RANDOM WALKS IN A QUEUEING NETWORK ENVIRONMENT

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Abstract

We propose a class of models of random walks in a random environment where an exact solution can be given for a stationary distribution. The environment is cast in terms of a Jackson/Gordon–Newell network although alternative interpretations are possible. The main tool is the detailed balance equations. The difference compared to earlier works is that the position of the random walk influences the transition intensities of the network environment and vice versa, creating strong correlations. The form of the stationary distribution is closely related to the well-known product formula.

Keywords: Continuous-time Markov process; queueing network; Jackson network; simple exclusion; zero range; reversibility; stationary probability; product formula

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1. Introduction

In this paper we introduce exactly solvable reversible models of a random walk interacting with a random environment of a queueing-network type. The environment stems from a symmetric Jackson network or its closed version (a Gordon–Newell network); cf. [7], [10], [11]. The walking particles can be interpreted as distinguished customers (DCs). Depending on the site of the network that a DC is in, he/she may prefer to stay in the site if the queue size is large (or small) and in turn encourages more (or less) tasks to come to this site. Further development is where DCs interact with each other: here we consider the case of a symmetric simple exclusion; the zero-range model is also included. By analyzing the detailed balance equations (DBEs), the equilibrium distribution is derived, closely related to the product-form distribution; cf. [1], [17], [19]. Our approach can be considered as a further development of earlier papers [5], [22], where possible applications have been outlined in the area of communication networks.

Other areas of applications may cover problems of random trapping/localization and condensation; cf. [2], [6], [8], [21], and the references therein.

An immediate problem would be to identify a Lyapunov function assessing the speed of convergence to the stationary (equilibrium) distribution under subcriticality conditions; cf. [9], [16], and references therein.

A possible step would be to introduce similar models in a quasi-reversible setting [12]. In this direction, it would be interesting to treat a wider class of service rules for tasks and the DC. Namely, a processor-sharing discipline seems a natural choice, involving a branching formalism; see, e.g. [18].

Our models also show some potential in the direction of Markov processes with local interaction. An immediate goal can be to develop models in an infinite-volume configuration space [3], [4], [20].

2. Description of a basic model

2.1. A symmetric open Jackson network

As a model for an environment, we take a symmetric and homogeneous Jackson job-shop network; see [10], [11]. The model is defined by the following ingredients:

(i) a finite collection Λ of sites with a single-server system assigned to each $k \in \Lambda$;

(ii) two positive numbers: $\lambda > 0$, the intensity of an exogenous input flow to a given site, and $\mu > 0$, the intensity of the flow from a given site out of the network;

(iii) a nonnegative symmetric matrix of transmission intensities,

$$\boldsymbol{B} = (\beta_{ik}, \, i, k \in \Lambda), \qquad \beta_{ik} = \beta_{ki}, \, \beta_{ik} \geq 0, \, \beta_{ii} = 0. \tag{2.1}$$

The value λ gives the rate of independent Poisson processes of exogenous tasks arriving at sites in Λ and $\boldsymbol{\mu}_i = \mu + \boldsymbol{\beta}_i$ is the rate of servicing the queue at site $i \in \Lambda$, where $\boldsymbol{\beta}_i = \sum_{k \in \Lambda} \beta_{ik}$. After completing service at site i, a task leaves the network with probability $\mu/\boldsymbol{\mu}_i$ and jumps to site k with probability $\beta_{ik}/\boldsymbol{\mu}_i$. Condition $\beta_{ii} = 0$ is used for convenience. In contrast, the symmetry property $\beta_{ik} = \beta_{ki}$ in (2.1) is essential but rather restrictive and hopefully could be weakened in the future.

The above description gives rise to a continuous-time Markov process (MP) with states $\boldsymbol{n} = (n_i, \, i \in \Lambda) \in \mathbb{Z}_+^\Lambda$, where $n_i \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. The generator matrix $\boldsymbol{Q} = (Q(\boldsymbol{n}, \boldsymbol{n}'))$ of the process has nonzero entries corresponding to the following transitions:

(i) $Q(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}^i) = \lambda$ an exogenous arrival of a task at site i;

(ii) $Q(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}^i) = \mu \, \mathbf{1}_{\{n_i \geq 1\}}$ a task exits from site i out of the network;

(iii) $Q(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}^{i \to k}) = \beta_{ik} \, \mathbf{1}_{\{n_i \geq 1\}}$ a task jumps from site i to k.

Here, $\boldsymbol{e}^i = (e_l^i, \, l \in \Lambda) \in \mathbb{Z}_+^\Lambda$ has $e_l^i = \delta_{il}$, $\boldsymbol{e}^{i \to k}$ denotes the difference $\boldsymbol{e}^k - \boldsymbol{e}^i$, and $\mathbf{1}$ denotes the indicator.

Assuming the subcriticality condition $\lambda/\mu < 1$, the invariant measure π is the product of geometric distributions with parameter λ/μ. Here, the probability $\pi(\boldsymbol{n})$ of having n_i tasks at sites $i \in \Lambda$ is of the form

$$\pi(\boldsymbol{n}) = \left(1 - \frac{\lambda}{\mu}\right)^{\#\Lambda} \prod_{i \in \Lambda} \left(\frac{\lambda}{\mu}\right)^{n_i}, \qquad \boldsymbol{n} = (n_i) \in \mathbb{Z}_+^\Lambda. \tag{2.2}$$

In fact, assuming that matrix $\boldsymbol{B}$ is irreducible (i.e. $\boldsymbol{B}^t$ has strictly positive entries for some $t \in \mathbb{Z}_+$), the process is positive recurrent (PR).

Equation (2.2) follows from the DBEs for probabilities $\pi(\boldsymbol{n})$, $\boldsymbol{n} \in \mathbb{Z}_+^\Lambda$:

$$\pi(\boldsymbol{n})Q(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}^i) = \pi(\boldsymbol{n} + \boldsymbol{e}^i)Q(\boldsymbol{n} + \boldsymbol{e}^i, \boldsymbol{n}),$$

$$\pi(\boldsymbol{n})Q(\boldsymbol{n}, \boldsymbol{n} - \boldsymbol{e}^j) = \pi(\boldsymbol{n} - \boldsymbol{e}^j)Q(\boldsymbol{n} - \boldsymbol{e}^j, \boldsymbol{n}), \qquad n_j \geq q,$$

$$\pi(\boldsymbol{n})Q(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{e}^{k \to l}) = \pi(\boldsymbol{n} + \boldsymbol{e}^{k \to l})Q(\boldsymbol{n} + \boldsymbol{e}^{k \to l}, \boldsymbol{n}), \qquad n_k \geq 1,$$

which are easy to check. Note that the probabilities $\pi(\boldsymbol{n})$ in (2.2) do not refer to matrix $\boldsymbol{B}$.

Throughout this paper, Λ denotes a finite set, and sites of Λ are marked by i, j, k, l, p, q, r, s, and j'.

2.2. Random walk in a Jackson environment

We now give the description of the model with interaction. A new ingredient is the presence of a random particle (a distinguished customer (DC)) walking over set Λ. Parameters of the Jackson network are changed only for a site where the DC is located; we call it a *loaded* site and denote by j. A site $i \neq j$ (free of a DC) is called *unloaded*. In general, i, k, and l denote sites in Λ which may be loaded or unloaded.

