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RUIN PROBABILITIES IN A FINITE-HORIZON RISK MODEL WITH INVESTMENT AND REINSURANCE

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Abstract

A finite-horizon insurance model is studied where the risk/reserve process can be controlled by reinsurance and investment in the financial market. Our setting is innovative in the sense that we describe in a unified way the timing of the events, that is, the arrivals of claims and the changes of the prices in the financial market, by means of a continuous-time semi-Markov process which appears to be more realistic than, say, classical diffusion-based models. Obtaining explicit optimal solutions for the minimizing ruin probability is a difficult task. Therefore we derive a specific methodology, based on recursive relations for the ruin probability, to obtain a reinsurance and investment policy that minimizes an exponential bound (Lundberg-type bound) on the ruin probability.

Keywords: Risk process; semi-Markov process; optimal reinsurance and investment; Lundberg-type bound

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1. Introduction

We consider a discrete-time insurance risk/reserve process which can be controlled by reinsurance and investment in the financial market, and we study the ruin probability problem in the finite-horizon case. Although controlling a risk/reserve process is a very active area of research (see Chen *et al.* (2000), Wang *et al.* (2004), Schmidli (2008), Huang *et al.* (2009), and the references therein), obtaining explicit optimal solutions minimizing the ruin probability is in general a difficult task even for the classical Cramér–Lundberg risk process. Thus, an alternative method commonly used in ruin theory is to derive inequalities for ruin probabilities. The inequalities can be used to obtain upper bounds for the ruin probabilities (see Wilmot and Lin (2001), Grandell (1991, pp. 1–32), and Schmidli (2002)), and this is the approach followed in the present paper. The basis of this approach is the well-known fact that in the classical Cramér–Lundberg model if the claim sizes have finite exponential moments then the ruin probability decays exponentially as the initial surplus increases (see, for instance, the book by Asmussen (2000, pp. 97–129)). For the heavy-tailed claims case, it is also shown to decay with a rate depending on the distribution of the claim size; see, e.g. Gaier *et al.* (2003). Paulsen (1998) reviewed general processes for the ruin problem when the insurance company invests in

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a risky asset. Xiong and Yang (2011) gave conditions for the ruin probability to be equal to 1 for any initial endowment and without any assumption on the distribution of the claim size as long as it was not identically zero.

Control problems for risk/reserve processes are commonly formulated in continuous time. Schäl (2004) introduced a formulation of the problem where events (arrivals of claims and asset price changes) occur at discrete points in time that may be deterministic or random, but their total number is fixed. Diasparra and Romera (2009) considered a similar formulation in discrete time. Having a fixed total number of events implies that in the case of random time points the horizon is random as well. In the present paper we follow an approach inspired by Edoli and Runggaldier (2010), who claimed that a more natural way to formulate the problem in the case of random time points is to consider a given fixed time horizon so that the number of event times also becomes random, making the problem nonstandard. Accordingly, it is reasonable to assume that the control decisions (level of reinsurance and amount invested) also correspond to these random time points. Note that this formulation can be seen equivalently in discrete or continuous time.

The stochastic elements that affect the evolution of the risk/reserve process are thus the timing and size of the claims as well as the dynamics of the prices of the assets in which the insurer is investing. This evolution is controlled by the sequential choice of the reinsurance and investment levels.

Claims occur at random points in time and their sizes are also random, while asset price evolutions are usually modeled as continuous-time processes. On small time scales, prices actually change at discrete random time points and vary by tick size. In the proposed model we also let asset prices change only at discrete random time points with their sizes being random as well. This will allow us to consider the timing of the events, namely the arrivals of claims and the changes of the asset prices, to be triggered by the same continuous-time semi-Markov process, i.e. a stochastic process where the embedded jump chain (the discrete process registering what values the process takes) is a Markov chain, and where the holding times (the times between jumps) are random variables, whose distribution function may depend on the two states between which the move is made. Since between event times the situation for the insurer does not change, we will consider controls only at event times.

Our bounds are derived mainly for the purpose of obtaining a reinsurance and investment policy that possibly minimizes the ruin probability. These bounds may not be particularly useful as bounds for the ruin probability itself since they are not guaranteed to be less than 1; however, they make it possible to actually derive a reinsurance and investment policy that, by minimizing the bounds, may be considered as a reasonable policy in view of minimizing the ruin probability. This is confirmed by the simulation results in Piscitello (2012), where one can also see that, in line with the recent literature, investing in the financial market may be dangerous. The main computational advantage of minimizing the bounds is that, while the actual optimal policy may turn out to be of the form of a closed-loop feedback policy and, thus, difficult to determine computationally, the bound minimizing policy is of the myopic type requiring to minimize locally the bound at each event time and it is thus of a much simpler structure.

The rest of the paper is organized as follows. In Section 2 we describe the model and, in particular, the risk process. Section 3 is devoted to deriving recursive relations for the ruin probability. On the basis of these recursive relations, in Section 4 we obtain exponential (Lundberg-type) bounds on the ruin probability. In Section 5 we then discuss a policy iteration approach which yields insurance and investment levels that minimize these bounds. Finally, Section 6 contains some concluding remarks.

2. The model

We consider a finite time horizon $T > 0$. More precisely, to model the timing of the events (arrival of claims and asset price changes), inspired by Schäl (2005) we introduce the process $\{K_t\}_{t>0}$ for $t \leq T$, a continuous-time semi-Markov process (SMP) on $\{0, 1\}$, where $K_t = 0$ holds for the *arrival of a claim*, and $K_t = 1$ for a *change in the asset price*. The embedded Markov chain, i.e. the jump chain associated to the SMP $\{K_t\}_{t>0}$, evolves according to a transition probability matrix $P = \|p_{ij}\|_{i,j\in\{0,1\}}$ that is supposed to be given, and the holding times (the times between jumps) are random variables whose probability distribution function may depend on the two states between which the move is made. We return to this point in Subsection 2.1.

