Proceedings of the Royal Society of Edinburgh, **148A**, 1097–1113, *2018*
DOI:10.1017/S0308210517000324

Spectral flow and bifurcation for a class of strongly indefinite elliptic systems

Nils Waterstraat
School of Mathematics, Statistics and Actuarial Science,
University of Kent, Canterbury, Kent CT2 7NF, UK
(n.waterstraat@kent.ac.uk)

(MS received 13 December 2015; accepted 28 October 2016)

We consider bifurcation of solutions from a given trivial branch for a class of strongly indefinite elliptic systems via the spectral flow. Our main results establish bifurcation invariants that can be obtained from the coefficients of the systems without using explicit solutions of their linearizations at the given branch. Our constructions are based on a comparison principle for the spectral flow and a generalization of a bifurcation theorem due to Szulkin.

Keywords: variational bifurcation theory; spectral flow; elliptic systems

2010 *Mathematics subject classification:* Primary 35J57
Secondary 58J30; 35J61

1. Introduction

Let Ω be a domain in $\mathbb{R}^N$, for some $N \in \mathbb{N}$, that we assume to have a smooth boundary. Let $a, b, c \colon I \times \bar{\Omega} \to \mathbb{R}$ and $G \colon I \times \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ be C^2-functions, where $I := [0, 1]$ denotes – here and throughout the paper – the unit interval. We denote by G'_u and G'_v the partial derivatives of G with respect to the components in $\mathbb{R}^2$, respectively, we assume that $G'_u(\lambda, x, 0, 0) = G'_v(\lambda, x, 0, 0) = 0$ for all $(\lambda, x) \in I \times \Omega$ and we consider the systems of elliptic partial differential equations

$$\left.\begin{aligned} -\Delta u &= b_\lambda(x)u + c_\lambda(x)v + G'_v(\lambda, x, u, v) && \text{in } \Omega, \\ -\Delta v &= a_\lambda(x)u + b_\lambda(x)v + G'_u(\lambda, x, u, v) && \text{in } \Omega, \\ u &= v = 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

depending on the parameter $\lambda \in I$. Clearly, under the mentioned assumptions the constant function $(u, v) \equiv 0$ is a solution of (1.1) for all values of λ and the aim of this article is to investigate bifurcation from this trivial branch of solutions $I \times \{0\}$. Here, a bifurcation point of (1.1) is an instant $\lambda^* \in I$ for which there is a sequence $\{(\lambda_n, u_n, v_n)\}_{n\in\mathbb{N}}$ such that $(u_n, v_n) \neq 0$ is a weak solution of (1.1) for λ_n, $\lambda_n \to \lambda^*$ and $u_n, v_n \to 0$ in the Sobolev space $H^1_0(\Omega, \mathbb{R})$ for $n \to \infty$. Our methods are based on a bifurcation theorem for critical points of families of functionals due to Fitzpatrick *et al*. [5], which was recently improved by Pejsachowicz and the author in [11]. In order to explain this theorem briefly, let $f \colon I \times H \to \mathbb{R}$ be a family of C^2

functionals that are defined on a Hilbert space H and such that $0 \in H$ is a critical point of all $f_\lambda := f(\lambda, \cdot)\colon H \to \mathbb{R}$. If we represent the second derivatives $D_0^2 f_\lambda$ of f at the critical point 0 against the scalar product of H, then we obtain a path $L = \{L_\lambda\}_{\lambda \in I}$ of self-adjoint operators.

The *spectral flow* is an integer-valued homotopy invariant for paths of self-adjoint Fredholm operators, which has been used in global analysis for about 40 years. Its relevance for bifurcation of critical points for families of functionals was clarified in [5]: if the self-adjoint operators L_λ, which are induced by the Hessians of f at 0, are Fredholm, then a non-vanishing spectral flow is a sufficient condition for the existence of a bifurcation of critical points of f. Let us point out that if the operators L_λ have finite Morse indices, then the spectral flow of L is just the difference of the Morse indices of L_0 and L_1, and so the bifurcation theorem [5] is a classical assertion in variational bifurcation theory in this case. In contrast, it is often hard to compute the spectral flow of a given path of operators when the Morse indices are infinite (see, for example, [14]).

The aim of this article is to show that for the indefinite elliptic systems (1.1), where the Morse indices of the corresponding operators L_λ are indeed infinite, the spectral flow can be computed, or at least estimated, so that [5] can be used to derive bifurcation criteria. To the best of our knowledge, such easily computable bifurcation invariants that are induced by the spectral flow have not been obtained for partial differential equations before.

In the following section, we introduce a family of C^2 functionals $f\colon I \times E \to \mathbb{R}$ that is defined on the Sobolev space $E := H_0^1(\Omega, \mathbb{R}^2)$ and is such that the critical points of $f_\lambda := f(\lambda, \cdot)\colon E \to \mathbb{R}$ are precisely the weak solutions of (1.1). In particular, $0 \in E$ is a critical point of each f_λ and we can deduce the existence of a bifurcation from the zero-solution of (1.1) by considering bifurcation of critical points from 0 for the family of functionals f. We will state below conditions on the map G that ensure that the Hessians $D_0^2 f_\lambda$ of f_λ at $0 \in E$ exist, and that elements in the kernel of the representations L_λ of $D_0^2 f_\lambda$ on E are the solutions of the linearized equations

$$\left.\begin{aligned} -\Delta u &= b_\lambda(x)u + c_\lambda(x)v && \text{in } \Omega, \\ -\Delta v &= a_\lambda(x)u + b_\lambda(x)v && \text{in } \Omega, \\ u &= v = 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

Since the operators L_λ are readily seen to be Fredholm in this case, we can use the abstract bifurcation theorem [5], and consequently we will be concerned with the spectral flow of the corresponding path $L = \{L_\lambda\}_{\lambda \in I}$. In theorem 4.2, which we consider to be the main result of this paper, we estimate the spectral flow in terms of the coefficients of (1.2) at $\lambda = 0$ and $\lambda = 1$, which is enough to conclude that it does not vanish and so implies the existence of a bifurcation of solutions for the nonlinear equations (1.1).

Another objective of this paper is to consider the special case in which the maps a, b and c do not depend on $x \in \Omega$, i.e.

$$\left.\begin{aligned} -\Delta u &= b_\lambda u + c_\lambda v + G_v'(\lambda, x, u, v) && \text{in } \Omega, \\ -\Delta v &= a_\lambda u + b_\lambda v + G_u'(\lambda, x, u, v) && \text{in } \Omega, \\ u &= v = 0 && \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.3}$$

For these equations, we compute the spectral flow of the corresponding path of operators L exactly in terms of an integral index that can be constructed from the coefficients of the linearized equations

$$\left.\begin{aligned} -\Delta u &= b_\lambda u + c_\lambda v && \text{in } \Omega, \\ -\Delta v &= a_\lambda u + b_\lambda v && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

for $\lambda = 0$ and $\lambda = 1$. The idea of this index goes back to Li and Liu [9], who used a similar construction in their study of existence of periodic solutions of asymptotically quadratic Hamiltonian systems. Their index was later applied in bifurcation theory for periodic solutions of Hamiltonian systems, for example, by Szulkin in [17], and by Fitzpatrick *et al*. in [6] who in particular used it to compute the spectral flow for autonomous Hamiltonian systems. It has some interest in itself that we compute in our theorem 3.2 the spectral flow for (1.4) by way of an index that is very much reminiscent of Li and Liu's index from [9]. The adaption of Li and Liu's index for Hamiltonian systems to the elliptic systems (1.4) closely follows Szulkin's work [17, § 5] (see also [8, § 9]), who investigated the bifurcation problem for (1.3) in the special case in which a, b, c and G depend linearly on λ, i.e.

$$\left.\begin{aligned} -\Delta u &= \lambda(bu + cv + G'_v(x, u, v)) && \text{in } \Omega, \\ -\Delta v &= \lambda(au + bv + G'_u(x, u, v)) && \text{in } \Omega, \\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.5}$$

by using infinite-dimensional Morse theory for strongly indefinite functionals. We reobtain Szulkin's results in corollary 3.4 as a consequence of our theorem 3.2, and we will also assess our main theorem (theorem 4.2) for (1.5) below.