In addition to λ, μ, and $\boldsymbol{B} = (\beta_{ik})$ (see items (b) and (c) in Section 2.1), we need more ingredients and rules. The queue of tasks at the loaded site j has an exogenous arrival intensity $e^\varphi \lambda$, where $\varphi \in \mathbb{R}$, while for the remaining sites the intensity remains λ. The exit flow intensity from all sites of the network equals μ as before. The intensities β_{ik} for $i \neq j \neq k$ (task jumps from one unloaded site to another) are as before (and satisfy (2.1)). Next, we introduce symmetric matrices $\boldsymbol{\theta} = (\theta_{ik}, i, k \in \Lambda)$ and $\boldsymbol{T} = (\tau_{ik}, i, k \in \Lambda)$, where

$$\theta_{ik} = \theta_{ki}, \qquad \theta_{ik} \geq 0, \qquad \theta_{ii} = 0; \qquad \tau_{ik} = \tau_{ki}, \qquad \tau_{ik} \geq 0, \qquad \tau_{ii} = 0. \qquad (2.3)$$

(Again, condition $\theta_{ii} = \tau_{ii} = 0$ is used for convenience.) For the loaded site j, the intensity of the task flow from j to $k \neq j$ equals θ_{jk}. For an unloaded site $k \neq j$, the intensity of the task flow from k to j equals $e^\varphi \theta_{kj}$. Finally, the intensity of a leap of the DC from a loaded site j where there are n_j tasks to a site $j' \neq j$ is taken to be $e^{-\varphi n_j} \tau_{jj'}$. As with intensities β_{ik}, symmetry equations $\theta_{ik} = \theta_{ki}$ and $\tau_{ik} = \tau_{ki}$ are essential (in modified forms, they reappear in the rest of the paper), and it would be interesting to replace these conditions with weaker ones.

Matrices $\boldsymbol{\theta}$ and $\boldsymbol{T}$ and parameter $\varphi \in \mathbb{R}$ are additional ingredients of the model. (In the future, we also refer to $\boldsymbol{B}, \boldsymbol{\theta}$, and $\boldsymbol{T}$ as arrays or collections of intensities.) The state of the emerging MP is a pair $(j, \boldsymbol{n})$, where $j \in \Lambda$ and $\boldsymbol{n} = (n_i, i \in \Lambda) \in \mathbb{Z}_+^\Lambda$. As was said, the entry j indicates the loaded site, and n_i, as before, gives the number of tasks at site $i \in \Lambda$. In accordance with the above description, the generator $\boldsymbol{R} = \{R[(j, \boldsymbol{n}); (j', \boldsymbol{n}')]\}$ has the following entries (here and below, zero-rate transitions are not shown):

(R2.1) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^k)] = \lambda e^{\varphi \delta_{jk}}$ exogenous arrival of a task;

(R2.2) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} - \boldsymbol{e}^k)] = \mu \mathbf{1}_{\{n_k \geq 1\}}$ exit of a task out of the network;

(R2.3) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \beta_{kl} \mathbf{1}_{\{n_k \geq 1\}}$ jump of a task for $k \neq j \neq l$;

(R2.4) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{j \to l})] = \theta_{jl} \mathbf{1}_{\{n_j \geq 1\}}$ a task jumps from the loaded site for $l \neq j$;

(R2.5) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{k \to j})] = \theta_{kj} e^\varphi \mathbf{1}_{\{n_k \geq 1\}}$ a task jumps to the loaded site for $k \neq j$;

(R2.6) $R[(j, \boldsymbol{n}); (j', \boldsymbol{n})] = e^{-\varphi n_j} \tau_{jj'}$ leap of a DC, from j to $j' \neq j$.

Pictorially, the DC is served or provides service affecting task arrivals (both exogenous and intrinsic) at the loaded site, after which it moves to another site. Conversely, the DC departure from the loaded site is influenced by the number of tasks accumulated in the queue. If $\varphi < 0$ then the presence of the DC suppresses task arrivals, whereas $\varphi > 0$ means the opposite. Likewise, the intensity of a DC leap increases with n_j for $\varphi < 0$ and decreases for $\varphi > 0$. Mnemonically, if the DC attracts/repells tasks to/from the site where it currently is then it does so for both exogenous arrivals and arrivals from other sites, increasing/reducing the corresponding 'nominal' rate by the same factor e^{φ}. At the same time, the DC would like to stay for longer/shorter at a given site, by reducing/increasing the rate of its leap by factor $e^{-n_j \varphi}$.

3. An exact solution for a single DC

3.1. The basic case

In this section we assume that the rate collections $\boldsymbol{B}, \boldsymbol{\theta}$, and $\boldsymbol{T}$ satisfy (2.1), (2.3), and are irreducible (i.e. matrices $\boldsymbol{B}^u$, $\boldsymbol{\theta}^u$, and $\boldsymbol{T}^u$ have strictly positive entries for some positive integer u).

Theorem 3.1. *Assume that the subcriticality condition holds true: λ/μ, $\lambda e^{\varphi}/\mu < 1$. Then the MP on $\Lambda \times \mathbb{Z}_+^{\Lambda}$ with generator $\boldsymbol{R}$ is positive recurrent and reversible (PRR). The stationary probability (SP) $\pi(j, \boldsymbol{n})$, $j \in \Lambda$, $\boldsymbol{n} = (n_i) \in \mathbb{Z}_+^{\Lambda}$, of locating the DC at site j and having n_i tasks in sites of Λ is of the form*

$$\pi(j, \boldsymbol{n}) = (\#\Lambda)^{-1}\left(1 - \frac{\lambda e^{\varphi}}{\mu}\right)\left(1 - \frac{\lambda}{\mu}\right)^{\#\Lambda - 1}\left(\frac{\lambda}{\mu}\right)^{\sum_{i \in \Lambda} n_i} e^{\varphi n_j}. \qquad (3.1)$$

Proof. Probabilities $\pi(j, \boldsymbol{n})$ from (3.1) satisfy the following DBEs:

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^k)] = \pi(j, \boldsymbol{n} + \boldsymbol{e}^k)R[(j, \boldsymbol{n} + \boldsymbol{e}^k); (j, \boldsymbol{n})], \qquad k \neq j, \quad (3.2a)$$

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^j)] = \pi(j, \boldsymbol{n} + \boldsymbol{e}^j)R[(j, \boldsymbol{n} + \boldsymbol{e}^j); (j, \boldsymbol{n})], \qquad (3.2b)$$

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \pi(j, \boldsymbol{n} + \boldsymbol{e}^{k \to l})R[(j, \boldsymbol{n} + \boldsymbol{e}^{k \to l}); (j, \boldsymbol{n})],$$
$$k \neq j \neq l \neq k, \, n_k > 0, \quad (3.2c)$$

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{j \to k})] = \pi(j, \boldsymbol{n} + \boldsymbol{e}^{j \to k})R[(j, \boldsymbol{n} + \boldsymbol{e}^{j \to k}); (j, \boldsymbol{n})],$$
$$j \neq k, \, n_j > 0, \quad (3.2d)$$

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j', \boldsymbol{n})] = \pi(j', \boldsymbol{n})R[(j', \boldsymbol{n}); (j, \boldsymbol{n})], \qquad j \neq j'. \quad (3.2e)$$

In fact, substituting (R2.1)–(R2.6), omitting constant factors, and canceling the common term $(\lambda/\mu)^{\sum_i n_i}$ yields the identities

$$e^{\varphi n_j}\lambda = \left(\frac{\lambda}{\mu}\right)e^{\varphi n_j}\mu, \quad k \neq j, \qquad e^{\varphi n_j}\lambda e^{\varphi} = \left(\frac{\lambda}{\mu}\right)e^{\varphi(n_j+1)}\mu,$$

$$e^{\varphi n_j}\beta_{kl} = e^{\varphi n_j}\beta_{lk}, \qquad k \neq j \neq l \neq k, \, n_k > 0,$$

$$e^{\varphi n_j}\theta_{jk} = e^{\varphi(n_j-1)}\theta_{kj}e^{\varphi}, \quad j \neq k, \, n_j > 0, \qquad e^{\varphi n_j}e^{-\varphi n_j}\tau_{jj'} = e^{\varphi n_{j'}}e^{-\varphi n_{j'}}\tau_{j'j}, \quad j \neq j'.$$

Finally, under the irreducibility assumption, the process is PR. $\qquad \square$

Note that the SP distribution π in (3.1) does not involve $\boldsymbol{B}$, $\boldsymbol{\theta}$, or $\boldsymbol{T}$. The same pattern will be observed in the generalizations of the basic model below.