Let T_n be the random time of the nth event, $n \geq 1$, and let the counting process N_t denote the number of events having occurred up to time t, defined as

$$N_t = \sum_{j=1}^{\infty} \mathbf{1}_{\{T_j \leq t\}},$$

and so $T_n = \min[t \geq 0 \mid N_t = n]$.

2.1. Risk process

In this section we introduce the dynamics of the controlled risk process X_t for $t \in [0, T]$ with T a given fixed horizon. For this purpose, let Y_n be the nth ($n \geq 1$) claim payment represented by a sequence of independent and identically distributed (i.i.d.) random variables with common probability distribution function (PDF) $F(y)$ having support in the positive half-line. Let Z_n be the random variable denoting the time between the occurrence of the $(n-1)$th and nth ($n \geq 1$) jumps of the SMP $\{K_t\}_{t>0}$. We assume that $\{Z_n\}$ is a sequence of i.i.d. random variables with PDF $G(\cdot)$. From this we may consider that the transition probabilities of the SMP $\{K_t\}_{t>0}$ are

$$\mathrm{P}\{K_{T_{n+1}} = j,\ Z_{n+1} \leq s \mid K_{T_n} = i\} = p_{ij} G(s).$$

Note that, for a full SMP model, the distribution function $G(\cdot)$ also depends on i and j, and the results derived below go through in the same way for this more general case. Since in many cases of interest (see Example 1 below) $G(\cdot)$ is independent of i and j, for simplicity of presentation, we will restrict ourselves to such a situation.

Example 1. A specific form of SMP, which we will also refer to later, arises, for example, as follows. Let N_t^0 and N_t^1 be independent Poisson processes with intensities λ_0 and λ_1, respectively. We may think of N_t^0 as counting the number of claims and of N_t^1 as counting the number of price changes, with $N_t = N_t^0 + N_t^1$ again a Poisson process of intensity $\lambda = \lambda_0 + \lambda_1$. We may then set

$$K_{T_n} = \begin{cases} 0 & \text{if at } T_n \text{ a jump of } N_t^0 \text{ has occurred } (\textit{claim}), \\ 1 & \text{if at } T_n \text{ a jump of } N_t^1 \text{ has occurred } (\textit{price change}). \end{cases}$$

It then easily follows that

$$\begin{aligned} \mathrm{P}\{K_{T_{n+1}} &= j,\ Z_{n+1} \leq s \mid K_{T_n} = i\} \\ &= \mathrm{P}\{K_{T_{n+1}} = j \mid Z_{n+1} \leq s,\ K_{T_n} = i\}\,\mathrm{P}\{Z_{n+1} \leq s \mid K_{T_n} = i\} \\ &= p_{ij}\,\mathrm{P}\{Z_1 \leq s\} \\ &= \frac{\lambda_j}{\lambda}[1 - \mathrm{e}^{-\lambda s}], \end{aligned}$$

so that, in this case

$$p_{ij} = \frac{\lambda_j}{\lambda} := p_j \quad \text{for all } i, \qquad G(s) = [1 - \mathrm{e}^{-\lambda s}].$$

Remark 1. Our model, in particular the specific case of Example 1, does not allow for simultaneous jumps (claims and price changes may however occur very close to one another). On the other hand, we could include simultaneous jumps into our model by extending K_t to take three possible values, namely,

$$K_{T_n} = \begin{cases} 0 & \text{for a claim,} \\ 1 & \text{for a price change,} \\ 2 & \text{for simultaneous claim and price changes.} \end{cases}$$

The risk process is controlled by reinsurance and investment. In general, this means that we may choose adaptively at the event times T_{N_t} (they correspond to the jump times of N_t), the retention level (or proportionality factor or risk exposure) b_{N_t} of a reinsurance contract as well as the amount δ_{N_t} to be invested in the risky asset, namely in S_{N_t} with S_t denoting discounted prices. For the values b that the various b_{N_t} may take, we assume that $b \in [b_{\min}, 1] \subset (0, 1]$, where $b_{\min}$ will be introduced below and, for the values of δ for the various δ_{N_t}, we assume that $\delta \in [\underline{\delta}, \overline{\delta}]$ with $\underline{\delta} \le 0$ and $\overline{\delta} > 0$ exogenously given. Note that this condition also allows for negative values of δ, meaning (see also Schäl (2004)) that short selling of stocks is allowed. On the other hand, with an exogenously given upper bound $\overline{\delta}$, it might occasionally happen that $\delta_{N_t} > X_{N_t}$, implying a temporary debt of the agent beyond his/her current wealth in order to invest optimally in the financial market. By choosing a policy that minimizes the ruin probability, this debt is however only instantaneous and, with high probability, leads to a positive wealth at the next event time.

Assume that prices change only according to

$$\frac{S_{N_t+1} - S_{N_t}}{S_{N_t}} = (\mathrm{e}^{W_{N_t+1}} - 1) K_{T_{N_t+1}},$$

where W_n is a sequence of i.i.d. random variables taking values in $[\underline{w}, \overline{w}]$ with $-\infty < \underline{w} < 0 < \overline{w} < +\infty$ and with PDF $H(w)$. For simplicity and without loss of generality, we consider only one asset to invest in. An immediate generalization would be to also allow for investment in the money market account.

