*Euro. Jnl of Applied Mathematics* (2005), *vol.* 16, *pp.* 335–359.  
doi:10.1017/S0956792505006200 Printed in the United Kingdom

# Homogenization of a degenerate triple porosity model with thin fissures

B. AMAZIANE,[1] M. GONCHARENKO[2] and L. PANKRATOV[1,2]

[1] *Laboratoire de Mathématiques Appliquées, CNRS–UMR5142, Université de Pau, av. de l'Université, 64000 Pau, France*
*email*: brahim.amaziane@univ-pau.fr
[2] *Mathematical Division, B.Verkin Institute for Low Temperature Physics and Engineering (FTINT), 47, Lenin av., 61103, Kharkov, Ukraine*

(*Received 20 April 2004; revised 31 March 2005*)

We consider the problem of modelling the flow of a slightly compressible fluid in a periodic fractured medium assuming that the fissures are thin with respect to the block size. As a starting point we used a formulation applied to a system comprising a fractured porous medium made of blocks and fractures separated by a thin layer which is considered as an interface. The inter-relationship between these three characteristics comprise the triple porosity model. The microscopic model consists of the usual equation describing Darcy flow with the permeability being highly discontinuous. Over the matrix domain, the permeability is scaled by $(\varepsilon\delta)^2$, where $\varepsilon$ is the size of a typical porous block, with $\delta$ representing the relative size of the fracture. We then consider a model with Robin type transmission conditions: a jump of the density across the interface block-fracture is taken into account and proportional to the flux by the mean of a function $(\varepsilon\delta)^{-\gamma}$, where $\gamma$ is a parameter. Using two-scale convergence, we get homogenized models which govern the global behaviour of the flow as $\varepsilon$ and $\delta$ tend to zero. The resulting homogenized problem is a dual-porosity type model that contains a term representing memory effects for $\gamma \leqslant 1$, and it is a single porosity model with effective coefficients for $\gamma > 1$.

## 1 Introduction

Modelling of flow in fractured media is a subject of intensive research in many engineering disciplines, such as petroleum engineering, water resources management, civil engineering. More recently, fractured rock domains corresponding to the so-called Excavation Damaged Zone (EDZ) receives increasing attention in connection with the behaviour of geological isolation of radioactive waste after the drilling of the wells or shafts. A fissured medium is a structure consisting of a porous and permeable matrix which is interlaced on a fine scale by a system of highly permeable fissures. The majority of fluid transport will occur along flow paths through the fissure system, and the relative volume and storage capacity of the porous matrix is much larger than that of the fissure system. When the system of fissures is so well developed that the matrix is broken into individual blocks or cells that are isolated from each other, there is consequently no flow directly from cell to cell, but only an exchange of fluid between each cell and the surrounding fissure system. Therefore the large-scale description will have to incorporate the two different flow mechanisms. For some permeability ratios and fissure widths, the large-scale description is achieved by introducing the so-called double porosity model. It was introduced first for describing the global behaviour of fractured porous media by Barenblatt *et al.* [7]. It has been since used in a wide range of engineering specialities related to geohydrology, petroleum reservoir engineering, civil engineering or soil science.

Within the framework of the homogenization approach the usual double porosity model assumes that the width or opening of the fractures [34] containing highly permeable porous media is of the same order as the block size, and the ratio of the permeability in the matrix blocks and the fissures system is of the order $\varepsilon^2$, leading to a high contrast for the corresponding characteristic times [25]. The double porosity problem was first studied in Arbogast *et al.* [5], and was then revisited in the mathematical literature by many other authors [10, 11, 25, 29, 31, 35].

In this paper we investigate models assuming different scale ratios. For this we consider the fissured part to be a porous medium crossed by many small fissures but behaving like a porous medium with permeability of order 1, and the porous blocks (or matrix) made of porous material but with a small permeability. Then we introduce an additional small parameter $\delta$ quantifying the ratio between the thickness of the fissured part and the matrix diameter (see Figure 1). We consider, then, the family of models corresponding to a range of permeability ratios allowing the fissures thickness $\varepsilon\delta$ to become very small. If we denote by $\varepsilon$ the size of a typical block of porous materiel, then in order to have the same characteristic time scale for a parabolic evolution in one block and for the flow through the entire system of fractures, it is necessary to assume a ratio of permeability in the blocks and in the fissures to be of order $(\varepsilon\delta)^2$. This time ratio, $(\varepsilon\delta)^2$, is exactly the one leading to the dual-porosity model. If the ratio is smaller than that of order $(\varepsilon\delta)^2$, then there is no contribution from the blocks to the global continuity system of equations in the limit model, which is obtained as a homogenization limit of the system of fissures only.

In the present paper we consider a single phase flow of a slightly compressible fluid through a periodic fractured–porous medium made of a set of porous blocks with permeability of order $(\varepsilon\delta)^2$, where $0 < \varepsilon \ll \delta \ll 1$; these porous blocks are surrounded by a system of connected fissures with permeability of order 1. We also suppose that the fissures and the matrix are separated by a very thin, so called colmated layer (e.g. see Sophocleous [33] or Delleur [18] (Chapter 1), where it appears as a fractured-porous layer of sedimentary deposits). The model will be developed in Appendix A on the basis of physical intuition when the thickness of this layer tends to zero. The model is described by a linear parabolic equation in each part with Robin type transmission conditions plus appropriate initial and boundary conditions. We consider the continuity of the flux at the interfaces, but there is a jump of the density, which is proportional to the flux, by means of a function $(\varepsilon\delta)^{-\gamma}$ where $\gamma$ is a parameter. Following Bourgeat *et al.* [14] and Amaziane *et al.* [4], our homogenization process will be based on two main steps. In the first step we fix $\delta$ and apply the Laplace transform to the boundary value problem to reduce our parabolic problem to an elliptic one. We then pass to the limit, as $\varepsilon$ tends to zero, using two-scale convergence [2] leading to a $\delta$-model, i.e. a boundary value problem considered in a homogeneous domain with coefficients still depending on the parameter $\delta$. In the second step we then pass to the limit as $\delta$ tends to zero and we obtain a final homogenized model with no dependence on either on $\varepsilon$ or on $\delta$.

Similar questions, with different parameters and different scope, have been considered by several authors [4, 12, 14, 30]. In Pankratov & Rybalko [30], this type of microstructure, but with a fixed relative fissure size, was modeled with only one parameter $\varepsilon$. In contrast with the present paper the continuity of the density and the flux were assumed to be satisfied in these papers. Let us mention also that homogenization problems involving Robin type interface conditions have been studied in past years [3, 19, 20, 23, 24, 25, 27]. More details about the physics of the problem can be found in Panfilov [29]. For a more general discussion of the homogenization method used to establish the results of this paper, we refer to Cioranescu & Donato [15] and Cioranescu & Saint Jean Paulin [16].

The outline of the rest of the paper is as follows. § 2 contains the equations of the microscopic model and the main results of the paper which correspond to different values of the parameter $\gamma$ as $\varepsilon$ and $\delta$ tend to zero. There are three typical different behaviours for $\gamma$ equal to, strictly greater than or strictly less than one. The resulting homogenized problem is a dual-porosity type model that contains a term representing memory effects which could be seen as source term or as a time delay for $\gamma \leqslant 1$, and it is a single porosity model with effective coefficients for $\gamma > 1$. § 3 is devoted to the proof of the convergence result for $\gamma = 1$. The result when $\gamma > 1$ is proved in § 4. The proof of the convergence result for $\gamma < 1$ is carried out in § 5. Note that the expression of the exchange kernel is different for the two homogenized models corresponding to $\gamma = 1$ and $\gamma < 1$. Additional conclusions are drawn in § 6. In Appendix A, we derive, by physical arguments, the model used in this study by considering a double porosity model with a thin weakly permeable layer between fissures and blocks. The model is obtained when the thickness of this layer tends to zero.

## 2 Formulation of the problem and the main results

In this section, we describe a microscopic double porosity model with Robin interface conditions in a periodic fractured medium. We consider a reservoir $\Omega \subset \mathbb{R}^3$ to be a bounded connected domain with a periodic structure. More precisely, we will scale this periodic structure by a parameter $\varepsilon$ which represents the ratio of the cell size to the size of the whole region $\Omega$ and we will assume that $\varepsilon$ is a parameter tending to zero.

Let $\mathscr{Y} = ]0,1[^3$ represent the microscopic domain of the basic cell of a fractured porous medium. For the sake of simplicity and without loss of generality, we assume that $\mathscr{Y}$ is made up of two homogeneous porous media $\mathscr{M}^\delta$ and $\mathscr{F}^\delta$ corresponding to the parties of the microscopic domain occupied by the matrix block and the fracture, respectively. We assume that $\mathscr{M}^\delta$ is an open cube centered at the same point as $\mathscr{Y}$ with length equal to $(1-\delta)$, where $0 < \delta < 1$. Thus $\mathscr{Y} = \mathscr{M}^\delta \cup \Gamma^\delta_{mf} \cup \mathscr{F}^\delta$, where $\Gamma^\delta_{mf}$ denotes the interface between the two media.

Let $\Omega^{\varepsilon,\delta}_i$ with $i = m$ or $f$ denote an open set filled with the porous medium $i$. Then $\Omega = \Omega^{\varepsilon,\delta}_m \cup \Gamma^{\varepsilon,\delta}_{mf} \cup \Omega^{\varepsilon,\delta}_f$, where $\Gamma^{\varepsilon,\delta}_{mf} = \partial\Omega^{\varepsilon,\delta}_m \cap \partial\Omega^{\varepsilon,\delta}_f$ and the subscripts $m$ and $f$ refer to the matrix and fracture, respectively (see Figure 1). Let $\Omega_T = ]0,T[\times\Omega$, where $T > 0$ is given. For the sake of simplicity we assume that $\partial\Omega^{\varepsilon,\delta}_m \cap \partial\Omega = \emptyset$. Since the measure of the set $\mathscr{F}^\delta$ is given by

$$|\mathscr{F}^\delta| = 1 - (1-\delta)^3 = 3\delta - 3\delta^2 + \delta^3 = 3\delta + o(\delta) \tag{2.1}$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

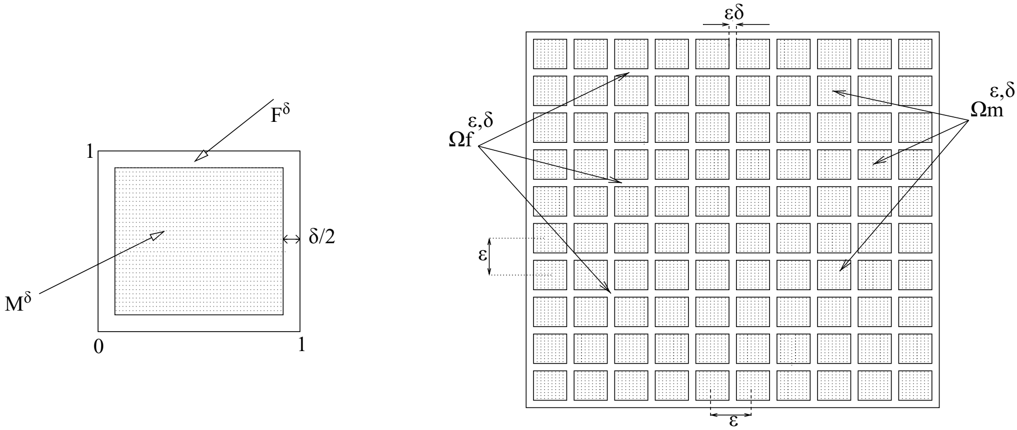


FIGURE 1. A fractured periodic domain with thin fissures.