3.2. A direct generalization

In this section the rates $R[(j, \boldsymbol{n}), (j', \boldsymbol{n}')]$ are as follows (cf. (R2.1)–(R2.6)):

(R3.1) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^k)] = \lambda_k(n_k; j)$, $k \neq j$;

(R3.2) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^j)] = \lambda_j(n_j; j)\gamma_j(n_j)$;

(R3.3) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} - \boldsymbol{e}^k)] = \mu_k(n_k, k)\,\mathbf{1}_{\{n_k \geq 1\}}$;

(R3.4) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \beta_{kl}(n_k, n_l; j)\,\mathbf{1}_{\{n_k \geq 1\}}$, $k \neq j \neq l \neq k$;

(R3.5) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{j \to l})] = \theta_{jl}(n_j, n_l)\,\mathbf{1}_{\{n_j \geq 1\}}$, $l \neq j$;

(R3.6) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + \boldsymbol{e}^{k \to j})] = \theta_{kj}(n_k, n_j)\gamma_j(n_j)\,\mathbf{1}_{\{n_k \geq 1\}}$, $k \neq j$;

(R3.7) $R[(j, \boldsymbol{n}); (j', \boldsymbol{n})] = [\boldsymbol{\gamma}_j(n_j)]^{-1}\tau_{jj'}(\boldsymbol{n})$, $j \neq j'$.

Let us comment on the form of these rates. As one can see, we take into account the number of tasks in related sites and the position of the DC. To start with, we deal in (R3.1)–(R3.7) with values $\lambda_i(n; j)$ and $\mu_i(n; j)$, $n \in \mathbb{Z}_+$, $i, j \in \Lambda$. For $i \neq j$, values $\lambda_i(n; j)$, $\mu_i(n; j)$ give the intensities of exogenous arrival and exit of tasks, depending on i, the site location, $n(= n_i)$, the current number of tasks at the site, and on j, the current loaded site. For $i = j$, $\mu_j(n; j)$ yields an exit intensity from a loaded site while $\lambda_j(n; j)$ represents a nominal arrival intensity which will be modified via a gauge function $\gamma_j(n)$. Examples are

$$\lambda_i(n; j) = \lambda^{\mathrm{U}}\,\mathbf{1}_{\{n < C\}}, \qquad \mu_i(n, j) = \mu^{\mathrm{U}}\min[n, K], \quad j \neq i,$$
$$\lambda_j(n; j) = \lambda^{\mathrm{L}}\,\mathbf{1}_{\{n < C\}}, \qquad \mu_j(n; j) = \mu^{\mathrm{L}}\min[n, K],$$

where λ^{U}, λ^{L}, μ^{U}, μ^{L}, C, and K are positive constants (which can be made by varying the site) with 'U' and 'L' denoting unloaded and loaded. It means that the arrival at a given (unloaded) site is blocked if the number of tasks reaches C, and the pre-exit service is done by a K-server device, with intensities depending on the site.

For simplicity, we assume that

$$\sup[\lambda_i(n; j)\colon n \geq 0, i; j \in \Lambda] < +\infty, \qquad \inf[\mu_i(n; j)\colon n \geq 1, i; j \in \Lambda] > 0,$$

and that $\lambda_i(n; j) > 0$ if $\lambda_i(n + 1; j) > 0$. Throughout, we also set

$$\boldsymbol{\lambda}_i(0; j) = \boldsymbol{\mu}_i(0; j) = 1, \tag{3.3}$$

$$\boldsymbol{\lambda}_i(n; j) = \prod_{0 \leq m < n} \lambda_i(m; j), \qquad \boldsymbol{\mu}_i(n; j) = \prod_{1 \leq m \leq n} \mu_i(m; j), \quad n > 0. \tag{3.4}$$

We also work with values $\gamma_i(n) \geq 0$, $i \in \Lambda$, $n \in \mathbb{Z}_+$, assuming that $\gamma_i(0) = 1$ and $\gamma_i(n) > 0$ if $\gamma_i(n + 1) > 0$. These values are used to modify intensities of task arrival and jump at loaded site j. Namely,

$$\gamma_j(0) = 1, \qquad \gamma_j(n) = \frac{\mathrm{e}^{\varphi(j)}}{n} \quad \text{or} \quad \gamma_j(n) = n\mathrm{e}^{\varphi(j)}, \qquad n \geq 1,$$

where $\varphi(j)$ is a given real parameter depending upon j. Next, we let, again throughout,

$$\gamma_i(0) = 1, \qquad \gamma_i(n) = \prod_{0 \le m < n} \gamma_i(m), \quad n \ge 1. \tag{3.5}$$

Furthermore, the intensities β_{ik} depend on j and n_i, n_k, and so we can write $\boldsymbol{B}(\boldsymbol{n}; j) = (\beta_{ik}(n_i, n_k; j))$. Consequently, we modify the symmetry assumptions in (2.1). For $i \ne k$,

$$\frac{\lambda_i(n_i - 1; j)}{\mu_i(n_i; j)} \beta_{ik}(n_i, n_k; j) = \frac{\lambda_k(n_k; j)}{\mu_k(n_k + 1; j)} \beta_{ki}(n_i + 1, n_k - 1; j) \tag{3.6}$$

and set $\beta_{ii}(n_i, n_i; j) = 0$. Similar conditions are imposed on array $\boldsymbol{\theta}(\boldsymbol{n}; j) = (\theta_{ik}(n_i, n_k; j))$. For $i \ne k$,

$$\frac{\lambda_i(n_i - 1; j)}{\mu_i(n_i; j)} \theta_{ik}(n_i, n_k; j) = \frac{\lambda_k(n_k; j)}{\mu_k(n_k + 1; j)} \theta_{ki}(n_k + 1, n_i - 1; j) \tag{3.7}$$

and $\theta_{ii}(n_i, n_i; j) = 0$.

Finally, intensities τ_{ik} depend upon $\boldsymbol{n}$ yielding an array $\boldsymbol{T}(\boldsymbol{n}) = (\tau_{ik}(\boldsymbol{n}), i, k \in \Lambda)$. In this section we assume that $\tau_{ii}(\boldsymbol{n}) = 0$ and for $i \ne k$ and $\boldsymbol{n} = (n_q, q \in \Lambda) \in \mathbb{Z}_+^\Lambda$,

$$\tau_{ik}(\boldsymbol{n}) \prod_{q \in \Lambda} \frac{\lambda_q(n_q; i)}{\mu_q(n_q; i)} = \tau_{ki}(\boldsymbol{n}) \prod_{q \in \Lambda} \frac{\lambda_q(n_q; k)}{\mu_q(n_q; k)}. \tag{3.8}$$

We continue referring to (3.6)–(3.8) as symmetry conditions; one of our primary future tasks should be their replacement with less restrictive assumptions.

In Theorem 3.2 we assume that conditions (3.6)–(3.8) hold (these conditions will be recast in Section 4 in a more general situation) and suppose that arrays $\boldsymbol{B}(\boldsymbol{n}; j)$, $\boldsymbol{\theta}(\boldsymbol{n}; j)$, and $\boldsymbol{T}(\boldsymbol{n})$ have off-diagonal entries greater than 0. (We keep referring to this property as irreducibility.) The subcriticality condition for the unloaded and loaded site reads

$$U_{ij} := \sum_{n \in \mathbb{Z}_+} \frac{\lambda_i(n; j)}{\mu_i(n; j)} < +\infty \quad \text{for all } i, j \in \Lambda, \ i \ne j, \tag{3.9}$$

$$L_j := \sum_{n \in \mathbb{Z}_+} \frac{\lambda_j(n; j) \gamma_j(n)}{\mu_j(n; j)} < \infty \quad \text{for all } j \in \Lambda. \tag{3.10}$$

Theorem 3.2. *Under (3.9) and (3.10), the MP on $\Lambda \times \mathbb{Z}_+^\Lambda$ with generator $\boldsymbol{R} = (R[(j, \boldsymbol{n}); (j', \boldsymbol{n}')])$ as in (R3.1)–(R3.7) is PRR. The SP $\pi(j, \boldsymbol{n})$, of having the DC at site j and n_q tasks at $q \in \Lambda$, is of the form*

$$\pi(j, \boldsymbol{n}) = \frac{1}{\Xi_\Lambda} \prod_{q \in \Lambda} \frac{\lambda_q(n_q; j)}{\mu_q(n_q; j)} \gamma_j(n_j), \qquad j \in \Lambda, \ \boldsymbol{n} = (n_q) \in \mathbb{Z}_+^\Lambda \tag{3.11}$$

with $\Xi_\Lambda = \sum_{j \in \Lambda} L_j \prod_{i \in \Lambda \setminus \{j\}} U_{ij}$. Here, Ξ_Λ is the partition function of the model.