Let c be the premium rate (income) paid by the customer to the company, fixed in the contract. Since the insurer pays the reinsurer a premium rate, which depends on the retention level b_{N_t} chosen at the various event times T_{N_t}, we denote by $C(b_{N_t})$ the net income rate of the insurer at time $t \in [0, T]$. For $b \in [b_{\min}, 1]$, we let $h(b, Y)$ represent the part of the generic claim Y paid by the insurer and in what follows we take the function $h(b, Y)$ to be of the form $h(b, Y) = bY$ (proportional reinsurance). We will call a sequence $\pi = (b_n, \delta_n)$ of *control actions* a *policy*, where (b_n, δ_n) is short for (b_{T_n}, δ_{T_n}). Control actions over a single period will be denoted by $\phi_n = (b_n, \delta_n)$. According to the expected value principle with safety loading θ of the reinsurer, for a given starting time $t < T$, the income rate function $C(b)$ can be chosen as follows:

$$C(b) := c - (1 + \theta) \frac{\mathrm{E}\{Y_1 - h(b, Y_1)\}}{\mathrm{E}\{Z_1 \wedge (T - t)\}}, \qquad 0 < t < T. \tag{1}$$

Note that $C(b)$ depends on the starting and the terminal times t and T only via $T - t$ in the denominator of (1); the value of $T - t$ can however be considered as given so that $C(b)$ can indeed be considered as depending only on the current value b of the retention level (it is in fact a rate over the interval $[t, T]$).

We use Z_1 and Y_1 in the above formula since, by our i.i.d. assumption, the various Z_n and Y_n are all independent copies of Z_1 and Y_1. Note also that, in order to keep (1) simple and possibly similar to standard usage, in the denominator on the right-hand side we have considered the random time Z_1 between two successive events, while more correctly we should have taken the random time between two successive claims, which is larger. For this, we can however play with the safety loading factor. In fact, if we denote by $\bar{Z}$ the average time between successive claims before T and, for a given θ, put $\bar{\theta} = (1+\theta)\bar{Z}/\operatorname{E}\{Z_1 \wedge (T-t)\} - 1$, we have $(1+\theta)/\operatorname{E}\{Z_1 \wedge (T-t)\} = (1+\bar{\theta})/\bar{Z}$. Since in this way $1+\bar{\theta} = (1+\theta)\bar{Z}/\operatorname{E}\{Z_1 \wedge (T-t)\}$ and $\bar{Z} > \operatorname{E}\{Z_1 \wedge (T-t)\}$, we are assured that $1+\bar{\theta} > 1$. We can now define

$$b_{\min} := \min[b \in [b^*, 1] \mid c \geq C(b) \geq c^*], \tag{2}$$

where $c^* \geq 0$ denotes the minimal value of the premium considered by the insurer and $b^* > 0$ is the minimal value for the proportion of the claim paid by the insurer. It follows that $C(b_{\min}) = \max[c^*, C(b^*)]$. We make the following assumption.

Assumption 1. *Suppose that the following statements hold.*

(i) *The tuple $(Z_n, Y_n, W_n)_{n\geq 1}$ is formed by independent sequences of i.i.d. random variables.*

(ii) *Defining $\bar{r} := \sup\{r \geq 0 \mid \operatorname{E}\{e^{rY_1}\} < +\infty\}$, assume that $\bar{r} > 0$ and that $\operatorname{E}\{e^{\bar{r}Y_1}\} = +\infty$.*

(iii) $c - (1+\theta)\operatorname{E}\{Y_1\}/\operatorname{E}\{Z_1 \wedge T\} > 0$.

Remark 2. (i) Since $b \leq 1$, Assumption 1(ii) implies that, for all $b \in [b_{\min}, 1]$, we have $\operatorname{E}\{e^{rbY_1}\} < +\infty$ for $r \in [0, \bar{r})$ and $\operatorname{E}\{e^{rbY_1}\} = +\infty$ for $r = \bar{r}/b_{\min}$.

(ii) Since the support of Y_1 is in the positive half-line, we have $\lim_{r\uparrow\bar{r}} \operatorname{E}\{e^{rY_1}\} = +\infty$. Note that $\bar{r}$ may be equal to $+\infty$, e.g. if the support of Y_1 is bounded.

(iii) For $h(b, Y) = bY$, Assumption 1(iii) implies that $c \geq C(b) \geq c^* \geq 0$ for all $b \in [b_{\min}, 1]$ and that, furthermore, $c \geq 0$.

(iv) From the definition of $C(b)$ in (1), the definition of $b_{\min}$ in (2), Assumption 1(iii), and the fact that $C(b_{\min}) = \max[c^*, C(b^*)]$, it follows that

$$b_{\min} = \begin{cases} 1 - (c - c^*)\dfrac{\operatorname{E}\{Z_1 \wedge (T-t)\}}{(1+\theta)\operatorname{E}\{Y_1\}} & \text{if } c^* > C(b^*) > 0, \\ b^* & \text{if } 0 < c^* \leq C(b^*). \end{cases}$$

In the given setting, for the insurance risk process (surplus) X, we obtain the following one-step transition dynamics between the generic random times T_n and T_{n+1} when at T_n a control action $\phi = (b, \delta)$ is taken for a certain $b \in [b_{\min}, 1] \subset (0, 1]$ and $\delta \in [\underline{\delta}, \overline{\delta}]$:

$$X_{T_{n+1}} = X_{T_n} + C(b)Z_{n+1} - (1 - K_{T_{n+1}})h(b, Y_{n+1}) + K_{T_{n+1}}\delta(e^{W_{n+1}} - 1). \tag{3}$$

Definition 1. Letting $U := [b_{\min}, 1] \times [\underline{\delta}, \overline{\delta}]$, we say that a control action $\phi = (b, \delta)$ is *admissible* if $(b, \delta) \in U$. Note that U is compact. A policy π will be called admissible if it implies admissible control actions.