Let us finally point out that a more general index for self-adjoint operators was recently introduced by Wang and Liu in [18], which accordingly might be applicable to bifurcation theory along the lines of this paper. Moreover, Gołębiewska and Rybicki discussed in [7] equivariant decompositions of $H_0^1(\Omega)$ and obtained global bifurcation results for equations that are similar to (1.5).

2. Spectral flow and bifurcation for (1.1)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. In what follows we assume that:

(A1) $a, b, c \colon I \times \bar{\Omega} \to \mathbb{R}$ and $G \colon I \times \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ are C^2-functions;

(A2) G'_u and G'_v are bounded and

$$|G'_u(\lambda, x, u, v)| + |G'_v(\lambda, x, u, v)| = o(|u| + |v|)$$

as $|u| + |v| \to 0$ uniformly in $(\lambda, x) \in I \times \bar{\Omega}$;

(A3) $D^2G(\lambda, x, 0, 0) = 0$ for all $(\lambda, x) \in I \times \Omega$, where $D^2G(\lambda, x, u, v)$ denotes the Hessian matrix of $G(\lambda, x, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ at $(u, v) \in \mathbb{R}^2$.

Moreover, if $N > 1$, we shall also assume that

(A4) there exists $C \geqslant 0$ such that

$$\|D^2G(\lambda, x, u, v)\| \leqslant C(1 + |u| + |v|)^{p-1}, \quad (\lambda, x) \in I \times \bar{\Omega},\ u, v \in \mathbb{R},$$

where $1 \leqslant p < (N+2)/(N-2)$ if $N > 2$ and $1 \leqslant p < \infty$ if $N = 2$.

Note that the constant function $(u, v) \equiv 0$ is a solution of (1.1) for all $\lambda \in I$ by (A2), and the aim of this article is to study bifurcation of (weak) solutions of (1.1) from this trivial branch.

Now let $H^1_0(\Omega, \mathbb{R})$ be the usual Sobolev space with scalar product

$$\langle u_1, u_2 \rangle_{H^1_0(\Omega,\mathbb{R})} = \int_\Omega \langle \nabla u_1, \nabla u_2 \rangle \, \mathrm{d}x$$

and we set $E := H^1_0(\Omega, \mathbb{R}) \times H^1_0(\Omega, \mathbb{R})$, which is a Hilbert space with respect to

$$\langle (u_1, v_1), (u_2, v_2) \rangle_E = \langle u_1, u_2 \rangle_{H^1_0(\Omega,\mathbb{R})} + \langle v_1, v_2 \rangle_{H^1_0(\Omega,\mathbb{R})}.$$

We consider the map $f\colon I \times E \to \mathbb{R}$ given by

$$\begin{aligned} f_\lambda(z) = \int_\Omega \langle \nabla u, \nabla v \rangle \, \mathrm{d}x - \frac{1}{2} \int_\Omega a_\lambda(x) u^2 + 2b_\lambda(x) uv + c_\lambda(x) v^2 \, \mathrm{d}x \\ - \int_\Omega G(\lambda, x, u, v) \, \mathrm{d}x, \end{aligned} \tag{2.1}$$

where $z = (u, v) \in E$, and we note that f is C^2 under assumptions (A1), (A2) and (A4) (see [8, 15]). The critical points of f_λ are precisely the weak solutions of (1.1), and in particular $0 \in E$ is a critical point of all functionals f_λ. We say that $\lambda^* \in I$ is a *bifurcation point* of weak solutions for (1.1) if every neighbourhood of $(\lambda^*, 0) \in I \times E$ contains some $(\lambda, z) \neq (\lambda, 0)$, where z is a weak solution of (1.1), or, equivalently, a critical point of f_λ. Consequently, in order to investigate bifurcation of (1.1) from the trivial branch of solutions we need to study bifurcation of critical points of (2.1) from the branch $I \times \{0\} \subset I \times E$. For this we consider the Hessians of f_λ at $0 \in E$, which are given by

$$\begin{aligned} D^2_0 f_\lambda(z, \bar{z}) = \int_\Omega \langle \nabla u, \nabla \bar{v} \rangle \, \mathrm{d}x + \int_\Omega \langle \nabla \bar{u}, \nabla v \rangle \, \mathrm{d}x \\ - \int_\Omega a_\lambda(x) u\bar{u} + b_\lambda(x)(\bar{u}v + u\bar{v}) + c_\lambda(x) v\bar{v} \, \mathrm{d}x, \\ z = (u, v),\ \bar{z} = (\bar{u}, \bar{v}), \end{aligned} \tag{2.2}$$

where we use assumption (A3). Let us denote by L_λ the Riesz representations of $D^2_0 f_\lambda$, i.e. the bounded self-adjoint operators on E defined by

$$\langle L_\lambda z, \bar{z} \rangle_E = D^2_0 f(z, \bar{z}), \quad z, \bar{z} \in E. \tag{2.3}$$

Then $L_\lambda = T + K_\lambda$, where $T\colon E \to E$ is the self-adjoint invertible operator given by

$$Tz = T(u, v) = (v, u), \quad z = (u, v) \in E. \tag{2.4}$$

Moreover, the operator K_λ, which is uniquely determined by

$$\langle K_\lambda z, \bar{z}\rangle_E = -\int_\Omega a_\lambda(x)u\bar{u} + b_\lambda(x)(\bar{u}v + u\bar{v}) + c_\lambda(x)v\bar{v}\,\mathrm{d}x, \quad z = (u,v),\ \bar{z} = (\bar{u},\bar{v}), \tag{2.5}$$

is compact since the right-hand side of (2.5) extends to a bounded quadratic form on $L^2(\Omega,\mathbb{R}^2)$ and E is compactly embedded in this space (see, for example, [20, lemma 3.1]). Consequently, $L = \{L_\lambda\}_{\lambda\in I}$ is a path of self-adjoint Fredholm operators to which we can assign the *spectral flow*.

Now let H be an arbitrary separable real Hilbert space. The spectral flow is an integer-valued index for paths $L = \{L_\lambda\}_{\lambda\in I}$ of self-adjoint Fredholm operators L_λ on H, which we denote by $\mathrm{sf}(L, I)$. It was introduced in the 1970s by Atiyah *et al*. [1] and since then it has reappeared in many different areas of geometry and analysis (we refer the reader to [11] for a detailed list of references). Here we introduce it along the lines of [5] and discuss an application to bifurcation of critical points of families of functionals from [5, 11]. In what follows, we call a path of self-adjoint Fredholm operators *admissible* if its endpoints are invertible. Moreover, we denote by $\Phi_S(H)$ the space of all bounded self-adjoint Fredholm operators equipped with the norm topology. In order to shorten the presentation, we use an axiomatic description from [4]. Accordingly, the spectral flow is the unique map that assigns to each admissible path $L = \{L_\lambda\}_{\lambda\in I}$ in $\Phi_S(H)$ an integer such that the following hold.

- *Normalization:* if L_λ is invertible for all $\lambda \in I$, then
$$\mathrm{sf}(L, I) = 0.$$
- *Additivity:* if $H = H_1 \oplus H_2$ and $L_\lambda(H_i) \subset H_i$ for all $\lambda \in I$ and $i = 1, 2$, then
$$\mathrm{sf}(L, I) = \mathrm{sf}(L\mid_{H_1}, I) + \mathrm{sf}(L\mid_{H_2}, I).$$
- *Homotopy:* if $\{h_{(\lambda,s)}\}_{(\lambda,s)\in I\times I}$ is a family in $\Phi_S(H)$ such that $h(0, s)$ and $h(1, s)$ are invertible for all $s \in I$, then
$$\mathrm{sf}(h(\cdot, 0), I) = \mathrm{sf}(h(\cdot, 1), I).$$
- *Dimension:* if $\dim H < \infty$, then
$$\mathrm{sf}(L, I) = \mu_{\mathrm{Morse}}(L_0) - \mu_{\mathrm{Morse}}(L_1),$$
where μ_{Morse} denotes the Morse index, i.e. the number of negative eigenvalues counted with multiplicities.

