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AN EIGENVALUE CHARACTERISATION OF THE DUAL EDM CONE

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Abstract

We show that the elements of the dual of the Euclidean distance matrix cone can be described via an inequality on a certain weighted sum of its eigenvalues.

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Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix. Given a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n \backslash \{0\}$, we can associate a Euclidean distance matrix (EDM) $\Delta = \Delta^v$ (of embedding dimension one) by declaring the entries of Δ to be $\Delta_{ij} = (v_i - v_j)^2$. Let $(\cdot,\cdot)_{\mathrm{F}}$ denote the Frobenius pairing on $\mathbb{R}^{n\times n}$ given by $(A, B)_{\mathrm{F}} := \mathrm{tr}(AB^t)$. If $\mathbb{EDM}_n$ denotes the cone of Euclidean distance matrices in $\mathbb{R}^{n\times n}$, we denote by $\mathbb{EDM}_n^*$ the cone dual to $\mathbb{EDM}$, namely,

$$\mathbb{EDM}_n^* := \{A \in \mathbb{R}^{n\times n} : (A, B)_{\mathrm{F}} \geq 0 \text{ for all } B \in \mathbb{EDM}_n\}.$$

The purpose of this note is to prove the following result.

THEOREM 1. *Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then $A \in \mathbb{EDM}_n^*$ if and only if*

$$\lambda_1 \geq \sum_{k=2}^{n} r_k \lambda_k$$

for all Perron weights $0 \leq r_k \leq 1$.

REMARK 2. Let us explain the meaning of the nonstandard terminology *Perron weights*. A symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ is said to be a Euclidean distance matrix if there is a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ such that $\Sigma_{ij} = (v_i - v_j)^2$. The well-known Schoenberg criterion [3] states that a symmetric hollow matrix Σ (that is, a matrix with nonzero entries on its diagonal) is a Euclidean distance matrix if and only if it is

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negative semi-definite on the hyperplane $H = \{x \in \mathbb{R} : x^t\mathbf{e} = 0\}$, where $\mathbf{e} = (1, \ldots, 1)^t$. Of course, a Euclidean distance matrix is, in particular, a nonnegative matrix in the sense that each entry of the matrix is a nonnegative real number. As a consequence, the Perron–Frobenius theorem asserts that the largest eigenvalue of the Euclidean distance matrix Σ is positive and occurs with eigenvector in the nonnegative orthant $\mathbb{R}^n_{\geq 0}$. This eigenvalue is often called the Perron root of Σ, denoted $r = r(\Sigma)$. Therefore, if $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ denote the eigenvalues of a nontrivial Euclidean distance matrix Σ (that is, a Euclidean distance matrix with $\delta_1 > 0$), then $\delta_1 > 0$ and $\delta_2, \ldots, \delta_n \leq 0$. The Perron weights are then defined by

$$r_k := -\frac{\delta_k}{\delta_1} \in [0, 1] \quad \text{for } 2 \leq k \leq n.$$

REMARK 3. Theorem 1 can be described in terms of zonahedra. Given a set of vectors $w_1, \ldots, w_k \in \mathbb{R}^d$, the zonahedron generated by $w_1, \ldots, w_k$ is the Minkowski sum of line segments connecting each of the points. That is, for $\vartheta_1, \ldots, \vartheta_k \in [0, 1]$, the zonahedron generated by $w_1, \ldots, w_k$ is the set

$$Z(w_1, \ldots, w_k) := \left\{ \sum_{i=1}^{k} \vartheta_i w_i : \vartheta_i \in [0, 1] \right\}.$$

The dual EDM cone, therefore, consists of those symmetric matrices whose largest eigenvalue lies on the right of the zonahedron formed by the remaining eigenvalues.

PROOF OF THEOREM 1. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of the symmetric matrix $A \in \mathbb{R}^{n\times n}$ and let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ denote the eigenvalues of a Euclidean distance matrix Δ. If $A = U^t \text{diag}(\lambda) U$ and $\Delta = V^t \text{diag}(\delta) V$ denote the eigenvalue decompositions for A and Δ,

$$\begin{aligned} \text{tr}(A\Delta) &= \text{tr}(U^t \text{diag}(\lambda) U V^t \text{diag}(\delta) V) = \text{tr}(V U^t \text{diag}(\lambda) U V^t \text{diag}(\delta)) \\ &= \text{tr}(Q^t \text{diag}(\lambda) Q \text{diag}(\delta)) \\ &= \sum_{i,j} \lambda_i \delta_j Q_{ij}^2, \end{aligned}$$

where $Q = UV^t$ is orthogonal. The Hadamard square (that is, the matrix $Q \circ Q$ with entries Q_{ij}^2) of an orthogonal matrix is doubly stochastic (see, for example, [2]). The class of $n \times n$ doubly stochastic matrices forms a convex polytope, the Birkhoff polytope $\mathcal{B}^n$. The minimum of $\text{tr}(A\Delta)$ is given by

$$\min_{S \in \mathcal{B}^n} \sum_{i,j=1}^{n} \lambda_i \delta_j S_{ij}.$$

This function is linear in S and, therefore, achieves its minimum on the boundary of the Birkhoff polytope. The Birkhoff–von Neumann theorem tells us that $\mathcal{B}^n$ is the convex hull of the set of permutation matrices and, moreover, the vertices of $\mathcal{B}^n$ are precisely

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the permutation matrices. Hence,

$$\min_{S\in\mathcal{B}^n}\sum_{i,j=1}^{n}\lambda_i\delta_j S_{ij} = \min_{\sigma\in S_n}\sum_{i=1}^{n}\lambda_i\delta_{\sigma(i)},$$

where S_n denotes the symmetric group on n letters. An elementary argument (by induction, for instance) shows that

$$\min_{\sigma\in S_n}\sum_{i=1}^{n}\lambda_i\delta_{\sigma(i)} = \sum_{i=1}^{n}\lambda_i\delta_i.$$

From the discussion in Remark 2, this completes the proof. □

Applications to graph theory. In [1], I addressed the problem of when a weighted finite graph (with possibly negative weights) has nonnegative Dirichlet energy. I showed that the Dirichlet energy was nonnegative if and only if the matrix describing the weighting was an element of the dual EDM cone. If the matrix is symmetric, the graph is said to be directed. The main theorem of this note has the following corollary.

COROLLARY 4. *Let (G, A) be a directed weighted graph. Let $V(G) = \{x_1, \ldots, x_n\}$ denote the vertex set of G. The Dirichlet energy*

$$\mathcal{E}(f) := \sum_{i,j=1}^{n} A_{ij}(f(x_i) - f(x_j))^2$$

is nonnegative for all graph functions $f : V(G) \to \mathbb{R}$ if and only if the largest eigenvalue of A dominates the Perron-weighted average of the remaining eigenvalues.

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