J. Fluid Mech. (2002), *vol.* 468, *pp.* 317–326. © 2002 Cambridge University Press
DOI: 10.1017/S0022112002001696 Printed in the United Kingdom

Exact second-order structure-function relationships

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(Received 9 April 2002 and in revised form 12 June 2002)

Equations that follow from the Navier–Stokes equation and incompressibility but with no other approximations are 'exact'. Exact equations relating second- and third-order structure functions are studied, as is an exact incompressibility condition on the second-order velocity structure function. Opportunities for investigations using these equations are discussed. Precisely defined averaging operations are required to obtain exact averaged equations. Ensemble, temporal and spatial averages are all considered because they produce different statistical equations and because they apply to theoretical purposes, experiment and numerical simulation of turbulence. Particularly simple exact equations are obtained for the following cases: (i) the trace of the structure functions, (ii) DNS that has periodic boundary conditions, and (iii) an average over a sphere in $\boldsymbol{r}$-space. Case (iii) introduces the average over orientations of $\boldsymbol{r}$ into the structure-function equations. The energy dissipation rate ε appears in the exact trace equation without averaging, whereas in previous formulations ε appears after averaging and use of local isotropy. The trace mitigates the effect of anisotropy in the equations, thereby revealing that the trace of the third-order structure function is expected to be superior for quantifying asymptotic scaling laws. The orientation average has the same property.

1. Introduction

Equations relating statistics for turbulence studies, such as Kolmogorov's (1941) equation, are asymptotic. This has required experimenters to seek turbulence that satisfies the criteria of the asymptotic state. The present approach is to derive exact statistical equations. These can be used to determine all effects contributing to the balance of statistical equations. By 'exact' we mean that the equations follow from the Navier–Stokes equation and the incompressibility condition with no additional approximations.

Exact equations have the potential to detect the limitations of direct numerical simulation (DNS) and of experiments and to study the approach to local homogeneity and local isotropy and scaling laws (Hill 2001). For those purposes, the averaging operation must be exactly defined and implemented; that is done here. The methods developed here can be used on the exact structure-function equations of all orders N; those equations are in Hill (2001). It is useful to further investigate the exact second-order ($N = 2$) equation, which relates second- and third-order structure functions, because it has special simplifications that the higher-order equations ($N > 2$) do not possess and because the second-order equation is the most familiar. Exact equations satisfy the need perceived by Yaglom (1998) for careful derivation of dynamic-theory equations and the perceived value placed by Sreenivasan & Antonia (1997) on aspects of turbulence that can be understood precisely. Experimental data have been used to evaluate the balance of Kolmogorov's equation (Antonia, Chambers & Browne 1983; Chambers & Antonia 1984) and generalizations of it (Lindborg 1999; Danaila *et al.* 1999*a*, *b*; Antonia *et al.* 2000). This report supports such experimental work as well as precise use of DNS by giving exact equations to be used in such evaluations. The connection between the derivations presented here and any experiment or DNS is important because the equations relate several statistics and therefore are most revealing when data are substituted into them.

The equations derived here are exact for every flow, whether laminar or turbulent, provided that no forces act on the fluid at the points of measurement, denoted below by $\boldsymbol{x}$ and $\boldsymbol{x}'$. Forces can be applied near the point of measurement; e.g. the equations are exact for hot-wire anemometer supports just downstream of the measurement points. The cases of forces at the points of measurement and throughout the fluid are considered in Hill (2002).

The ensemble average is typically used for theoretical studies, the temporal average for experimental data, and the spatial average for data from DNS; thus all three are employed here. Ensemble, time and space averages are not interchangeable because the averages commute differently with differential operators within the dynamical equations. For the homogeneous case and infinite averaging volume, the spatially averaged equation (3.9) and the ensemble-averaged equation (3.1) reduce to the same form, and similarly for the temporally averaged equation for the stationary case and infinite averaging time.

Ongoing interest in turbulence intermittency includes accurate evaluation of inertial-range exponents of structure functions, for which purpose precise definition of an observed inertial range is needed. The third-order structure function can serve this purpose because it has a well-known inertial-range power law and the 4/5 coefficient (Kolmogorov 1941). Deviations from the 4/5 coefficient are observed in experiments; this casts doubt on the precision with which measured exponents apply to the intermittency phenomenon (Sreenivasan & Dhruva 1998). The equations derived here, when evaluated with data, can reveal all effects contributing to the deviation from Kolmogorov's 4/5 law.

The plan of the paper is to develop the mathematics in § 2 and § 3; § 2 contains necessary definitions and unaveraged equations; §§ 3.1–3.3 contain the definition of averaging operations and their application to produce averaged equations. Section 3.4 contains the simplifications for the case of spatially periodic DNS. Section 3.5 defines the sphere average in $\boldsymbol{r}$-space and its associated orientation average and relates these to the work of Kolmogorov (1962) and Obukhov (1962); Kolmogorov's equation is derived in § 3.6 as a useful point of reference. Discussion of opportunities that these equations present for future investigations is in § 4.