as $\delta \to 0$, the measure of the fissure set $\Omega_f^{\varepsilon,\delta}$ satisfies

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} |\Omega_f^{\varepsilon,\delta}| = 0. \tag{2.2}$$

Let us now introduce the permeability coefficient and the porosity of the porous medium $\Omega$. We set

$$K^{\varepsilon,\delta}(x) = k_m(\varepsilon\delta)^2 \cdot \mathbf{1}_m^{\varepsilon,\delta}(x) + k_f \cdot \mathbf{1}_f^{\varepsilon,\delta}(x); \tag{2.3}$$

$$\omega^{\varepsilon,\delta}(x) = \omega_m \cdot \mathbf{1}_m^{\varepsilon,\delta}(x) + \omega_f \cdot \mathbf{1}_f^{\varepsilon,\delta}(x), \tag{2.4}$$

where $k_f$ is the permeability of the fissures, $k_m$ is the permeability of the blocks, $\omega_f$ is the porosity of the fissures, $\omega_m$ is the porosity of the blocks; $\mathbf{1}_f^{\varepsilon,\delta} = \mathbf{1}_f^{\varepsilon,\delta}(x)$ and $\mathbf{1}_m^{\varepsilon,\delta} = \mathbf{1}_m^{\varepsilon,\delta}(x)$ denote the characteristic periodic functions of the sets $\Omega_f^{\varepsilon,\delta}$ and $\Omega_m^{\varepsilon,\delta}$, respectively. Here $0 < k_f, k_m, \omega_f, \omega_m < +\infty$. As in the classical double porosity model, the critical process in any naturally fractured porous medium is the transfer of fluid between the matrix and fractures. Homogenized models which do not preserve this transfer in some sense as $\varepsilon \to 0$, as shown in Arbogast *et al.* [6], are not reasonable. The traditional scaling means that the square of the fissure thickness is of the same order as the permeability coefficient of the matrix set. The same result was obtained in the case when the fissures had a low bulk volume (see Pankratov & Rybalko [30] and Amaziane *et al.* [4]). The homogenization of a fractured media was obtained in Amaziane *et al.* [4] for a ratio between permeabilities of blocks and fractures equal to $(\varepsilon\delta)^2$.

A popular model of the flow of a single, slightly compressible fluid in an incompressible porous media is described by the mass balance equation combined with Darcy's law, leading to the following diffusion equations [8]:

$$\omega_f \frac{\partial u_f^{\varepsilon,\delta}}{\partial t} - k_f\, \Delta u_f^{\varepsilon,\delta} = Q(x) \quad \text{in} \quad ]0,T[\times \Omega_f^{\varepsilon,\delta}; \tag{2.5}$$

$$\omega_m \frac{\partial u_m^{\varepsilon,\delta}}{\partial t} - k_m(\varepsilon\delta)^2\, \Delta u_m^{\varepsilon,\delta} = 0 \quad \text{in} \quad ]0,T[\times \Omega_m^{\varepsilon,\delta}, \tag{2.6}$$

where $u_i^{\varepsilon,\delta}$, $i = f, m$, is the fluid density in $\Omega_i^{\varepsilon,\delta}$ and $Q \in L^2(\Omega)$ represent external sources. For simplicity we have neglected the gravity effect, the fluid viscosity and the compressibility factor are taken to be equal to one. We assume that in such a geometrical configuration of $\Omega$, the transmission conditions described in Appendix A are of the form

$$\begin{cases} k_f \nabla u_f^{\varepsilon,\delta} \cdot \vec{\nu} = (\varepsilon\delta)^2 k_m \nabla u_m^{\varepsilon,\delta} \cdot \vec{\nu} & \text{on} \quad \Gamma_{mf}^{\varepsilon,\delta}; \\ (\varepsilon\delta)^2 k_m \nabla u_m^{\varepsilon,\delta} \cdot \vec{\nu} = \sigma\, (\varepsilon\delta)^\gamma (u_f^{\varepsilon,\delta} - u_m^{\varepsilon,\delta}) & \text{on} \quad \Gamma_{mf}^{\varepsilon,\delta}, \end{cases} \tag{2.7}$$

where $\vec{\nu}$ is the normal vector on $\Gamma_{mf}^{\varepsilon,\delta}$ (exterior to $\Omega_f^{\varepsilon,\delta}$), $\gamma \in \mathbf{R}$ is a parameter and $\sigma$ is a positive constant.

Consider now the scaling of the interface conditions. The factor $\Sigma$ in (A 7) (see Appendix A) describes the jump in the pressure with respect to the fluid flux which is continuous. Then it follows from the equation of state satisfied by the density that the transmission conditions could be written in the form (2.7). Moreover the jump of the density at the interface depends on several geometrical and hydraulic parameters and is scaled by $\sigma(\varepsilon\delta)^\gamma$, where $\sigma$ is a positive constant independent of $\varepsilon, \delta$ and $\gamma$ is a parameter, $\gamma \in ]0, +\infty[$. In the framework of the scaling procedure used in this paper $\gamma$ is "responsible" for the exchange process between the fissures system and the matrix set. It is shown that if $\gamma < 1$ then we have no density jump on the interface fissure–matrix and the influence of the colmated layer is negligible. This case corresponds to the usual double porosity model. If $\gamma = 1$ it is necessary to take into account this layer. Finally if $\gamma > 1$ the layer is totally impermeable to the fluid. From a mathematical point of view this difference comes from the asymptotic behaviour of the surface term. Moreover, $\gamma = 1$ corresponds to the critical case when the limit of this surface term is nontrivial.

The system (2.5)–(2.7) is completed by boundary and initial conditions:

$$\begin{cases} u_f^{\varepsilon,\delta} = 0 & \text{on} \quad \partial\Omega; \\ u_f^{\varepsilon,\delta}(0, x) = 0 & \text{in} \quad \Omega_f^{\varepsilon,\delta}; \\ u_m^{\varepsilon,\delta}(0, x) = 0 & \text{in} \quad \Omega_m^{\varepsilon,\delta}. \end{cases} \tag{2.8}$$

In what follows we use standard notations for Sobolev spaces. Let us define the energy space

$$H^1(\Omega_f^{\varepsilon,\delta}, \Omega_m^{\varepsilon,\delta}) \equiv (H^1(\Omega_f^{\varepsilon,\delta}) \cap H_0^1(\Omega)) \times H^1(\Omega_m^{\varepsilon,\delta}).$$

Then following the lines of Clark & Showalter [17] and using Showalter [32] [Chapter III] we can prove that for each $\varepsilon, \delta \in ]0, 1[$ there exists a unique solution $u^{\varepsilon,\delta} = (u_f^{\varepsilon,\delta}, u_m^{\varepsilon,\delta}) \in C(0, T; H^1(\Omega_f^{\varepsilon,\delta}, \Omega_m^{\varepsilon,\delta}))$ of problem (2.5)–(2.8).

Due to the vanishing measure of the fissure, we should define the convergence of sequences according to the singularity of the fissure measure. For this, inspired by Bouchitté & Fragala [9], Bourgeat *et al.* [13], Cioranescu & Saint Jean Paulin [16], Panasenko [28] and Zhikov [35] we define:

**Definition 2.1** *A sequence* $(v^{\varepsilon,\delta}) \subset L^2(\Omega_f^{\varepsilon,\delta})$ *is said to* $L_{\varepsilon,\delta}$*–converge to a function* $v \in L^2(\Omega)$ *if*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{|\Omega_f^{\varepsilon,\delta}|}\|v^{\varepsilon,\delta}-v\|^2_{L^2(\Omega_f^{\varepsilon,\delta})}=0.$$

**Definition 2.2** *Let* $\Omega^\varepsilon$ *be any sub-domain of the domain* $\Omega$*, we will say that the sequence* $(v^\varepsilon) \subset L^2(\Omega^\varepsilon)$ *converges in the space* $L^2(\Omega^\varepsilon)$ *to a function* $v \in L^2(\Omega)$ *if*

$$\lim_{\varepsilon\to 0}\|v^\varepsilon - v\|_{L^2(\Omega^\varepsilon)}=0.$$

It is already known [13] that the $\varepsilon$ and $\delta$ limits commute. We then choose to study the asymptotic behaviour of $u^{\varepsilon,\delta}$ the solution of problem (2.5)–(2.8) as $\varepsilon \to 0$ and $\delta \to 0$.

At the present time two main approaches were developed to study double porosity type problems with thin fissures [4, 14, 30]. The first one involves only one small parameter but it requires some special notions of extension and convergence. The second one involves two small parameters and it is based on the ideas of Cioranescu & Saint Jean Paulin [16] where thin reticulated structures are studied by such a method. Notice that a singular double porosity model was considered in Bourgeat *et al.* [13]. In this paper we use the same homogenization approach considered in Amaziane *et al.* [4]. It involves two small parameters $\varepsilon, \delta$. We also make use of the Laplace transform to reduce our parabolic problem to an elliptic one. The homogenization process is then achieved in three main steps. On the first step we fix $\delta$ and apply the Laplace transform to the boundary value problem (2.5)–(2.8). We study then the asymptotic behaviour of $u_\lambda^{\varepsilon,\delta}$ solutions of the corresponding stationary boundary value problem as $\varepsilon \to 0$. For different values of the parameter $\gamma$ we obtain then stationary boundary value problems considered in the whole domain $\Omega$ but with the coefficients depending on the parameter $\delta$. In the second step we pass to the limit as $\delta \to 0$ and obtain a stationary homogenized problem independent of $\varepsilon, \delta$. Finally, on the third step we make use of the $L_{\varepsilon,\delta}$–convergence (see Definition 2.1) of $u_\lambda^{\varepsilon,\delta}$ to $u_\lambda^*$ solution of the stationary homogenized problem. Then we prove that the inverse Laplace transform of $u_\lambda^*$, denoted $u^*$, is the solution of the macroscopic model that it will be specified later.

Macroscopic models corresponding to the various situations are given by the following convergence results:

**Theorem 2.3** *Let* $\gamma = 1$ *in* (2.7) *and* $u^{\varepsilon,\delta} = (u_f^{\varepsilon,\delta}, u_m^{\varepsilon,\delta})$ *be the solution of* (2.5)–(2.8)*. Then, for any* $t \in ]0, T[$, $u_f^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$*-converges to* $u^*$*, the solution of a global model with an additional source term* $S(u^*)$ *and the fracture porosity as effective porosity:*

$$\begin{cases} \omega_f \dfrac{\partial u^*}{\partial t} - \dfrac{2}{3}k_f\Delta u^* = Q(x) + S(u^*), & \text{in} \quad \Omega_T; \\ u^*(t,x) = 0 & \text{on} \quad ]0,T[\times\partial\Omega; \\ u^*(0,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.9}$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

*where*

$$S(u^*) = -2\sigma \int_0^t u_t^*(t-\tau)\exp(\mu^2\tau)\,\mathrm{erfc}\,(\mu\sqrt{\tau})\,d\tau \tag{2.10}$$

*with*

$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)\,dt$$

*and*

$$\mu = \frac{\sigma}{\sqrt{k_m\omega_m}}.$$

**Remark 1** The exchange kernel in (2.10) contains the function erfc which usually appears as a solution of the $1D$ diffusion problem on a half axis. The appearance of this function in the macroscopic model (2.9)–(2.10) could be interpreted by some physical arguments. In the model under consideration, the width of the fissure is much smaller than the dimension of the adjacent block. Therefore this block may be considered as semi–infinite for the thin fissure in the direction orthogonal to the interface between the fissure and the block. Moreover, the density stabilization in a fracture happens very rapidly as the fracture permeability is very high. So the density along the block boundary is practically uniform, which suggests that no flow will be observed in the direction parallel to the block boundary. Thus, within a block we obtain a $1D$ diffusion problem from the interface with the fracture in the direction normal to the block boundary. This process may be described by the erfc–function which determines the structure of convolution kernels entering in the macro-scale model.

