that the subject is ready for a new standard introductory text. The present book shares all the features that helped its predecessor become such a standard thirty years ago, and at the same time, it is modern, and it is relevant to today's state of the field. The subject will be well-served by it.

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Picture yourself a country with trails, grasslands, etc., inhabited by beautiful unicorns, in quite a number. In fact, most of the trails avoid those unicorns, but a few of them have the astonishing particularity of hosting herds of unicorns. Similarly, the unicorns are quite scarce on many grasslands, while on others they do appear but only along the trails, and on still other grasslands they appear in surprisingly large numbers throughout. For a long time, geographers could not really understand what was so special about the geography of those trails and grasslands, that were fully inhabited by unicorns, despite an attractive and convincing suggestion by unicorn ecologists. Remarkably, this inspired nereid ecologists to wonder whether a similar suggestion could explain the population of nereids in some rare rivers and ponds of a neighboring country.

This small book aims at unveiling a similar mathematical mystery.

Our mathematical countries, no less fantastic but absolutely real, are the Abelian varieties and the Shimura varieties, named after the mathematicians Niels Abel (1802–1829) and Goro Shimura (1930–). There is less poetry, however, in the name given to our magical beasts, respectively torsion points or special points.

By *varieties*, we mean here algebraic varieties, that is, loci defined by polynomial equations, say, with complex coefficients. Our trails are curves, our grasslands, surfaces, etc.

The most elementary examples of *Abelian varieties* are given by elliptic curves, each of them being the set of solutions in the projective plane of some cubic equation with nonzero discriminant. By Weierstrass's theory of bi-periodic functions, elliptic curves can also be described as the quotient of the complex plane $\mathbf{C}$ by a lattice $\mathbf{Z} + \mathbf{Z}\tau$, where τ, a complex number of positive imaginary part, is an element of Poincaré's upper half plane $\mathfrak{h}$.

More generally, Abelian varieties are those irreducible varieties which are endowed with a group law, defined by polynomials as well, and are, moreover, "compact" or, more precisely, projective; they can also be understood from the point of view of complex function theory, where they appear as (particular) complex tori, quotients of a complex affine space $\mathbf{C}^g$ by a lattice Λ. Torsion points are then defined as in group theory. A basic property is that an Abelian variety of dimension g contains n^{2g} points a such that $n \cdot a = 0$, for every integer $n \geq 1$; these are the images modulo Λ of the points of $n^{-1}\Lambda$.

Around 1960, Yuri Manin and David Mumford had conjectured that irreducible subvarieties of an Abelian variety which contain a *dense* set of torsion points must be Abelian subvarieties, or the image of such a subvariety under that translation by a torsion point. By "dense", we mean that those points are not contained in a subvariety of a smaller dimension—on remarkable grasslands, unicorns are not soleley populated along a few trails. This conjecture has been proved by Michel Raynaud in 1983 and many new beautiful proofs have been given since.

The simplest example of a *Shimura variety* is the *modular curve*, which parameterizes elliptic curves. Namely, it is just the quotient of the upper half plane by identifying two elements τ and τ' for which the lattices $\mathbf{Z} + \mathbf{Z}\tau$ and $\mathbf{Z} + \mathbf{Z}\tau'$ give rise to the same elliptic curve. It comes out that this corresponds to quotienting the upper half-plane $\mathfrak{h}$ by the group $\mathrm{SL}(2, \mathbf{Z})$ acting by homographies. In this case, the Jacobi j-function identifies this quotient with the complex plane $\mathbf{C}$. More generally, Shimura varieties are relatively easily defined from the point of view of complex function theory, where they appear as quotients of "symmetric

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hermitian domains" (very symmetric open subsets of some complex space $\mathbf{C}^m$) by the action of discrete subgroups of holomorphic automorphisms. However, the possibility of viewing them as algebraic varieties is a subtle theorem due to Walter Baily and Armand Borel.

One may roughly think of *Shimura varieties* as parameter spaces for Abelian varieties (or related objects), and this is one of the motivations for their study. In the important case of the full moduli space of Abelian varieties, *special points* correspond to Abelian varieties with *complex multiplication*, those Abelian varieties whose algebra of endomorphisms is as large as it can be. In the case of elliptic curves, it means that the complex number τ belongs to an imaginary quadratic field or, equivalently, by Theodor Schneider's theorem, that both τ and $j(\tau)$ be algebraic numbers.

In analogy with the conjecture of Manin–Mumford, Yves André and Frans Oort conjectured around 1990 that an irreducible subvariety of a Shimura variety which contains a dense set of special points must be a Shimura subvariety, or a connected component of the image of such a Shimura subvariety by the action of a Hecke correspondence.

André proved this conjecture in the special case of $\mathbf{C}^2$: the product of the modular curve by itself. This says that if an irreducible algebraic curve V in $\mathbf{C}^2$ contains infinitely many pairs of the form $(j(\tau_1), j(\tau_2))$ where τ_1 and τ_2 are imaginary quadratic complex numbers, then either it is parallel to the coordinate axes, or there exists an integer $n \geq 1$ such that $(j(\tau), j(n\tau))$ belongs to V for every $\tau \in \mathfrak{h}$.

Bas Edixhoven, partly in joint work with Andrei Yafaev, developed remarkable results in the direction of the general conjecture, but their approach required the Generalized Riemann Hypothesis.