2. Exact unaveraged two-point equations

The equations given here relate two-point quantities and are obtained from the Navier–Stokes equations and incompressibility. The two spatial points are denoted $\boldsymbol{x}$ and $\boldsymbol{x}'$; they are independent variables. They have no relative motion; e.g. anemometers at $\boldsymbol{x}$ and $\boldsymbol{x}'$ are fixed relative to one another. To be concise, velocities are denoted $u_i = u_i(\boldsymbol{x}, t)$, $u_i' = u_i(\boldsymbol{x}', t)$, energy dissipation rates $\varepsilon = \varepsilon(\boldsymbol{x}, t)$, $\varepsilon' = \varepsilon(\boldsymbol{x}', t)$, etc; p is the pressure divided by the density (density is constant), ν is kinematic viscosity, and ∂ denotes partial differentiation with respect to its subscript variable. Summation is implied by repeated Roman indices; e.g. $\partial_{x_n}\partial_{x_n}$ is the Laplacian operator. For brevity, define:

$$d_{ij} \equiv (u_i - u_i')(u_j - u_j'), \tag{2.1}$$

$$d_{ijn} \equiv (u_i - u_i')(u_j - u_j')(u_n - u_n'), \tag{2.2}$$

$$\tau_{ij} \equiv (\partial_{x_i} p - \partial_{x_i'} p')(u_j - u_j') + (\partial_{x_j} p - \partial_{x_j'} p')(u_i - u_i'), \tag{2.3}$$

$$e_{ij} \equiv (\partial_{x_n} u_i)(\partial_{x_n} u_j) + (\partial_{x_n'} u_i')(\partial_{x_n'} u_j'), \tag{2.4}$$

$$F_{ijn} \equiv (u_i - u_i')(u_j - u_j')\tfrac{1}{2}(u_n + u_n'). \tag{2.5}$$

Change variables from $\boldsymbol{x}$ and $\boldsymbol{x}'$ to the sum and difference independent variables:

$$\boldsymbol{X} \equiv (\boldsymbol{x} + \boldsymbol{x}')/2 \quad \text{and} \quad \boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}' \quad \text{and} \quad \text{define } r \equiv |\boldsymbol{r}|.$$

The derivatives ∂_{X_i} and ∂_{r_i} are related to ∂_{x_i} and $\partial_{x_i'}$ by

$$\partial_{x_i} = \partial_{r_i} + \tfrac{1}{2}\partial_{X_i}, \quad \partial_{x_i'} = -\partial_{r_i} + \tfrac{1}{2}\partial_{X_i}, \quad \partial_{X_i} = \partial_{x_i} + \partial_{x_i'}, \quad \partial_{r_i} = \tfrac{1}{2}(\partial_{x_i} - \partial_{x_i'}). \tag{2.6}$$

For any functions $f(\boldsymbol{x}, t)$ and $g(\boldsymbol{x}', t)$, (2.6) gives

$$\partial_{r_i}[f(\boldsymbol{x}, t) \pm g(\boldsymbol{x}', t)] = \partial_{X_i}[f(\boldsymbol{x}, t) \mp g(\boldsymbol{x}', t)]/2. \tag{2.7}$$

Use of (2.6) in (2.3) and in the trace of (2.4) and rearranging terms gives

$$\tau_{ij} = -2(p - p')(s_{ij} - s_{ij}') + \partial_{X_i}[(p - p')(u_j - u_j')] + \partial_{X_j}[(p - p')(u_i - u_i')], \tag{2.8}$$

$$e_{ii} = \nu^{-1}(\varepsilon + \varepsilon') + \partial_{X_n}\partial_{X_n}(p + p'), \tag{2.9}$$

where

$$s_{ij} \equiv (\partial_{x_i} u_j + \partial_{x_j} u_i)/2 \quad \text{and} \quad \varepsilon \equiv 2\nu s_{ij} s_{ij}. \tag{2.10}$$

To obtain (2.9) we used Poisson's equation, $\partial_{x_n}\partial_{x_n} p = -\partial_{x_i} u_j \partial_{x_j} u_i$. Incompressibility requires $s_{ii} = 0$; thus, the trace of (2.8) is

$$\tau_{ii} = 2\partial_{X_i}[(p - p')(u_i - u_i')]. \tag{2.11}$$

Of course, all quantities above are local and instantaneous quantities.

2.1. *Use of the Navier–Stokes equation*

The Navier–Stokes equation and incompressibility give

$$\partial_t d_{ij} + \partial_{X_n} F_{ijn} + \partial_{r_n} d_{ijn} = -\tau_{ij} + 2\nu(\partial_{r_n}\partial_{r_n} d_{ij} + \tfrac{1}{4}\partial_{X_n}\partial_{X_n} d_{ij} - e_{ij}). \tag{2.12}$$

As a check, one sees that (2.12) can be obtained from equation (2.9) of Hill (2001). The trace of (2.12) and substitution of (2.9) and (2.11) gives

$$\partial_t d_{ii} + \partial_{X_n} F_{iin} + \partial_{r_n} d_{iin} = 2\nu\partial_{r_n}\partial_{r_n} d_{ii} - 2(\varepsilon + \varepsilon') + w, \tag{2.13}$$

where

$$w = -2\partial_{X_i}[(p - p')(u_i - u_i')] + \tfrac{1}{2}\nu\partial_{X_n}\partial_{X_n} d_{ii} - 2\nu\partial_{X_n}\partial_{X_n}(p + p'). \tag{2.14}$$

The first term in (2.14) is $-\tau_{ii}$ from (2.11) and the last term in (2.14) arises from e_{ii} in (2.9); the disparate terms in (2.14) are given the symbol w for subsequent convenience and brevity. The limit $r \to 0$ applied to (2.13) recovers the definition of ε in (2.10).