Similar results with a 1D exchange process between fractures and blocks described by an erfc–function (or a Gauss-function) have been obtained at the physical level of study in Panfilov [29] [pp. 47–48] for the case of non–thin fractures and very low permeable blocks. It has been shown that the overall mass exchange is localized within a narrow boundary layer in the neighborhood of the block-fracture interface which causes finally the appearance of the erfc–function typical for the 1D diffusion problems.

**Theorem 2.4** *Let* $\gamma > 1$ *in* (2.7) *and* $u^{\varepsilon,\delta} = (u_f^{\varepsilon,\delta}, u_m^{\varepsilon,\delta})$ *be the solution of* (2.5)–(2.8). *Then, for any* $t \in ]0,T[$, $u_f^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$*–converges to* $u^*$, *the solution of a single porosity model with effective constant porosity and permeability:*

$$\begin{cases} \omega_f \dfrac{\partial u^*}{\partial t} - \dfrac{2}{3}k_f\Delta u^* = Q(x), & \text{in} \quad \Omega_T; \\ u^*(t,x) = 0 & \text{on} \quad ]0,T[\times\partial\Omega; \\ u^*(0,x) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.11}$$

**Theorem 2.5** *Let* $\gamma < 1$ *in* (2.7) *and* $u^{\varepsilon,\delta} = (u_f^{\varepsilon,\delta}, u_m^{\varepsilon,\delta})$ *be the solution of* (2.5)–(2.8). *Then, for any* $t \in ]0,T[$, $u_f^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$*–converges to* $u^*$, *the solution of a global model with an additional*

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

*source term $S(u^*)$ and the fracture porosity as effective porosity:*

$$\begin{cases} \omega_f \dfrac{\partial u^*}{\partial t} - \dfrac{2}{3} k_f \Delta u^* = Q(x) + S(u^*), & \text{in} \quad \Omega_T; \\ u^*(t,x) = 0 & \text{on} \quad ]0,T[\times \partial\Omega; \\ u^*(0,x) = 0 & \text{in} \quad \Omega, \end{cases} \tag{2.12}$$

*where*

$$S(u^*) = -\frac{2\sqrt{k_m \omega_m}}{\sqrt{\pi}} \int_0^t \frac{u_t^*(x,\tau)}{\sqrt{t-\tau}} \, d\tau. \tag{2.13}$$

**Remark 2** Notice that the homogenized model (2.12)–(2.13) describes the well known Lauwerier phenomenon [26].

The homogenized models obtained in the paper can be written in the following form:

$$\omega_f \frac{\partial u^*}{\partial t} - \frac{2}{3} k_f \Delta u^* = Q(x) + S(u^*) \quad \text{in} \quad \Omega_T; \tag{2.14}$$

plus appropriate initial and boundary conditions, where

$$S(u^*) = -\int_0^t u_t^*(x,\tau) B_\gamma(t-\tau) \, d\tau \tag{2.15}$$

and the exchange kernel $B_\gamma$ is given in (2.10) when $\gamma = 1$, in (2.13) when $\gamma < 1$, and $B_\gamma \equiv 0$ when $\gamma > 1$. Homogenized models of type (2.14) were studied in the literature (e.g. see Hornung & Showalter [22] or Panfilov [29] Chapter 1). In these works the exchange kernel is calculated by the solution of an auxiliary boundary value problem with constant coefficients considered on the rescaled matrix inclusion. Furthermore, it could be found explicitly for some special geometry of the block. In contrast, in our case the corresponding boundary value problem depends on the small parameter $\delta$ and the kernel is calculated from the asymptotic expansion of the solution. Moreover, this solution appears to be exponentially small inside the inclusion and it is non-negligible only in a thin boundary layer. In particular, this means that in the traditional models we have an exchange between all the matrix blocks and the fissures. In our case we have an exchange only between the boundary of the block and the fissure.

## 3 Proof of Theorem 2.3

Knowing from Bourgeat *et al.* [13] that the limit as $(\varepsilon, \delta) \to 0$ does not depend on the order, Theorem 2.3 will be proved in three main steps. On the first step, fixing $\delta$, we apply the Laplace transform to the boundary value problem (2.5)–(2.8). We study the asymptotic behaviour of $u_\lambda^{\varepsilon,\delta}$ solutions of the corresponding stationary boundary value problem as $\varepsilon \to 0$. We obtain then a stationary boundary value problem considered in the

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

whole domain $\Omega$ but with the coefficients depending on the parameter $\delta$. In the second step we pass to the limit as $\delta \to 0$ and obtain a stationary homogenized problem, i.e. the problem independent of $\varepsilon, \delta$. Finally, in the third step we make use of the $L_{\varepsilon,\delta}$-convergence of $u^{\varepsilon,\delta}_{f\lambda}$ to $u^*_\lambda$ the solution of the stationary homogenized problem to complete the proof of Theorem 2.3.

### 3.1 Step 1: Passage to the limit as $\varepsilon \to 0$

Let us fix $\delta$. We consider $u^{\varepsilon,\delta}_\lambda = (u^{\varepsilon,\delta}_{f\lambda}, u^{\varepsilon,\delta}_{m\lambda})$ the Laplace transform of $u^{\varepsilon,\delta}$ solution of (2.5)–(2.8) with $\gamma = 1$ and then study the corresponding boundary value problem:

$$\begin{cases} \omega_f \lambda u^{\varepsilon,\delta}_{f\lambda} - k_f \,\Delta u^{\varepsilon,\delta}_{f\lambda} = \dfrac{1}{\lambda} Q & \text{in} \quad \Omega^{\varepsilon,\delta}_f; \\ \omega_m \lambda u^{\varepsilon,\delta}_{m\lambda} - k_m (\varepsilon\delta)^2 \,\Delta u^{\varepsilon,\delta}_{m\lambda} = 0 & \text{in} \quad \Omega^{\varepsilon,\delta}_m; \\ k_f \nabla u^{\varepsilon,\delta}_{f\lambda} \cdot \vec{v} = (\varepsilon\delta)^2 k_m \nabla u^{\varepsilon,\delta}_{m\lambda} \cdot \vec{v} & \text{on} \quad \Gamma^{\varepsilon,\delta}_{mf}; \\ (\varepsilon\delta)^2 k_m \nabla u^{\varepsilon,\delta}_{m\lambda} \cdot \vec{v} = \sigma\,(\varepsilon\delta)(u^{\varepsilon,\delta}_{f\lambda} - u^{\varepsilon,\delta}_{m\lambda}) & \text{on} \quad \Gamma^{\varepsilon,\delta}_{mf}; \\ u^{\varepsilon,\delta}_{f\lambda} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where $\vec{v}$ is the normal vector to $\Gamma^{\varepsilon,\delta}_{mf}$ (exterior to $\Omega^{\varepsilon,\delta}_f$) and $\lambda > 0$. By standard arguments we can prove that for each $\varepsilon \in ]0,1[$, problem (3.1) has a unique solution $u^{\varepsilon,\delta}_\lambda \in H^1(\Omega^{\varepsilon,\delta}_f, \Omega^{\varepsilon,\delta}_m)$.

The next result relies on the two-scale approach [2]. For the reader's convenience, let us recall the definition of the two-scale convergence.

We denote, by $C^\infty_\#(\mathscr{Y})$, the space of infinitely differentiable functions in $\mathbf{R}^3$ which are periodic of period $\mathscr{Y}$ and, by $\mathscr{D}(\Omega; C^\infty_\#(\mathscr{Y}))$, the space of infinitely smooth and compactly supported functions in $\Omega$ with values in the space $C^\infty_\#(\mathscr{Y})$.

**Definition 3.1** *A sequence of functions $v^\varepsilon$ in $L^2(\Omega)$ two–scale converges to $v(x,y)$ belonging to $L^2(\Omega \times \mathscr{Y})$ if, for any function $\varphi(x,y)$ in $\mathscr{D}(\Omega; C^\infty_\#(\mathscr{Y}))$, it satisfies*

$$\lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega\times\mathscr{Y}} v(x,y)\varphi(x,y)\, dx\, dy.$$

*We denote $v^\varepsilon(x) \xrightarrow{2s} v(x,y)$ two-scale in $L^2(\Omega \times \mathscr{Y})$.*

The asymptotic behaviour of $u^{\varepsilon,\delta}_\lambda$ as $\varepsilon \to 0$ is given by the following proposition.

**Proposition 3.2** *Let $u^{\varepsilon,\delta}_\lambda = (u^{\varepsilon,\delta}_{f\lambda}, u^{\varepsilon,\delta}_{m\lambda})$ be the solution of* (3.1). *Then $u^{\varepsilon,\delta}_\lambda$ two–scale converges as follows:*

$$\mathbf{1}^{\varepsilon,\delta}_f\, u^{\varepsilon,\delta}_{f\lambda} \xrightarrow{2s} \mathbf{1}^\delta_f(y)\, u^\delta_{f\lambda}(x); \quad \mathbf{1}^{\varepsilon,\delta}_m\, u^{\varepsilon,\delta}_{m\lambda} \xrightarrow{2s} \mathbf{1}^\delta_m(y)\, u^\delta_{m\lambda}(x,y),$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

*where* $u_\lambda^\delta = (u_{f\lambda}^\delta, u_{m\lambda}^\delta)$ *is the unique solution of*

$$\begin{cases} |\mathscr{F}^\delta|\omega_f \lambda u_{f\lambda}^\delta - \operatorname{div}_x (K^\delta \nabla u_{f\lambda}^\delta) = S(u_{f\lambda}^\delta, u_{m\lambda}^\delta) + |\mathscr{F}^\delta| \frac{1}{\lambda} Q & \text{in} \quad \Omega; \\ u_{f\lambda}^\delta(x) = 0 & \text{on} \quad \partial\Omega; \\ \omega_m \lambda u_{m\lambda}^\delta - \delta^2 k_m \Delta_y u_{\lambda m}^\delta = 0 & \text{in} \quad \Omega \times \mathscr{M}^\delta; \\ \delta^2 k_m \nabla_y u_{m\lambda}^\delta \cdot \vec{\nu} = \sigma\delta(u_{f\lambda}^\delta(x) - u_{m\lambda}^\delta(x,y)) & \text{on} \quad \Omega \times \Gamma_{mf}^\delta; \end{cases} \tag{3.2}$$

*where* $K^\delta = (k_{ij}^\delta)$ *is the homogenized permeability tensor defined by:*

$$k_{ij}^\delta = k_f \int_{\mathscr{F}^\delta} (\nabla_y w_i + \vec{e}_i) \cdot (\nabla_y w_j + \vec{e}_j)\, dy \tag{3.3}$$

*with* $w_i$ *being the unique solution in* $H^1_\#(\mathscr{F}^\delta) \setminus \mathbf{R}$ *of*

$$\begin{cases} -\Delta w_i = 0, & \text{in} \quad \mathscr{F}^\delta; \\ (\nabla_y w_i + \vec{e}_i) \cdot \vec{\nu} = 0, & \text{on} \quad \Gamma_{mf}^\delta; \\ y \to w_i(x,y) & \mathscr{Y} - \text{periodic}; \end{cases} \tag{3.4}$$

*where* $H^1_\#(\mathscr{F}^\delta)$ *denotes the Hilbert space*

$$H^1_\#(\mathscr{F}^\delta) = \{\varphi \in H^1_{loc}(\mathbf{R}^3), \quad \varphi \text{ is } \mathscr{Y} - \textit{periodic} \text{ in } y\}.$$

*The effective source term is given by*

$$S(u_{f\lambda}^\delta, u_{m\lambda}^\delta) = -\delta\sigma \int_{\Gamma_{mf}^\delta} (u_{f\lambda}^\delta(x) - u_{m\lambda}^\delta(x,y))\, ds_y. \tag{3.5}$$

*Moreover* $u_{f\lambda}^{\varepsilon,\delta}$ *converges to* $u_{f\lambda}^\delta$ *in* $L^2(\Omega_f^{\varepsilon,\delta})$ *as* $\varepsilon \to 0$.