Proof. Probabilities $\pi(j, \boldsymbol{n})$ from (3.11) and rates $R[(j, \boldsymbol{n}), (j', \boldsymbol{n}')]$ from (R3.1)–(R3.7) satisfy the DBEs (3.2a)–(3.2e). In fact, after omitting the factor $1/\Xi_\Lambda$ and canceling common

terms in $\prod_{i \in \Lambda}(\lambda_i(n_i; j)/\boldsymbol{\mu}_i(n_i; j))$, the DBEs are again verified with the help of symmetry conditions. For $j', j, k, l \in \Lambda$, with $k \neq j \neq l \neq k$, $j \neq j'$,

$$\boldsymbol{\gamma}_j(n_j)\lambda_k(n_k; j) = \frac{\lambda_k(n_k; j)\boldsymbol{\gamma}_j(n_j)}{\mu_k(n_k + 1; j)}\mu_k(n_k + 1; j),$$

$$\boldsymbol{\gamma}_j(n_j)\lambda_j(n_j; j)\gamma_j(n_j) = \frac{\lambda_j(n_j; j)\boldsymbol{\gamma}_j(n_j + 1)}{\mu_j(n_j + 1; j)}\mu_j(n_j + 1; j),$$

$$\frac{\lambda_k(n_k - 1; j)}{\mu_k(n_k; j)}\boldsymbol{\gamma}_j(n_j)\beta_{kl}(n_k, n_l; j) = \frac{\lambda_l(n_l; j)\boldsymbol{\gamma}_j(n_j)}{\mu_l(n_l + 1; j)}\beta_{lk}(n_l + 1, n_k - 1; j), \qquad n_k \geq 1,$$

$$\frac{\lambda_j(n_j - 1; j)\boldsymbol{\gamma}_j(n_j)}{\mu_j(n_j; j)}\theta_{jk}(n_j, n_k; j) = \frac{\lambda_k(n_k; j)\boldsymbol{\gamma}_j(n_j - 1)}{\mu_k(n_k + 1; j)}\theta_{kj}(n_k + 1, n_j - 1; j)$$
$$\times \gamma_j(n_j - 1), \qquad n_j \geq 1,$$

$$\boldsymbol{\gamma}_j(n_j)\boldsymbol{\gamma}_j(n_j)^{-1}\tau_{jj'}(\boldsymbol{n}) = \boldsymbol{\gamma}_{j'}(n_{j'})\boldsymbol{\gamma}_{j'}(n_{j'})^{-1}\tau_{j'j}(\boldsymbol{n}).$$

As before, the process is PR under the irreducibility assumption. $\square$

3.3. A closed-network version

This version arises when the number of tasks in the network remain fixed. Correspondingly, we drop the three first rates in (R3.1)–(R3.7), i.e.

(R3.4) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + e^{k \to l})] = \beta_{kl}(n_k, n_l; j)\, \mathbf{1}_{\{n_k \geq 1\}}, \; k \neq j \neq l,$

(R3.5) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + e^{j \to l})] = \theta_{jl}(n_j, n_l)\, \mathbf{1}_{\{n_j \geq 1\}},$

(R3.6) $R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + e^{k \to j})] = \theta_{kj}(n_k, n_j)\gamma_j(n_j)\, \mathbf{1}_{\{n_k \geq 1\}},$

(R3.7) $R[(j, \boldsymbol{n}); (j', \boldsymbol{n})] = [\boldsymbol{\gamma}_j(n_j)]^{-1}\tau_{jj'}(\boldsymbol{n}), \; j \neq j'.$

In Theorem 3.3 we assume a modification of condition (3.8):

$$\tau_{jj'}(\boldsymbol{n}) = \tau_{j'j}(\boldsymbol{n}). \tag{3.12}$$

Theorem 3.3. *Fix N, the number of tasks in the network. Given* $\boldsymbol{n} \in \mathbb{Z}_+^\Lambda$, *set* $|\boldsymbol{n}| = \sum_{s \in \Lambda} n_s$. *The MP on* $\{(j, \boldsymbol{n}) \in \Lambda \times \mathbb{Z}_+^\Lambda : |\boldsymbol{n}| = N\}$ *with generator* $\boldsymbol{R} = (R[(j, \boldsymbol{n}); (j', \boldsymbol{n}')])$ *as in (R3.4), (R3.5), (R3.6), and (R3.7) is PRR. The SPs* $\pi(j, \boldsymbol{n})$ *take the form*

$$\pi(j, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{n}| = N\}}}{\Xi_{N,\Lambda}}\, \boldsymbol{\gamma}_j(n_j),$$

where $\Xi_{N,\Lambda} = \sum_{\boldsymbol{n} = (n_s) \in \mathbb{Z}_+^\Lambda : |\boldsymbol{n}| = N} \sum_{l \in \Lambda} \boldsymbol{\gamma}_l(n_l).$

Proof. Again, we use the DBEs verified with the help of the corresponding symmetry conditions

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + e^{i \to k})] = \pi(j, \boldsymbol{n} + e^{i \to k})R[(j, \boldsymbol{n} + e^{i \to k}); (j, \boldsymbol{n})],$$
$$i \neq j \neq k, \, n_i > 0,$$

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j, \boldsymbol{n} + e^{j \to k})] = \pi(j, \boldsymbol{n} + e^{j \to k})R[(j, \boldsymbol{n} + e^{j \to k}); (j, \boldsymbol{n})],$$
$$j \neq k, \, n_j > 0,$$

$$\pi(j, \boldsymbol{n})R[(j, \boldsymbol{n}); (j', \boldsymbol{n})] = \pi(j', \boldsymbol{n})R[(j', \boldsymbol{n}); (j, \boldsymbol{n})], \qquad j \neq j'. \qquad \square$$

4. Simple exclusion in a Jackson-type environment

4.1. A closed–open network

The simple exclusion model was introduced in [20] (where the corresponding term was coined). The model was extensively studied thereafter: see [13]–[15]. Here, the state of the MP is a pair $(\boldsymbol{y}, \boldsymbol{n})$, where $\boldsymbol{y} = (y_s, s \in \Lambda) \in \{0, 1\}^{\Lambda}$ and $\boldsymbol{n} = (n_i, i \in \Lambda) \in \mathbb{Z}_+^{\Lambda}$. Let us set $|\boldsymbol{y}| = \sum_{s \in \Lambda} y_s$. We also write $j \in \boldsymbol{y}$ when $y_j = 1$, and $j \notin \boldsymbol{y}$ when $y_j = 0$. In the case of a closed–open network, the sum $M = |\boldsymbol{y}|$ remains constant. The rates (now denoted by $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ are specified as

(R4.1) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^p)] = \lambda_p(n_p; \boldsymbol{y})[\gamma_p(n_p)]^{y_p}$;

(R4.2) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} - \boldsymbol{e}^p)] = \mu_p(n_p; \boldsymbol{y}) \mathbf{1}_{\{n_p \geq 1\}}$;

(R4.3) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \beta_{kl}(n_k, n_l; \boldsymbol{y})$;

(R4.4) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{i \to j})] = \epsilon_{ij}(n_i, n_j; \boldsymbol{y})$;

(R4.5) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{i \to l})] = \theta_{il}(n_i, n_l; \boldsymbol{y})$;

(R4.6) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to j})] = \theta_{kj}(n_k, n_j; \boldsymbol{y})\gamma_j(n_j)$;

(R4.7) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n})] = [\gamma_j(n_j)]^{-1} \tau_{jj'}(\boldsymbol{n}; \boldsymbol{y})$;

assuming

(i) $i \neq j, i, j \in \boldsymbol{y}, k \neq l, k, l, j' \notin \boldsymbol{y}$, and

(ii) $n_i, n_k \geq 1$.