2.2. *Exact second-order incompressibility relationships*

Because $\boldsymbol{x}$ and $\boldsymbol{x}'$ are independent variables, $\partial_{x_i} u'_j = 0$, and $\partial_{x'_i} u_j = 0$. Then, incompressibility gives $\partial_{X_n} u_n = 0$, $\partial_{X_n} u'_n = 0$, $\partial_{r_n} u_n = 0$, $\partial_{r_n} u'_n = 0$, so $\partial_{X_n}(u_n - u'_n) = 0$, and $\partial_{r_n}(u_n - u'_n) = 0$. The combined use of those incompressibility relations and (2.7) gives

$$\partial_{r_n}[(u_j - u'_j)(u_n - u'_n)] = \partial_{X_n}[(u_j + u'_j)(u_n - u'_n)]/2, \tag{2.15}$$

$$\partial_{r_j}\partial_{r_n}[(u_j - u'_j)(u_n - u'_n)] = \partial_{X_j}\partial_{X_n}[(u_j + u'_j)(u_n + u'_n)]/4. \tag{2.16}$$

3. Exact averaged two-point equations

3.1. *Ensemble average: exact equations*

The ensemble average is defined at each point $(\boldsymbol{X}, \boldsymbol{r}, t)$ as the arithmetical average over the ensemble. We denote the ensemble average by angle brackets and subscript $E : \langle \cdot \rangle_E$. Because the ensemble averaging operation is a summation, it commutes with differential operators; the average of (2.12) is

$$\partial_t \langle d_{ij} \rangle_E + \partial_{X_n} \langle F_{ijn} \rangle_E + \partial_{r_n} \langle d_{ijn} \rangle_E = -\langle \tau_{ij} \rangle_E + 2\nu\left(\partial_{r_n}\partial_{r_n} \langle d_{ij} \rangle_E + \tfrac{1}{4}\partial_{X_n}\partial_{X_n} \langle d_{ij} \rangle_E - \langle e_{ij} \rangle_E\right). \tag{3.1}$$

The argument list for each tensor in (3.1) is $(\boldsymbol{X}, \boldsymbol{r}, t)$; the ensemble average does not eliminate dependence on any independent variable. The average of (2.13) is

$$\partial_t \langle d_{ii} \rangle_E + \partial_{X_n} \langle F_{iin} \rangle_E + \partial_{r_n} \langle d_{iin} \rangle_E = 2\nu \partial_{r_n}\partial_{r_n} \langle d_{ii} \rangle_E - 2\langle \varepsilon + \varepsilon' \rangle_E + \langle w \rangle_E. \tag{3.2}$$

Exact incompressibility conditions on the second-order velocity structure function are given by the ensemble averages of (2.15) and (2.16):

$$\partial_{r_n} \langle d_{jn} \rangle_E = \partial_{X_n} \langle (u_j + u'_j)(u_n - u'_n) \rangle_E / 2, \tag{3.3}$$

$$\partial_{r_j}\partial_{r_n} \langle d_{jn} \rangle_E = \partial_{X_j}\partial_{X_n} \langle (u_j + u'_j)(u_n + u'_n) \rangle_E / 4. \tag{3.4}$$

3.2. *Temporal average: exact equations*

Because nearly continuous temporal sampling is typical, we represent the temporal average by an integral, but all results are valid for the sum of discrete points as well. The temporal average is most useful when the turbulence is nearly statistically stationary. Let t_0 be the start time of the temporal average of duration T. The operator effecting the temporal average of any quantity Q is denoted by $\langle \cdot \rangle_T$, which has argument list $(\boldsymbol{X}, \boldsymbol{r}, t_0, T)$; that is,

$$\langle Q \rangle_T \equiv \frac{1}{T} \int_{t_0}^{t_0+T} Q(\boldsymbol{X}, \boldsymbol{r}, t)\, \mathrm{d}t. \tag{3.5}$$

The argument list $(\boldsymbol{X}, \boldsymbol{r}, t_0, T)$ is suppressed. The temporal average of (2.12)–(2.16) gives equations that are the same form as (3.1)–(3.4) with one exception: ∂_t does not commute with the integral operator (3.5) such that $\langle \partial_t d_{ij} \rangle_T$ appears, whereas $\partial_t \langle d_{ij} \rangle_E$ appears in (3.1), and similarly for the trace equation (3.2). Because data are taken at $\boldsymbol{x}$ and $\boldsymbol{x}'$ in the rest frame of the anemometers, and ∂_t is the time derivative for that reference frame, it follows that

$$\langle \partial_t d_{ij} \rangle_T \equiv \frac{1}{T} \int_{t_0}^{t_0+T} \partial_t d_{ij} \mathrm{d}t = [d_{ij}(\boldsymbol{X}, \boldsymbol{r}, t_0 + T) - d_{ij}(\boldsymbol{X}, \boldsymbol{r}, t_0)]/T. \tag{3.6}$$

This shows that it is easy to evaluate $\langle \partial_t d_{ij} \rangle_T$ using experimental data because only the first (at $t = t_0$) and last (at $t = t_0 + T$) data in the time series are used. One can make $\langle \partial_t d_{ij} \rangle_T$ as small as one desires by allowing T to be very large provided that $d_{ij}(\boldsymbol{X}, \boldsymbol{r}, t_0 + T)$ does not differ greatly from $d_{ij}(\boldsymbol{X}, \boldsymbol{r}, t_0)$. This is aided by judicious choice of t_0 and $t_0 + T$ for the stationary case, but is not possible in all cases.