**Proof of Proposition 3.2.** First we establish the following lemma.

**Lemma 3.3** *Let* $u_\lambda^{\varepsilon,\delta}$ *be the solution of the problem* (3.1). *Then there exist* $u_{f\lambda}^\delta \in H_0^1(\Omega)$, $U_{f\lambda}^\delta \in L^2(\Omega; H^1_\#(\mathscr{F}) \setminus \mathbf{R})$, $u_{m\lambda}^\delta \in L^2(\Omega; H^1_\#(\mathscr{Y}))$, *and a subsequence of solutions of* (3.1) *that two–scale converges as follows:*

$$\mathbf{1}_f^{\varepsilon,\delta}\, u_{f\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_f^\delta(y)\, u_{f\lambda}^\delta(x); \quad \mathbf{1}_f^{\varepsilon,\delta}\, \nabla u_{f\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_f^\delta(y) \left[\nabla_x u_{f\lambda}^\delta(x) + \nabla_y U_{f\lambda}^\delta(x,y)\right];$$

$$\mathbf{1}_m^{\varepsilon,\delta}\, u_{m\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_m^\delta(y)\, u_{m\lambda}^\delta(x,y); \quad \varepsilon \mathbf{1}_m^{\varepsilon,\delta}\, \nabla u_{m\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_m^\delta(y)\, \nabla_y u_{m\lambda}^\delta(x,y).$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

The proof of the Lemma is based on the two-scale results in Allaire [2] and the following a priori estimates for the family $(u_\lambda^{\varepsilon,\delta})_{\varepsilon,\delta>0}$:

$$\|u_{f\lambda}^{\varepsilon,\delta}\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} + \|u_{m\lambda}^{\varepsilon,\delta}\|^2_{L^2(\Omega_m^{\varepsilon,\delta})} + \|\nabla u_{f\lambda}^{\varepsilon,\delta}\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} + (\varepsilon\delta)^2\|\nabla u_{m\lambda}^{\varepsilon,\delta}\|^2_{L^2(\Omega_m^{\varepsilon,\delta})}$$
$$+ (\varepsilon\delta)\sigma \int_{\Gamma_{mf}^{\varepsilon,\delta}} (u_{f\lambda}^{\varepsilon,\delta} - u_{m\lambda}^{\varepsilon,\delta})^2\, ds \leqslant C, \tag{3.6}$$

where $C$ is a constant independent of $\varepsilon, \delta$.

We now define a variational formulation of the problem (3.1):

$$\omega_f\lambda \int_{\Omega_f^{\varepsilon,\delta}} u_{f\lambda}^{\varepsilon,\delta}(x)v_f(x)\,dx + \omega_m\lambda \int_{\Omega_m^{\varepsilon,\delta}} u_{m\lambda}^{\varepsilon,\delta}(x)\phi_m\,dx$$
$$+ \int_{\Omega_f^{\varepsilon,\delta}} k_f\nabla u_{f\lambda}^{\varepsilon,\delta}\cdot\nabla v_f\,dx + (\varepsilon\delta)^2 \int_{\Omega_m^{\varepsilon,\delta}} k_m\nabla u_{m\lambda}^{\varepsilon,\delta}\cdot\nabla\phi_m\,dx$$
$$+ (\varepsilon\delta)\sigma \int_{\Gamma_{mf}^{\varepsilon,\delta}} (u_{f\lambda}^{\varepsilon,\delta} - u_{m\lambda}^{\varepsilon,\delta})(v_f - \phi_m)\,ds = \int_{\Omega_f^{\varepsilon,\delta}} \frac{1}{\lambda}Q(x)v_f(x)\,dx, \tag{3.7}$$

where $v_f(x) = \phi_f(x) + \varepsilon\zeta(x, \frac{x}{\varepsilon})$ with $\phi_f \in C^1(\Omega)$ and $\zeta \in C^1(\Omega; C^1_\#(\mathscr{Y}))$; $\phi_m \in C^1(\Omega; C^1_\#(\mathscr{Y}))$.

Then we want to pass to the limit as $\varepsilon \to 0$ in the equation (3.7). Consider first the surface term in the left-hand side of (3.7).

The next result relies on the two–scale approach for sequences of functions which are defined on periodic surfaces. For the reader's convenience, let us recall the definition of the two-scale convergence on periodic surfaces [3, 27].

Let $\Gamma$ be a smooth $(n-1)$–dimensional manifold compactly included in $\mathscr{Y}$. Let $\Gamma^\varepsilon$ be the union of all $\varepsilon(\Gamma^\varepsilon + l_i\vec{e}_i)$, $l_i \in \mathbf{Z}$ which are contained in $\Omega$.

**Definition 3.4** *A sequence of functions $w^\varepsilon$ in $L^2(\Gamma^\varepsilon)$ two-scale converges to $w(x,y)$ belonging to $L^2(\Omega\times\Gamma)$ if, for any function $\varphi \in \mathscr{D}(\Omega; C^\infty_\#(\Gamma))$, it satisfies*

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Gamma^\varepsilon} w^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) ds = \int_{\Omega\times\Gamma} w(x,y)\varphi(x,y)\,dx\,ds_y.$$

Now the asymptotic behaviour of the term involving $\Gamma_{mf}^{\varepsilon,\delta}$ is given by the following Lemma.

**Lemma 3.5** *Let*

$$I^\varepsilon[u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta}] = (\varepsilon\delta)\,\sigma \int_{\Gamma_{mf}^{\varepsilon,\delta}} (u_{f\lambda}^{\varepsilon,\delta} - u_{m\lambda}^{\varepsilon,\delta})\left\{\phi_f(x) + \varepsilon\zeta\left(x, \frac{x}{\varepsilon}\right) - \phi_m\left(x, \frac{x}{\varepsilon}\right)\right\} ds.$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

*Then*

$$\lim_{\varepsilon\to 0} I^{\varepsilon}[u^{\varepsilon,\delta}_{f\lambda}, u^{\varepsilon,\delta}_{m\lambda}] = \sigma\delta \int_{\Omega}\int_{\Gamma^{\delta}_{mf}} (u^{\delta}_{f\lambda}(x) - u^{\delta}_{m\lambda}(x,y))(\phi_f(x) - \phi_m(x,y))\, ds_y\, dx.$$

The proof of the Lemma is a direct consequence of Theorem 2.1 and Proposition 2.6 of Allaire *et al.* [3].

By using this result and standard arguments of the two-scale convergence method, we can pass to the two-scale limit in (3.7) as $\varepsilon \to 0$ and obtain the variational formulation of (3.2).

It remains to show the convergence of $u^{\varepsilon,\delta}_{f\lambda}$ to $u^{\delta}_{f\lambda}$ in $L^2(\Omega^{\varepsilon,\delta}_f)$ as $\varepsilon \to 0$. We have from Acerbi *et al.* [1] that there exists an extension $\tilde{u}^{\varepsilon,\delta}_{f\lambda}$ of $u^{\varepsilon,\delta}_{f\lambda}$ from the set $\Omega^{\varepsilon,\delta}_f$ to $\Omega$ such that

$$\|\tilde{u}^{\varepsilon,\delta}_{f\lambda}\|_{H^1(\Omega)} \leqslant C\|u^{\varepsilon,\delta}_{f\lambda}\|_{H^1(\Omega^{\varepsilon,\delta}_f)},$$

where $C$ is a constant independent of $\varepsilon$. Then the desired convergence result easily follows from the *a priori* estimate (3.6) and Lemma 3.3. This completes the proof of Proposition 3.2. □

**Remark 3** The convergence result in Proposition 3.2 can be reformulated as follows. Let $\tilde{u}^{\varepsilon,\delta}_{f\lambda}$ be an extension of $u^{\varepsilon,\delta}_{f\lambda}$ from the set $\Omega^{\varepsilon,\delta}_f$ to $\Omega$ which exists as shown in Acerbi *et al.* [1]. Then it follows that $\tilde{u}^{\varepsilon,\delta}_{f\lambda}$ converges to $u^{\delta}_{f\lambda}$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

**Remark 4** The homogenization result of Proposition 3.2 remains true when the matrix blocks $\Omega^{\varepsilon,\delta}_m$ form a connected set in $\Omega$, provided that the fissure system remains also connected. In this case, the interface condition for the function $u^{\delta}_{m\lambda}$ becomes

$$\delta^2 k_m \nabla_y u^{\delta}_{m\lambda} \cdot \vec{\nu} = \sigma\delta(u^{\delta}_{f\lambda}(x) - u^{\delta}_{f\lambda}(x,y))$$

when $y \in \Gamma^{\delta}_{mf} \setminus \partial\mathscr{Y}$ and $u^{\delta}_{m\lambda}(x,y)$ is $\mathscr{Y}$–periodic in $y$.

### 3.2 Step 2: Passage to the limit as $\delta \to 0$

Now we pass to the limit as $\delta \to 0$ in (3.2). The asymptotic behaviour of $u^{\varepsilon,\delta}_{\lambda}$ as $\varepsilon, \delta \to 0$ is given by the following proposition.

**Proposition 3.6** *Let $u^{\varepsilon,\delta}_{\lambda} = (u^{\varepsilon,\delta}_{f\lambda}, u^{\varepsilon,\delta}_{m\lambda})$ be the solution of (3.1). Then the sequence $(u^{\varepsilon,\delta}_{f\lambda})$ $L_{\varepsilon,\delta}$-converges to $u^*_{\lambda}$ the solution of*

$$\begin{cases} \omega_f \lambda u^*_{\lambda} - \dfrac{2}{3} k_f \Delta u^*_{\lambda} + 2\sigma\lambda u^*_{\lambda} \dfrac{1}{\sqrt{\lambda}(\mu + \sqrt{\lambda})} = \dfrac{1}{\lambda} Q & \text{in} \quad \Omega; \\ u^*_{\lambda}(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.8}$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

**Proof of Proposition 3.6.** First we consider the effective source term $S(u^\delta_{f\lambda}, u^\delta_{m\lambda})$ given by (3.5):

$$S(u^\delta_{f\lambda}, u^\delta_{m\lambda}) = -\delta\sigma \int\limits_{\Gamma^\delta_{mf}} (u^\delta_{f\lambda}(x) - u^\delta_{m\lambda}(x,y))\, ds_y.$$

Here the function $u^\delta_{m\lambda} = u^\delta_{m\lambda}(x,y)$ satisfies the following boundary value problem:

$$\begin{cases} \omega_m \lambda u^\delta_{m\lambda} - \delta^2 k_m \Delta_y u^\delta_{\lambda m} = 0 & \text{in} \quad \Omega \times \mathscr{M}^\delta; \\ \delta^2 k_m \nabla_y u^\delta_{m\lambda} \cdot \vec{\nu} = \sigma\delta(u^\delta_{f\lambda}(x) - u^\delta_{m\lambda}(x,y)) & \text{on} \quad \Omega \times \Gamma^\delta_{mf}, \end{cases} \tag{3.9}$$

where $\mathscr{M}^\delta$ is an open cube with length equal $(1-\delta)$. It is clear that

$$u^\delta_{m\lambda}(x,y) = U^\delta_{m\lambda}(y)\, u^\delta_{f\lambda}(x), \tag{3.10}$$

where $U^\delta_{m\lambda}$ is the unique solution of

$$\begin{cases} \omega_m \lambda U^\delta_{m\lambda} - \delta^2 k_m \Delta_y U^\delta_{m\lambda} = 0 & \text{in} \quad \mathscr{M}^\delta; \\ \delta^2 k_m \nabla_y U^\delta_{m\lambda} \cdot \vec{\nu} = \sigma\delta(1 - U^\delta_{m\lambda}(y)) & \text{on} \quad \Gamma^\delta_{mf}. \end{cases} \tag{3.11}$$

Therefore, we have:

$$S(u^\delta_{f\lambda}, u^\delta_{m\lambda}) = -\delta\sigma u^\delta_{f\lambda}(x) \int\limits_{\Gamma^\delta_{mf}} (1 - U^\delta_{m\lambda}(x,y))\, ds_y \equiv -\delta\sigma u^\delta_{f\lambda}(x) C(\lambda,\delta). \tag{3.12}$$

Let us study the asymptotic behaviour of the integral $C(\lambda,\delta)$.

