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A REMARK ON THE SEPARABLE QUOTIENT PROBLEM FOR TOPOLOGICAL GROUPS

SIDNEY A. MORRIS

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Abstract

The Banach–Mazur separable quotient problem asks whether every infinite-dimensional Banach space B has a quotient space that is an infinite-dimensional separable Banach space. The question has remained open for over 80 years, although an affirmative answer is known in special cases such as when B is reflexive or even a dual of a Banach space. Very recently, it has been shown to be true for dual-like spaces. An analogous problem for topological groups is: Does every infinite-dimensional (in the topological sense) connected (Hausdorff) topological group G have a quotient topological group that is infinite dimensional and metrisable? While this is known to be true if G is the underlying topological group of an infinite-dimensional Banach space, it is shown here to be false even if G is the underlying topological group of an infinite-dimensional locally convex space. Indeed, it is shown that the free topological vector space on any countably infinite k_ω-space is an infinite-dimensional toplogical vector space which does not have any quotient topological group that is infinite dimensional and metrisable. By contrast, the Graev free abelian topological group and the Graev free topological group on any infinite connected Tychonoff space, both of which are connected topological groups, are shown here to have the tubby torus $\mathbb{T}^\omega$, which is an infinite-dimensional metrisable group, as a quotient group.

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1. Introduction and notation

The Banach–Mazur separable quotient problem, which asks whether every infinite-dimensional Banach space B has a quotient space that is a separable infinite-dimensional Banach space, has remained unsolved for 85 years. An affirmative answer is known in many cases, including when B is a reflexive Banach space. As an extension of this, Spiros Argyros, Pandelis Dodos and Vassilis Kanellopoulos [1], in 2008, proved that if B is the Banach dual of any infinite-dimensional Banach space, then B has a separable infinite-dimensional quotient Banach space.

An analogous problem for topological groups is: *Does every connected (Hausdorff) topological group G have a quotient topological group that is infinite dimensional as a topological space and metrisable?* In [6], a positive answer to this question was given by S. Gabriyelyan and the author for the case where G is the underlying topological group of an infinite-dimensional Banach space or even the underlying topological group of a locally convex space that has an infinite-dimensional Fréchet space as a subspace. Indeed, G has the tubby torus group $\mathbb{T}^\omega$, which is infinite dimensional and metrisable, as a quotient group.

All topological spaces, topological groups and topological vector spaces considered here are assumed to Hausdorff.

Definition 1.1. The *free topological vector space* $\mathbb{V}(X)$ over a Tychonoff space X is a pair consisting of a topological vector space $\mathbb{V}(X)$ and a continuous mapping $i : X \to \mathbb{V}(X)$ such that every continuous mapping f from X to a topological vector space E gives rise to a unique continuous linear operator $\bar{f} : \mathbb{V}(X) \to E$ with $f = \bar{f} \circ i$.

Definition 1.2. The *free locally convex space* $L(X)$ over a Tychonoff space X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i : X \to L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$.

Definition 1.3. The *Graev free topological group* $F_G(X)$ over a Tychonoff space X with distinguished point $e \in X$ is a pair consisting of a topological group $F_G(X)$ and a continuous mapping $i : X \to F_G(X)$ such that $i(e)$ is the identity element of $F_G(X)$ with the property that every continuous mapping f from X to a topological group G with $f(e)$ equal to the identity element of G gives rise to a unique continuous homomorphism $\bar{f} : F_G(X) \to G$ with $f = \bar{f} \circ i$.

Definition 1.4. The *Graev free abelian topological group* $A_G(X)$ over a Tychonoff space X with distinguished point $e \in X$ is a pair consisting of an abelian topological group $A_G(X)$ and a continuous mapping $i : X \to A_G(X)$ such that $i(e)$ is the identity element of $A_G(X)$ with the property that every continuous mapping f from X to an abelian topological group G with $f(e)$ equal to the identity element of G gives rise to a unique continuous homomorphism $\bar{f} : A_G(X) \to G$ with $f = \bar{f} \circ i$.

A useful reference for k_ω-spaces is [3]. We mention that a compact Hausdorff space and a countably infinite discrete space is a k_ω-space, as are the topological group of all real numbers $\mathbb{R}$ and the compact circle group $\mathbb{T}$. If the topological group G is a k_ω-group, then every quotient group of G is also a k_ω-group. If G is also metrisable, then it is locally compact.

We state some well-known results from [2, 4, 5, 11]. For any Tychonoff space X, the free topological vector space on X exists and is unique up to topological vector space isomorphism. The same is true for the free locally convex space on X, the Graev free topological group on X and the Graev free abelian topological group on X. If X is a k_ω-space, then $\mathbb{V}(X)$ is a k_ω-space. If X is the discrete countably infinite space, then $\mathbb{V}(X)$, which is usually denoted by φ, is locally convex and, indeed, is the free locally convex space on X. If X is any other infinite k_ω-space, then $\mathbb{V}(X)$ is not locally convex, and so neither is $L(X)$.

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For a subset A of a vector space E and a natural number $n \in \mathbb{N}$, we denote by $\mathrm{sp}_n(A)$ the following subset of E:

$$\mathrm{sp}_n(A) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \in [-n, n],\ x_i \in A,\ \forall i = 1, \ldots, n\}.$$

We shall need the following theorem.

