Appendix D
Spinor fields

In this appendix we record the basics of spinor fields. We start with the properties of Dirac matrices in Euclidean space–time. The four Dirac matrices γ_μ, $\mu = 1, 2, 3, 4$, are 4×4 matrices with the properties

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\nu = 2\delta_{\mu\nu} \mathbb{1}. \tag{D.1}$$

So they anticommute: $\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu$, $\mu \neq \nu$. They can be chosen Hermitian and unitary, $\gamma_\mu^\dagger = \gamma_\mu = \gamma_\mu^{-1}$. The matrix

$$\gamma_5 \equiv -\gamma_1 \gamma_2 \gamma_3 \gamma_4 \tag{D.2}$$

anticommutes with the γ_μ, $\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu$, and it is also Hermitian and unitary, $\gamma_5 = \gamma_5^\dagger$, $\gamma_5^2 = \mathbb{1}$. A realization can be given in terms of tensor products of the 2×2 Pauli matrices σ_k, $k = 1, 2, 3$, and $\sigma_0 \equiv \mathbb{1}_{2\times 2}$: $\gamma_k = -\sigma_2 \otimes \sigma_k$, $\gamma_4 = \sigma_1 \otimes \sigma_0$, $\gamma_5 = \sigma_3 \otimes \sigma_0$. Usually one does not need a realization as almost all relations follow from the basic anticommutation relations (D.1). Other realizations are related by unitary transformations, which preserve the Hermiticity and unitarity of the Dirac matrices, but not the behavior under complex conjugation or transposition. It can be shown that, in every such realization, there is an antisymmetric unitary 4×4 matrix C, called the charge-conjugation matrix, which relates γ_μ to its transpose:

$$\gamma_\mu^{\mathrm{T}} = -C^\dagger \gamma_\mu C, \quad C^{\mathrm{T}} = -C, \quad C^\dagger C = \mathbb{1}, \tag{D.3}$$

$$\Rightarrow \gamma_5^{\mathrm{T}} = \gamma_5^* = C^\dagger \gamma_5 C. \tag{D.4}$$

In the above realization a possible C is given by $C = \sigma_3 \otimes \sigma_2$. The matrices $\Gamma = \mathbb{1}$, γ_μ, $(-i/2)[\gamma_\mu, \gamma_\nu]$, $i\gamma_\mu\gamma_5$ and γ_5 form a complete set of 16 independent Hermitian 4×4 matrices with the properties $\Gamma^2 = \mathbb{1}$, $\mathrm{Tr}\,\Gamma = 0$ except for $\Gamma = \mathbb{1}$, $\mathrm{Tr}\,(\Gamma\Gamma') = 0$ for $\Gamma \neq \Gamma'$. Useful relations are

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furthermore $\gamma_5\gamma_\kappa = \epsilon_{\kappa\lambda\mu\nu}\gamma_\lambda\gamma_\mu\gamma_\nu$, with $\epsilon_{\kappa\lambda\mu\nu}$ the completely antisymmetric Levi-Civita tensor, $\epsilon_{1234} = +1$, the trace of an odd number of γ_μ's is zero, $\mathrm{Tr}\,(\gamma_5\gamma_\mu\gamma_\nu) = 0$, and

$$\mathrm{Tr}\,(\gamma_\mu\gamma_\nu) = 4\delta_{\mu\nu}, \tag{D.5}$$

$$\mathrm{Tr}\,(\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu) = 4(\delta_{\kappa\lambda}\delta_{\mu\nu} - \delta_{\kappa\mu}\delta_{\lambda\nu} + \delta_{\kappa\nu}\delta_{\lambda\mu}), \tag{D.6}$$

$$\mathrm{Tr}\,(\gamma_5\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu) = -4\epsilon_{\kappa\lambda\mu\nu}. \tag{D.7}$$

More trace relations are given in most textbooks on relativistic field theory.