Here, we deal with intensities $\lambda_{\bullet}(\cdot; \boldsymbol{y})$ and $\mu_{\bullet}(\cdot; \boldsymbol{y})$ depending on $\boldsymbol{y}$. Such a generalization is extended to arrays $\boldsymbol{B}(\boldsymbol{n}; \boldsymbol{y}) = (\beta_{kl}(n_k, n_l; \boldsymbol{y}))$, $\boldsymbol{\theta}(\boldsymbol{n}; \boldsymbol{y}) = (\theta_{kl}(n_k, n_l; \boldsymbol{y}))$, and $\boldsymbol{T}(\boldsymbol{n}, \boldsymbol{y}) = (\tau_{jj'}(\boldsymbol{n}; \boldsymbol{y}))$. We assume symmetry conditions similar to (3.6)–(3.8). For $j \in \boldsymbol{y}$ and $k \neq l, k, l, j' \notin \boldsymbol{y}$,

$$\frac{\lambda_k(n_k - 1; \boldsymbol{y})}{\mu_k(n_k; \boldsymbol{y})} \beta_{kl}(n_k, n_l; \boldsymbol{y}) = \frac{\lambda_l(n_l; \boldsymbol{y})}{\mu_l(n_l + 1; \boldsymbol{y})} \beta_{lk}(n_l + 1, n_k - 1; \boldsymbol{y}), \qquad n_k \geq 1, \quad (4.1)$$

$$\frac{\lambda_j(n_j - 1; \boldsymbol{y})}{\mu_j(n_j; \boldsymbol{y})} \theta_{jk}(n_j, n_k; \boldsymbol{y}) = \frac{\lambda_k(n_k; \boldsymbol{y})}{\mu_k(n_k + 1; \boldsymbol{y})} \theta_{kj}(n_k + 1, n_j - 1; \boldsymbol{y}), \qquad n_j \geq 1, \quad (4.2)$$

and (see (3.3) and (3.4))

$$\tau_{jj'}(\boldsymbol{n}; \boldsymbol{y}) \prod_{q \in \Lambda} \frac{\lambda_q(n_q; \boldsymbol{y})}{\mu_q(n_q; \boldsymbol{y})} = \tau_{j'j}(\boldsymbol{n}; \boldsymbol{y} + \boldsymbol{e}^{j \to j'}) \prod_{q \in \Lambda} \frac{\lambda_q(n_q; \boldsymbol{y} + \boldsymbol{e}^{j \to j'})}{\mu_q(n_q; \boldsymbol{y} + \boldsymbol{e}^{j \to j'})}. \qquad (4.3)$$

We also have a new collection of jump rates $\boldsymbol{E}(\boldsymbol{n}; \boldsymbol{y}) = (\epsilon_{ij}(n_i, n_j; \boldsymbol{y}))$ satisfying the following symmetry property. For $i \neq j, i, j \in \boldsymbol{y}$, and $n_i \geq 1$,

$$\frac{\lambda_i(n_i - 1; \boldsymbol{y})\gamma_i(n_i - 1)}{\mu_i(n_i; \boldsymbol{y})} \epsilon_{ij}(n_i, n_j; \boldsymbol{y}) = \frac{\lambda_j(n_j; \boldsymbol{y})\gamma_j(n_j)}{\mu_j(n_j + 1; \boldsymbol{y})} \epsilon_{ji}(n_j + 1, n_i - 1; \boldsymbol{y}). \quad (4.4)$$

Until the end of Section 4 we work with irreducible collections $\boldsymbol{B}(\boldsymbol{n}; \boldsymbol{y}), \boldsymbol{\theta}(\boldsymbol{n}; \boldsymbol{y}), \boldsymbol{E}(\boldsymbol{n}; \boldsymbol{y})$, and $\boldsymbol{T}(\boldsymbol{n}; \boldsymbol{y})$. Furthermore, assumptions (4.1)–(4.4) are adopted in Sections 4.1 and 4.2.

The interpretation is that we have a 1-0 configuration $\boldsymbol{y}$ of loaded sites occupied by DCs, with a total number of DCs M; each of them influences task arrivals and task jumps as indicated, independently for different sites. In addition, each DC can jump from a loaded to an unloaded site, again independently.

With $\boldsymbol{\gamma}_j(n)$ as in (3.5), the SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ can be expressed as

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{y}|=M\}}}{\Xi_{\Lambda,M}} \prod_{q \in \Lambda} \frac{\lambda_q(n_q; \boldsymbol{y})}{\mu_q(n_q; \boldsymbol{y})} \prod_{j \in \Lambda:\, y_j=1} \boldsymbol{\gamma}_j(n_j) \tag{4.5}$$

with partition function

$$\Xi_{\Lambda,M} = \sum_{\boldsymbol{y}=(y_s)\in\{0,1\}^\Lambda:\, |\boldsymbol{y}|=M} \prod_{r:\, y_r=1} L_r(\boldsymbol{y}) \prod_{l:\, y_l=0} U_l(\boldsymbol{y}), \qquad M \leq \#\Lambda,$$

and

$$U_l(\boldsymbol{y}) = \sum_{n \in \mathbb{Z}_+} \frac{\lambda_l(n; \boldsymbol{y})}{\mu_l(n; \boldsymbol{y})}, \qquad L_r(\boldsymbol{y}) = \sum_{n \in \mathbb{Z}_+} \frac{\lambda_r(n; \boldsymbol{y}) \boldsymbol{\gamma}_r(n)}{\mu_r(n; \boldsymbol{y})}. \tag{4.6}$$

The subcriticality conditions emerging from (4.6) are, for all $\boldsymbol{y} \in \{0, 1\}^\Lambda$ with $|\boldsymbol{y}| = M$,

$$U_l(\boldsymbol{y}) < +\infty, \qquad L_j(\boldsymbol{y}) < +\infty \quad \text{for all } l, j \in \Lambda \tag{4.7}$$

with $y_l = 0$ and $y_j = 1$.

Theorem 4.1. *The MP with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as in (R4.1)–(R4.7) on state space $\{(\boldsymbol{y}, \boldsymbol{n}) \in \{0, 1\}^\Lambda \times \mathbb{Z}_+^\Lambda : |\boldsymbol{y}| = M\}$ is PRR. The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ are given by (4.5).*

Proof. As before, the proof is based on DBEs. These are now as follows. For $j, j', k, l \in \Lambda$, with $k \neq l$,

$$\pi(\boldsymbol{y}, \boldsymbol{n})R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^k)] = \pi(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^k)R[(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^k); (\boldsymbol{y}, \boldsymbol{n})], \qquad k \notin \boldsymbol{y}, \tag{4.8a}$$

$$\pi(\boldsymbol{y}, \boldsymbol{n})R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^j)] = \pi(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^j)R[(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^j); (\boldsymbol{y}, \boldsymbol{n})], \qquad j \in \boldsymbol{y}, \tag{4.8b}$$

$$\pi(\boldsymbol{y}, \boldsymbol{n})R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \pi(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l})R[(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l}); (\boldsymbol{y}, \boldsymbol{n})],$$
$$y_k = y_l,\, n_k \geq 1, \tag{4.8c}$$

$$\pi(\boldsymbol{y}, \boldsymbol{n})R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to k})] = \pi(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to k})R[(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to k}); (\boldsymbol{y}, \boldsymbol{n})],$$
$$j \in \boldsymbol{y},\, k \notin \boldsymbol{y},\, n_j \geq 1, \tag{4.8d}$$

$$\pi(\boldsymbol{y}, \boldsymbol{n})R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n})] = \pi(\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n})R[(\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n})],$$
$$j \in \boldsymbol{y},\, j' \notin \boldsymbol{y}. \tag{4.8e}$$

The verification is still direct. For definiteness, we show the equation emerging in (4.8c), when $k, l \in \boldsymbol{y}$ (other cases have been effectively considered earlier). Upon omitting the factor $1/\Xi_\Lambda$ and canceling common terms in the products

$$\prod_{j \in \Lambda:\, y_j=1} \boldsymbol{\gamma}_j(n_j) \quad \text{and} \quad \prod_{q \in \Lambda} \frac{\lambda_q(n_q; j)}{\mu_q(n_q; j)},$$

this equation becomes (4.4). □

4.2. An open–open network

Now the rates (R4.1)–(R4.7) are complemented with

(R4.8) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n})] = \xi_k \boldsymbol{\gamma}_k(n_k) \, \mathbf{1}_{\{y_k=0\}}$, arrival of a DC at site k;

(R4.9) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} - \boldsymbol{e}^i, \boldsymbol{n})] = \eta_i \, \mathbf{1}_{\{y_i=1\}}$, exit of a DC from site i.