3.3. *Spatial average: exact equations*

Because nearly continuous spatial sampling is typical of DNS, we represent the spatial average by an integral, but all results can be generalized to the case of a sum of discrete points. Let the spatial average be over a region $\mathbb{R}$ in $\boldsymbol{X}$-space. The spatial average of any quantity Q is denoted by $\langle Q \rangle_{\mathbb{R}}$ which has argument list $(\boldsymbol{r}, t, \mathbb{R})$; that is,

$$\langle Q \rangle_{\mathbb{R}} \equiv \frac{1}{V} \iiint_{\mathbb{R}} Q(\boldsymbol{X}, \boldsymbol{r}, t) \, \mathrm{d}\boldsymbol{X}, \tag{3.7}$$

where V is the volume of the space region $\mathbb{R}$. The argument list $(\boldsymbol{r}, t, \mathbb{R})$ is suppressed. The spatial average commutes with $\boldsymbol{r}$ and t differential and integral operations, and with ensemble, time, and $\boldsymbol{r}$-space averages, but not with ∂_{X_n}. Given any vector q_n, the divergence theorem relates the volume average of $\partial_{X_n} q_n$ to the surface average; that is,

$$\langle \partial_{X_n} q_n \rangle_{\mathbb{R}} \equiv \frac{1}{V} \iiint \partial_{X_n} q_n \mathrm{d}\boldsymbol{X} = \frac{S}{V} \left(\frac{1}{S} \iint \check{N}_n q_n \mathrm{d}S \right) \equiv \frac{S}{V} \oint_{\boldsymbol{X}_n} q_n, \tag{3.8}$$

where S is the surface area bounding $\mathbb{R}$, $\mathrm{d}S$ is the differential of surface area, and $\check{N}_n$ is the unit vector oriented outward and normal to the surface. As seen on the right-hand side of (3.8), we adopt, for brevity, the integral-operator notation

$$\oint_{\boldsymbol{X}_n} \equiv \frac{1}{S} \iint \check{N}_n \mathrm{d}S.$$

The spatial average of (2.12) is

$$\begin{aligned} \partial_t \langle d_{ij} \rangle_{\mathbb{R}} &+ \frac{S}{V} \oint_{\boldsymbol{X}_n} F_{ijn} + \partial_{r_n} \langle d_{ijn} \rangle_{\mathbb{R}} \\ &= -\langle \tau_{ij} \rangle_{\mathbb{R}} + 2\nu \left(\partial_{r_n} \partial_{r_n} \langle d_{ij} \rangle_{\mathbb{R}} + \frac{1}{4} \frac{S}{V} \oint_{\boldsymbol{X}_n} \partial_{X_n} d_{ij} - \langle e_{ij} \rangle_{\mathbb{R}} \right). \end{aligned} \tag{3.9}$$

The spatial average of (2.13) is

$$\partial_t \langle d_{ii} \rangle_{\mathbb{R}} + \frac{S}{V} \oint_{\boldsymbol{X}_n} F_{iin} + \partial_{r_n} \langle d_{iin} \rangle_{\mathbb{R}} = 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ii} \rangle_{\mathbb{R}} - 2 \langle \varepsilon + \varepsilon' \rangle_{\mathbb{R}} + \langle w \rangle_{\mathbb{R}}, \tag{3.10}$$

where

$$\langle w \rangle_{\mathbb{R}} \equiv \frac{S}{V} \oint_{\boldsymbol{X}_n} \left[-2(p - p')(u_n - u_n') + \tfrac{1}{2} \nu \partial_{X_n} d_{ij} - 2\nu \partial_{X_n} (p + p') \right].$$

The spatial average of the incompressibility condition (2.15) is

$$\partial_{r_n} \langle d_{jn} \rangle_{\mathbb{R}} = \frac{S}{2V} \oint_{\boldsymbol{X}_n} (u_n - u_n')(u_j + u_j'), \tag{3.11}$$

which is, on the right-hand side, a surface flux of a quantity that depends on large-scale structures in the flow. Similarly, (2.16) gives

$$\partial_{r_j}\partial_{r_n}\langle d_{jn}\rangle_{\mathbb{R}} = \frac{S}{4V}\oint_{X_n}\partial_{X_j}[(u_n+u'_n)(u_j+u'_j)]. \tag{3.12}$$

3.4. *Spatial average: DNS with periodic boundary conditions*

The spatial average is particularly relevant to DNS. DNS that is used to investigate turbulence at small scales often has periodic boundary conditions. For such DNS, consider the spatial average over the entire DNS domain. Contributions to $\oint_{X_n} q_n$ from opposite sides of the averaging volume cancel for that case such that $\oint_{X_n} q_n = 0$ and therefore $\langle\partial_{X_n} q_n\rangle_{\mathbb{R}} = 0$. In (3.9) we then have $\oint_{X_n} F_{ijn} = 0$ and $\oint_{X_n}\partial_{X_n} d_{ij} = 0$. In (3.10) we have $\oint_{X_n} F_{iin} = 0$ and $\langle w\rangle_{\mathbb{R}} = 0$. In (3.11), the right-hand side vanishes so that $\partial_{r_n}\langle d_{jn}\rangle_{\mathbb{R}} = 0$. Thus, in the DNS case described above, we have