Clearly, by reparametrizing, the spectral flow can also be defined for paths that are parametrized by a general compact interval $[\lambda_0, \lambda_1]$. If $L = \{L_\lambda\}_{\lambda\in[\lambda_0,\lambda_1]}$ is an admissible path of self-adjoint Fredholm operators, then we denote its spectral flow by $\mathrm{sf}(L, [\lambda_0, \lambda_1])$, and we note the following property for later reference.

- *Concatenation:* if $\lambda_0 < \lambda_1 < \lambda_2$ and L_{λ_1} is invertible, then
$$\mathrm{sf}(L, [\lambda_0, \lambda_2]) = \mathrm{sf}(L, [\lambda_0, \lambda_1]) + \mathrm{sf}(L, [\lambda_1, \lambda_2]).$$

https://doi.org/10.1017/S0308210517000324 Published online by Cambridge University Press

Let us now consider continuous maps $f\colon I \times H \to \mathbb{R}$ of C^2 functionals $f_\lambda := f(\lambda,\cdot)\colon H \to \mathbb{R}$ such that the derivatives Df_λ and $D^2 f_\lambda$ depend continuously on λ, and let us assume that $D_0 f_\lambda = 0$, i.e. $0 \in H$ is a critical point of f_λ for all $\lambda \in I$. Recall that a *bifurcation point* of critical points of f is an instant $\lambda^* \in I$ such that every neighbourhood of $(\lambda^*, 0)$ in $I \times H$ contains elements (λ, u), where $u \neq 0$ is a critical point of f_λ. The main theorems in [5, 11] state that if the Riesz representations L_λ of $D_0^2 f_\lambda$ are Fredholm for all λ and L_0, L_1 are invertible, then there is a bifurcation of critical points of f if $\mathrm{sf}(L, I) \neq 0$. Even though the spectral flow is an invariant that arose in elliptic topology, it is still unknown how this bifurcation theorem relates to the classical topological bifurcation theory from, for example, [10] and [2].

Let us now come back to the differential equations (1.1) and the functionals (2.1) on the Hilbert space E for which the corresponding operators L_λ are the Riesz representations of (2.2). By standard regularity theory, it follows that the kernels of L_λ consist of solutions of the linearized equations (1.2). Since L_λ is Fredholm and self-adjoint, its Fredholm index vanishes, and so we conclude that L_λ is non-invertible if and only if (1.2) has a non-trivial solution. Let us mention in passing that it is readily seen from the implicit function theorem that L_{λ^*} is not invertible if λ^* is a bifurcation point, which provides information about the location of possible bifurcation points. Finally, we can summarize the previous discussion as follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain having a smooth boundary and let the functions a, b, c and G in (1.1) satisfy (A1)–(A4). If the linear systems (1.2) have no non-trivial solution for $\lambda = 0, 1$ and $\mathrm{sf}(L, I) \neq 0$, then there is a bifurcation point $\lambda^* \in (0, 1)$ for the family of equations (1.1).*

Of course, the difficult point when applying theorem 2.1 to the equations (1.1) is to compute $\mathrm{sf}(L, I)$, or at least to find conditions that ensure its non-triviality. In the remainder of this article we will be concerned with this problem. At first, we want to review a method for computing spectral flows that has been applied several times in the past in other settings (for example, for Hamiltonian systems in [6] and for partial differential equations in [12–14, 19]).

Let us assume for the remainder of this section that the path $\{L_\lambda\}_{\lambda\in I}$ is C^1 in $\mathcal{L}(E)$. We call an instant $\lambda_0 \in I$ a crossing if L_{λ_0} is non-invertible, which is, as we have already observed, the case if and only if (1.2) has a non-trivial solution. Given a crossing λ_0, we obtain a quadratic form on $\ker L_{\lambda_0}$ by

$$\Gamma(L, \lambda_0)\colon \ker L_{\lambda_0} \to \mathbb{R}, \quad \Gamma(L, \lambda_0)[u] = \left\langle \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=\lambda_0} L_\lambda u, u \right\rangle_H,$$

and we say that a crossing is *regular* if $\Gamma(L, \lambda_0)$ is non-degenerate. Now let λ_0 be a regular crossing of L. One can show that regular crossings are isolated, and hence there is an $\varepsilon > 0$ such that L_λ is invertible for all λ in the punctured neighbourhood $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \setminus \{\lambda_0\}$. We obtain from the previously mentioned bifurcation theorem [5] that there is a bifurcation point for f in $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ if $\mathrm{sf}(L, [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]) \neq 0$. As L_λ is not invertible at bifurcation points by the implicit-function theorem, it follows in this case that the obtained bifurcation point is λ_0. By a theorem due to Robbin and Salamon [16] (see also [5, 21]), $\mathrm{sf}(L, [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon])$ is given by the signature of the quadratic form $\Gamma(L, \lambda_0)$ on $\ker L_{\lambda_0}$. From (2.2) and (2.3) we obtain that

$$\Gamma(L, \lambda_0)[z] = -\int_\Omega \dot{a}_{\lambda_0}(x)u^2 + 2\dot{b}_{\lambda_0}(x)uv + \dot{c}_{\lambda_0}(x)v^2 \,\mathrm{d}x, \quad z = (u, v) \in \ker L_{\lambda_0},$$

where the dot denotes the derivative with respect to the parameter λ. If we use that a quadratic form is non-degenerate and of non-vanishing signature if it is positive or negative definite, we obtain from Sylvester's criterion the following result.

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain having a smooth boundary and let the functions a, b, c and G in (1.1) satisfy (A1)–(A4). If the linear systems (1.2) have a non-trivial solution for $\lambda = \lambda_0 \in (0, 1)$, $\dot{a}_{\lambda_0}(x) \neq 0$ for all $x \in \Omega$ and*

$$\dot{a}_{\lambda_0}(x)\dot{c}_{\lambda_0}(x) - \dot{b}^2_{\lambda_0}(x) > 0, \quad x \in \Omega, \tag{2.6}$$

then λ_0 is a bifurcation point for (1.1).

Equation (2.6) is a convenient criterion for the existence of bifurcation points. However, we want to point out a drawback of this approach: the non-triviality of $\ker L_{\lambda_0}$, and so the existence of non-trivial solutions of (1.2), needs to be known. The aim of the following sections is to present approaches to the bifurcation problem of (1.1) that only use information about the coefficients of (1.2) and not about possible solutions for parameter values $\lambda \in (0, 1)$.

3. Index and bifurcation for (1.3)

In this section we consider (1.3), where we again assume throughout that (A1)–(A4) hold. Our first aim is to construct an invariant for the equations (1.4), which we will use below to compute the spectral flow of the associated path $L = \{L_\lambda\}_{\lambda \in I}$ introduced in (2.3) in order to obtain the existence of bifurcation from theorem 2.1. The following construction is based on Li and Liu's work [9] for Hamiltonian systems, which was adapted to (1.5) by Szulkin in [17].

Let $\{e_k\}_{k \in \mathbb{N}}$ be an orthonormal basis of $H^1_0(\Omega, \mathbb{R})$ such that $-\Delta e_k = \lambda_k e_k$, and let us recall that the eigenvalues λ_k are all positive and $\lambda_k \to \infty$ for $k \to \infty$. Now

$$\left\{ \frac{1}{\sqrt{2}}(e_k, -e_k), \frac{1}{\sqrt{2}}(e_k, e_k) \right\}_{k \in \mathbb{N}}$$

is an orthonormal basis of E and we get an orthonormal decomposition

$$E = H^1_0(\Omega, \mathbb{R}) \oplus H^1_0(\Omega, \mathbb{R}) = \bigoplus_{k \in \mathbb{N}} E_k,$$

where E_k is the two-dimensional space generated by $(e_k, -e_k)$ and (e_k, e_k). Since $T(u, v) = (v, u)$ for all $(u, v) \in E$ (see (2.4)), we see that $T(E_k) \subset E_k$. Also, by the following lemma, the operators K_λ in (2.5) leave the spaces E_k invariant.