We now want to express the one-step dynamics in (3) when starting from a generic time instant $t < T$ with a capital x. For this purpose, note that if, for a given $t < T$, we have $N_t = n$, the time T_{N_t} is the random time of the nth event and $T_n \leq t \leq T_{n+1}$. Since, when standing at time t, we observe the time that has elapsed since the last event in T_{N_t}, it is not restrictive to assume that $t = T_{N_t}$ (see the comment following (4) below). A further justification for letting $t = T_{N_t}$ can be given in the case of Example 1: the random variable there has a negative exponential distribution and this distribution is memoryless. Furthermore, since Z_n, Y_n, and W_n are i.i.d., in the one-step random dynamics for the risk process X_t we may replace the generic $(Z_{n+1}, Y_{n+1}, W_{n+1})$ by (Z_1, Y_1, W_1). We may thus write

$$X_{N_t+1} = x + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(\mathrm{e}^{W_1} - 1) \tag{4}$$

for $0 < t < T$, $T > 0$, and with $X_t = x \geq 0$ (recall that we assumed that $t = T_{N_t}$). Note that, if we had $t \neq T_{N_t}$ and, therefore, $t > T_{N_t}$, the second term on the right-hand side of (4) would become $C(b)[Z_1 - (t - T_{N_t})]$ and (4) could then be rewritten as

$$X_{N_t+1} = [x - C(b)(t - T_{N_t})] + C(b)Z_1 - (1 - K_{T_{N_t+1}})h(b, Y_1) + K_{T_{N_t+1}}\delta(\mathrm{e}^{W_1} - 1),$$

with the quantity $[x - C(b)(t - T_{N_t})]$, which is known at time t, replacing x. This is the sense in which we mentioned above that it is not restrictive to assume that $t = T_{N_t}$. In what follows we will work with the risk process X_t (or X_{N_t}), as defined by (4). For convenience, we will denote by (b_n, δ_n) the values of $\phi = (b, \delta)$ at $t = T_{N_t}$. Accordingly, we will also write (b_{N_t}, δ_{N_t}) for $(b_{T_{N_t}}, \delta_{T_{N_t}})$.

Objective. Determine an admissible reinsurance and investment policy π so as to minimize a suitable bound on the ruin probability, where by 'ruin' we mean the event when the wealth process first becomes 0.

Following Schmidli (2008, pp. 147–194) we will also introduce an absorbing (cemetery) state that we take as the state $x = 0$; in fact, once the company has been ruined, it is reasonable to assume that its wealth remains at the level zero.

3. Recursions

We start this section by specifying some notation and introducing the basic definitions concerning our ruin probabilities.

3.1. Notation and definitions

In view of our ultimate goal of obtaining bounds on the ruin probability over a given finite time horizon $[t, T]$, we first introduce the notion of probability of ruin before T within the first n events. Recalling that ruin occurs as soon as $X_t = 0$ and that X_t remains at level zero after ruin, we give the following more formal definition.

Definition 2. Assume that we are standing at time $t < T$ with a surplus value of $X_t = X_{T_{N_t}} = x > 0$ and with $K_t = K_{T_{N_t}} = k \in \{0, 1\}$. Given an admissible reinsurance and investment policy π, we will denote by $\psi_n^\pi(t, x; k)$ the probability of ruin before T within the first n events, defined by

$$\psi_n^\pi(t, x, k) = \mathrm{P}_{t,x,k}^\pi\{X_n \leq 0,\ T_n \leq T\}, \tag{5}$$

where $\mathrm{P}_{t,x;k}^\pi$ denotes the probability conditional on $X_t = x > 0$, $K_t = k \in \{0, 1\}$ and for a given admissible policy π.

Our first aim in the next subsection is to obtain a recursive relation for $\psi_n^\pi(t, x; k)$.

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3.2. Recursive relations

In (5) it was implicit that $t < T$. In what follows we write $\psi_{n-1}^{\pi}(T_1, X_1; K_1)$ to mean the random variable, as a function of $(T_1, X_1; K_1)$, given by

$$\psi_{n-1}^{\pi}(T_1, X_1; K_1) = \mathrm{P}_{T_1, X_1, K_1}^{\pi}\{X_n \le 0,\ T_n \le T\} \tag{6}$$

on the event $T_1 \le T$.

We now have the following recursion result.

Proposition 1. *For an initial surplus x at a given time $t \in [0, T]$, as well as an initial event $K_{T_{N_t}} = k$ and a given admissible policy π, and any $n \ge 1$, we have*

$$\psi_n^{\pi}(t, x, k) \le \psi_1^{\pi}(t, x, k) + \mathrm{P}\{T_1 \le T\}\, \mathrm{E}_{t,x,k}^{\pi}\{\psi_{n-1}^{\pi}(T_1, X_1, K_1)\}.$$

Proof. We have the decomposition

$$\begin{aligned}
\psi_n^{\pi}(t, x, k) &= \mathrm{P}_{t,x,k}^{\pi}\{X_n \le 0,\ T_n \le T\} \\
&= \mathrm{P}_{t,x,k}^{\pi}\{X_n \le 0,\ T_n \le T,\ X_1 \le 0,\ T_1 \le T\} \\
&\quad + \mathrm{P}_{t,x,k}^{\pi}\{X_n \le 0,\ T_n \le T,\ X_1 > 0,\ T_1 \le T\}.
\end{aligned}$$