Here, $\xi_k > 0$ and $\eta_i > 0$ do not depend on $\boldsymbol{n}$ or $\boldsymbol{y}$. (This assumption can be generalized but a certain independence should be maintained.) Furthermore, in this section we assume that (4.1) holds. In addition, we assume here that, for all $\boldsymbol{n} \in \mathbb{Z}_+^\Lambda$, the product

$$V(\boldsymbol{n}) = \prod_{q \in \Lambda} \frac{\boldsymbol{\lambda}_q(n_q; \boldsymbol{y})}{\boldsymbol{\mu}_q(n_q; \boldsymbol{y})} \tag{4.9}$$

does not depend on configuration $\boldsymbol{y} \in \{0, 1\}^\Lambda$. Accordingly, (4.3) has to be replaced with

$$\tau_{jj'}(\boldsymbol{n}; \boldsymbol{y}) \frac{\xi_j}{\eta_j} = \tau_{j'j}(\boldsymbol{n}; \boldsymbol{y} + \boldsymbol{e}^{j \to j'}) \frac{\xi_{j'}}{\eta_{j'}}, \qquad j \in \boldsymbol{y}, \; j' \notin \boldsymbol{y}. \tag{4.10}$$

The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ then become

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{V(\boldsymbol{n})}{\Xi_\Lambda} \prod_{j \in \Lambda: \, y_j=1} \frac{\xi_j \boldsymbol{\gamma}_j(n_j)}{\eta_j}. \tag{4.11}$$

Here,

$$\Xi_\Lambda = \sum_{\boldsymbol{y} \in \{0,1\}^\Lambda} \sum_{\boldsymbol{n} \in \mathbb{Z}_+^\Lambda} V(\boldsymbol{n}) \prod_{j \in \Lambda: \, y_j=1} \frac{\xi_j \boldsymbol{\gamma}_j(n_j)}{\eta_j} = \sum_{\boldsymbol{y} \in \{0,1\}^\Lambda} \prod_{r: y_r=1} \frac{\xi_r L_r(\boldsymbol{y})}{\eta_r} \prod_{l \in \Lambda: \, y_l=0} U_l(\boldsymbol{y}),$$

$L_r(\boldsymbol{y})$, and $U_l(\boldsymbol{y})$ are as in (4.6). The subcriticality condition reads $\Xi_\Lambda < \infty$, or

$$U_l(\boldsymbol{y}) < +\infty, \qquad L_r(\boldsymbol{y}) < +\infty \quad \text{for all } \boldsymbol{y} \in \{0, 1\}^\Lambda \tag{4.12}$$

and $l, r \in \Lambda$ with $y_l = 0$ and $y_r = 1$, and can be treated as a slight modification of (4.7).

Theorem 4.2. *Under (4.12), the MP on $\{0, 1\}^\Lambda \times \mathbb{Z}_+^\Lambda$ with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as in (R4.1)–(R4.9), is PRR. The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ are given by (4.11).*

Proof. As before, we check the DBEs (4.8a)–(4.8e) completed with

$$\pi(\boldsymbol{y}, \boldsymbol{n}) R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n})] = \pi(\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n}) R[(\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n})], \qquad k \notin \boldsymbol{y}. \tag{4.13}$$

In view of (4.9), the latter holds. $\qquad\square$

4.3. A closed–closed network

Here, we keep $|\boldsymbol{y}|$ and $|\boldsymbol{n}|$ fixed: $|\boldsymbol{y}| = M$, $|\boldsymbol{n}| = N$. The rates are as in (R4.3)–(R4.7). The following conditions are assumed. For $i \neq j$, $i, j \in \boldsymbol{y}$ and $k \neq l$, $k, l, j' \notin \boldsymbol{y}$,

$$\beta_{kl}(n_k, n_l; \boldsymbol{y}) = \beta_{lk}(n_l + 1, n_k - 1; \boldsymbol{y}), \tag{4.14}$$

$$\theta_{jk}(n_j, n_k; \boldsymbol{y}) = \theta_{kj}(n_k + 1, n_j - 1; \boldsymbol{y}), \qquad n_k, n_j \geq 1, \tag{4.15}$$

$$\epsilon_{ij}(n_i, n_j; \boldsymbol{y}) \gamma_i(n_i - 1) = \gamma_j(n_j) \epsilon_{ji}(n_j + 1, n_i - 1; \boldsymbol{y}), \qquad n_i \geq 1, \tag{4.16}$$

and

$$\tau_{jj'}(\boldsymbol{n}; \boldsymbol{y}) = \tau_{j'j}(\boldsymbol{n}; \boldsymbol{y} - \boldsymbol{e}^j + \boldsymbol{e}^{j'}), \tag{4.17}$$

which can be viewed as a modification of (4.1)–(4.4) and (3.12).

The SP distribution resembles (3.11):

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{y}|=M, |\boldsymbol{n}|=N\}}}{\Xi_{N,\Lambda,M}} \prod_{j \in \Lambda} [\boldsymbol{\gamma}_j(n_j)]^{y_j}, \tag{4.18}$$

$$\Xi_{N,\Lambda,M} = \sum_{\substack{\boldsymbol{y}=(y_s)\in\{0,1\}^\Lambda, \\ \boldsymbol{n}=(n_s)\in\mathbb{Z}_+^\Lambda : |\boldsymbol{y}|=M, |\boldsymbol{n}|=N}} \prod_{l \in \Lambda} [\boldsymbol{\gamma}_l(n_l)]^{y_l}. \tag{4.19}$$

Theorem 4.3. *Given integer $M, N \geq 1$, the MP on $\{(\boldsymbol{y}, \boldsymbol{n}) \in \{0,1\}^\Lambda \times \mathbb{Z}_+^\Lambda : |\boldsymbol{y}| = M, |\boldsymbol{n}| = N\}$ with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as specified in this subsection is PRR. The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ are given by (4.18) and (4.19).*

Proof. Again by means of suitable DBEs (4.8a)–(4.8e), the proof is complete with the help of (4.14)–(4.16). □

4.4. An open–closed network

In this version of the model, N, the number of tasks, is fixed, but the number of DCs varies due to arrivals and exits. A part of the transition rates $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')]$ come from (R4.1)–(R4.7):

(R4.10) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \beta_{kl}(n_k, n_l; \boldsymbol{y}) \mathbf{1}_{\{n_k \geq 1\}}, \ k \neq l, \ k, l \notin \boldsymbol{y}$;

(R4.11) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{i \to j})] = \epsilon_{ij}(n_i, n_j; \boldsymbol{y}) \mathbf{1}_{\{n_i \geq 1\}}, \ i \neq j, \ i, j \in \boldsymbol{y}$;

(R4.12) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to l})] = \theta_{jl}(n_j, n_l; \boldsymbol{y}) \mathbf{1}_{\{n_j \geq 1\}}, \ j \in \boldsymbol{y}, \ l \notin \boldsymbol{y}$;

(R4.13) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to j})] = \theta_{kj}(n_k, n_j; \boldsymbol{y}) \gamma_j(n_j) \mathbf{1}_{\{n_k \geq 1\}}, \ j \in \boldsymbol{y}, \ k \notin \boldsymbol{y}$;

(R4.14) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n})] = [\boldsymbol{\gamma}_j(n_j)]^{-1} \tau_{jj'}(\boldsymbol{n}; \boldsymbol{y}), \ j \in \boldsymbol{y}, \ j' \notin \boldsymbol{y}$.