LEMMA 3.1. *Let P_k and P_l denote the orthogonal projections in E onto E_k and E_l, respectively. If $k \neq l$, then*

$$P_k K_\lambda P_l = 0, \quad \lambda \in I.$$

Proof. If $z, \bar{z} \in E$, then $P_k z$ and $P_l \bar{z}$ are linear combinations of (e_k, e_k), $(-e_k, e_k)$ and (e_l, e_l), $(-e_l, e_l)$, respectively. Since the coefficients a, b, c do not depend on $x \in \Omega$, it follows from (2.5) that

$$\langle P_k K_\lambda P_l z, \bar{z}\rangle_E = \langle K_\lambda P_l z, P_k \bar{z}\rangle_E = m \int_\Omega e_k e_l \,\mathrm{d}x$$

for some number $m \in \mathbb{R}$. However,

$$\int_\Omega e_k e_l \,\mathrm{d}x = -\frac{1}{\lambda_k}\int_\Omega (\Delta e_k) e_l \,\mathrm{d}x = \frac{1}{\lambda_k}\langle e_k, e_l\rangle_{H_0^1(\Omega,\mathbb{R})} = 0$$

by Green's formula and so $P_k K_\lambda P_l = 0$ for $k \neq l$. □

We now define

$$L_\lambda^k := P_k L_\lambda P_k = P_k(T + K_\lambda)P_k|_{E_k} = T + K_\lambda|_{E_k} \colon E_k \to E_k, \quad k \in \mathbb{N},$$

where $P_k \colon E \to E$ denotes the orthogonal projection onto E_k. If we set

$$z = (u, v) = \frac{\alpha}{\sqrt{2}}(e_k, -e_k) + \frac{\beta}{\sqrt{2}}(e_k, e_k), \quad \alpha, \beta \in \mathbb{R},$$

then

$$D_0^2 f_\lambda(z, z) = (\beta^2 - \alpha^2) - \frac{1}{2\lambda_k}((a_\lambda - 2b_\lambda + c_\lambda)\alpha^2 + 2(a_\lambda - c_\lambda)\alpha\beta + (a_\lambda + 2b_\lambda + c_\lambda)\beta^2),$$

where we use that $\int_\Omega e_k^2 \,\mathrm{d}x = 1/\lambda_k$, $k \in \mathbb{N}$. We obtain

$$L_\lambda^k = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2\lambda_k}\begin{pmatrix} a_\lambda - 2b_\lambda + c_\lambda & a_\lambda - c_\lambda \\ a_\lambda - c_\lambda & a_\lambda + 2b_\lambda + c_\lambda \end{pmatrix}, \quad \lambda \in I, \tag{3.1}$$

with respect to the orthonormal basis

$$\left\{\frac{1}{\sqrt{2}}(e_k, -e_k), \frac{1}{\sqrt{2}}(e_k, e_k)\right\}$$

of E_k. In particular, since $\lambda_k \to \infty$ as $k \to \infty$, there exists $k_0 \in \mathbb{N}$ such that L_λ^k is an isomorphism and

$$\operatorname{sgn}(L_\lambda^k) = \mu_{\mathrm{Morse}}(-L_\lambda^k) - \mu_{\mathrm{Morse}}(L_\lambda^k) = 0 \quad \text{for all } k \geqslant k_0 \text{ and all } \lambda \in I.$$

Hence we can define for all $\lambda \in I$ an *index* of the coefficient matrix

$$A_\lambda := \begin{pmatrix} a_\lambda & b_\lambda \\ b_\lambda & c_\lambda \end{pmatrix}$$

of (1.4) by

$$i(A_\lambda) = \frac{1}{2}\sum_{k=1}^{\infty} \operatorname{sgn}(L_\lambda^k).$$

Note that if L_λ is invertible, then L_λ^k is invertible for all $k \in \mathbb{N}$ and so $\operatorname{sgn}(L_\lambda^k)$ is either -2, 0 or 2. Hence $i(A_\lambda)$ is an integer if L_λ is invertible, whereas it is only a half-integer in general. The main theorem of this section reads as follows.

THEOREM 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain having a smooth boundary and let us assume that (A1)–(A4) hold. If (1.4) has only the trivial solution for* $\lambda = 0, 1$ *and*

$$i(A_0) \neq i(A_1),$$

then there exists a bifurcation point for (1.3) in $(0, 1)$.

Proof. Let us recall that L_λ is of the form $L_\lambda = T + K_\lambda$, where the operators on the right-hand side were introduced in (2.4) and (2.5), respectively. Moreover, L_0 and L_1 are invertible since, by assumption, (1.4) has no non-trivial solutions for these parameter values.

We denote by $Q_n := \sum_{k=1}^n P_k$ the orthogonal projection onto $\bigoplus_{k=1}^n E_k$ and by $Q_n^\perp$ the corresponding complementary projection, i.e. $Q_n^\perp = I_E - Q_n$. We note that

$$\begin{aligned} L_\lambda &= T + Q_n K_\lambda Q_n + Q_n^\perp K_\lambda Q_n + Q_n K_\lambda Q_n^\perp + Q_n^\perp K_\lambda Q_n^\perp \\ &= T + Q_n K_\lambda Q_n + Q_n^\perp K_\lambda Q_n^\perp, \quad n \in \mathbb{N}, \end{aligned} \tag{3.2}$$

where the second equality is a simple consequence of lemma 3.1. We now claim that there are $n_0 \in \mathbb{N}$ and a constant $C > 0$ such that, for all $n \geqslant n_0$,

$$\|Tu + Q_n K_\lambda Q_n u\| \geqslant 2C\|u\|, \quad u \in E, \ \lambda = 0, 1, \tag{3.3}$$

and

$$\|Q_n^\perp K_\lambda Q_n^\perp\| \leqslant C, \quad \lambda \in [0, 1]. \tag{3.4}$$

It is a well-known fact that if $\{S_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}(E)$ that converges strongly to some $S \in \mathcal{L}(H)$, and K_λ, $\lambda \in [0, 1]$, is a continuous family of compact operators, then $S_n K_\lambda$ converges in norm to SK_λ as $n \to \infty$, and the convergence is uniform in λ. Consequently, since $Q_n^\perp$ converges strongly to 0 as $n \to \infty$ and $\|Q_n^\perp\| = 1$, we infer that

$$\|Q_n^\perp K_\lambda Q_n^\perp\| \leqslant \|Q_n^\perp K_\lambda\| \to 0, \quad n \to \infty. \tag{3.5}$$

Since L_λ is invertible for $\lambda = 0, 1$, there is a $C > 0$ such that $\|L_\lambda u\| \geqslant 3C\|u\|$ for $u \in E$ and $\lambda = 0, 1$. We obtain from (3.2) and (3.5) that there is an n_0 such that $\|Tu + Q_n K_\lambda Q_n u\| \geqslant 2C\|u\|$ for all $n \geqslant n_0$, $u \in E$ and $\lambda = 0, 1$, which is (3.3). After possibly increasing n_0, we can assume that (3.4) holds for the same constant $C > 0$, where we use that the convergence in (3.5) is uniform in λ.