**Lemma 3.7** *Let $C(\lambda,\delta)$ be the integral defined in (3.12). Then*

$$C(\lambda,\delta) = 6\lambda \left( \frac{1}{\sqrt{\lambda}(\mu + \sqrt{\lambda})} + o(1) \right) \tag{3.13}$$

*as $\delta \to 0$, where*

$$\mu = \frac{\sigma}{\sqrt{k_m \omega_m}}.$$

**Proof of Lemma 3.7.** Consider the boundary value problem (3.11). Changing variables (without changing notation for the solution) we obtain that $U^\delta_{m\lambda}$ is the unique solution of

$$\begin{cases} \omega_m \lambda U^\delta_{m\lambda} - \dfrac{\delta^2}{(1-\delta)^2} k_m \Delta_y U^\delta_{m\lambda} = 0 & \text{in} \quad \mathscr{M}; \\ \delta^2 k_m \nabla_y U^\delta_{m\lambda} \cdot \vec{\nu} = \sigma\delta(1-\delta)(1 - U^\delta_{m\lambda}(y)) & \text{on} \quad \partial\mathscr{M}, \end{cases} \tag{3.14}$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

where $\mathscr{M}$ is the open unit cube in $\mathbf{R}^3$. Then

$$C(\lambda,\delta) = (1-\delta)^2 \int\limits_{\partial\mathscr{M}} (1 - U^\delta_{m\lambda}(y))\, ds_y. \tag{3.15}$$

Let us rewrite (3.14) as follows:

$$\begin{cases} \beta^\delta_\lambda U^\delta_{m\lambda} - \delta^2 \Delta_y U^\delta_{m\lambda} = 0 & \text{in} \quad \mathscr{M}; \\ \delta^2 k_m \nabla_y U^\delta_{m\lambda} \cdot \vec{\nu} = \sigma\delta(1-\delta)(1 - U^\delta_{m\lambda}(y)) & \text{on} \quad \partial\mathscr{M}, \end{cases} \tag{3.16}$$

where

$$\beta^\delta_\lambda = \lambda \frac{\omega_m}{k_m}(1-\delta)^2. \tag{3.17}$$

Let us introduce the functions $v^\pm_l = v^\pm_l(y_l)$ $(l = 1,2,3)$:

$$v^\pm_l(y_l) = \frac{\sigma}{\sigma + \sqrt{\lambda}\sqrt{\omega_m k_m}} \exp\left\{ \pm \frac{\sqrt{\beta^\delta_\lambda}}{\delta} \left( y_l \mp \frac{1}{2} \right) \right\}. \tag{3.18}$$

It is clear that these functions verify the differential equation in (3.16) and the boundary condition on the corresponding surfaces of the cube $\mathscr{M}$. For example, the function $v^+_1$ satisfies the boundary condition on $\mathscr{M}^+_{y_1}$, where $\mathscr{M}^+_{y_1} = \{y \in \mathscr{M} : y_1 = 1/2\}$.

Let us introduce the function $\Phi^\delta = \Phi^\delta(y)$ in the following way:

$$\Phi^\delta(y) = \sum_{k=1}^{3} (v^+_k(y_k) + v^-_k(y_k)). \tag{3.19}$$

It is clear that $\Phi^\delta$ satisfies the first equation in (3.16) but it satisfies the boundary condition with some exponentially small residual $\epsilon^\delta = \epsilon^\delta(y)$, i.e. the function $(U^\delta_{m\lambda} - \Phi^\delta)(y)$ satisfies the following boundary value problem:

$$\begin{cases} \beta^\delta_\lambda (U^\delta_{m\lambda} - \Phi^\delta) - \delta^2 \Delta_y (U^\delta_{m\lambda} - \Phi^\delta) = 0 & \text{in} \quad \mathscr{M}; \\ \delta^2 k_m \nabla_y (U^\delta_{m\lambda} - \Phi^\delta) \cdot \vec{\nu} = \sigma\delta(1-\delta)(\epsilon^\delta(y) - (U^\delta_{m\lambda} - \Phi^\delta)(y)) & \text{on} \quad \partial\mathscr{M}. \end{cases} \tag{3.20}$$

Using the variational formulation of the problem (3.20), we can show that

$$\int\limits_{\partial\mathscr{M}} |U^\delta_{\lambda m} - \Phi^\delta|^2\, ds_y \leqslant \int\limits_{\partial\mathscr{M}} |\epsilon^\delta(y)|^2\, ds_y. \tag{3.21}$$

Moreover, the integral in the right hand side of (3.21) tends to zero as $\delta \to 0$. This means that we can replace the surface integral of the function $U^\delta_{m\lambda}$ by the integral of the function $\Phi^\delta$. Now the statement of the lemma follows from simple calculations. This proves Lemma 3.7. □

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

Equation (3.12) shows that the system (3.2) could be decoupled. Plugging the expression of the source term $S(u^{\delta}_{f\lambda}, u^{\delta}_{m\lambda})$ given by (3.12) into equation (3.2) directly gives:

$$\begin{cases} |\mathscr{F}^{\delta}|\omega_f \lambda u^{\delta}_{f\lambda} - \operatorname{div}_x (K^{\delta}\nabla u^{\delta}_{f\lambda}) + C(\lambda,\delta)\sigma\delta u^{\delta}_{f\lambda} = |\mathscr{F}^{\delta}|\dfrac{1}{\lambda}Q & \text{in} \quad \Omega; \\ u^{\delta}_{f\lambda}(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.22}$$

where $C(\lambda,\delta)$ is defined in (3.12). Thus the limit of $u^{\delta}_{f\lambda}$ as $\delta \to 0$ will give a global behaviour of the system.

Notice that all the coefficients in (3.22) are of order $\delta$. Then following the arguments of Cioranescu & Saint Jean Paulin [16] [Chapter 2] and using (2.1) one can show that $u^{\delta}_{f\lambda}$ converges strongly in $H^1_0(\Omega)$ to $u^*_{\lambda}$ the solution of (3.8).

Let us now show that $u^{\varepsilon,\delta}_{f\lambda}$ $L_{\varepsilon,\delta}$-converges to $u^*_{\lambda}$ the solution of (3.8). As in Bourgeat *et al.* [14], we have

$$\frac{1}{|\Omega^{\varepsilon,\delta}_f|}\|u^{\varepsilon,\delta}_{f\lambda} - u^*_{\lambda}\|^2_{L^2(\Omega^{\varepsilon,\delta}_f)} \leqslant C\left(\|\tilde{u}^{\varepsilon,\delta}_{f\lambda} - u^{\delta}_{f\lambda}\|^2_{L^2(\Omega)} + \|u^{\delta}_{f\lambda} - u^*_{\lambda}\|^2_{L^2(\Omega)}\right), \tag{3.23}$$

where $\tilde{u}^{\varepsilon,\delta}_{f\lambda}$ is the extension of $u^{\varepsilon,\delta}_{f\lambda}$ from the set $\Omega^{\varepsilon,\delta}_f$ to $\Omega$ and $C$ is a constant independent of $\varepsilon, \delta$. Now the $L_{\varepsilon,\delta}$–convergence of $u^{\varepsilon,\delta}_{f\lambda}$ to $u^*_{\lambda}$ easily follows from Remark 3 and the strong convergence in $L^2(\Omega)$ of the sequence $(u^{\delta}_{f\lambda})$ to $u^*_{\lambda}$. This completes the proof of Proposition 3.6. □

Now we are in position to complete the proof of Theorem 2.3.

### 3.3 Step 3: Proof of the convergence result in Theorem 2.3

Consider the boundary value problem (2.5)–(2.8). Using standard parabolic theory one can obtain the following uniform estimates:

$$\|u^{\varepsilon,\delta}(t)\|^2_{L^2(\Omega^{\varepsilon,\delta}_f)} + \|u^{\varepsilon,\delta}(t)\|^2_{L^2(\Omega^{\varepsilon,\delta}_m)} + \|\nabla u^{\varepsilon,\delta}(t)\|^2_{L^2(\Omega^{\varepsilon,\delta}_f)} + (\varepsilon\delta)^2\|\nabla u^{\varepsilon}(t)\|^2_{L^2(\Omega^{\varepsilon,\delta}_m)}$$

$$+ \int_0^t \left(\|u^{\varepsilon}_t(\tau)\|^2_{L^2(\Omega^{\varepsilon,\delta}_f)} + \|u^{\varepsilon}_t(\tau)\|^2_{L^2(\Omega^{\varepsilon,\delta}_m)}\right) d\tau \leqslant C_1|\Omega^{\varepsilon,\delta}_f|, \tag{3.24}$$

where $C_1$ is a constant independent of $t, \varepsilon, \delta$. Then, for any $t \in ]0, T[$, the function $u^{\varepsilon,\delta}_f = u^{\varepsilon,\delta}_f(t, x)$ $L_{\varepsilon,\delta}$-converges to a function $v = v(t, x)$, i.e.

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \mu^{\varepsilon,\delta} \int_{\Omega^{\varepsilon,\delta}_f} u^{\varepsilon,\delta}(t, x)\varphi(x)\, dx = \int_{\Omega} v(t, x)\varphi(x)\, dx \tag{3.25}$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

for any $\varphi(x) \in L^{\infty}(\Omega)$. Here the constant $\mu^{\varepsilon,\delta}$ is defined by

$$\mu^{\varepsilon,\delta} = \frac{|\Omega|}{|\Omega_f^{\varepsilon,\delta}|}. \tag{3.26}$$

Let us show that $v = v(t,x)$ is the solution of problem (2.9)–(2.10). Let $u_\lambda^{\varepsilon,\delta}$ be the solution of the boundary value problem (3.1) with an arbitrary complex $\lambda$ such that $\arg\lambda \neq \pi$. Then $u_{f\lambda}^{\varepsilon,\delta}$ is an analytic function in the complex $\lambda$–plane $\mathbb{C} \setminus \{\arg\lambda = \pi\}$ and

$$\|u_{f\lambda}^{\varepsilon,\delta}\|^2_{L^2(\Omega_f^{\varepsilon,\delta})} \leqslant C_2 \frac{|\Omega_f^{\varepsilon,\delta}|}{|\lambda|^4}, \quad |\arg\lambda - \pi| \geqslant \vartheta_0 > 0, \tag{3.27}$$

where $C_2$ is a constant independent of $\varepsilon, \delta$. Moreover, $u_f^{\varepsilon,\delta}$ may be represented by the inverse Laplace transform as follows

$$u_f^{\varepsilon,\delta}(t,x) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \exp(\lambda t) u_{f\lambda}^{\varepsilon,\delta}(x)\, d\lambda, \quad \theta > 0, \tag{3.28}$$

where $u_{f\lambda}^{\varepsilon,\delta}$ is the first component of the solution of problem (3.1) with an arbitrary complex $\lambda$ such that $\arg\lambda \neq \pi$.