In addition, we use the rates (R4.8) and (R4.9). The DBEs read, for $i, j, j', k, l \in \Lambda$ with $k \neq l$, $j \in \boldsymbol{y}$,

$$\pi(\boldsymbol{y}, \boldsymbol{n}) R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l})] = \pi(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l}) R[(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k \to l}); (\boldsymbol{y}, \boldsymbol{n})],$$
$$y_k = y_l, \ n_k \geq 1, \quad (4.20a)$$

$$\pi(\boldsymbol{y}, \boldsymbol{n}) R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to k})] = \pi(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to k}) R[(\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j \to k}); (\boldsymbol{y}, \boldsymbol{n})],$$
$$k \notin \boldsymbol{y}, \ n_j \geq 1, \quad (4.20b)$$

$$\pi(\boldsymbol{y}, \boldsymbol{n}) R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n})] = \pi(\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n}) R[(\boldsymbol{y} + \boldsymbol{e}^{j \to j'}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n})], \quad j' \notin \boldsymbol{y}, \tag{4.20c}$$

$$\pi(\boldsymbol{y}, \boldsymbol{n}) R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n})] = \pi(\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n}) R[(\boldsymbol{y} + \boldsymbol{e}^k, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n})], \quad k \notin \boldsymbol{y}. \tag{4.20d}$$

We now assume conditions (4.10) and (4.14)–(4.16). The SPs and subcriticality condition read as

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{n}|=N\}}}{\Xi_{N,\Lambda}} \prod_{q \in \Lambda} \left[\frac{\xi_q}{\eta_q} \boldsymbol{\gamma}_q(n_q)\right]^{y_q}, \tag{4.21}$$

$$\Xi_{N,\Lambda} = \sum_{\substack{\boldsymbol{y}=(y_s)\in\{0,1\}^\Lambda, \\ \boldsymbol{n}=(n_s)\in\mathbb{Z}_+^\Lambda: \, |\boldsymbol{n}|=N}} \prod_{q \in \Lambda} \left[\frac{\xi_q}{\eta_l} \boldsymbol{\gamma}_q(n_q)\right]^{y_q} < \infty. \tag{4.22}$$

Theorem 4.4. *Fix $N \in \mathbb{Z}_+$. If $\Xi_{N,\Lambda} < \infty$, the MP with generator $\boldsymbol{R} = (R[(\boldsymbol{y}; \boldsymbol{n}); (\boldsymbol{y}'; \boldsymbol{n}')])$ as above is PRR on state space $\{(\boldsymbol{y}, \boldsymbol{n}) \in \{0, 1\}^\Lambda \times \mathbb{Z}_+^\Lambda : |\boldsymbol{n}| = N\}$. The SPs are given by (4.21) and (4.22).*

Proof. The proof follows from using the DBEs, now from (4.20a)–(4.20d). $\qquad\square$

5. A zero-range system in a Jackson-type environment

A zero-range modification arises when we allow the DCs to accumulate in sites $i \in \Lambda$. Here, $\boldsymbol{y} = (y_s, s \in \Lambda) \in \mathbb{Z}_+^\Lambda$; we again set $|\boldsymbol{y}| = \sum_{s \in \Lambda} y_s$ and write $j \in \boldsymbol{y}$ when $y_j \geq 1$ and $l \notin \boldsymbol{y}$ when $y_l = 0$.

5.1. A closed–open network

In this section, $M := |\boldsymbol{y}|$ is a conserved quantity. The rates are similar to (R4.1)–(R4.7). For $i, j, k, l, p \in \Lambda$,

(R5.1) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^p)] = \lambda_p(n_p; \boldsymbol{y})[\gamma_p(n_p)]^{y_p}$;

(R5.2) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} - \boldsymbol{e}^p)] = \mu_p(n_p; \boldsymbol{y}) \mathbf{1}_{\{n_p \geq 1\}}$;

(R5.3) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k\to l})] = \beta_{kl}(n_k, n_l; \boldsymbol{y}) \mathbf{1}_{\{n_k \geq 1\}}, \, k, l \notin \boldsymbol{y}, \, k \neq l$;

(R5.4) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{i\to j})] = \epsilon_{ij}(n_i, n_j; \boldsymbol{y}) \mathbf{1}_{\{n_i \geq 1\}}, \, i, j \in \boldsymbol{y}, \, i \neq j$;

(R5.5) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{j\to l})] = \theta_{jl}(n_j, n_l; \boldsymbol{y}) \mathbf{1}_{\{n_j \geq 1\}}, \, j \in \boldsymbol{y}, \, l \notin \boldsymbol{y}$;

(R5.6) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + \boldsymbol{e}^{k\to j})] = \theta_{kj}(n_k, n_j; \boldsymbol{y})[\gamma_j(n_j)]^{y_j} \mathbf{1}_{\{n_k \geq 1\}}, \, j \in \boldsymbol{y}, \, k \notin \boldsymbol{y}$;

(R5.7) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^{j\to j'}, \boldsymbol{n})] = [\gamma_j(n_j)]^{-y_j} \tau_{jj'}(\boldsymbol{n}; \boldsymbol{y})[\gamma_{j'}(n_{j'})]^{-y_{j'}}, \, j \in \boldsymbol{y}, \, j' \neq j$.

Until the end of Section 5, we suppose that irreducibility of collections $\boldsymbol{B}(\boldsymbol{n}; \boldsymbol{y})$, $\boldsymbol{\theta}(\boldsymbol{n}; \boldsymbol{y})$, $\boldsymbol{E}(\boldsymbol{n}; \boldsymbol{y})$, and $\boldsymbol{T}(\boldsymbol{n}; \boldsymbol{y})$. In Sections 5.1 and 5.2, intensities $\epsilon_{ij}(n_i, n_j; \boldsymbol{y})$ are supposed to obey the following. For all $i \neq j, i, j \in \boldsymbol{y}$,

$$\frac{\lambda_i(n_i - 1; \boldsymbol{y})[\gamma_i(n_i - 1)]^{y_i}}{\mu_i(n_i; \boldsymbol{y})} \epsilon_{il}(n_i, n_j; \boldsymbol{y})$$
$$= \frac{\lambda_j(n_j; \boldsymbol{y})[\gamma_j(n_j)]^{y_j}}{\mu_j(n_j + 1; \boldsymbol{y})} \epsilon_{ji}(n_j + 1, n_i - 1; \boldsymbol{y}), \qquad n_i \geq 1, \tag{5.1}$$

which replaces (4.4). Conditions (4.1)–(4.3) remain in place in both Sections 5.1 and 5.2.

The present model gives rise to SPs,

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{y}|=M\}}}{\Xi_{\Lambda, M}} \prod_{p \in \Lambda} \frac{\boldsymbol{\lambda}_p(n_p; \boldsymbol{y})}{\boldsymbol{\mu}_p(n_p; \boldsymbol{y})} \prod_{q \in \Lambda} [\boldsymbol{\gamma}_q(n_q)]^{y_q}, \qquad (5.2)$$

$$\Xi_{\Lambda, M} = \sum_{\boldsymbol{y}=(y_s) \in \mathbb{Z}_+^{\Lambda}: \, |\boldsymbol{y}|=M} \prod_{r: \, y_r \geq 1} C_r(\boldsymbol{y}) \prod_{l: \, y_l=0} U_l(\boldsymbol{y}), \qquad (5.3)$$

and

$$U_l(\boldsymbol{y}) = \sum_{n \in \mathbb{Z}_+} \frac{\boldsymbol{\lambda}_l(n_l; \boldsymbol{y})}{\boldsymbol{\mu}_l(n; \boldsymbol{y})}, \qquad C_r(\boldsymbol{y}) = \sum_{n \geq 0} \frac{\boldsymbol{\lambda}_r(n; \boldsymbol{y})}{\boldsymbol{\mu}_r(n; \boldsymbol{y})} \boldsymbol{\gamma}_r(n)^{y_r}. \qquad (5.4)$$

The subcriticality conditions read as follows. For all $\boldsymbol{y} \in \mathbb{Z}_+^{\Lambda}$ with $|\boldsymbol{y}| = M$,

$$U_l(\boldsymbol{y}) < +\infty, \qquad C_r(\boldsymbol{y}) < +\infty \quad \text{for all } l, r \in \Lambda \qquad (5.5)$$

with $y_l = 0$ and $y_r \geq 1$.