Theorem 1.5 [5, Theorem 3.1]. *Assume X is a k_ω-space and $X = \bigcup_{n\in\mathbb{N}} C_n$ is a k_ω-decomposition of X. Then $\mathbb{V}(X)$ is a k_ω-space and $\mathbb{V}(X) = \bigcup_{n\in\mathbb{N}} \mathrm{sp}_n(C_n)$ is a k_ω-decomposition of $\mathbb{V}(X)$.*

2. Results

In this section, we prove that if φ is the free locally convex space on a countably infinite discrete space, then φ does not have a quotient topological group that is metrisable and infinite dimensional as a topological space. (This extends a result in [6].) Indeed, if $\mathbb{V}(X)$ is the free topological vector space on any countably infinite k_ω-space, then the infinite-dimensional topological vector space $\mathbb{V}(X)$ does not have a quotient group that is metrisable and infinite dimensional as a topological space.

Theorem 2.1. *Let X be a countably infinite k_ω-space and let $\mathbb{V}(X)$ be the free topological vector space on X. Then the topological vector space $\mathbb{V}(X)$ is an infinite-dimensional (in the topological sense) connected topological group that does not have any infinite-dimensional (in the topological sense) metrisable quotient group. In particular, if X is the countably infinite discrete space, then $\mathbb{V}(X) = \varphi = L(X)$ and is a locally convex space.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n, \ldots\}$, G be any metrisable topological group and let ψ be a continuous open homomorphism of $\mathbb{V}(X)$ onto G. We are required to prove that G is a finite-dimensional topological group. By Theorem 1.5, $\mathbb{V}(X)$ is a k_ω-space and so its quotient group G is also a k_ω-space. As G is metrisable, it is a connected locally compact abelian group. Therefore, by [10, Theorem 26], G is topologically isomorphic to $\mathbb{R}^n \times K$, where K is a connected compact abelian group and n is a nonnegative integer.

Suppose G is infinite dimensional. Then K must be infinite dimensional. By [10, Theorem 34], the dual group $\widehat{K}$ is a discrete torsion-free abelian group with infinite rank. By the remark after Proposition 8.15 of [8], K has the torus $\mathbb{T}^m$ as a quotient group, where m is the rank of $\widehat{G}$. So G has the tubby torus $\mathbb{T}^\omega$ as a quotient group. Thus we have a continuous homomorphism θ of $\mathbb{V}(X)$ onto the tubby torus $\mathbb{T}^\omega$.

Clearly, $\mathbb{V}(X)$ is a countably infinite union of finite-dimensional subspaces $\mathbb{R}^n$ for $n \in \mathbb{N}$, where each such subspace is the span of the set $\{x_1, x_2, \ldots, x_n\}$. Then the restriction θ_n of θ to $\mathbb{R}^n$ induces a one-to-one continuous homomorphism δ_n from $\mathbb{R}^n/\ker(\theta_n)$ into $\mathbb{T}^\omega$, where $\ker(\theta_n)$ is the kernel of the homomorphism θ_n. By [7, Theorem 9.11], the group $\mathbb{R}^n/\ker(\theta_n)$ is topologically isomorphic to $\mathbb{R}^{a_n} \times \mathbb{T}^{b_n}$

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for some nonnegative integers a_n and b_n. So we have a one-to-one continuous homomorphism $\delta_n : \mathbb{R}^{a_n} \times \mathbb{T}^{b_n} \to \mathbb{T}^\omega$. It follows that

$$\mathbb{T}^\omega = \theta(\mathbb{V}(X)) = \bigcup_{n\geq 1} \Big(\bigcup_{p\in\mathbb{N}} \delta_n([-p, p]^{a_n} \times \mathbb{T}^{b_n}) \Big). \tag{2.1}$$

As δ_n is a homeomorphism on the compact finite-dimensional space $[-p, p]^{a_n} \times \mathbb{T}^{b_n}$, the equality (2.1) implies that $\theta(\mathbb{V}(X)) = \mathbb{T}^\omega$ is a countable-dimensional topological space. However, the metrisable group $\mathbb{T}^\omega$ is not a countable-dimensional topological space by [12, Corollary 3.13.6]. So we have a contradiction and our supposition that $\mathbb{V}(X)$ has an infinite-dimensional metrisable quotient group is false. □

We conclude with a contrasting result.

THEOREM 2.2. *Let $F_G(X)$ and $A_G(X)$ be the Graev free topological group and the Graev free abelian topological group, respectively, on an infinite connected compact Hausdorff space. Then the connected topological groups $F_G(X)$ and $A_G(X)$ have the tubby torus $\mathbb{T}^\omega$ as a quotient group.*

PROOF. As X is an infinite connected compact (Tychonoff) space, there exists a continuous mapping φ of X onto the closed (compact) unit interval $\mathbb{I} = [0, 1]$. Note that there exists a continuous mapping of $\mathbb{I}$ onto $\mathbb{I}^\omega$. Further, $\mathbb{I}^\omega$ obviously maps continuously onto the tubby torus $\mathbb{T}^\omega$. Hence there is an open continuous map of the compact space X onto the tubby torus $\mathbb{T}^\omega$. By [9, Lemma 2.1], the extension of a surjective continuous open mapping of the generating space X of a free abelian topological group $A_G(X)$ onto a topological group is an open continuous homomorphism. So we see that $A_G(X)$ has the tubby torus $\mathbb{T}^\omega$ as a quotient group. As $A_G(X)$ is a quotient group of $F_G(X)$, it immediately follows that $F_G(X)$ also has $\mathbb{T}^\omega$ as a quotient group. □

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SIDNEY A. MORRIS, Department of Mathematics and Statistics,
La Trobe University, Melbourne, Victoria, 3086, Australia
and
Center for Informatics and Applied Optimization,
Federation University Australia, PO Box 663,
Ballarat, Victoria, 3353, Australia
e-mail: morris.sidney@gmail.com

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