}(E + \mathbf{C}),$$

$$\Gamma := \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_S \end{pmatrix}, \quad E := \begin{pmatrix} E_0 \\ \vdots \\ E_S \end{pmatrix}, \quad \mathbf{C} := \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \quad \mathbf{Q} := \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix},$$

$$M_{y\Gamma} := \begin{bmatrix} \Phi & \Lambda^{0A} & & \\ & \ddots & \ddots & \\ & 0 & \Phi & \Lambda^{0A} \\ & & & \Phi \end{bmatrix}, \quad \Phi := \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix},$$

$$\Lambda^{0A} := \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix}.$$

Note that because Φ is invertible, $M_{y\Gamma}$ is also clearly invertible. Hence,

$$\begin{aligned} 0 &= \Gamma + M_{y\Gamma} \backslash \mathbf{Q}(E + \mathbf{C}) \\ &= M_{\Gamma\Pi} \Pi + M_{y\Gamma} \backslash \mathbf{Q}(E + \mathbf{C}), \end{aligned}$$

where (7) is used to derive the second line. Hence,

$$0 = \Pi + M_{\Pi E}(E + \mathbf{C}), \quad \textbf{(A.18a)}$$

$$M_{\Pi E} := \left(M_{y\Gamma} M_{\Gamma\Pi}\right) \backslash \mathbf{Q}. \quad \textbf{(A.18b)}$$

In the following, we compute E and Π *column by column*:

$$\Pi_{.i} = M_{\Pi E}\,(E_{.i} + \mathbf{C}_{.i}).$$

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Remember that some elements in $\Pi_{.i}$ are zero due to imperfect information, whereas some elements in $E_{.i}$ are nonzero. For example,

$$0 = \begin{pmatrix} \Pi_{1,i} \\ \vdots \\ \Pi_{k,i} \, (= 0) \\ \vdots \\ \Pi_{M(S+1),i} \end{pmatrix} + M_{\Pi E} \left(\begin{pmatrix} 0 \\ \vdots \\ E_{ji} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} C_{.i} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right). \qquad \textbf{(A.19)}$$

From the kth set of equations in (A.19),

$$0 = \left[M_{\Pi E} \right]_{kj} E_{ji} + \left[M_{\Pi E} \right]_{kj} \mathbf{C}_{ji} + \left[M_{\Pi E} \right]_{k\neg j} \mathbf{C}_{\neg ji}.$$

Hence, assuming $[M_{\Pi E}]_{kj}$ is invertible,

$$E_{ji} = -\left[M_{\Pi E} \right]_{kj} \backslash \left[M_{\Pi E} \right]_{k\neg j} \mathbf{C}_{\neg ji} - \mathbf{C}_{ji}.$$

From the other equations in (A.19), we eliminate the expectation errors E_{ji}:

$$\begin{aligned} \Pi_{\neg ki} &= [M_{\Pi E}]_{\neg kj} \left([M_{\Pi E}]_{kj} \backslash [M_{\Pi E}]_{k\neg j} \mathbf{C}_{\neg ji} + \mathbf{C}_{ji} \right) \\ &\quad - [M_{\Pi E}]_{\neg kj} \mathbf{C}_{ji} - [M_{\Pi E}]_{\neg k\neg j} \mathbf{C}_{\neg ji} \\ &= \left([M_{\Pi E}]_{\neg kj} \left([M_{\Pi E}]_{kj} \backslash [M_{\Pi E}]_{k\neg j} \right) - [M_{\Pi E}]_{\neg k\neg j} \right) \mathbf{C}_{\neg ji} \\ &= -\left[M_{\Pi E}^{-1} \right]_{\neg j\neg k} \backslash \mathbf{C}_{\neg ji}. \end{aligned}$$

The vector $\Pi_{\neg ki}$ and $\Pi_{ki} = 0$ can be vertically merged to recover $\Pi_{.i}$, and the vectors $\Pi_{.i}$ are horizontally concatenated to recover the full Π matrix. Note that an invertible $[M_{\Pi E}]_{kj}$ implies an invertible $\left[M_{\Pi E}^{-1} \right]_{\neg j\neg k}$. Not surprisingly, $\mathbf{C}_{ji}$ does not affect the coefficient matrix $\Pi_{.i}$, because the jth set of equations does not hold for the ith innovation in any case; it only affects the expectation error E_{ji}.

To obtain the J and G matrices, from (7),

$$\Gamma := \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi} \Pi.$$

From the A matrix in a given model (3),

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix},$$

which always satisfies (A.11a). It can be shown that the jth rows in D are zeros if the jth equation does not include $t+1$ dynamic jump variables (see the next section).