$$\partial_t\langle d_{ij}\rangle_{\mathbb{R}} + \partial_{r_n}\langle d_{ijn}\rangle_{\mathbb{R}} = -\langle\tau_{ij}\rangle_{\mathbb{R}} + 2\nu(\partial_{r_n}\partial_{r_n}\langle d_{ij}\rangle_{\mathbb{R}} - \langle e_{ij}\rangle_{\mathbb{R}}), \tag{3.13}$$

$$\partial_t\langle d_{ii}\rangle_{\mathbb{R}} + \partial_{r_n}\langle d_{iin}\rangle_{\mathbb{R}} = 2\nu\partial_{r_n}\partial_{r_n}\langle d_{ii}\rangle_{\mathbb{R}} - 2\langle\varepsilon+\varepsilon'\rangle_{\mathbb{R}}, \tag{3.14}$$

$$\partial_{r_n}\langle d_{jn}\rangle_{\mathbb{R}} = 0, \qquad \partial_{r_n}\langle e_{jn}\rangle_{\mathbb{R}} = 0. \tag{3.15}$$

Proof of $\partial_{r_n}\langle e_{jn}\rangle_{\mathbb{R}} = 0$ is given in Hill (2002).

Performing the $\boldsymbol{r}$-space divergence of (3.13) and using (3.15), we have

$$\partial_{r_j}\partial_{r_n}\langle d_{ijn}\rangle_{\mathbb{R}} = -\partial_{r_j}\langle\tau_{ij}\rangle_{\mathbb{R}}. \tag{3.16}$$

This exact result is analogous to the asymptotic result in Frisch (1995), Lindborg (1996), and Hill (1997).

Using the Taylor series of ε and ε' around the point $\boldsymbol{X}$, Hill (2002) obtains the following exact result for the periodic DNS case considered:

$$-2\langle\varepsilon+\varepsilon'\rangle_{\mathbb{R}} = -4\langle\varepsilon(\boldsymbol{X},t)\rangle_{\mathbb{R}}, \qquad -\langle e_{ij}\rangle_{\mathbb{R}} = -4\nu\langle[(\partial_{x_n}u_i)(\partial_{x_n}u_j)]_{\boldsymbol{x}=\boldsymbol{X}}\rangle_{\mathbb{R}}, \tag{3.17}$$

where the subscript $\boldsymbol{x}=\boldsymbol{X}$ means that the derivatives are evaluated at the point $\boldsymbol{X}$. An important aspect of (3.17) is that the right-hand sides depend only on t. Of course, none of (3.13)–(3.16) depends on $\boldsymbol{X}$ because of the spatial average over $\boldsymbol{X}$.

No approximations have been used to obtain the above equations for the spatially periodic DNS case considered.

3.5. *Averages over an r-space sphere*

The energy dissipation rate averaged over a sphere in $\boldsymbol{r}$-space has been a recurrent theme in small-scale similarity theories since its introduction by Obukhov (1962) and Kolmogorov (1962). By averaging our equations for the trace, we can, for the first time, produce an exact dynamical equation containing the sphere-averaged energy dissipation rate. The volume average over an $\boldsymbol{r}$-space sphere of radius r_S of a quantity Q is denoted

$$\langle Q\rangle_{\boldsymbol{r}\text{-sphere}} \equiv (4\pi r_S^3/3)^{-1}\iiint_{|\boldsymbol{r}|\leqslant r_S} Q(\boldsymbol{X},r,t)\,\mathrm{d}\boldsymbol{r}. \tag{3.18}$$

The orientation average over the surface of the $\boldsymbol{r}$-space sphere of radius r_S of a vector $q_n(\boldsymbol{X},\boldsymbol{r},t)$ is denoted by the following integral-operator notation:

$$\oint_{r_n} q_n \equiv (4\pi r_S^2)^{-1}\iiint_{|\boldsymbol{r}|=r_S}\frac{r_n}{r}q_n(\boldsymbol{X},\boldsymbol{r},t)\,\mathrm{d}s, \tag{3.19}$$

where ds is the differential of surface area, and r_n/r is the unit vector oriented outward and normal to the surface of the $\boldsymbol{r}$-space sphere. Note that $(4\pi r_S^2)^{-1}\,\mathrm{d}s = \mathrm{d}\Omega/4\pi$ where dΩ is the differential of solid angle from the sphere's centre. Both $\langle Q\rangle_{\boldsymbol{r}\text{-sphere}}$ and $\oint_{r_n} q_n$ have the argument list $(\boldsymbol{X}, r_S, t)$, which is suppressed. The divergence theorem is

$$\langle \partial_{r_n} q_n\rangle_{\boldsymbol{r}\text{-sphere}} = (3/r_S)\oint_{r_n} q_n. \tag{3.20}$$

Because $\boldsymbol{r}$, $\boldsymbol{X}$, and t are independent variables, the $\boldsymbol{r}$-space volume and orientation averages commute with time and $\boldsymbol{X}$-space averages and with $\boldsymbol{X}$- and t-differential operators, and, of course, with the ensemble average as well. For instance,

$$\langle \partial_t \langle d_{ii}\rangle_{\mathbb{R}}\rangle_{\boldsymbol{r}\text{-sphere}} = \partial_t\langle\langle d_{ii}\rangle_{\mathbb{R}}\rangle_{\boldsymbol{r}\text{-sphere}} = \langle\langle \partial_t d_{ii}\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_{\mathbb{R}} = \partial_t\langle\langle d_{ii}\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_{\mathbb{R}}, \text{etc.}$$