The Dirac matrices are used to describe covariance under (in our case) Euclidean rotations, which are elements of the group $SO(4)$. A rotation in the μ–ν plane over a small angle $\omega_{\mu\nu}$ can be written as

$$R_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu} + O(\omega^2), \quad \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{D.8}$$

$$= \delta_{\mu\nu} + i\tfrac{1}{2}\omega_{\kappa\lambda}(M_{\kappa\lambda})_{\mu\nu} + \cdots, \tag{D.9}$$

$$(M_{\kappa\lambda})_{\mu\nu} = -i(\delta_{\kappa\mu}\delta_{\lambda\nu} - \delta_{\kappa\nu}\delta_{\lambda\mu}). \tag{D.10}$$

The antisymmetry of $\omega_{\mu\nu}$ ensures that $R_{\mu\nu}$ is orthogonal, $R_{\kappa\mu}R_{\lambda\mu} = \delta_{\kappa\lambda}$, with $\det R = 1$. The $M_{\kappa\lambda}$ are the generators of $SO(4)$ in the defining representation. The structure constants $C^{\rho\sigma}_{\kappa\lambda\mu\nu}$ defined by $[M_{\kappa\lambda}, M_{\mu\nu}] = C^{\rho\sigma}_{\kappa\lambda\mu\nu}\, M_{\rho\sigma}$ are easily worked out.

The 4×4 *spinor representation* of these rotations can be written in terms of Dirac matrices as

$$\Lambda = e^{i\frac{1}{2}\omega_{\mu\nu}\Sigma_{\mu\nu}} = \mathbb{1} + i\tfrac{1}{2}\omega_{\mu\nu}\Sigma_{\mu\nu} + \cdots, \tag{D.11}$$

$$\Sigma_{\mu\nu} = -i\tfrac{1}{4}\,[\gamma_\mu, \gamma_\nu], \tag{D.12}$$

where the $\Sigma_{\mu\nu}$ are the generators in the spinor representation. They satisfy the same commutation relations as the $M_{\mu\nu}$, as follows from the basic relations (D.1). The matrices Λ are unitary,

$$\Lambda^\dagger = \Lambda^{-1}, \quad \text{Euclid.} \tag{D.13}$$

They form a unitary representation up to a sign, e.g. for a rotation over an angle 2π in the 1–2 plane, $\omega_{12} = -\omega_{21} = 2\pi$, and in the realization of the Dirac matrices introduced above, $\Lambda = \exp(\tfrac{1}{4}\omega_{\mu\nu}\gamma_\mu\gamma_\nu) = \exp(i\pi\sigma_0 \otimes \sigma_3) = -\mathbb{1}$.

The representation Λ is reducible, as follows from the fact that Λ commutes with γ_5, $[\Lambda, \gamma_5] = 0$. Introducing the projectors $P_{\mathrm{R,L}}$ onto the eigenspaces ± 1 of γ_5,

$$P_{\rm R} = \tfrac{1}{2}(\mathbb{1} + \gamma_5), \quad P_{\rm L} = \tfrac{1}{2}(\mathbb{1} - \gamma_5), \quad P_{\rm L}^2 = P_{\rm L}, \quad P_{\rm R}^2 = P_{\rm R},$$
$$P_{\rm L} P_{\rm R} = 0, \quad P_{\rm L} + P_{\rm R} = \mathbb{1}, \tag{D.14}$$

we can decompose Λ into two components $\Lambda_{\rm L}$ and $\Lambda_{\rm R}$ as

$$\Lambda = \Lambda P_{\rm L} + \Lambda P_{\rm R} \equiv \Lambda_{\rm L} + \Lambda_{\rm R}. \tag{D.15}$$

The $\Lambda_{\rm L}$ and $\Lambda_{\rm R}$ are inequivalent irreducible representations (up to a sign) of $SO(4)$. They are essentially two-dimensional, because the subspace of $\gamma_5 = 1$ or -1 is two-dimensional, but we shall keep them as 4×4 matrices. The Λ's are real up to equivalence,

$$\Lambda^* = e^{\frac{1}{4}\omega_{\mu\nu}\gamma_\mu^* \gamma_\nu^*} = e^{\frac{1}{4}\omega_{\mu\nu}\gamma_\mu^{\rm T} \gamma_\nu^{\rm T}} = C^\dagger e^{\frac{1}{4}\omega_{\mu\nu}\gamma_\mu \gamma_\nu} C$$
$$= C^\dagger \Lambda\, C, \tag{D.16}$$
$$\Lambda_{\rm L,R}^* = C^\dagger \Lambda_{\rm L,R}\, C. \tag{D.17}$$