We now assume that n_0 in (3.3) and (3.4) is sufficiently large such that $\operatorname{sgn}(L_\lambda^k) = 0$ for all $\lambda \in I$ and all $k \geqslant n_0$, and we consider for some $n \geqslant n_0$ the homotopy $h\colon [0, 1] \times [0, 1] \to \Phi_S(E)$ defined by

$$h(t, \lambda) = T + Q_n K_\lambda Q_n + tQ_n^\perp K_\lambda Q_n^\perp.$$

By (3.3) and (3.4), we conclude that

$$\|h(t, \lambda)u\| \geqslant C\|u\|, \quad u \in E, \ \lambda = 0, 1,$$

and hence $h(t, 0)$ and $h(t, 1)$ are invertible for all $t \in [0, 1]$ since they are Fredholm of index 0. The homotopy invariance property of the spectral flow yields

$$\operatorname{sf}(L, I) = \operatorname{sf}(\{T + Q_n K_\lambda Q_n\}_{\lambda\in I}, I). \tag{3.6}$$

By lemma 3.1 we have

$$Q_nK_\lambda Q_n=\sum_{k,l=1}^{n}P_kK_\lambda P_l=\sum_{k=1}^{n}P_kK_\lambda P_k,$$

and since T is also reduced by the projections P_k it follows likewise that

$$Q_nTQ_n=\sum_{k,l=1}^{n}P_kTP_l=\sum_{k=1}^{n}P_kTP_k.$$

We obtain

$$\begin{aligned}T+Q_nK_\lambda Q_n&=Q_nTQ_n+Q_nK_\lambda Q_n+Q_n^\perp TQ_n^\perp\\&=\sum_{k=1}^{n}(P_k(T+K_\lambda)P_k)+Q_n^\perp TQ_n^\perp.\end{aligned}$$

Now the additivity and normalization properties of the spectral flow yield

$$\begin{aligned}\operatorname{sf}(\{T+Q_nK_\lambda Q_n\}_{\lambda\in I},I)&=\operatorname{sf}\left(\left\{\sum_{k=1}^{n}(P_k(T+K_\lambda)P_k)+Q_n^\perp TQ_n^\perp\right\}_{\lambda\in I},I\right)\\&=\sum_{k=1}^{n}\operatorname{sf}(\{P_k(T+K_\lambda)P_k\}_{\lambda\in I},I)\\&=\sum_{k=1}^{n}\operatorname{sf}(L^k,I),\end{aligned}$$

where we use that $Q_n^\perp TQ_n^\perp$ is an invertible operator on the image of $Q_n^\perp$. By the dimension property of the spectral flow, we obtain

$$\sum_{k=1}^{n}\operatorname{sf}(L^k,I)=\sum_{k=1}^{n}(\mu_{\mathrm{Morse}}(L_0^k)-\mu_{\mathrm{Morse}}(L_1^k)).\tag{3.7}$$

As L_0 and L_1 are invertible by assumption, we see that L_0^k and L_1^k are invertible for all $k\in\mathbb{N}$. Since the signature and the Morse index of an invertible symmetric (2×2)-matrix B are related by $\frac{1}{2}\operatorname{sgn}B=1-\mu_{\mathrm{Morse}}(B)$, we can rewrite the right-hand side in (3.7) by

$$\begin{aligned}\sum_{k=1}^{n}(\mu_{\mathrm{Morse}}(L_0^k)-\mu_{\mathrm{Morse}}(L_1^k))&=\sum_{k=1}^{n}\tfrac{1}{2}(\operatorname{sgn}(L_1^k)-\operatorname{sgn}(L_0^k))\\&=\frac{1}{2}\sum_{k=1}^{n}\operatorname{sgn}(L_1^k)-\frac{1}{2}\sum_{k=1}^{n}\operatorname{sgn}(L_0^k)\\&=i(A_1)-i(A_0),\end{aligned}$$

where we have used in the last step that $\operatorname{sgn}(L_\lambda^k)=0$ for all $\lambda\in I$ and all $k\geqslant n_0$ by our choice of n_0. Consequently, we have shown that

$$\operatorname{sf}(L,I)=i(A_1)-i(A_0),\tag{3.8}$$

and now the assertion follows from theorem 2.1. □

Remark 3.3. Let us point out that we have derived in the proof of theorem 3.2 in (3.8) a spectral flow formula for the path $\{L_\lambda\}_{\lambda\in I}$, which is of independent interest. The spectral flow can also be defined for paths of unbounded self-adjoint Fredholm operators (see, for example, [3, 21]). Let us consider on $L^2(\Omega,\mathbb{R}^2)$ the differential operators $\mathcal{A}_\lambda$ on the domain $W = H^2(\Omega,\mathbb{R}^2)\cap H^1_0(\Omega,\mathbb{R}^2)$ defined by

$$\mathcal{A}_\lambda \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} -\Delta v \\ -\Delta u \end{pmatrix} + \begin{pmatrix} a_\lambda & b_\lambda \\ b_\lambda & c_\lambda \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Note that elements in the kernel of $\mathcal{A}_\lambda$ are the solutions of (1.4). It can be shown that the spectral flow of the path $\mathcal{A} = \{\mathcal{A}_\lambda\}_{\lambda\in I}$ coincides with the spectral flow of the corresponding path $L = \{L_\lambda\}_{\lambda\in I}$ in (2.3) (see [20, theorem 2.6]), and so (3.8) yields also a spectral flow formula for the differential operators $\mathcal{A}_\lambda$.

As announced in the previous section, theorem 3.2 uses only the coefficients of (1.2) and no information about solutions of the linearizations (1.4) for $\lambda \in (0, 1)$.

Let us now consider (1.5), where $A_\lambda = \lambda A$ depends linearly on the parameter λ. Here we want to change the setting slightly and instead of restricting λ to the unit interval I, we consider the case in which $\lambda \in \mathbb{R}$. As before, we have for each $\lambda \in \mathbb{R}$ the integral number $i(A_\lambda)$. We obtain from theorem 3.2 the following result, which was proved by Szulkin in [17, § 5].

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain having a smooth boundary and let us assume that (A1)–(A4) hold, where $a_\lambda(x) = \lambda a$, $b_\lambda(x) = \lambda b$ and $c_\lambda(x) = \lambda c$ for some real numbers a, b, c and $\lambda \in \mathbb{R}$. If $i(\lambda A)$ jumps at some $\lambda^* \in \mathbb{R}$, then λ^* is a bifurcation point.*

Proof. We first note that the operators L_λ are of the form $T + \lambda K$, where T is invertible and K is compact and does not depend on λ. Hence, by the spectral theory of compact operators, the set of all $\lambda \in \mathbb{R}$ for which L_λ is not invertible is discrete. Secondly, if L^k_λ is non-invertible for some $k \in \mathbb{N}$, then L_λ is non-invertible as well.

Let us now assume that $i(\lambda A)$ jumps at some λ^*. Then there is a $k \in \mathbb{N}$ such that $L^k_{\lambda^*}$ is not invertible. Hence, L_{λ^*} is not invertible and there is an $\varepsilon > 0$ such that L_λ is invertible if $\lambda \in (\lambda^* - 2\varepsilon, \lambda^* + 2\varepsilon) \setminus \{\lambda_0\}$. Consequently, $L_{\lambda^*-\varepsilon}$ and $L_{\lambda^*+\varepsilon}$ are invertible, and since $i(A_{\lambda^*-\varepsilon}) \neq i(A_{\lambda^*+\varepsilon})$ the assertion follows from theorem 3.2. □

4. Bifurcation by comparison

For the considerations of this section, we want to introduce at first a theorem about the spectral flow that was proved in [11]. Beforehand, we need to extend the definition of the spectral flow, which we recalled in the second section, to paths $L = \{L_\lambda\}_{\lambda\in I}$ in $\Phi_S(H)$ that do not have invertible endpoints, i.e. that are not admissible. Since 0 is an isolated eigenvalue of finite multiplicity (see, for example, [19, lemma 2.2]), there exists a $\delta \geqslant 0$ such that $L_0 + \mu I_H$ and $L_1 + \mu I_H$ are invertible for all $0 < \mu \leqslant \delta$, where I_H denotes the identity operator on H. We set

$$\mathrm{sf}(L, I) := \mathrm{sf}(L + \delta I_H, I).$$

https://doi.org/10.1017/S0308210517000324 Published online by Cambridge University Press

Of course, if L is admissible, then this definition coincides with the previous one by the homotopy invariance property. In what follows, we write $T \leqslant S$ for $T, S \in \Phi_S(H)$ if

$$\langle Tz, z\rangle_E \leqslant \langle Sz, z\rangle_H, \quad z \in H.$$

A proof of the following proposition can be found in [11, § 7].