The $\boldsymbol{r}$-sphere average (3.18) can operate on all of the above structure-function equations; it can operate on unaveraged equations (2.12) and (2.13) as well. These equations have terms of the form $\partial_{r_n} q_n$; e.g. $q_n = \langle d_{ijn}\rangle_{\mathbb{R}}$, $\partial_{r_n}\langle d_{ii}\rangle_{\mathbb{R}}$, $\langle d_{iin}\rangle_E$, $\langle d_{ijn}\rangle_T$, $\partial_{r_n}\langle d_{ii}\rangle_T$, etc. By means of (3.20), the volume average in $\boldsymbol{r}$-space of any term of the form $\partial_{r_n} q_n$ produces the orientation average of q_n within the subject equation. After operating on (3.2) with the volume average in $\boldsymbol{r}$-space (3.18), the term $-2\langle \varepsilon + \varepsilon'\rangle_E$ in that equation produces $-2\langle\langle \varepsilon + \varepsilon'\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_E$. Now, $\langle\langle \varepsilon + \varepsilon'\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_E/2$ is the sphere-averaged energy dissipation rate defined in the third equations of both Obukhov (1962) and Kolmogorov (1962).

The result of the $\boldsymbol{r}$-space sphere average of any of our equations will be clear from operating on (3.14). The average of (3.14) over a sphere in $\boldsymbol{r}$-space of radius r_S and multiplication by $r_S/3$ and use of (3.17) gives

$$\frac{r_S}{3}\partial_t\langle\langle d_{ii}\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_{\mathbb{R}} + \oint_{r_n}\langle d_{iin}\rangle_{\mathbb{R}} = 2\nu\oint_{r_n}\partial_{r_n}\langle d_{ii}\rangle_{\mathbb{R}} - \frac{4r_S}{3}\langle\langle \varepsilon\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_{\mathbb{R}}. \tag{3.21}$$

The terms have argument list (r_S, t), but $\langle\langle \varepsilon\rangle_{\boldsymbol{r}\text{-sphere}}\rangle_{\mathbb{R}}$ depends only on t. Of course, none of the quantities in (3.21) depends on $\boldsymbol{X}$ because of the $\boldsymbol{X}$-space average. Despite its simplicity, (3.21) has been obtained without approximations for the freely decaying spatially periodic DNS case considered; (3.21) applies to inhomogeneous and anisotropic DNS that have periodic boundary conditions. Nie & Tanveer (1999) define a structure function $\tilde{S}_3$ using time, space, and solid-angle averages acting on d_{iin}, and consider the asymptotic inertial-range case to obtain that $\tilde{S}_3 = -(4/3)\epsilon r$ without use of local isotropy. An analogous result can be obtained by applying inertial-range asymptotics to (3.21); namely, neglect the time-derivative term on the basis of local stationarity and neglect the term proportional to ν.

3.6. *Kolmogorov's equation derived from* (3.21)

Most readers are familiar with Kolmogorov's (1941) famous equation that is valid for locally isotropic turbulence. A useful point of reference is to derive it from (3.21). This helps elucidate (3.21). An index 1 denotes projection in the direction of $\boldsymbol{r}$, and indices 2 and 3 denote orthogonal directions perpendicular to $\boldsymbol{r}$. For locally isotropic turbulence we recall that the only non-zero components of $\langle d_{ijn}\rangle_{\mathbb{R}}$ are $\langle d_{111}\rangle_{\mathbb{R}}$, $\langle d_{221}\rangle_{\mathbb{R}} = \langle d_{331}\rangle_{\mathbb{R}}$, and of $\langle d_{ij}\rangle_{\mathbb{R}}$ are $\langle d_{11}\rangle_{\mathbb{R}}$, and $\langle d_{22}\rangle_{\mathbb{R}} = \langle d_{33}\rangle_{\mathbb{R}}$. These components depend only on r such that there is no distinction in an $\boldsymbol{r}$-space sphere average between r_S and r; thus, we simplify the notation by replacing r_S with r. The isotopic-tensor formula for $\langle d_{ijn}\rangle_{\mathbb{R}}$ gives $\langle d_{iin}\rangle_{\mathbb{R}} = (r_n/r)(\langle d_{111}\rangle_{\mathbb{R}} + 2\langle d_{221}\rangle_{\mathbb{R}}) = (r_n/r)\langle d_{ii1}\rangle_{\mathbb{R}}$, substitution of which into (3.19) gives $\oint_{r_n}\langle d_{iin}\rangle_{\mathbb{R}} = (r_n/r)\langle d_{iin}\rangle_{\mathbb{R}} = (r_n/r)(r_n/r)\langle d_{ii1}\rangle_{\mathbb{R}} = \langle d_{ii1}\rangle_{\mathbb{R}}$.

Since $(\partial_{r_n} r) = (r_n/r)$, we have $\oint_{r_n} \partial_{r_n} \langle d_{ii} \rangle_{\mathbb{R}} = (r_n/r)(\partial_{r_n} r)\partial_r \langle d_{ii} \rangle_{\mathbb{R}} = \partial_r \langle d_{ii} \rangle_{\mathbb{R}}$. Kolmogorov (1941) considered the locally stationary case such that he neglected the time-derivative term; thus we also neglect that term to obtain from (3.21) that

$$\langle d_{ii1} \rangle_{\mathbb{R}} = 2\nu \partial_r \langle d_{ii} \rangle_{\mathbb{R}} - \tfrac{4}{3} \langle \varepsilon \rangle_{\mathbb{R}}\, r. \tag{3.22}$$