For the first term on the right-hand side, we have

$$\begin{aligned}
&\mathrm{P}_{t,x,k}^{\pi}\{X_n \le 0,\ T_n \le T,\ X_1 \le 0,\ T_1 \le T\} \\
&\quad = \mathrm{P}_{t,x,k}^{\pi}\{T_n \le T \mid X_1 \le 0,\ T_1 \le T\}\, \mathrm{P}_{t,x,k}^{\pi}\{X_1 \le 0,\ T_1 \le T\} \\
&\quad = \mathrm{P}\{T_n \le T \mid T_1 \le T\}\psi_1^{\pi}(t, x, k) \\
&\quad = \frac{\mathrm{P}\{T_n \le T\}}{\mathrm{P}\{T_1 \le T\}}\psi_1^{\pi}(t, x, k) \\
&\quad = \frac{\mathrm{P}\{N_T - N_{T_1} \ge n - 1\}}{\mathrm{P}\{T_1 \le T\}}\psi_1^{\pi}(t, x, k) \\
&\quad = \frac{\mathrm{P}\{N_T - N_t \ge n\}}{\mathrm{P}\{T_1 \le T\}}\psi_1^{\pi}(t, x, k) \\
&\quad \le \psi_1^{\pi}(t, x, k),
\end{aligned}$$

where we have used the facts that

$$\mathrm{P}\{N_T - N_t \ge n\} \le \mathrm{P}\{N_T - N_t \ge 1\} = \mathrm{P}\{Z_1 \le T - t\} = \mathrm{P}\{T_1 \le t\}$$

and the probability of events related to T_n (equivalently to N_t) does not depend on either π or (t, x, k).

Taking into account (6) and the fact that it holds on the event $T_1 \le T$, for the second term, we have instead (by $\mathcal{F}_1^{T,X,K}$ we denote the σ-algebra generated by (T_m, X_m, K_m) up to $t = T_1$)

$$\begin{aligned}
&\mathrm{P}_{t,x,k}^{\pi}\{X_n \le 0,\ T_n \le T,\ X_1 > 0,\ T_1 \le T\} \\
&\quad \le \mathrm{P}_{t,x,k}^{\pi}\{X_n \le 0,\ T_n \le T,\ T_1 \le T\} \\
&\quad = \mathrm{E}_{t,x,k}^{\pi}\{\mathrm{E}^{\pi}\{\mathbf{1}_{\{X_n \le 0,\, T_n \le T\}}\mathbf{1}_{\{T_1 \le T\}} \mid \mathcal{F}_1^{T,X,K}\}\} \\
&\quad = \mathrm{E}_{t,x,k}^{\pi}\{\mathbf{1}_{\{T_1 \le T\}}\, \mathrm{E}^{\pi}\{\mathbf{1}_{\{X_n \le 0,\, T_n \le T\}} \mid \mathcal{F}_1^{T,X,K}\}\} \\
&\quad = \mathrm{E}_{t,x,k}^{\pi}\{\mathbf{1}_{\{T_1 \le T\}}\, \mathrm{P}_{T_1, X_1, K_1}^{\pi}\{X_n \le 0,\ T_n \le T\}\} \\
&\quad = \mathrm{P}\{T_1 \le T\}\, \mathrm{E}_{t,x,k}^{\pi}\{\psi_{n-1}^{\pi}(T_1, X_1, K_1)\},
\end{aligned}$$

where we have also used the fact that T_1 is independent of the other random variables.

4. Bounds

In this section we derive bounds on the ruin probability in a general setting and in Section 5 we then minimize them with respect to the reinsurance and investment policy. We base our analysis on the results given in Diasparra and Romera (2009) and Diasparra and Romera (2010) that are here extended to the general setup of the present paper. To stress the fact that the process X defined in (3) corresponds to the choice of a specific policy π, in what follows we will use the notation X^π.

Given an admissible policy $\pi_t = (b_t, \delta_t)$ and defining, for $t \in [0, T]$, the random variable

$$V_t^\pi := C(b)(Z_1 \wedge (T-t)) - \mathbf{1}_{\{Z_1 \le T-t\}}[(1 - K_{T_{N_t}+1})bY_1 - K_{T_{N_t}+1}\delta(\mathrm{e}^{W_1} - 1)], \tag{7}$$

where $b = b_t$ and $\delta = \delta_t$, let, for $r \in [0, \bar{r}/b_{\min})$ and $k \in \{0, 1\}$,

$$l_r^\pi(t, k) := \mathrm{E}_{t,x,k}\{\mathrm{e}^{-rV_t^\pi}\} - 1, \tag{8}$$

which does not depend on x and where, for reasons that should become clear below, we distinguish the dependence of l^π on r from that on (t, k).

Remark 3. Note that (see Remark 2(i) and (ii)) $\lim_{r \uparrow \bar{r}/b_{\min}} l_r^\pi(t, k) = +\infty$ for all (t, k).

Definition 3. We will call a policy π *strongly admissible* and denote its set by $\mathcal{A}$ if at each $t \in [0, T]$ the corresponding control action $(b_t, \delta_t) \in U$ and, for any $(t, k) \in [0, T] \times \{0, 1\}$, it holds that $\mathrm{E}_{t,k}^\pi\{V_t^\pi\} > 0$.

Note that $\mathcal{A}$ is nonempty since, see Assumption 1(iii), it contains at least the stationary policy $(b_{N_t}, \delta_{N_t}) \equiv (b_{\min}, 0)$.