The γ_μ are vector matrices in the sense that

$$\Lambda^\dagger\, \gamma_\mu\, \Lambda = R_{\mu\nu} \gamma_\nu. \tag{D.18}$$

This follows from the basic anticommutation relations between the γ's, as can easily be checked for infinitesimal rotations. Products $\gamma_\mu \gamma_\nu \cdots$ transform as tensors. Because $\gamma_\mu P_{\rm R,L} = P_{\rm L,R} \gamma_\mu$, the projected relations have the form $\Lambda_{\rm R}^\dagger\, \gamma_\mu\, \Lambda_{\rm L} = R_{\mu\nu} \gamma_\nu P_{\rm L}$, and similarly for L $\leftrightarrow$ R. It follows that

$$R_{\mu\nu} = \tfrac{1}{2}\, {\rm Tr}\, (\Lambda_{\rm R}^\dagger \gamma_\mu \Lambda_{\rm L} \gamma_\nu), \tag{D.19}$$

which illustrates the relation

$$SO(4) \simeq SU(2) \times SU(2)/Z_2 \tag{D.20}$$

(interpreted as 2×2 matrices, $\Lambda_{\rm L,R}$ are elements of $SU(2)$, and $Z_2 = \{1, -1\}$ compensates for $\Lambda_{\rm L,R}$ and $-\Lambda_{\rm L,R}$ giving the same R).

We can enlarge $SO(4)$ to $O(4)$ by adding reflections to the set of R's, which have determinant -1. An important one is parity $P \equiv {\rm diag}(-1, -1, -1, 1)$. Its spinor representation can be taken as $\Lambda_P = \gamma_4$, which has the expected effect on the γ_μ:

$$\gamma_4\, \gamma_\mu\, \gamma_4 = P_{\mu\nu}\, \gamma_\nu, \tag{D.21}$$

and it has therefore also the required effect on the generators $\Sigma_{\mu\nu}$, such that we have a representation of $O(4)$. Because $\gamma_4 P_{\rm L,R} \gamma_4 = P_{\rm R,L}$ we

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have $\gamma_4\Lambda_{\mathrm{L,R}}\gamma_4 = \Lambda_{\mathrm{R,L}}$. So we need both irreps L and R in order to be able to incorporate parity transformations.

Vector fields $V_\mu(x)$ transform under $SO(4)$ rotations as

$$V'_\mu(x) = R_{\mu\nu}V_\nu(R^{-1}x), \quad (R^{-1}x)_\mu = R_{\nu\mu}x_\nu, \tag{D.22}$$

which can be understood by drawing a vector field in two dimensions on a sheet of paper and seeing how it changes under rotations. Spinor fields $\psi(x)$ transform according to

$$\psi'_\alpha(x) = \Lambda_{\alpha\beta}\,\psi_\beta(R^{-1}x), \tag{D.23}$$

where α and β are matrix indices ('Dirac indices'). The fields can be decomposed into irreducible components as

$$\psi_{\mathrm{L}}(x) = P_{\mathrm{L}}\psi(x), \quad \psi_{\mathrm{R}}(x) = P_{\mathrm{R}}\psi(x). \tag{D.24}$$

It is customary to introduce a separate notation $\bar\psi$ for fields transforming with the inverse $\Lambda^\dagger$ as

$$\bar\psi'(x) = \bar\psi(R^{-1}x)\,\Lambda^\dagger \tag{D.25}$$

(so ψ is a column vector and $\bar\psi$ a row vector in the matrix sense). Under parity we have

$$\psi'(x) = \gamma_4\psi(Px), \quad \bar\psi'(x) = \bar\psi(Px)\gamma_4. \tag{D.26}$$

In general ψ and $\bar\psi$ are independent fields, but with the help of the charge-conjugation matrix C we can make a $\bar\psi$-type object out of ψ and vice-versa:

$$\begin{aligned} \bar\psi^{(\mathrm{c})} &\equiv -(C^\dagger\psi)^{\mathrm{T}} = \psi^{\mathrm{T}}C^\dagger, \quad \bar\psi^{(\mathrm{c})\prime}(x) = \bar\psi^{(\mathrm{c})}(R^{-1}x)\Lambda^{-1} \\ \psi^{(\mathrm{c})} &\equiv (\bar\psi C)^{\mathrm{T}} = -C\bar\psi^{\mathrm{T}}, \quad \psi^{(\mathrm{c})\prime}(x) = \Lambda\psi^{(\mathrm{c})}(R^{-1}x). \end{aligned} \tag{D.27}$$

The fields $\bar\psi^{(\mathrm{c})}$ and $\psi^{(\mathrm{c})}$ are called the charge conjugates of ψ and $\bar\psi$, respectively.