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A.5. A COMMENT ON THE D MATRIX

The direct derivation of the D matrix from (A.11a) is a bit tricky, and requires careful attention concerning the nonsquare matrices $\Lambda^A_{s\phi}$ and $Q^H_{s.}$. Also, it is perhaps not intuitive. In this article, we exploit an ex post relationship (21), and here we show that it always satisfies (A.11a), which, in turn, reveals an important intuition.

First, we define dynamic and nondynamic jump variables: $\phi_{t+1} = [(\phi^d_{t+1})^T \ (\phi^n_{t+1})^T]^T$. Note that the coefficients on the nondynamic jump variables ϕ^n_{t+1} in A matrix must be zero by the definition of "nondynamic".

$$Ay_{t+1} := \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{pmatrix} \kappa_{t+1} \\ \phi^d_{t+1} \\ \phi^n_{t+1} \end{pmatrix},$$

where ϕ^d_{t+1} is the vector of dynamic variables, such as consumption in the Euler equation. The submatrices in G_0 and Q^H are defined as

$$\tilde{G}_0 := \begin{bmatrix} 0 \\ G_0 \end{bmatrix} := \begin{bmatrix} 0 \\ G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix},$$

$$Q^H := \begin{bmatrix} Q^H_{s.} \\ Q^H_{\phi.} \end{bmatrix}, \quad Q^H_{s.} := \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix}, \quad Q^H_{u.} := \begin{bmatrix} Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix},$$

where u^f and u^i imply finite and infinite unstable roots, respectively.

Focusing on the second term of (A.11a)

$$Q^H_{s.}D = Q^H_{s.}A\tilde{G}_0 = \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix}$$
$$= \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) G_{0,\phi^d.} \tag{A.24}$$

For the first term of (A.11a) note that $\Lambda^A_{s\phi}$ is the $s\phi$th elements in $\Omega^A Z^H$; i.e.,

$$\Lambda^A_{s\phi} = \left[\Omega^A Z^H\right]_{s\phi} = \left[QQ^H\Omega^A Z^H\right]_{s\phi} = [QA]_{s\phi}$$
$$= \left[\begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \\ Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \right]_{s\phi}$$
$$= \left[\begin{bmatrix} * & \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d}\right) & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix} \right]_{s\phi}$$
$$= \left[\left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d}\right) \quad 0 \right],$$

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where $*$ elements are irrelevant to our current concern. Hence,

$$\Lambda^A_{s\phi} G_0 = \left[\left(Q^H_{s\kappa} A_{\kappa\phi^d} + Q^H_{s\phi^d} A_{\phi^d\phi^d} + Q^H_{s\phi^n} A_{\phi^n\phi^d} \right) \quad 0 \right] \begin{bmatrix} G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix}$$

$$= \left(Q^H_{s\kappa} A_{\kappa\phi^d} + Q^H_{s\phi^d} A_{\phi^d\phi^d} + Q^H_{s\phi^n} A_{\phi^n\phi^d} \right) G_{0,\phi^d.} \qquad \textbf{(A.25)}$$

(A.24) and (A.25) show that (A.11a) satisfies (21). The key to the solution is a sort of zero restriction: the A matrix has zero columns by the definition of "nondynamic" variables.

A further question is the consistency of D (i.e., whether the computed D always has zeros at the proper positions). Specifically, if the jth equation does not have ϕ^d_{t+1}, it should not have an expectation error due to ξ_{t+1}, and hence the row vector $D_{j.}$ must be zero; this zero restriction on D is analogous to that on E. This is surely satisfied because the rows corresponding to nondynamic equations in D $(= A\tilde{G}_0)$ are always zero by the construction of A; i.e., the jth row in A is zero if the jth equation does not include dynamic jump variables ϕ^d_{t+1}. For example, in the standard RBC model, all but the Euler equation have zero rows in A and hence in D.

What this section discusses is the correspondence between expectation errors and the source of such errors. If, for example, expectation errors with respect to full information up to time t appear in the equations without dynamic jump variables, then it is a logical contradiction (expectation errors without their causes), and hence (A.11a) is not satisfied. Conceptually, the consistency of the D matrix is parallel to the invertibility of $[M_{\Pi E}]_{kj}$. As mentioned in the main text, the noninvertibility of $[M_{\Pi E}]_{kj}$ implies an incorrect specification of the information structure with respect to $\xi_{t+\tau}$ $(\tau = 0, 1, \ldots, S)$. Similarly, an inconsistent D (or the nonexistence of a consistent D) implies an incorrect specification of information structure with respect to ξ_{t+1}. Such inconsistency/nonexistence happens, for example, if a researcher puts an expectation operator on the evolution of capital, rather than on the consumption Euler equation.