To eliminate $\langle d_{22} \rangle_{\mathbb{R}}$ and $\langle\langle d_{221} \rangle_{\mathbb{R}}\rangle$ from the expressions $\langle d_{ii} \rangle_{\mathbb{R}} = \langle d_{11} \rangle_{\mathbb{R}} + 2\langle d_{22} \rangle_{\mathbb{R}}$ and $\langle d_{ii1} \rangle_{\mathbb{R}} = \langle d_{111} \rangle_{\mathbb{R}} + 2\langle d_{221} \rangle_{\mathbb{R}}$, we use the incompressibility conditions $\frac{1}{2} r \partial_r \langle d_{11} \rangle_{\mathbb{R}} + \langle d_{11} \rangle_{\mathbb{R}} - \langle d_{22} \rangle_{\mathbb{R}} = 0$, and $r\partial_r \langle d_{111} \rangle_{\mathbb{R}} + \langle d_{111} \rangle_{\mathbb{R}} - 6\langle d_{221} \rangle_{\mathbb{R}} = 0$, which are valid for local isotropy (Hill 1997), and were also used by Kolmogorov (1941). Then (3.22) becomes, after multiplying by $3r^{-1}$, $\partial_r \langle d_{111} \rangle_{\mathbb{R}} + (4/r)\langle d_{111} \rangle_{\mathbb{R}} = 6\nu [\partial_r^2 \langle d_{11} \rangle_{\mathbb{R}} + (4/r)\langle d_{11} \rangle_{\mathbb{R}}] - 4\langle \varepsilon \rangle_{\mathbb{R}}$; this is Kolmogorov's (1941) third equation. After multiplication by r^4 and integrating from 0 to r, we have Kolmogorov's (1941) equation

$$\langle d_{111} \rangle_{\mathbb{R}} = 6\nu \partial_r \langle d_{11} \rangle_{\mathbb{R}} - \tfrac{4}{5} \langle \varepsilon \rangle_{\mathbb{R}}\, r. \tag{3.23}$$

Kolmogorov's inertial-range 4/5 law and the viscous-range law follow immediately from (3.23).

4. Examples of opportunities for using the exact equations

4.1. *Mitigating anisotropy to check asymptotic laws*

Consider homogeneous, anisotropic turbulence. Homogeneity causes ∂_{X_n} operating on a statistic to vanish (Hill 2001), so $\partial_{X_n} \langle F_{ijn} \rangle_E$ and $\partial_{X_n}\partial_{X_n} \langle d_{ij} \rangle_E$ vanish from (3.1), but $\langle \tau_{ij} \rangle_E$ becomes $-2\langle (p - p')(s_{ij} - s'_{ij}) \rangle_E$ (see (2.8)), which does not vanish. Under the more restrictive assumption of local isotropy, $\langle \tau_{ij} \rangle_E = 0$ (Hill 1997) such that the entire non-zero value of $\langle \tau_{ij} \rangle_E$ is a source of anisotropy in (3.1). For the locally stationary case, the anisotropy quantified by $\langle \tau_{ij} \rangle_E$ is approximately balanced by that from the term $\partial_{r_n} \langle d_{ijn} \rangle_E$ in (3.1) (Hill 1997, and exactly so for the stationary case). In contrast consider (3.2). Homogeneity causes $\partial_{X_n} F_{iin}$ and w to vanish from (3.2); equivalently, $\langle \tau_{ii} \rangle_E$ is absent from (3.2) because incompressibility gives $s_{ii} - s'_{ii} = 0$. Therefore, for the homogeneous, anisotropic case, an important source of anisotropy of $\partial_{r_n} \langle d_{ijn} \rangle_E$, namely $\langle \tau_{ij} \rangle_E$, is absent from $\partial_{r_n} \langle d_{iin} \rangle_E$. It therefore seems that $\langle d_{ii1} \rangle_E$ will more accurately show the asymptotic inertial-range power law than does $\langle d_{111} \rangle_E$ (or $\langle d_{221} \rangle_E$ or $\langle d_{331} \rangle_E$). This result for the homogeneous case extends to the locally homogeneous case as follows: For inhomogeneous turbulence, the non-vanishing part of $\langle \tau_{ii} \rangle_E$, i.e. $\langle \tau_{ii} \rangle_E = 2\partial_{X_i} \langle (p - p')(u_i - u'_i) \rangle_E$ (see (2.11)) is expected to approach zero rapidly as r decreases for two reasons. First, $\langle (p - p')(u_i - u'_i) \rangle_E$ vanishes on the basis of local isotropy. Second, the operator ∂_{X_i} causes $\partial_{X_i} \langle (p - p')(u_i - u'_i) \rangle_E$ to vanish on the basis of local homogeneity. From (2.14), $\langle w \rangle_E$ contains the terms $\nu \partial_{X_n}\partial_{X_n} \langle d_{ii} \rangle_E/2$ and $-2\nu \partial_{X_n}\partial_{X_n} \langle p + p' \rangle_E$; because of the operator $\partial_{X_n}\partial_{X_n}$, these terms vanish on the basis of local homogeneity. Thus, all terms in $\langle w \rangle_E$ are negligible for locally homogeneous turbulence. By performing the trace, it appears that anisotropy has been significantly reduced in (3.2) relative to in (3.1) for the high-Reynolds-number, locally homogeneous case such that the above hypothesis is extended to locally homogeneous turbulence. The hypothesis should be checked by comparison with anisotropic DNS. Evaluation of all terms in (3.1) and (3.2) is the basis for such an investigation. The above discussion holds for temporal and spatial averages as well.