Proposition 2. *For each $(t, k) \in [0, T] \times \{0, 1\}$ and each $\pi \in \mathcal{A}$, the following statements hold.*

(i) *As a function of $r \in [0, \bar{r}/b_{\min})$ with $\bar{r}$ such that Assumption 1(ii) is satisfied, $l_r^\pi(t, k)$ is convex with a negative slope at $r = 0$.*

(ii) *The equation $l_r^\pi(t, k) = 0$, seen as an equation in r, has a unique positive root in $(0, \bar{r}/b_{\min})$ that we denote by $R^\pi(t, k)$, so that the defining relation for $R^\pi(t, k)$ is*

$$l_{R^\pi(t,k)}^\pi(t, k) = 0 \quad \textit{for all } t \in [0, T],\ k \in \{0, 1\}. \tag{9}$$

Proof. Differentiating with respect to r under the expectation sign leads to

$$\frac{\partial}{\partial r}(l_r^\pi(t, k))\bigg|_{r=0} = \mathrm{E}_{t,x,k}\{-V_t^\pi\} < 0, \qquad \frac{\partial^2}{\partial r^2}(l_r^\pi(t, k)) = \mathrm{E}_{t,x,k}\{(V_t^\pi)^2\mathrm{e}^{-rV_t^\pi}\} > 0,$$

where the first inequality follows from the admissibility of π (see Definition 3), and so part (i) follows immediately. In view of (ii) note that from Assumption 1(ii) (see also Remark 2(i) and (ii)) we obtain $\lim_{r \uparrow \bar{r}/b_{\min}} l_r^\pi(t, k) = +\infty$. This fact combined with (i) leads to (ii).

Definition 4. For given $\pi \in \mathcal{A}$ and $t \in [0, T]$, let

$$R_t^\pi := \inf_{t \le s \le T} \min[R^\pi(s, 0), R^\pi(s, 1)]. \tag{10}$$

In relation to this definition see also Remark 7 below.

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Remark 4. Note that by Remark 3 we may always assume that $R^{\pi}(t,k) < \bar{r}/b_{\min}$ for all $(t,k) \in [0,T] \times \{0,1\}$ and, thus, also $R_t^{\pi} < \bar{r}/b_{\min}$.

We now derive our first bound.

Lemma 1. *Given an initial time $t \in [0,T]$ and an initial event $k \in \{0,1\}$, we have*

$$\psi_1^{\pi}(t,x,k) \leq \mathrm{e}^{-R_t^{\pi} x} \quad \textit{for all } x > 0,\ \pi \in \mathcal{A},$$

where R_t^{π} is as in Definition 4 (see (10)).

Proof. Note first that, from (3) and definition (7), the random variable V_t^{π} represents the increment of the risk process X_t between two successive event times, provided that these event times occur within the interval $[t,T]$. According to Definition 2 we then obtain

$$\psi_1^{\pi}(t,x,k) = \mathrm{P}_{t,x,k}^{\pi}\{X_1 < 0, T_1 \leq T\} \leq \mathrm{P}_{t,x,k}^{\pi}\{X_1 < 0\} = \mathrm{P}_{t,x,k}^{\pi}\{V_t^{\pi} \leq -x\}.$$

On the other hand, by Chebyshev's inequality we have, for all $r > 0$,

$$\mathrm{P}_{t,x,k}^{\pi}\{V_t^{\pi} \leq -x\} = \mathrm{P}_{t,x,k}^{\pi}\{\mathrm{e}^{-rV_t^{\pi}} \geq \mathrm{e}^{rx}\} \leq \mathrm{e}^{-rx}\,\mathrm{E}_{x,k}^{\pi}\{\mathrm{e}^{-rV_t^{\pi}}\}.$$

For $r = R^{\pi}(t,k)$, using the previous two relations, we obtain

$$\psi_1^{\pi}(t,x,k) \leq \mathrm{e}^{-R^{\pi}(t,k)x}\,\mathrm{E}_{t,x,k}^{\pi}\{\mathrm{e}^{-R^{\pi}(t,k)V_t^{\pi}}\} \leq \mathrm{e}^{-R^{\pi}(t,k)x} \leq \mathrm{e}^{-R_t^{\pi}x},$$

where in the last relation we have used Definition 4 and Proposition 2(ii).

Next set $\gamma := \mathrm{P}\{N_T - N_t \geq 1\} = \mathrm{P}\{Z_1 \leq T - t\} = \mathrm{P}\{T_1 \leq T\}$, and note that this γ is independent of (π,x,k) and it holds that $\gamma < 1$.

Lemma 2. *Given an initial surplus $x > 0$ at a given time $t \in [0,T]$, we have, for all $n \in \mathbb{N}$, any initial event $k \in \{0,1\}$, and all $\pi \in \mathcal{A}$,*

$$\psi_n^{\pi}(t,x,k) \leq \bigg(\sum_{m=0}^{n-1}\gamma^m\bigg)\mathrm{e}^{-R_t^{\pi}x}.$$

Proof. The proof is by induction. By Lemma 1, the statement is true for $n = 1$. Assume that it is true for $n-1$, namely, $\psi_{n-1}^{\pi}(t,x,k) \leq (\sum_{m=0}^{n-2}\gamma^m)\mathrm{e}^{-R_t^{\pi}x}$ for any $x > 0$, $k \in \{0,1\}$, and $\pi \in \mathcal{A}$. Given the definition of R_t^{π} in (4), we may formulate the induction hypothesis as follows:

$$\psi_{n-1}^{\pi}(T-s, X_1, K_1) \leq \bigg(\sum_{m=0}^{n-2}\gamma^m\bigg)\mathrm{e}^{-R_t^{\pi}X_1} \quad \text{for all } s \in [0, T-t].$$