Now let $u_\lambda^*$ be the solution of problem (3.8) with an arbitrary complex $\lambda$ such that $\arg\lambda \neq \pi$. The solution $u_\lambda^*$ of this problem is an analytic function with respect to $\lambda$ in the complex $\lambda$-plane $\mathbb{C} \setminus \{\arg\lambda = \pi\}$ and

$$\|u_\lambda^*\|^2_{L^2(\Omega)} \leqslant \frac{C_3}{|\lambda|^4} \tag{3.29}$$

for $|\arg\lambda - \pi| \geqslant \vartheta_0 > 0$. Moreover, the solution of problem (2.9)–(2.10) can be represented as follows

$$u^*(t,x) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \exp(\lambda t) u_\lambda^*(x)\, d\lambda, \quad \theta > 0, \tag{3.30}$$

where $u_\lambda^*$ is the solution of problem (3.8) with an arbitrary complex $\lambda$ such that $\arg\lambda \neq \pi$.

Now it follows from (3.27)–(3.30) that

$$\mu^{\varepsilon,\delta} \int_{\Omega_f^{\varepsilon,\delta}} u_f^{\varepsilon,\delta}(t,x)\varphi(x)\, dx = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \left\{ \mu^{\varepsilon,\delta} \int_{\Omega_f^{\varepsilon,\delta}} u_{f\lambda}^{\varepsilon,\delta}(x)\varphi(x)\, dx \right\} d\lambda \tag{3.31}$$

and

$$\int_{\Omega} u^*(t,x)\varphi(x)\, dx = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \left\{ \int_{\Omega} u_\lambda^*(x)\varphi(x)\, dx \right\} d\lambda. \tag{3.32}$$

From Proposition 3.6, for any $\lambda > 0$, we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mu^{\varepsilon,\delta}\int_{\Omega_f^{\varepsilon,\delta}} u_{f\lambda}^{\varepsilon,\delta}(x)\varphi(x)\,dx = \int_{\Omega} u_\lambda^*(x)\varphi(x)\,dx. \tag{3.33}$$

Here $u_{f\lambda}^{\varepsilon,\delta} = u_{f\lambda}^{\varepsilon,\delta}(x)$ is an analytic function with respect to $\lambda$ and is uniformly bounded with respect to $\varepsilon, \delta$. Therefore, (3.33) is valid for any complex $\lambda$ and the limit is achieved uniformly with respect to $\lambda$ for any compact set in the domain $|\arg\lambda - \pi| \geqslant \vartheta_0 > 0$. Now from (3.31)–(3.33) we obtain that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mu^{\varepsilon,\delta}\int_{\Omega_f^{\varepsilon,\delta}} u_f^{\varepsilon,\delta}(t,x)\varphi(x)\,dx = \int_{\Omega} u^*(t,x)\varphi(x)\,dx, \tag{3.34}$$

for any $t \in ]0,T[$. Comparing (3.25) and (3.34) we conclude that $v(t,x) = u^*(t,x)$ and Theorem 2.3 is proved.

## 4 Sketch of the proof of Theorem 2.4

The main lines of the proof of Theorem 2.4 are similar to those of Theorem 2.3.

On the first step we fix $\delta$ and then pass to the limit as $\varepsilon \to 0$. For this we consider $u_\lambda^{\varepsilon,\delta} = (u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta})$ the Laplace transform of $u^{\varepsilon,\delta}$ the solution of (2.5)–(2.8) with $\gamma > 1$ and then we study the corresponding boundary value problem:

$$\begin{cases} \omega_f\lambda u_{f\lambda}^{\varepsilon,\delta} - k_f\,\Delta u_{f\lambda}^{\varepsilon,\delta} = \dfrac{1}{\lambda}Q & \text{in} \quad \Omega_f^{\varepsilon,\delta}; \\ \omega_m\lambda u_{m\lambda}^{\varepsilon,\delta} - k_m(\varepsilon\delta)^2\,\Delta u_{m\lambda}^{\varepsilon,\delta} = 0 & \text{in} \quad \Omega_m^{\varepsilon,\delta}; \\ k_f\nabla u_{f\lambda}^{\varepsilon,\delta}\cdot\vec{\nu} = (\varepsilon\delta)^2 k_m\nabla u_{m\lambda}^{\varepsilon,\delta}\cdot\vec{\nu} & \text{on} \quad \Gamma_{mf}^{\varepsilon,\delta}; \\ (\varepsilon\delta)^2 k_m\nabla u_{m\lambda}^{\varepsilon,\delta}\cdot\vec{\nu} = \sigma\,(\varepsilon\delta)^\gamma(u_{f\lambda}^{\varepsilon,\delta} - u_{m\lambda}^{\varepsilon,\delta}) & \text{on} \quad \Gamma_{mf}^{\varepsilon,\delta}; \\ u_{f\lambda}^{\varepsilon,\delta} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.1}$$

where $\lambda > 0$.

By standard arguments we can prove that for each $\varepsilon \in ]0,1[$ problem (4.1) has a unique solution $u_\lambda^{\varepsilon,\delta} \in H^1(\Omega_f^{\varepsilon,\delta}, \Omega_m^{\varepsilon,\delta})$.

The asymptotic behaviour of $u_\lambda^{\varepsilon,\delta}$ as $\varepsilon \to 0$ is given by the following proposition.

**Proposition 4.1** *Let* $u_\lambda^{\varepsilon,\delta} = (u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta})$ *be the solution of* (4.1) *with* $\gamma > 1$. *Then* $u_\lambda^{\varepsilon,\delta}$ *two–scale converges as follows:*

$$\mathbf{1}_f^{\varepsilon,\delta}\, u_{f\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_f^{\delta}(y)\, u_{f\lambda}^{\delta}(x); \quad \mathbf{1}_m^{\varepsilon,\delta}\, u_{m\lambda}^{\varepsilon,\delta} \xrightarrow{2s} 0,$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

*where $u^{\delta}_{f\lambda}$ is the unique solution of*

$$\begin{cases} |\mathscr{F}^{\delta}|\omega_f \lambda u^{\delta}_{f\lambda} - \mathrm{div}_x\,(K^{\delta}\nabla u^{\delta}_{f\lambda}) = |\mathscr{F}^{\delta}|\dfrac{1}{\lambda}Q & \text{in} \quad \Omega; \\ u^{\delta}_{f\lambda}(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.2}$$

*where $K^{\delta} = (k^{\delta}_{ij})$ is the homogenized permeability tensor defined by* (3.3)–(3.4).

*Moreover, $u^{\varepsilon,\delta}_{f\lambda}$ converges to $u^{\delta}_{f\lambda}$ in $L^2(\Omega^{\varepsilon,\delta}_f)$ as $\varepsilon \to 0$.*

**Proof of Proposition 4.1.** Again, as in the previous case, the variational formulation of problem (4.1) is given by:

$$\omega_f\lambda \int_{\Omega^{\varepsilon,\delta}_f} u^{\varepsilon,\delta}_{f\lambda}(x)v_f(x)\,dx + \omega_m\lambda \int_{\Omega^{\varepsilon,\delta}_m} u^{\varepsilon,\delta}_{m\lambda}(x)\phi_m\,dx + \int_{\Omega^{\varepsilon,\delta}_f} k_f\nabla u^{\varepsilon,\delta}_{f\lambda}\cdot\nabla v_f\,dx + (\varepsilon\delta)^2 \int_{\Omega^{\varepsilon,\delta}_m} k_m\nabla u^{\varepsilon,\delta}_{m\lambda}\cdot\nabla\phi_m\,dx$$

$$+ (\varepsilon\delta)^{\gamma}\sigma \int_{\Gamma^{\varepsilon,\delta}_{mf}} (u^{\varepsilon,\delta}_{f\lambda} - u^{\varepsilon,\delta}_{m\lambda})(v_f - \phi_m)\,ds = \int_{\Omega^{\varepsilon,\delta}_f} \frac{1}{\lambda}Q(x)v_f(x)\,dx, \tag{4.3}$$

where $v_f(x) = \phi_f(x) + \varepsilon\zeta(x, \frac{x}{\varepsilon})$ with $\phi_f \in C^1(\Omega)$ and $\zeta \in C^1(\Omega; C^1_{\#}(\mathscr{Y}))$; $\phi_m \in C^1(\Omega; C^1_{\#}(\mathscr{Y}))$.

Then we want to pass to the limit as $\varepsilon \to 0$ in equation (4.3). Consider first the surface term in the left-hand side of (4.3). In the framework of the definition of two-scale convergence on periodic surfaces (see Definition 3.4) we have that

$$\lim_{\varepsilon\to 0}(\varepsilon\delta)^{\gamma}\,\sigma \int_{\Gamma^{\varepsilon,\delta}_{mf}} (u^{\varepsilon,\delta}_{f\lambda} - u^{\varepsilon,\delta}_{m\lambda})\left\{\phi_f(x) + \varepsilon\zeta\left(x, \frac{x}{\varepsilon}\right) - \phi_m\left(x, \frac{x}{\varepsilon}\right)\right\}\,ds = 0$$

for $\gamma > 1$.

As in the previous section, the two-scale convergence results from Lemma 3.3 permit to pass to the limit for the other terms in (4.3). In particular, we obtain that the function $u^{\delta}_{m\lambda}$ satisfies the following boundary value problem:

$$\begin{cases} \omega_m\lambda u^{\delta}_{m\lambda} - \delta^2 k_m\,\Delta_y u^{\delta}_{\lambda m} = 0 & \text{in} \quad \mathscr{M}^{\delta}; \\ \delta^2 k_m\nabla_y u^{\delta}_{m\lambda}\cdot\vec{\nu} = 0 & \text{on} \quad \Gamma^{\delta}_{mf}. \end{cases} \tag{4.4}$$

This means that $u^{\delta}_{m\lambda} \equiv 0$. The limit of the other terms gives the variational formulation of (4.2). The second convergence result in Proposition 4.1 is obtained by the same arguments as in the previous section. This completes the proof of Proposition 4.1. □

**Remark 5** The convergence result in Proposition 4.1 can be reformulated as follows. Let $\tilde{u}^{\varepsilon,\delta}_{f\lambda}$ be an extension of $u^{\varepsilon,\delta}_{f\lambda}$ from the set $\Omega^{\varepsilon,\delta}_f$ to $\Omega$ which exists as shown in Acerbi *et al.* [1]. Then it follows that $\tilde{u}^{\varepsilon,\delta}_{f\lambda}$ converges to $u^{\delta}_{f\lambda}$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

### 4.1 Step 2: Passage to the limit as $\delta \to 0$

Now we pass to the limit as $\delta \to 0$ in (4.2). The asymptotic behaviour of $u_\lambda^{\varepsilon,\delta}$ as $\varepsilon, \delta \to 0$ is given by the following proposition.

**Proposition 4.2** *Let* $u_\lambda^{\varepsilon,\delta} = (u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta})$ *be the solution of* (4.1). *Then the sequence* $(u_{f\lambda}^{\varepsilon,\delta})$ $L_{\varepsilon,\delta}$*–converges to* $u_\lambda^*$ *which is the solution of*

$$\begin{cases} \omega_f \lambda u_\lambda^* - \dfrac{2}{3} k_f \Delta u_\lambda^* = \dfrac{1}{\lambda} Q & \text{in} \quad \Omega; \\ u_\lambda^*(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.5}$$

**Proof of Proposition 4.2.** Following the arguments of Cioranescu & Saint Jean Paulin [16] [Chapter 2] and using (2.1) we can show that $u_{f\lambda}^\delta$ converges strongly in $H_0^1(\Omega)$ to $u_\lambda^*$ the solution of (4.5).