PROPOSITION 4.1. *Let $L = \{L_\lambda\}_{\lambda\in I}$ and $M = \{M_\lambda\}_{\lambda\in I}$ be paths in $\Phi_S(H)$ such that $L_\lambda - M_\lambda$ is compact for all $\lambda \in I$. If*

$$L_0 \leqslant M_0 \quad \text{and} \quad M_1 \leqslant L_1,$$

then

$$\operatorname{sf}(M, I) \leqslant \operatorname{sf}(L, I).$$

Let us now again consider (1.1). Note that the operators K_λ in (2.5) can be written as

$$\langle K_\lambda z, \bar z\rangle_E = \int_\Omega \langle A_\lambda(x) z, \bar z\rangle \,\mathrm{d}x,$$

where

$$A_\lambda(x) := -\begin{pmatrix} a_\lambda(x) & b_\lambda(x) \\ b_\lambda(x) & c_\lambda(x)\end{pmatrix}$$

is a symmetric matrix. Each $A_\lambda(x)$ has two real eigenvalues $\mu^1_\lambda(x)$, $\mu^2_\lambda(x)$, which depend continuously on $(\lambda, x) \in I \times \bar\Omega$. We set, for $\lambda \in I$,

$$\alpha_\lambda := \inf_{x\in\bar\Omega}\{\mu^1_\lambda(x), \mu^2_\lambda(x)\} = \inf_{x\in\bar\Omega}\inf_{\|w\|=1}\langle A_\lambda(x) w, w\rangle_{\mathbb{R}^2},$$
$$\beta_\lambda := \sup_{x\in\bar\Omega}\{\mu^1_\lambda(x), \mu^2_\lambda(x)\} = \sup_{x\in\bar\Omega}\sup_{\|w\|=1}\langle A_\lambda(x) w, w\rangle_{\mathbb{R}^2},$$

and note that these numbers can be easily obtained since $\mu^1_\lambda(x)$, $\mu^2_\lambda(x)$ are just the zeros of quadratic polynomials. For example, for (1.3) we have

$$\left.\begin{aligned}\alpha_\lambda &= -\frac{a_\lambda + c_\lambda}{2} - \sqrt{\tfrac14(a_\lambda - c_\lambda)^2 + b_\lambda^2},\\ \beta_\lambda &= -\frac{a_\lambda + c_\lambda}{2} + \sqrt{\tfrac14(a_\lambda - c_\lambda)^2 + b_\lambda^2}.\end{aligned}\right\} \tag{4.1}$$

Let us recall that we denote by $\{\lambda_k\}_{k\in\mathbb{N}}$ the sequence of Dirichlet eigenvalues of the domain Ω and that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. The main theorem of this section reads as follows.

THEOREM 4.2. *Let Ω be a bounded domain having a smooth boundary and let us assume that (A1)–(A4) hold and that the linear equations (1.2) have only the trivial solution for $\lambda = 0, 1$.*

(i) *If $\beta_0 < \alpha_1$ and there exists $k \in \mathbb{N}$ such that*

$$\beta_0 < \lambda_k < \alpha_1 \quad \text{or} \quad \beta_0 < -\lambda_k < \alpha_1, \tag{4.2}$$

then there is a bifurcation point for (1.1).

https://doi.org/10.1017/S0308210517000324 Published online by Cambridge University Press

(ii) *If $\beta_1 < \alpha_0$ and there exists $k \in \mathbb{N}$ such that*

$$\beta_1 < \lambda_k < \alpha_0 \quad \textit{or} \quad \beta_1 < -\lambda_k < \alpha_0, \tag{4.3}$$

then there is a bifurcation point for (1.1).

Let us point out again that no knowledge about solutions of the systems (1.2) for $\lambda \in (0,1)$ is used in theorem 4.2.

Proof. By definition of α_0, α_1, β_0, β_1, we have the inequalities

$$\alpha_0 I_2 \leqslant A_0(x) \leqslant \beta_0 I_2, \quad \alpha_1 I_2 \leqslant A_1(x) \leqslant \beta_1 I_2, \quad x \in \Omega, \tag{4.4}$$

where I_2 denotes the 2×2 identity matrix. Let us now consider the paths of matrices $\{B_\lambda\}_{\lambda\in I}$ and $\{C_\lambda\}_{\lambda\in I}$ given by

$$B_\lambda = (\beta_0 + \lambda(\alpha_1 - \beta_0))I_2 \quad \text{and} \quad C_\lambda = (\alpha_0 + \lambda(\beta_1 - \alpha_0))I_2.$$

We obtain associated paths $M = \{M_\lambda\}_{\lambda\in I}$ and $N = \{N_\lambda\}_{\lambda\in I}$ in $\Phi_S(E)$ by setting

$$\langle M_\lambda z, \bar z\rangle_E := \langle Tz, \bar z\rangle_E + \int_\Omega \langle B_\lambda z, \bar z\rangle \,\mathrm{d}x,$$
$$\langle N_\lambda z, \bar z\rangle_E := \langle Tz, \bar z\rangle_E + \int_\Omega \langle C_\lambda z, \bar z\rangle \,\mathrm{d}x$$

and we note that, by (2.2), $L_\lambda - M_\lambda$ and $L_\lambda - N_\lambda$ are compact for all $\lambda \in I$. Since

$$\langle (L_\lambda - M_\lambda)z, \bar z\rangle_E = \int_\Omega \langle (A_\lambda(x) - B_\lambda)z, \bar z\rangle \,\mathrm{d}x, \quad z, \bar z \in E,$$

and

$$\langle (L_\lambda - N_\lambda)z, \bar z\rangle_E = \int_\Omega \langle (A_\lambda(x) - C_\lambda)z, \bar z\rangle \,\mathrm{d}x, \quad z, \bar z \in E,$$

we obtain from (4.4) and proposition 4.1 that

$$\mathrm{sf}(M, I) \leqslant \mathrm{sf}(L, I) \leqslant \mathrm{sf}(N, I). \tag{4.5}$$

Because L_0 and L_1 are invertible by the assumption that (1.2) has no non-trivial solutions for $\lambda = 0$ and $\lambda = 1$, the assertion follows from theorem 2.1 if we can prove that $\mathrm{sf}(M, I) > 0$ under the assumptions of (i), and $\mathrm{sf}(N, I) < 0$ under the assumptions of (ii), respectively.

Let us first consider the path M. Since M_0 and M_1 are not necessarily invertible, we have by definition $\mathrm{sf}(M, I) = \mathrm{sf}(M^\delta, I)$, where $M^\delta := \{M_\lambda + \delta I_E\}_{\lambda\in I}$ for an arbitrarily small $\delta > 0$. From the results in § 3, we know that there is a decomposition of E into two-dimensional subspaces E_k, $k \in \mathbb{N}$, such that the operator T is reduced by this decomposition. Clearly, the E_k reduce M^δ too, and moreover it is readily seen that

$$M^\delta_\lambda|_{E_k} = \begin{pmatrix} -1+\delta & 0 \\ 0 & 1+\delta \end{pmatrix} + \frac{\beta_0 + \lambda(\alpha_1 - \beta_0)}{\lambda_k} I_2, \quad \lambda \in I.$$

By (3.8) we know that

$$\mathrm{sf}(M,I)=\frac{1}{2}\sum_{k=1}^{\infty}\mathrm{sgn}(M_1^{\delta}|_{E_k})-\frac{1}{2}\sum_{k=1}^{\infty}\mathrm{sgn}(M_0^{\delta}|_{E_k}). \tag{4.6}$$

Let us now consider at first

$$M_1^{\delta}|_{E_k}=\begin{pmatrix}-1+\delta+\dfrac{\alpha_1}{\lambda_k} & 0\\ 0 & 1+\delta+\dfrac{\alpha_1}{\lambda_k}\end{pmatrix}.$$