By Proposition 1, as well as Lemma 1, and taking into account the definitions of R_t^{π} in (10) and V_t^{π} in (7), we then obtain

$$\begin{aligned}
\psi_n^{\pi}(t,x,k) &\leq \mathrm{e}^{-R_t^{\pi}x} + \mathrm{P}\{T_1 \leq T\}\bigg(\sum_{m=0}^{n-2}\gamma^m\bigg)\mathrm{E}_{t,x,k}^{\pi}\{\mathrm{e}^{-R_t^{\pi}X_1}\} \\
&\leq \mathrm{e}^{-R_t^{\pi}x} + \gamma\bigg(\sum_{m=0}^{n-2}\gamma^m\bigg)\mathrm{e}^{-R_t^{\pi}x}\,\mathrm{E}_{t,x,k}^{\pi}\{\mathrm{e}^{-R_t^{\pi}V_t^{\pi}}\} \\
&\leq \bigg(\sum_{m=0}^{n-1}\gamma^m\bigg)\mathrm{e}^{-R_t^{\pi}x},
\end{aligned}$$

where the last inequality follows from the definitions of $R^\pi(t,k)$ and R_t^π as well as the properties of the function $l_r^\pi(t,k)$ stated in Proposition 2. In fact, from $l^\pi_{R^\pi(t,k)}(t,k) = 0$ (see (9)) and $R_t^\pi \le R^\pi(t,k)$, it follows that $l^\pi_{R_t^\pi}(t,k) = \mathrm{E}^\pi_{t,x,k}\{\mathrm{e}^{-R_t^\pi V_t^\pi}\} - 1 \le 0$.

As an immediate consequence of Lemma 2, we now obtain our main result.

Theorem 1. *Given an initial surplus $x > 0$ at a given time $t \in [0, T]$, we have, for all $n \in \mathbb{N}$, any initial event $k \in \{0, 1\}$, and all $\pi \in \mathcal{A}$,*

$$\psi_n^\pi(t,x,k) \le \frac{1}{1-\gamma}\mathrm{e}^{-R_t^\pi x}. \tag{11}$$

Remark 5. Note that the bound in Theorem 1 will in many cases be larger than 1 and, thus, not useful as a numerical bound on the probability of ruin itself. The main purpose for deriving this bound however is to obtain a reinsurance and investment policy that keeps the ruin probability low, which in Section 5 we find to be the policy that minimizes the bound. With this goal in mind, there is no need to derive the tightest possible bound; what is important however is that the coefficient on the right-hand side of (11) is independent of (π, x, k). Note also that, while the true optimal policy is most likely of the form of a feedback policy and, thus, difficult to determine, the policy derived from minimizing the bound is of the form of a myopic policy, which is much easier to derive, as we will show in the next section.

Remark 6. Note that, see Remark 4, for $t \to T$, the right-hand side of (11) is bounded from below by $\mathrm{e}^{-\bar{r}x/b_{\min}}$, which is strictly positive unless $\bar{r} = +\infty$ (the latter happens whenever the support of Y_1 is bounded; see Remark 2(ii)). On the other hand, for $t \to T$, we always have $\psi_n^\pi(t,x,k) \to 0$. This hints at the possibility of viewing the bound in Theorem 1 as having the flavor of an infinite-time ruin probability in the spirit of the Lundberg bound, although we work on a finite time horizon.

5. Optimizing the bounds

As mentioned previously, it is in general a difficult task to obtain an explicit solution to the reinsurance–investment problem in order to minimize the ruin probability even for the classical risk process. We will thus choose the reinsurance and investment levels in order to minimize the bounds that we have derived. By Theorem 1, this amounts to choosing a strategy $\pi \in \mathcal{A}$ such that, for each $t \in [0, T]$, the value of R_t^π is as large as possible. In order to achieve this goal, note that, by Proposition 2, the function $l_r^\pi(t,k)$ is, as a function of $r \in [0, \bar{r}/b_{\min})$ (for every fixed $(t,k) \in [0,T] \times \{0,1\}$ and $\pi \in \mathcal{A}$), convex with a zero in $r = 0$ and (see Remark 4) a unique positive zero in $R^\pi(t,k) \in [0, \bar{r}/b_{\min})$. To obtain, for a given $t \in [0, T]$, the largest value of $R_t^\pi = \inf_{t \le s \le T} \min[R^\pi(s,0), R^\pi(s,1)]$, it thus suffices to choose $\pi \in \mathcal{A}$ that minimizes $l_r^\pi(t,k)$ at $r = R_t^\pi$. This also appeals to intuition in the sense that, by its definition in (8), minimizing $l_r^\pi(t,k)$ amounts to penalizing negative values of V_t^π and, thus, also of X_t^π, thereby minimizing the possibility of ruin. For this purpose, in Subsection 5.1 we propose a policy-improvement-type algorithm.

5.1. Policy improvement

Concerning the minimization of $l_r^\pi(t,k)$ at $r = R_t^\pi$, note that decisions concerning the control actions $\phi = (b, \delta)$ have to be made only at the event times T_n. The minimization of $l_r^\pi(t,k)$ with respect to $\pi \in \mathcal{A}$ has thus to be performed only for pairs (t,k) of the form (T_n, K_{T_n}), thus leading to a policy π with individual control actions $\phi_{T_n} = (b_{T_n}, \delta_{T_n})$.