Theorem 5.1. *Under conditions (5.5), the MP on $\{(\boldsymbol{y}, \boldsymbol{n}) \in \mathbb{Z}_+^{\Lambda} \times \mathbb{Z}_+^{\Lambda}: |\boldsymbol{y}| = M\}$ with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as in (R5.1)–(R5.7) is PRR. The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ are given by (5.2)–(5.4).*

Proof. The proof follows by still using the DBEs, now for rates (R5.1)–(R5.7). The DBEs are as in (4.8a)–(4.8e), and after cancellations are reduced to (4.1)–(4.3) and (5.1). □

5.2. An open–open network

In this model we allow both the tasks and DCs to arrive and depart. Correspondingly, the rates (R5.1)–(R5.7) are complemented in a manner similar to (R4.8) and (R4.9):

(R5.8) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + \boldsymbol{e}^p, \boldsymbol{n})] = \xi_p \boldsymbol{\gamma}_p(n_p)$;

(R5.9) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} - \boldsymbol{e}^p, \boldsymbol{n})] = \eta_p \mathbf{1}_{\{y_p \geq 1\}}$.

As in Section 4.2, we assume (4.1) and (4.2), and that (4.3) is replaced by (4.9). We also assume (4.10) holds. (In (4.9), $\boldsymbol{y} \in \{0, 1\}^{\Lambda}$ is replaced with $\boldsymbol{y} \in \mathbb{Z}_+^{\Lambda}$, and in (4.10) the restriction $j' \notin \boldsymbol{y}$ is removed.) As in Section 5.1, condition (5.1) is also in place.

Assuming $U_l(\boldsymbol{y})$ and $C_l(\boldsymbol{y})$ as in (5.4), the SPs and subcriticality condition now read as

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \Xi_{\Lambda}^{-1} \prod_{p \in \Lambda} \frac{\boldsymbol{\lambda}_p(n_p; \boldsymbol{y})}{\boldsymbol{\mu}_p(n_p; \boldsymbol{y})} \prod_{q \in \Lambda} \left[\frac{\xi_q}{\eta_q} \boldsymbol{\gamma}_q(n_q) \right]^{y_q}, \qquad (5.6)$$

$$\Xi_{\Lambda} = \sum_{\boldsymbol{y}=(y_s) \in \mathbb{Z}_+^{\Lambda}} \prod_{r: \, y_r \geq 1} C_r(\boldsymbol{y}) \prod_{l: \, y_l=0} U_l(\boldsymbol{y}) < \infty. \qquad (5.7)$$

Theorem 5.2. *If $\Xi_{\Lambda} < +\infty$, the MP on $\mathbb{Z}_+^{\Lambda} \times \mathbb{Z}_+^{\Lambda}$ with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as in (R5.1)–(R5.9) is PRR. The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ are given by (5.6) and (5.7).*

Proof. Using the DBEs again. The added equations (R5.8) and (R5.9) are treated similarly to (4.13). □

5.3. A closed–closed network

Let us now suppose that both $|\boldsymbol{y}|$ and $|\boldsymbol{n}|$ are fixed: $|\boldsymbol{y}| = M$ and $|\boldsymbol{n}| = N$. The rates are (R5.3)–(R5.7). In this section we assume conditions (4.14)–(4.16) and (4.17) hold, with specification $\boldsymbol{y} \in \{0, 1\}$ replaced by $\boldsymbol{y} \in \mathbb{Z}_+^\Lambda$ and condition $j' \notin \boldsymbol{y}$ removed. The only exception is that (4.16) is now replaced with

$$\epsilon_{ij}(n_i, n_j; \boldsymbol{y})[\gamma_i(n_i - 1)]^{y_i} = [\gamma_j(n_j)]^{y_j} \epsilon_{ji}(n_j + 1, n_i - 1; \boldsymbol{y}), \qquad i \neq j, \; i, j \in \boldsymbol{y}, \; n_i \geq 1. \tag{5.8}$$

The SP distribution mimics (4.18) and (4.19):

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{y}|=M, |\boldsymbol{n}|=N\}}}{\Xi_{N,\Lambda,M}} \prod_{j \in \Lambda} [\gamma_j(n_j)]^{y_j}, \tag{5.9}$$

$$\Xi_{N,\Lambda,M} = \sum_{\substack{\boldsymbol{y}=(y_s), \boldsymbol{n}=(n_s) \in \mathbb{Z}_+^\Lambda : \\ |\boldsymbol{y}|=M, |\boldsymbol{n}|=N}} \prod_{l \in \Lambda} [\gamma_l(n_l)]^{y_l}. \tag{5.10}$$

The above DBEs and symmetry conditions (including (5.8)) lead to Theorem 5.3.

Theorem 5.3. *The MP on state space $\{(\boldsymbol{y}, \boldsymbol{n}) \in \mathbb{Z}_+^\Lambda \times \mathbb{Z}_+^\Lambda : |\boldsymbol{y}| = M, |\boldsymbol{n}| = N\}$ with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as in (R5.3)–(R5.7) is PRR. The SPs $\pi(\boldsymbol{y}, \boldsymbol{n})$ are given by (5.9) and (5.10).*

5.4. An open–closed network

Here, as in Section 4.4, we only fix N. The rates follow (R4.8)–(R4.12) and consist of

(R5.10) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + e^{k \to l})] = \beta_{kl}(n_k, n_l; \boldsymbol{y}) \mathbf{1}_{\{n_k \geq 1\}}, \; k \neq l, \; k, l \notin \boldsymbol{y}$;

(R5.11) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + e^{i \to j})] = \epsilon_{ij}(n_i, n_j; \boldsymbol{y}) \mathbf{1}_{\{n_i \geq 1\}}, \; i \neq j, \; i, j \in \boldsymbol{y}$;

(R5.12) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + e^{j \to l})] = \theta_{jl}(n_j, n_l; \boldsymbol{y}) \mathbf{1}_{\{n_j \geq 1\}}, \; j \in \boldsymbol{y}, \; l \notin \boldsymbol{y}$;

(R5.13) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}, \boldsymbol{n} + e^{k \to j})] = \theta_{kj}(n_k, n_j; \boldsymbol{y})[\gamma_j(n_j)]^{y_j} \mathbf{1}_{\{n_k \geq 1\}}, \; j \in \boldsymbol{y}, \; k \notin \boldsymbol{y}$;

(R5.14) $R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y} + e^{j \to j'}, \boldsymbol{n})] = [\gamma_j(n_j)]^{-y_j} \tau_{jj'}(\boldsymbol{n}; \boldsymbol{y})[\gamma_{j'}(n_{j'})]^{-y_{j'}}, \; j \neq j', \; j \in \boldsymbol{y}$,

plus rates (R5.8) and (R5.9). As in Section 4.4, we now assume conditions (4.10) and (4.14)–(4.16) hold, modified like the above (including (5.9)). The SPs and subcriticality condition read as

$$\pi(\boldsymbol{y}, \boldsymbol{n}) = \frac{\mathbf{1}_{\{|\boldsymbol{n}|=N\}}}{\Xi_{N,\Lambda}} \prod_{l \in \Lambda} \left[\frac{\xi_l}{\eta_l} \gamma_l(n_l) \right]^{y_l}, \tag{5.11}$$

$$\Xi_{N,\Lambda} = \sum_{\boldsymbol{y}=(y_s), \boldsymbol{n}=(n_s) \in \mathbb{Z}_+^\Lambda : |\boldsymbol{n}|=N} \prod_{l \in \Lambda} \left[\frac{\xi_l}{\eta_l} \gamma_l(n_l) \right]^{y_l} < \infty. \tag{5.12}$$

The DBEs in this case yield the following theorem.

Theorem 5.4. *Fix $N \in \mathbb{Z}_+$ and consider the MP on $\{(\boldsymbol{y}, \boldsymbol{n}) \in \mathbb{Z}_+^\Lambda \times \mathbb{Z}_+^\Lambda : |\boldsymbol{n}| = N\}$ with generator $\boldsymbol{R} = (R[(\boldsymbol{y}, \boldsymbol{n}); (\boldsymbol{y}', \boldsymbol{n}')])$ as in (R5.10)–(R5.14). If $\Xi_{N,\Lambda} < \infty$, assuming (5.11), (5.12) it is PRR. The SPs are given by (5.11) and (5.12).*

Acknowledgements

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