If $\alpha_1\geqslant 0$, then $1+\delta+\alpha_1/\lambda_k>0$ for all $k\in\mathbb{N}$ and consequently $\mathrm{sgn}(M_1^{\delta}|_{E_k})$ is either 0 or 2. The latter case happens if and only if $-1+\delta+\alpha_1/\lambda_k>0$ and, since $\delta>0$ is arbitrarily small, this is equivalent to $-1+\alpha_1/\lambda_k\geqslant 0$, and so $\alpha_1\geqslant\lambda_k$. If, on the other hand, $\alpha_1<0$, then $-1+\delta+\alpha_1/\lambda_k<0$ for all $k\in\mathbb{N}$ and so $\mathrm{sgn}(M_1^{\delta}|_{E_k})$ is either 0 or -2. Here the latter case happens if $1+\delta+\alpha_1/\lambda_k<0$, which means that $\alpha_1<-\lambda_k$. In summary, we obtain

$$\frac{1}{2}\sum_{k=1}^{\infty}\mathrm{sgn}(M_1^{\delta}|_{E_k})=\begin{cases}\#\{k\in\mathbb{N}\colon\alpha_1\geqslant\lambda_k\} & \text{if } \alpha_1\geqslant 0,\\ -\#\{k\in\mathbb{N}\colon\alpha_1<-\lambda_k\} & \text{if } \alpha_1<0,\end{cases}$$

and by the very same argument we also get that

$$\frac{1}{2}\sum_{k=1}^{\infty}\mathrm{sgn}(M_0^{\delta}|_{E_k})=\begin{cases}\#\{k\in\mathbb{N}\colon\beta_0\geqslant\lambda_k\} & \text{if } \beta_0\geqslant 0,\\ -\#\{k\in\mathbb{N}\colon\beta_0<-\lambda_k\} & \text{if } \beta_0<0.\end{cases}$$

Consequently, it follows from (4.6) that

$$\mathrm{sf}(M,I)=\begin{cases}\#\{k\in\mathbb{N}\colon\alpha_1\geqslant\lambda_k\}-\#\{k\in\mathbb{N}\colon\beta_0\geqslant\lambda_k\} & \text{if } \alpha_1,\ \beta_0\geqslant 0,\\ -\#\{k\in\mathbb{N}\colon\alpha_1<-\lambda_k\}-\#\{k\in\mathbb{N}\colon\beta_0\geqslant\lambda_k\} & \text{if } \alpha_1<0,\ \beta_0\geqslant 0,\\ \#\{k\in\mathbb{N}\colon\alpha_1\geqslant\lambda_k\}+\#\{k\in\mathbb{N}\colon\beta_0<-\lambda_k\} & \text{if } \alpha_1\geqslant 0,\ \beta_0<0,\\ -\#\{k\in\mathbb{N}\colon\alpha_1<-\lambda_k\}+\#\{k\in\mathbb{N}\colon\beta_0<-\lambda_k\} & \text{if } \alpha_1,\ \beta_0<0,\end{cases} \tag{4.7}$$

which is positive if $\beta_0<\lambda_k<\alpha_1$ or $\beta_0<-\lambda_k<\alpha_1$ for some $k\in\mathbb{N}$. This finishes the proof of the first part of theorem 4.2.

For the second part we need to show that $\mathrm{sf}(N,I)=\mathrm{sf}(N^{\delta},I)<0$, where $N^{\delta}=\{N_{\lambda}+\delta I_E\}_{\lambda\in I}$ for an arbitrarily small $\delta>0$. We leave it to the reader to check that a similar argument as above shows that

$$\mathrm{sf}(N,I)=\begin{cases}\#\{k\in\mathbb{N}\colon\beta_1\geqslant\lambda_k\}-\#\{k\in\mathbb{N}\colon\alpha_0\geqslant\lambda_k\} & \text{if } \alpha_0,\ \beta_1\geqslant 0,\\ -\#\{k\in\mathbb{N}\colon\beta_1<-\lambda_k\}-\#\{k\in\mathbb{N}\colon\alpha_0\geqslant\lambda_k\} & \text{if } \beta_1<0,\ \alpha_0\geqslant 0,\\ \#\{k\in\mathbb{N}\colon\beta_1\geqslant\lambda_k\}+\#\{k\in\mathbb{N}\colon\alpha_0<-\lambda_k\} & \text{if } \beta_1\geqslant 0,\ \alpha_0<0,\\ -\#\{k\in\mathbb{N}\colon\beta_1<-\lambda_k\}+\#\{k\in\mathbb{N}\colon\alpha_0<-\lambda_k\} & \text{if } \beta_1,\ \alpha_0<0,\end{cases} \tag{4.8}$$

which is negative if $\beta_1<\lambda_k<\alpha_0$ or $\beta_1<-\lambda_k<\alpha_0$ for some $k\in\mathbb{N}$. □

As an example of theorem 4.2, let us consider once again (1.5), where the matrix A does not depend on $x \in \Omega$ and is linear in λ. Then we obtain from (4.1) that

$$\left.\begin{aligned} \alpha_0 &= \beta_0 = 0, \\ \alpha_1 &= -\frac{a+c}{2} - \sqrt{\tfrac{1}{4}(a-c)^2 + b^2}, \\ \beta_1 &= -\frac{a+c}{2} + \sqrt{\tfrac{1}{4}(a-c)^2 + b^2} \end{aligned}\right\} \tag{4.9}$$

and see that there is a bifurcation point for (1.5) in $(0, 1)$ if

$$-\frac{a+c}{2} - \sqrt{\tfrac{1}{4}(a-c)^2 + b^2} > \lambda_1 \quad \text{or} \quad -\frac{a+c}{2} + \sqrt{\tfrac{1}{4}(a-c)^2 + b^2} < -\lambda_1.$$

5. The $N = 1$ case

In this section we consider the special case in which $N = 1$, i.e. Ω is a bounded interval in $\mathbb{R}$. For the sake of simplicity, we restrict ourselves to the case in which $\Omega = (0, \pi)$ and so the systems (1.1) are of the form

$$\left.\begin{aligned} &-u'' = b_\lambda(x)u + c_\lambda(x)v + G'_v(\lambda, x, u, v) \quad \text{in } (0, \pi), \\ &-v'' = a_\lambda(x)u + b_\lambda(x)v + G'_u(\lambda, x, u, v) \quad \text{in } (0, \pi), \\ &u(0) = v(0) = u(\pi) = v(\pi) = 0. \end{aligned}\right\} \tag{5.1}$$

We want to show that our previous results can be used to obtain an estimate on the number of bifurcation points for (5.1). Let us note for later reference the corresponding linearized equations, which are

$$\left.\begin{aligned} &-u'' = b_\lambda(x)u + c_\lambda(x)v \qquad \text{in } (0, \pi), \\ &-v'' = a_\lambda(x)u + b_\lambda(x)v \qquad \text{in } (0, \pi), \\ &u(0) = v(0) = u(\pi) = v(\pi) = 0. \end{aligned}\right\} \tag{5.2}$$

Before we can state the main result of this section, we need to make another digression about a property of the spectral flow. Let us assume that $L = \{L_\lambda\}_{\lambda \in I}$ is a path of self-adjoint Fredholm operators such that L_λ is non-invertible only at the finite number of instants $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_m < 1$. Then, by the concatenation property of the spectral flow, there is an $\varepsilon > 0$ such that

$$\mathrm{sf}(L, I) = \sum_{i=1}^{m} \mathrm{sf}(L, [\lambda_i - \varepsilon, \lambda_i + \varepsilon]).$$

From the construction of the spectral flow in [5], it is intuitively clear (however, not trivial to prove rigorously (see [11, lemma 4.5])) that

$$|\mathrm{sf}(L, [\lambda_i - \varepsilon, \lambda_i + \varepsilon])| \leqslant \dim \ker L_{\lambda_i}. \tag{5.3}$$

Now let $f\colon I \times H \to \mathbb{R}$ be a family of functionals as in § 2 such that L_λ is the Riesz representation of $D_0^2 f_\lambda$ for $\lambda \in I$ as in (2.3). By the implicit function theorem, if λ^* is a bifurcation point for f, then $\lambda^* = \lambda_i$ for some $1 \leqslant i \leqslant m$. Moreover, λ_i is a bifurcation point if $\mathrm{sf}(L, [\lambda_i - \varepsilon, \lambda_i + \varepsilon]) \neq 0$ by theorem 2.1. From these facts, the following result is readily seen (see [11, theorem 2.1(ii)]).