Note the standard notation for the projected $\bar\psi$'s,

$$\bar\psi_{\mathrm{L}} = \bar\psi\,P_{\mathrm{R}}, \quad \bar\psi_{\mathrm{R}} = \bar\psi\,P_{\mathrm{L}}. \tag{D.28}$$

This looks unnatural here but it is natural in the operator formalism where $\hat{\bar\psi}_{\mathrm{L,R}} \equiv \hat\psi^\dagger_{\mathrm{L,R}}\gamma_4 = \hat{\bar\psi}P_{\mathrm{R,L}}$. In the path-integral formalism (in real as well as imaginary time) one introduces independent generators $\psi_\alpha(x)$ and $\psi^+_\alpha(x)$ of a Grassmann algebra, which are related by Hermitian conjugation, such that $\psi_{\mathrm{L,R}} = P_{\mathrm{L,R}}\,\psi$ implies $\psi^+_{\mathrm{L,R}} = \psi^+\,P_{\mathrm{L,R}}$, and then $\bar{\psi}_{\rm L,R} \equiv \psi^{+}_{\rm L,R}\,\gamma_4$ also gives (D.28). The fields $\bar{\psi}_{\rm L,R}$ transform in representations equivalent to $\Lambda_{\rm R,L}$:

$$\bar{\psi}_{\rm L} \to \bar{\psi}_{\rm L}\,\Lambda_{\rm R}^{\dagger} \Rightarrow (\bar{\psi}_{\rm L}C)^{\rm T} \to \Lambda_{\rm R}\,(\bar{\psi}_{\rm L}C)^{\rm T}, \tag{D.29}$$

$$\bar{\psi}_{\rm R} \to \bar{\psi}_{\rm R}\,\Lambda_{\rm L}^{\dagger} \Rightarrow (\bar{\psi}_{\rm R}C)^{\rm T} \to \Lambda_{\rm L}\,(\bar{\psi}_{\rm R}C)^{\rm T}, \tag{D.30}$$

where we used (D.17) and for clarity used the arrow notation for transformations, while suppressing the space–time index x.

An $O(4)$ invariant action which contains all the types of fields introduced so far with a minimum number (>0) of derivatives is given by

$$\begin{aligned} S &= -\int d^4x\,\bar{\psi}(m+\gamma_\mu\partial_\mu)\psi \qquad\qquad (\text{D.31})\\ &= -\int d^4x\,\big[m(\bar{\psi}_{\rm L}\psi_{\rm R}+\bar{\psi}_{\rm R}\psi_{\rm L})+\bar{\psi}_{\rm L}\gamma_\mu\partial_\mu\psi_{\rm L}+\bar{\psi}_{\rm R}\gamma_\mu\partial_\mu\psi_{\rm R}\big]. \end{aligned}$$

Finally, we can get corresponding formulas for Minkowski space–time by raising indices in contractions such that there is always a contraction between an upper and a lower index, e.g. $\omega_{\mu\nu}\Sigma_{\mu\nu} = \omega^{\mu\nu}\Sigma_{\mu\nu}$ (we do not make a distinction between upper and lower indices in Euclidean space–time), and substituting $x^4 = x_4 \to ix^0 = -ix_0$, $\omega^{4k} = \omega_{4k} \to i\omega^{0k} = -i\omega_{0k}$. This implies that $\partial_4 \to -i\partial_0$, $\partial_0 = \partial/\partial x^0$. It is then also expedient to use $\gamma^0 = -\gamma_0 = -i\gamma_4$. We have to be careful with Hermiticity properties of Λ, because after the substitution it is no longer unitary:

$$\Lambda^{-1} = \beta\Lambda^{\dagger}\beta, \quad \beta \equiv i\gamma^0, \quad \text{Minkowski.} \tag{D.32}$$

In Minkowski space–time $\mu = 0,1,2,3$ and indices are raised and lowered with the metric tensor $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1,1,1,1)$, e.g. $\partial^0 = -\partial_0$, $\partial_k = \partial^k = \partial/\partial x^k$.