As in the previous section we show that $u_{f\lambda}^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$-converges to $u_\lambda^*$ which is the solution of (4.5). □

We complete the proof of Theorem 2.4 through arguments similar to ones used in the proof of Theorem 2.3 (see Step 3 in the previous section). Theorem 2.4 is proved.

## 5 Sketch of the proof of Theorem 2.5

The main lines of the proof of Theorem 2.5 are similar to those of Theorem 2.3 or Theorem 2.4.

### 5.1 Step 1: Passage to the limit as $\varepsilon \to 0$

Let us fix $\delta$.

We consider $u_\lambda^{\varepsilon,\delta} = (u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta})$ the Laplace transform of $u^{\varepsilon,\delta}$ the solution of (2.5)–(2.8) and study then the boundary value problem (4.1) with $\gamma < 1$. It is clear that there exists a unique solution $u_\lambda^{\varepsilon,\delta} \in H^1(\Omega_f^{\varepsilon,\delta}, \Omega_m^{\varepsilon,\delta})$ of problem (4.1) when $\gamma < 1$.

The asymptotic behaviour of $u_\lambda^{\varepsilon,\delta}$ as $\varepsilon \to 0$ is given by the following proposition.

**Proposition 5.1** *Let* $u_\lambda^{\varepsilon,\delta} = (u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta})$ *be the solution of* (4.1) *with* $\gamma < 1$. *Then* $u_\lambda^{\varepsilon,\delta}$ *two-scale converges as follows:*

$$\mathbf{1}_f^{\varepsilon,\delta}\, u_{f\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_f^\delta(y)\, u_{f\lambda}^\delta(x); \quad \mathbf{1}_m^{\varepsilon,\delta}\, u_{m\lambda}^{\varepsilon,\delta} \xrightarrow{2s} \mathbf{1}_m^\delta(y)\, u_{m\lambda}^\delta(x,y),$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

*where* $u_\lambda^\delta = (u_{f\lambda}^\delta, u_{m\lambda}^\delta)$ *is the unique solution of*

$$\begin{cases} |\mathscr{F}^\delta|\omega_f \lambda u_{f\lambda}^\delta - \operatorname{div}_x (K^\delta \nabla u_{f\lambda}^\delta) = S(u_{f\lambda}^\delta, u_{m\lambda}^\delta) + |\mathscr{F}^\delta| \dfrac{1}{\lambda} Q & \text{in} \quad \Omega; \\ u_{f\lambda}^\delta(x) = 0 & \text{on} \quad \partial\Omega; \\ \omega_m \lambda u_{m\lambda}^\delta - \delta^2 k_m \Delta_y u_{\lambda m}^\delta = 0 & \text{in} \quad \Omega \times \mathscr{M}^\delta; \\ u_{m\lambda}^\delta(x,y) = u_{f\lambda}^\delta(x) & \text{on} \quad \Omega \times \Gamma_{mf}^\delta; \end{cases} \tag{5.1}$$

*where* $K^\delta = (k_{ij}^\delta)$ *is the homogenized permeability tensor defined by* (3.3)–(3.4)*; the effective source term is given by*

$$S(u_{f\lambda}^\delta, u_{m\lambda}^\delta) = -\delta^2 k_m \int_{\Gamma_{mf}^\delta} \nabla_y u_{m\lambda}^\delta \cdot \vec{v}\, ds_y. \tag{5.2}$$

*Moreover,* $u_{f\lambda}^{\varepsilon,\delta}$ *converges to* $u_{f\lambda}^\delta$ *in* $L^2(\Omega_f^{\varepsilon,\delta})$ *as* $\varepsilon \to 0$.

**Proof of Proposition 5.1.** As in the previous sections we first obtain a convergence result given by Lemma 3.3. Then we consider the variational formulation of the problem given by (4.3). The next step is to pass to the limit in equation (4.3) with $\gamma < 1$. We consider first the surface term in the left-hand side of (4.3). In the framework of the definition of two-scale convergence on periodic surfaces (see Definition 3.4) we show that the function $(u_{f\lambda}^{\varepsilon,\delta} - u_{m\lambda}^{\varepsilon,\delta})$ two-scale converges to 0 on the surface $\Gamma_{mf}^{\varepsilon,\delta}$. Therefore, taking $\phi_m(x,y) = \phi_f(x)$, for $y \in \mathscr{F}^\delta$, we pass to the two-scale limit in (4.3) taking into account that $\gamma < 1$ and we obtain the variational formulation of (5.1).

The second convergence result in Proposition 5.1 is obtained by the same arguments as in § 3. This completes the proof of Proposition 5.1. □

**Remark 6** The convergence result in Proposition 5.1 can be reformulated as follows. Let $\tilde{u}_{f\lambda}^{\varepsilon,\delta}$ be an extension of $u_{f\lambda}^{\varepsilon,\delta}$ from the set $\Omega_f^{\varepsilon,\delta}$ to $\Omega$ which exists as shown in Acerbi *et al.* [1]. Then it follows that $\tilde{u}_{f\lambda}^{\varepsilon,\delta}$ converges to $u_{f\lambda}^\delta$ in $L^2(\Omega)$ as $\varepsilon \to 0$.

**Remark 7** The homogenization result of Proposition 5.1 remains true when the matrix blocks $\Omega_m^{\varepsilon,\delta}$ form a connected set in $\Omega$, provided that the fissure system also remains connected. In this case the interface condition for the function $u_{m\lambda}^\delta$ becomes

$$u_{m\lambda}^\delta(x,y) = u_{f\lambda}^\delta(x)$$

where $y \in \Gamma_{mf}^\delta \setminus \partial\mathscr{Y}$ and $u_{m\lambda}^\delta(x,y)$ is $\mathscr{Y}$–periodic in $y$.

## 5.2 Step 2: Passage to the limit as $\delta \to 0$

Now we pass to the limit as $\delta \to 0$ in (5.1). The asymptotic behaviour of $u_\lambda^{\varepsilon,\delta}$ as $\varepsilon, \delta \to 0$ is given by the following proposition.

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

**Proposition 5.2** *Let $u_\lambda^{\varepsilon,\delta} = (u_{f\lambda}^{\varepsilon,\delta}, u_{m\lambda}^{\varepsilon,\delta})$ be the solution of* (4.1) *with $\gamma < 1$. Then the sequence $(u_{f\lambda}^{\varepsilon,\delta})$ $L_{\varepsilon,\delta}$-converges to $u_\lambda^*$ the solution of*

$$\begin{cases} \omega_f \lambda u_\lambda^* - \dfrac{2}{3} k_f \Delta u_\lambda^* + 2\sqrt{\lambda \omega_m k_m} u_\lambda^* = \dfrac{1}{\lambda} Q & \text{in} \quad \Omega; \\ u_\lambda^*(x) = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

**Proof of Proposition 5.2.** It is clear that the source term satisfies:

$$S(u_{f\lambda}^\delta, u_{m\lambda}^\delta) = -\delta^2 k_m \int_{\Gamma_{mf}^\delta} \nabla_y u_{m\lambda}^\delta \cdot \vec{v}\, ds_y = C_1(\lambda, \delta) u_{f\lambda}^\delta,$$

where

$$C_1(\lambda, \delta) = -\lambda \omega_m \int_{\mathscr{M}^\delta} U_{m\lambda}^\delta(y)\, dy \tag{5.4}$$

and $U_{m\lambda}^\delta$ is the unique solution of

$$\begin{cases} \omega_m \lambda U_{m\lambda}^\delta - \delta^2 k_m \Delta_y U_{\lambda m}^\delta = 0 & \text{in} \quad \mathscr{M}^\delta; \\ U_{m\lambda}^\delta(y) = 1 & \text{on} \quad \Gamma_{mf}^\delta. \end{cases} \tag{5.5}$$

Following Pankratov & Rybalko [30], the asymptotic behaviour of the integral $C_1(\lambda, \delta)$ is given by

$$C_1(\lambda, \delta) = -6\delta \sqrt{\lambda \omega_m k_m}\, (1 + o(1))$$

as $\delta \to 0$.

Finally, using the same arguments as in § 3, we can show that the system (5.1) could be decoupled. Thus the limit of $u_{f\lambda}^\delta$ as $\delta \to 0$ will give the global behaviour of the system. Moreover, following the arguments of Cioranescu & Saint Jean Paulin [16] [Chapter 2] and using (2.1) we can show that $u_{f\lambda}^\delta$ converges strongly in $H_0^1(\Omega)$ to $u_\lambda^*$ the solution of (5.3).

As in § 3 (see (3.23)) we prove that $u_{f\lambda}^{\varepsilon,\delta}$ $L_{\varepsilon,\delta}$-converges to $u_\lambda^*$ the solution of (5.3). □

We complete the proof of Theorem 2.5 repeating the arguments from § 3.3.

## 6 Concluding remarks

The problem of macroscopic behaviour for a double porosity model of single phase flow with Robin interface conditions has been formulated and analyzed in this paper. The convergence of the homogenization process in a suitable topology has been established. The main feature of the homogenized models is that the effective coefficients and the exchange kernel of the long time operator are obtained explicitly. This model may be of practical use for the numerical simulation of flow in fractured media. This study was intended as a first step to the homogenization of highly heterogeneous reservoirs with thin fissures. We are now investigating the homogenization problem of a coupled system modelling the flow and transport of contaminants in such porous media.

**Acknowledgements**

The work of B. Amaziane and L. Pankratov has been partially supported from the GdR MoMaS 2439 CNRS ANDRA BRGM CEA EDF whose support is gratefully acknowledged. This paper was completed when L. Pankratov was visiting the Applied Mathematics Laboratory of the University of Pau. He is grateful for the invitation and the hospitality. The authors are grateful to Professor M. Panfilov (*ENS de Géologie – INP de Lorraine, Nancy, France*) for stimulating discussions. The authors gratefully thank the anonymous referees for their insightful comments and suggestions.

## Appendix A A double porosity type model with Robin transmission conditions

In this section, we derive in a formal way a double porosity model with nonstandard transmission conditions at the interface between the matrix and fracture. Namely, we consider the continuity of the flux and a jump of the pressure proportional to the flux at the interface.

We consider a fractured porous medium $\Omega$ made of rectangular blocks $\Omega_m$ and fracture system $\Omega_f$ separated by a thin layer $\Omega_l$ (see Figure A 1). Notice that such a layer always exists in naturally fractured reservoirs because of the sedimentation process, and is usually called colmated layer [18, 33]. If we assume that the width of the layer $\Omega_l$ is very small in comparison with the width of the fissure, then we will characterize this layer as an interface between the block and the fissure. This is the reason why we call our model a degenerate triple porosity model. Let us mention that a similar approach was already used in Faille *et al.* [21] to treat the faults in geological basin modelling. We will describe briefly the procedure leading to the double porosity model with Robin transmission conditions at the interface between the matrix and fracture.

We denote by $K_f$ and $K_m$ the permeabilities of $\Omega_f$ and $\Omega_m$ respectively, which are functions of the space variable $x = (x_1, x_2)$. We assume that the thickness of the layer $\Omega_l$ is constant and we denote it by $h$. The permeability $K_l$ of $\Omega_l$ is scaled by $h$ and it is assumed to be anisotropic with constant components $K_l^1$, $K_l^2$. We suppose that the transverse permeability $K_l^1$ of the layer is much higher than the longitudinal layer permeability $K_l^2$ ($K_l^1 \gg K_l^2$), so that the flow inside the layer is oriented along the coordinate $x_1$ only. Moreover, this flow is assumed to be stationary.