Our problem to determine an investment and insurance policy to minimize the bounds on the ruin probability may thus be solved by solving the following subproblems:

(i) for a given policy $\bar{\pi} \in \mathcal{A}$, determine $l_r^{\bar{\pi}}(t,k)$ for pairs (t,k) of the form (T_n, K_{T_n});

(ii) determine $R^{\bar{\pi}}(T_n, K_{T_n})$, that is, a solution with respect to r of $l_r^{\bar{\pi}}(T_n, K_{T_n}) = 0$, and set $R_t^{\bar{\pi}} = \inf_{t \le T_n \le T} \min[R^{\bar{\pi}}(T_n, 0), R^{\bar{\pi}}(T_n, 1)]$;

(iii) improve the policy $\bar{\pi}$ by minimizing $l^{\pi}_{R^{\bar{\pi}}}(T_n, K_{T_n})$ with respect to $\pi \in \mathcal{A}$.

This leads to a policy-improvement-type approach, more precisely, one may proceed as follows:

(a) start from a given policy π^0 (e.g. the one requiring minimal reinsurance and no investment in the financial market);

(b) determine $R^{\pi^0}(T_n, K_{T_n})$ corresponding to π^0 for the various (T_n, K_{T_n}) as well as the ensuing R^{π^0};

(c) determine $\pi^1 \in \mathcal{A}$ that minimizes $l^{\pi}_{R^{\pi^0}}(T_n, K_{T_n})$ with respect to $\pi \in \mathcal{A}$ for the various (T_n, K_{T_n}); repeat the procedure until a stopping criterion is reached (note that, by the above procedure, $R^{\pi^n} > R^{\pi^{n-1}}$).

Note that the policy determined by this procedure is automatically strongly admissible.

A practical way to implement steps (b) and (c) above is to discretize the time interval $[0, T]$ and then register an event only at the end of the interval in which it occurred (multiple events in a same subinterval may be recorded at the end as a single event for each of the two categories: claims and price changes). The function $l_r^{\pi}(t,k)$ has then to be determined only for t corresponding to an endpoint of the various subintervals for each of the two possible values of k.

One crucial step in this procedure is to determine the function $l_r^{\pi}(t,k)$ corresponding to a given $\pi \in \mathcal{A}$ and, for this purpose, we can prove the following result.

Proposition 3. *Under the standing assumptions, we have*

$$l_r^{\pi}(t,k) = (1 - G(T-t))\mathrm{e}^{-rC(b)(T-t)} - 1 + \int_0^{T-t} \mathrm{e}^{-rC(b)z}\,\mathrm{d}G(z)\bigg[p_{k,0}\int_0^{\infty} \mathrm{e}^{rby}\,\mathrm{d}F(y) + p_{k,1}\int_{\underline{w}}^{\overline{w}} \mathrm{e}^{-r\delta(e^{w}-1)}\,\mathrm{d}H(w)\bigg].$$

Proof. It is an immediate consequence of (7) and (8).

In the specific case described in Example 1, where the Z_n are i.i.d., having a negative exponential distribution with parameter $\lambda = \lambda_0 + \lambda_1$ and where $p_{k,h} = \lambda_h/\lambda$, $h = 0, 1$, independently of k, we have the following immediate corollary.

Corollary 1. *In the case of Example 1, we have*

$$\begin{aligned} l_r^{\pi}(t,k) &= l_r^{\pi}(t) \\ &= \mathrm{e}^{-(\lambda + rC(b))(T-t)} - 1 + \frac{1}{\lambda + rC(b)}[1 - \mathrm{e}^{-(\lambda + rC(b))(T-t)}] \\ &\qquad \times \bigg[\lambda_0 \int_0^{\infty} \mathrm{e}^{rby}\,\mathrm{d}F(y) + \lambda_1 \int_{\underline{w}}^{\overline{w}} \mathrm{e}^{-r\delta(e^{w}-1)}\,\mathrm{d}H(w)\bigg] \end{aligned}$$

independently of the value of k.

Remark 7. It can furthermore be shown, see Piscitello (2012), that in the case of Example 1 the value of $R^{\pi}(t, k)$ is not only independent of k being a solution of $l_r^{\pi}(t) = 0$ but it is furthermore independent of t, so that the bound optimizing policy is stationary.

The expressions for $l_r^{\pi}(t, k)$ are thus easy to obtain provided the distributions of Y_1 and W_1 are such that the corresponding integrals are easy to compute. In any case they can be computed approximately by replacing the integrals with the corresponding Riemann–Stieltjes sums.

6. Conclusions

We have considered the problem of minimizing the ruin probability in an insurance model that allows us to dynamically choose the level of reinsurance and investment in the financial market. It is a general innovative model that describes in a unifying way the timing of the events, that is, the arrivals of claims and the changes of the prices in the market. It is based on a continuous-time semi-Markov process model and it is believed that this model represents reality more faithfully than, say, classical diffusion-based models. It leads also to some advantages when estimating the parameters in the model in the sense that it allows one to separate the information coming from observing the frequency of the individual events and that of the duration between successive events. Our insurance model is also general enough to contain particular cases ranging from the classical risk process to models with reinsurance, investment in the financial market, and dividends. It could possibly be extended to also include the recent general risk model with reinsurance in Eisenberg and Schmidli (2011) where, to prevent a negative surplus, the insurer may inject additional capital. We also recall that our bounds were derived for the main purpose of allowing one to actually determine a reinsurance and investment policy that keeps the ruin probability at low levels.

We developed a specific methodology to obtain a policy that minimizes the exponential bound on the ruin probability in Theorem 1. It is based on a policy iteration procedure. Besides being of interest in itself, the bound minimizing policy that is obtained may also serve as a benchmark policy with respect to which other standard policies may then be evaluated.

Contrary to many asymptotic (in time and in the initial surplus) approaches (see, e.g. Gaier *et al.* (2003), Paulsen (1998), and Hult and Lindskog (2011)) we obtained our results for a fixed, but arbitrary finite horizon and any given positive initial surplus.

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