Lemma 5.1. *Let $f\colon I\times H\to\mathbb{R}$ and $L=\{L_\lambda\}_{\lambda\in I}$ be as in § 2. We assume that L is admissible and L_λ is non-invertible for only a finite number of $\lambda\in(0,1)$. Then the number of bifurcation points for f is bounded from below by*

$$\frac{|\mathrm{sf}(L,I)|}{\max_{\lambda\in(0,1)}\dim\ker L_\lambda}.$$

We now introduce a natural number $\Gamma(\alpha,\beta)$ for any pair of real numbers $\alpha>\beta$ by

$$\Gamma(\alpha,\beta)=\begin{cases}\#\{k\in\mathbb{N}\colon \alpha\geqslant k^2\geqslant\beta\} & \text{if } \alpha,\beta\geqslant 0,\\ \#\{k\in\mathbb{N}\colon \alpha\geqslant k^2\}+\#\{k\in\mathbb{N}\colon \beta<-k^2\} & \text{if } \alpha\geqslant 0,\ \beta<0,\\ \#\{k\in\mathbb{N}\colon \beta<-k^2<\alpha\} & \text{if } \alpha,\beta<0,\end{cases}$$

which we need to state the main result of this section.

Theorem 5.2. *Let $\Omega=(0,\pi)\subset\mathbb{R}$ and let us assume that (A1)–(A3) hold. We suppose that there are only finitely many $\lambda\in(0,1)$ for which the linear equation (5.2) has a non-trivial solution and, moreover, we assume that there is only the trivial solution for $\lambda=0,1$.*

(i) *If $\alpha_1>\beta_0$, then there are at least $\frac{1}{2}\Gamma(\alpha_1,\beta_0)$ bifurcation points for (5.1).*

(ii) *If $\alpha_0>\beta_1$, then there are at least $\frac{1}{2}\Gamma(\alpha_0,\beta_1)$ bifurcation points for (5.1).*

Proof. In the proof of theorem 4.2 we constructed paths M and N such that $\mathrm{sf}(M,I)\leqslant\mathrm{sf}(L,I)$ and $\mathrm{sf}(L,I)\leqslant\mathrm{sf}(N,I)$, respectively. Note that the Dirichlet eigenvalues of the domain $\Omega=(0,\pi)$ are $\lambda_k=k^2$, $k\in\mathbb{N}$. If now $\alpha_1>\beta_0$, we obtain from (4.7) that $\mathrm{sf}(L,I)\geqslant\mathrm{sf}(M,I)=\Gamma(\alpha_1,\beta_0)$. If, however, $\alpha_0>\beta_1$, we get from (4.8) that $\mathrm{sf}(L,I)\leqslant\mathrm{sf}(N,I)=-\Gamma(\alpha_0,\beta_1)$ and so $|\Gamma(\alpha_0,\beta_1)|\leqslant|\mathrm{sf}(L,I)|$.

Finally, the result follows from lemma 5.1 if we note that $\dim\ker L_\lambda\leqslant 2$ as the kernel of L_λ consists of solutions of the two-dimensional system of linear ordinary equations (5.2). □

Finally, let us consider (1.5) on $\Omega=(0,\pi)$. It follows from the spectral theory of compact operators that the corresponding linearized equations (5.2) can only have a non-trivial solution for a finite number of values of the parameter λ. Hence, if we assume that there is only the trivial solution for $\lambda=1$, then we obtain that there are at least

$$\tfrac{1}{2}\max\left\{k\in\mathbb{N}\colon -\frac{a+c}{2}-\sqrt{\tfrac{1}{4}(a-c)^2+b^2}\geqslant k^2\right\}$$

or

$$\tfrac{1}{2}\max\left\{k\in\mathbb{N}\colon -\frac{a+c}{2}+\sqrt{\tfrac{1}{4}(a-c)^2+b^2}\geqslant -k^2\right\}$$

distinct bifurcation points for the nonlinear equations (1.5), where only one of these numbers can be non-zero. In particular, we can easily construct systems having an arbitrarily high number of bifurcation points in $(0,1)$.

References

1 M. F. Atiyah, V. K. Patodi and I. M. Singer. Spectral asymmetry and Riemannian geometry. III. *Proc. Camb. Phil. Soc.* **79** (1976), 71–99.

2 T. Bartsch. The global structure of the zero set of a family of semilinear Fredholm maps. *Nonlin. Analysis* **17** (1991), 313–331.

3 B. Booß-Bavnbek, M. Lesch and J. Phillips. Unbounded Fredholm operators and spectral flow. *Can. J. Math.* **57** (2005), 225–250.

4 E. Ciriza, P. M. Fitzpatrick and J. Pejsachowicz. Uniqueness of spectral flow. *Math. Comput. Modelling* **32** (2000), 1495–1501.

5 P. M. Fitzpatrick, J. Pejsachowicz and L. Recht. Spectral flow and bifurcation of critical points of strongly indefinite functionals. Part I. General theory. *J. Funct. Analysis* **162** (1999), 52–95.

6 P. M. Fitzpatrick, J. Pejsachowicz and L. Recht. Spectral flow and bifurcation of critical points of strongly indefinite functionals. Part II. Bifurcation of periodic orbits of Hamiltonian systems. *J. Diff. Eqns* **163** (2000), 18–40.

7 A. Gołębiewska and S. Rybicki. Global bifurcations of critical orbits of G-invariant strongly indefinite functionals. *Nonlin. Analysis* **74** (2011), 1823–1834.

8 W. Kryszewski and A. Szulkin. An infinite dimensional Morse theory with applications. *Trans. Am. Math. Soc.* **349** (1997), 3181–3234.

9 S. Li and J. Q. Liu. Morse theory and asymptotic linear Hamiltonian systems. *J. Diff. Eqns* **78** (1989), 53–73.

10 J. Pejsachowicz. K-theoretic methods in bifurcation theory. In *Fixed point theory and its applications (Berkeley, CA, 1986)*, Contemporary Mathematics, vol. 72, pp. 193–206 (Providence, RI: American Mathematical Society, 1988).

11 J. Pejsachowicz and N. Waterstraat. Bifurcation of critical points for continuous families of C^2 functionals of Fredholm type. *J. Fixed Point Theory Appl.* **13** (2013), 537–560.

12 A. Portaluri and N. Waterstraat. On bifurcation for semilinear elliptic Dirichlet problems and the Morse–Smale index theorem. *J. Math. Analysis Applic.* **408** (2013), 572–575.

13 A. Portaluri and N. Waterstraat. On bifurcation for semilinear elliptic Dirichlet problems on geodesic balls. *J. Math. Analysis Applic.* **415** (2014), 240–246.

14 A. Portaluri and N. Waterstraat. A Morse–Smale index theorem for indefinite elliptic systems and bifurcation. *J. Diff. Eqns* **258** (2015), 1715–1748.

15 P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65 (Providence, RI: American Mathematical Society, 1986).

16 J. Robbin and D. Salamon. The spectral flow and the Maslov index. *Bull. Lond. Math. Soc.* **27** (1995), 1–33.

17 A. Szulkin. Bifurcation for strongly indefinite functionals and a Liapunov-type theorem for Hamiltonian systems. *Diff. Integ. Eqns* **7** (1994), 217–234.

18 Q. Wang and C. Liu. A new index theory for linear self-adjoint operator equations and its applications. *J. Diff. Eqns* **260** (2016), 3749–3784.

19 N. Waterstraat. On bifurcation for semilinear elliptic Dirichlet problems on shrinking domains. In *Elliptic and parabolic equations*, Springer Proceedings in Mathematics & Statistics, vol. 119, pp. 273–291 (Springer, 2015).

20 N. Waterstraat. A family index theorem for periodic Hamiltonian systems and bifurcation. *Calc. Var. PDEs* **52** (2015), 727–753.

21 N. Waterstraat. Spectral flow, crossing forms and homoclinics of Hamiltonian systems. *Proc. Lond. Math. Soc.* **111** (2015), 275–304.