Let $\Gamma_{lm}$ be the interface between the block and the layer and let $\Gamma_{fl}$ be the interface between the fracture and the layer. We denote $\vec{n}_{fl}$ the normal vector to $\Gamma_{fl}$ oriented from $\Omega_f$ to $\Omega_l$, and $\vec{n}_{lm}$ is the normal vector to $\Gamma_{lm}$ oriented from $\Omega_l$ to $\Omega_m$.

Single phase, slightly compressible fluid flow through $\Omega$ is then described by Darcy's law and continuity subject to the classical conditions of pressure and flux continuity on $\Gamma_{fl}$ and $\Gamma_{lm}$:

$$\begin{cases} p_f = p_l & \text{on} \quad \Gamma_{fl}; \\ K_f \nabla p_f \cdot \vec{n}_{fl} = K_l \nabla p_l \cdot \vec{n}_{fl} & \text{on} \quad \Gamma_{fl}, \end{cases} \qquad \begin{cases} p_m = p_l & \text{on} \quad \Gamma_{lm}; \\ K_m \nabla p_m \cdot \vec{n}_{fl} = K_l \nabla p_l \cdot \vec{n}_{fl} & \text{on} \quad \Gamma_{lm}, \end{cases} \tag{A 1}$$

where $p_i$ denotes the pressure in $\Omega_i$, $i = f, l$ and $m$.

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

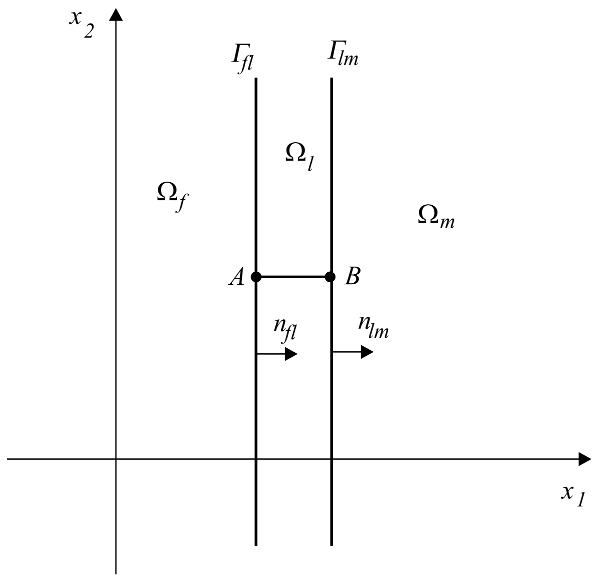


FIGURE A 1. A fractured domain with an intermediate layer.

Let $AB$ be a line between $\Gamma_{fl}$ and $\Gamma_{lm}$ such that $x_2 = constant$ (see Figure A 1) and let $\vec{v} = (v_1, v_2)$ be the flow velocity. Then the continuity equation and the Darcy law in the layer are:

$$v_2 \equiv 0, \quad \frac{\partial v_1}{\partial x_1} = 0, \quad v_1 = -K_l^1 \frac{\partial p_l}{\partial x_1}.$$

Therefore, we have for the pressure derivative along the transverse direction

$$K_l^1 \frac{\partial p_l}{\partial x_1} = C_0(x_2) \tag{A 2}$$

for any line $AB$ orthogonal to $\Gamma_{fl}$. This equation may be considered as a stationary model of flow through the interface layer. Moreover, since $K_l^1 \gg K_l^2$ we have:

$$K_l^1 \nabla p_l \cdot \vec{n}_{fl}\big|_A = K_l^1 \nabla p_l \cdot \vec{n}_{lm}\big|_B = C_0 \tag{A 3}$$

for any fixed line $AB$. Now the interface conditions (A 1) and (A 3) yield:

$$K_f \nabla p_f \cdot \vec{n}_{fl}|_A = K_m \nabla p_m \cdot \vec{n}_{lm}|_B, \quad A \in \Gamma_{fl}, B \in \Gamma_{lm}. \tag{A 4}$$

We integrate the equation (A 2) along the line $AB$ to obtain

$$p_A - p_B = C_0 \frac{h}{K_l^1}. \tag{A 5}$$

Then using (A 1) and (A 3) we get

$$p_A - p_B = \frac{h}{K_l^1} K_m \, \nabla p_m \cdot \vec{n}_{lm}|_B, \quad B \in \Gamma_{lm}. \tag{A 6}$$

https://doi.org/10.1017/S0956792505006200 Published online by Cambridge University Press

We have that $\Gamma_{fl} \to \Gamma_{lm} \equiv \Gamma_{mf}$ when $h \to 0$. We assume that the limit $K_l^1/h$ as $h \to 0$ exists and we denote it by $\Sigma$. Then (A 4) and (A 6) take the form:

$$\begin{cases} \Sigma(p_f - p_m) = K_m \nabla p_m \cdot \vec{n} & \text{on} \quad \Gamma_{mf}; \\ K_f \nabla p_f \cdot \vec{n} = K_m \nabla p_m \cdot \vec{n} & \text{on} \quad \Gamma_{mf}. \end{cases} \tag{A 7}$$

where $\vec{n}$ is the normal vector to $\Gamma_{mf}$. Conditions (A 7) represent the desired transmission conditions.

## References


[1] Acerbi, E., Chiadò Piat, V., Dal Maso, G. & Percival D. (1992) An extension theorem from connected sets, and homogenization in general periodic domains. *Nonlinear Anal.* **18**, 481–496.

[2] Allaire, G. (1992) Homogenization and two–scale convergence. *SIAM J. Math. Anal.* **28**, 1482–1518.

[3] Allaire, G., Damlamian, A. & Hornung, U. 1995 Two–scale convergence on periodic surfaces and applications, In: Bourgeat, A., Carasso, C., Luckhaus, S. and Mikelic, A. (editors), *Mathematical Modelling of Flow Through Porous Media*, pp. 15–25. World Scientific.

[4] Amaziane, B., Bourgeat, A., Goncharenko, M. & Pankratov L. (2004) Characterization of the flow for a single fluid in excavation zone. *Comptes Rendus Mécanique*, **332**, 79–84.

[5] Arbogast, T., Douglas, J. & Hornung, U. (1990) Derivation of the double porosity model of single phase flow via homogenization theory. SIAM J. Appl. Math. **21**, 823–826.

[6] Arbogast, T., Douglas, J. & Hornung, U. (1991) Modelling of naturally fractured reservoirs by formal homogenization technique. In: Dautray, R. (editor), *Frontiers in Pure and Applied Mathematics*, pp. 1–19. Elsevier.

[7] Barenblatt, G. I., Zheltov, Yu. P. & Kochina, I. N. (1960) Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks. *J. Appl. Math. & Mech.* **24**, 1286–1303.

[8] Bear, J. (1972) *Dynamics of Fluids in Porous Media*. New York. Dover.

[9] Bouchitté, G. & Fragala, I. (2001) Homogenization of thin structures by two-scale method with respect to measure. *SIAM J. Math. Anal.* **32**, 1198–1226.

[10] Bourgeat, A., Mikelic, A. & Piatnitski, A. (1998) Modèle de double porosité aléatoire. *Comptes Rendus de l'Académie des Sciences de Paris, Série 1*, **327**, 99–104.

[11] Bourgeat, A., Goncharenko, M., Panfilov, M. & Pankratov, L. (1999) A general double porosity model. *Comptes Rendus de l'Académie des Sciences de Paris, Série IIb*, **327**, 1245–1250.

[12] Bourgeat, A. (2002) Overall behaviour of fractured porous media versus fractures' size and permeability ratio. *Contemporary Math.* **295**, 75–92.

[13] Bourgeat, A., Chechkin, G. & Piatnitski, A. (2003) Singular double porosity model. *Applicable Anal.* **82**, 103–116.

[14] Bourgeat, A., Pankratov, L. & Panfilov, M. (2004) Study of the double porosity model versus the fissures thickness. *Asymptotic Anal.* **38**, 129–141.

[15] Cioranescu, D. & Donato, P. (1999) *An Introduction to Homogenization*. Oxford University Press.

[16] Cioranescu, D. & Saint Jean Paulin, J. (1999) *Homogenization of Reticulated Structures*. Springer-Verlag.

[17] Clark, G. W. & Showalter, R. E. (1999) Two-scale convergence of a model for flow in a partially fissured medium. *Electr. J. Diff. Eq.* **2**, 1–20.

[18] Delleur, J. W. (1998) *The Handbook of Groundwater Engineering*. CRC Press.

[19] Donato, P. & Monsurro, S. (2004) Homogenization of two heat conductors with an interfacial contact resistance. *Anal. & Applic.* **2**: 247–273.

[20] Ene, H. I. & Polisevski, D. (2002) Model of diffusion in partially fissured media. *Zeitschrift für angewandte Mathematik und Physik*, **53**, 1052–1059.

[21] Faille, I., Flauraud, E., Nataf, F., Pégaz-Fiornet, S., Scheneider, F. & Willien, F. (2002) A new fault model in geological basin modelling. Application of finite volume and domain decomposition methods. In: R. Herbin and D. Kröner (editors), *Finite Volumes for Complex Applications III: Problems and Perspectives*, pp. 543–550. Hermes Penton Science.

[22] Hornung, U. & Showalter, R. E. (1990) Diffusion models for fractured media. *J. Math. Anal. & Applic.* **147**, 69–80.

[23] Hornung, U., Jäger, W. & Mikelic, A. (1994) Reactive transport through an array of cells with semi-permeable membranes. *Math. Modelling & Numer. Anal.* **28**:1, 59–94.

[24] Hornung, U. (1996) Models for flow and transport through porous media derived by homogenization. In: M. F. Wheeler (editor), *Environmental Studies: Mathematical, Computational, and Statistical Analysis*, IMA Volumes on Mathematics, pp. 201–221. Springer.

[25] Hornung, U. (1997) *Homogenization and Porous Media.* Springer-Verlag.

[26] Lauwerier, H. A. (1955) The transport of heat in an oil layer caused by the injection of hot fluid. *Appl. Scientific Res.* 145-150

[27] Neuss–Radu, M. (1996) Some extensions of two-scale convergence. *Comptes Rendus de l'Académie des Sciences de Paris, Série 1*, **322**, 899–904.

[28] Panasenko, G. P. (1997) Homogenization of lattice–type domains: *L*–convergence, In: D. Cioranescu and J.-L. Lions (editors), *Nonlinear Partial Differential Equations, Collège de France Séminaire*, **XIII**. *Research Notes in Mathematics* N 220, Longman Scienifical and Technical, pp. 259–280.

[29] Panfilov, M. (2000) *Macroscale models of flow through highly heterogeneous porous media.* Kluwer.

[30] Pankratov, L. & Rybalko, V. (2003) Asymptotic analysis of a double porosity model with thin fissures. *Sbornik: Mathematics*, **194**, 121–146.

[31] Sandrakov, G. V. (1999) Homogenization of parabolic equations with contrasting coefficients. *Izvestiya: Mathematics*, **63**, 1015–1061.

[32] Showalter, R. E. (1997) *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations.* AMS.

[33] Sophocleous, M. (2002) Interaction between groundwater and surface water: the state of the science. *Hydrogeology J.* **10**, 52–67.

[34] Van Golf–Racht, T. D. (1982) *Fundamentals of Fractured Reservoir Engineering*. Elsevier.

[35] Zhikov, V. V. (2000) On one extension and application of the method of two–scale convergence. *Sbornik: Mathematics*, **191**, 31–72.