To determine scaling properties of the third-order structure function, past theory has used the isotropic formulas. One can use an equation like (3.2) or its temporal-average analogue without an assumption about the symmetry properties (e.g. isotropic) of the structure functions by means of the sphere average in $\boldsymbol{r}$-space. Without approximation, the $\boldsymbol{r}$-space sphere average produces the orientation-averaged third-order structure function. It would seem that the orientation average mitigates anisotropy effects. Thus, the orientation average $\oint_{r_n} \langle d_{iin} \rangle_E$ (or $\oint_{r_n} \langle d_{iin} \rangle_T$, or $\oint_{r_n} \langle d_{iin} \rangle_{\mathbb{R}}$) is expected to best exhibit scaling properties of locally isotropic turbulence, such as the inertial-range power law with the 4/3 coefficient in (3.21). This hypothesis should be checked by comparison with anisotropic DNS.

Consider the stationary, homogeneous case. From (3.2), $\partial_{r_n} \langle d_{iin} \rangle_E - 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ii} \rangle_E$ scales with $\langle \varepsilon + \varepsilon' \rangle_E$ because $(\partial_{r_n} \langle d_{iin} \rangle_E - 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ii} \rangle_E)/\langle \varepsilon + \varepsilon' \rangle_E = -2$, thereby ensuring K41 scaling of $\partial_{r_n} \langle d_{iin} \rangle_E - 2\nu \partial_{r_n} \partial_{r_n} \langle d_{ii} \rangle_E$ despite anisotropy. In contrast, (3.1) ensures that scaling only if local isotropy is invoked. Anisotropic DNS can be used to check whether or not K41 scaling is improved by performing the trace.

4.2. *Tests using DNS and experimental data*

The spatially periodic DNS case leads to especially simple equations. It seems that (3.13)–(3.14) offer an ideal opportunity to evaluate the contribution of $\partial_t \langle d_{ij} \rangle_{\mathbb{R}}$ for freely decaying turbulence, and of $\langle \tau_{ij} \rangle_{\mathbb{R}}$ for anisotropic turbulence, as well as the balance of the off-diagonal components of (3.13). Because we have not introduced a force generating the turbulence and because every point in the flow enters into the $\boldsymbol{X}$-space average, the DNS must be freely decaying. As shown in Hill (2002), it is straightforward to include forces in the equations. DNS can completely evaluate terms in the exact structure-function equations.

4.3. *Effect of inhomogeneity on incompressibility conditions*

Exact incompressibility relationships (3.3)–(3.4) are obtained that can be used to quantify the non-zero value of $\partial_{r_n} \langle d_{jn} \rangle_E$ (or of $\partial_{r_n} \langle d_{jn} \rangle_T$, or of $\partial_{r_n} \langle d_{jn} \rangle_{\mathbb{R}}$) caused by inhomogeneity. If inhomogeneity is only in the streamwise (say 1-axis) direction, then the time average gives $\partial_{r_n} \langle d_{jn} \rangle_T = \partial_{X_1} \langle (u_j + u'_j)(u_1 - u'_1) \rangle_T / 2$, which can be evaluated using anemometers. As $r \to 0$, (3.4) becomes the second derivative with respect to measurement location of the velocity variance and therefore clearly depends on flow inhomogeneity.

4.4. *Quantifying effects of inhomogeneity and anisotropy on scaling exponents*

Sreenivasan & Dhruva (1998) note that one could determine scaling exponents with greater confidence if one had a theory that exhibits not only the asymptotic power law but also the trend toward the power law. Such a theory must require difficult measurements or DNS to evaluate such trends. The equations given here are the required theory for the third-order structure function, given that data must be used to evaluate the equations in a manner analogous to previously cited evaluations. In fact, it is not possible that exact equations do not contain the physical effects discussed by Sreenivasan & Dhruva (1998). They discuss the fact that there is correlation of velocity increments with large-scale velocity in inhomogeneous turbulence, even for very large Reynolds numbers and r in the inertial range, but not so in isotropic turbulence. Our term $\partial_{X_n} \langle F_{iin} \rangle_E = \partial_{X_n} \langle |\boldsymbol{u} - \boldsymbol{u}'|^2 (u_n + u'_n)/2 \rangle_E$ in (3.2), and all such analogous terms in the other equations, explicitly contains such correlation, and the balance of the equations imparts that correlation effect to the other statistics; all such terms do vanish for isotropic turbulence. They also discuss the usefulness of graphing all three terms in (3.23) to discern the onset of the dissipation range. Our equations are exact there too.

4.5. *Quantifying effects of large-scale structure on small-scale structure*

Experimenters remove the mean from an anemometer's signal before calculating structure functions from the velocity fluctuations, whereas the exact dynamical equations contain statistics of the full velocity field. Hill (2002) applied the Reynolds decomposition to the above exact dynamical equations, and used inertial-range and viscous-range asymptotics to determine the approximate dynamical equations pertaining to statistics of fluctuations as well as all approximations that are required to obtain the approximate equations. The Reynolds decomposition produces terms that quantify the effect of the large-scale structure of turbulence on the small scales. For example, $\partial_{X_n}\langle F_{ijn}\rangle_E$ produces a generalization of the advective term discovered by Lindborg (1999). Hill (2002) contrasts the various definitions of local homogeneity and points out that the only definition that simplifies dynamical equations is that from Hill (2001).

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