Jonathan Pila and Umberto Zannier proposed a new strategy for proving the André–Oort conjecture and made it work in the framework of the Manin–Mumford conjecture. Soon, Pila was able to prove the André–Oort conjecture for arbitrary subvarieties of a product of modular curves, without any recourse to the Riemann Hypothesis. And in the next years, Jonathan Pila and Jacob Tsimerman on one side, and Bruno Klingler, Emmanuel Ullmo, and Andrei Yafaev on the other side, gave an almost definitive solution to the André–Ort conjecture. Only "almost", because the general case still requires the Riemann hypothesis; however, it is not needed in the case of subvarieties of the moduli space of Abelian varieties of dimensions ≤ 6, for example, and one may hope for further progress.

These remarkable advances have been made possible by the use of ingredients from "mathematical logic", namely *o-minimality*. O-minimal geometries were isolated around 1980 by Anand Pillay and Charles Steinhorn, and Lou van den Dries as an example of *tame geometries*. An o-minimal geometry is defined by families of subsets of $\mathbf{R}^n$, for all $n \in \mathbf{N}$, which are stable under union, complement, and projection, while the only allowed subsets of $\mathbf{R}$ are finite unions of intervals. Semi-algebraic sets, subanalytic sets, give rise to o-minimal geometries; a fundamental result of Alex Wilkie asserts that subsets of $\mathbf{R}^n$ defined using subanalytic functions and the exponential function also give rise to an o-minimal geometry.

This book, which stems out of a 2013 Instructional conference in Manchester, consists of 9 independent chapters, which, together, survey these developments as well as present some further results.

Before we detail their content, let us describe the strategy of Pila and Zannier. Let us thus consider a subvariety V of an ambient (Abelian or Shimura) variety A which contains a dense set Σ of "special" points; the goal is to prove that V is "special". One then introduces the complex uniformization $p\colon U \to A$, as described above—where the space U is an open subset of an affine space M. Defined by complex function theory, the map p is indeed highly transcendental, so that $p^{-1}(V)$ is an analytic subspace of U.

A first step consists in noting that special subvarieties are essentially those for which the irreducible components $\tilde{V}$ of $p^{-1}(V)$ can be defined in U by polynomial equations. If p were given by the exponential function, $(z_1, \ldots, z_n) \mapsto (\exp(z_1), \ldots, \exp(z_n))$, this would follow from a theorem of James Ax according to which if $f_1, \ldots, f_n$ are algebraic functions on a domain, linearly independent modulo constants, then $\exp(f_1), \ldots, \exp(f_n)$ are algebraically independent. This kind of result is viewed as a functional analogue of the Lindemann–Weierstrass theorem in transcendental number theory which states that

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exponentials of **Q**-linearly independent algebraic numbers are algebraically independent over **Q**—in particular, the relation $e^{2i\pi} = 1$ shows that π is a transcendental number.

How then can one detect from the set Σ the algebraic nature of $p^{-1}(V)$?

The second step incorporates number theory into this geometric picture. Namely, one observes that the points of $p^{-1}(\Sigma)$ have very particular coordinates in a good basis of M, either rational, or belonging to number fields of small degree. Moreover, their size (the precise term is *height*) is controlled. Since p is a covering map, each point of Σ gives rise to many points, and the arithmetic of Abelian varieties or the theory of complex multiplication can be exploited to produce further points on $p^{-1}(V)$. Then a theorem by Pila & Wilkie is invoked, that says that within a subset of $\mathbf{R}^n$ defined in an o-minimal geometry, the existence of "many points of bounded height" implies the existence of semi-algebraic subsets of positive dimension.

This last theorem is the main goal of chapter 2, written by Alex Wilkie. After a rapid introduction to o-minimal geometries, the proof of the Pila–Wilkie theorem is sketched. That proof follows a century-old pattern in transcendental number theory. In particular, it requires analytic estimates, the core of which follow from a reparametrization theorem due to Gromov and Yomdin. Chapter 8, by Gareth O. Jones, improves the bounds of the Pila–Wilkie theorem (from T^{ε} to $\log(T)^c$) in some cases.

Chapter 6, again written by Wilkie, provides a proof, due to Denef and van den Dries, that subanalytic subsets of $\mathbf{R}^n$ form an o-minimal geometry.

Chapters 1 and 4, respectively written by Philipp Habegger and Martin Orr, provide introductions to the theory of Abelian varieties with emphasis towards the proof of the Manin–Mumford conjecture. In particular, Habegger establishes the height bounds, due to David Masser, required by the strategy described above, while Orr proves the required theorem of Ax-Lindemann type.

In chapter 3, Jonathan Pila explains his proof of the functional theorem of Ax–Lindemann type which is required by his proof of the André–Oort conjecture for the product of modular curves. Actually, Pila also explains Ax's case of the exponential function. He also introduces Schanuel's conjecture, Ax's functional analogue, and Zilber's conjecture on intersection with tori. Jacob Tsimerman gives in chapter 9 a proof of the Ax–Schanuel theorem by o-minimality. (Ax's proof used differential algebra.)

The proof of the André–Oort conjecture following the strategy of Pila–Zannier is presented in chapter 5, written by Christopher Daw. By necessity, the theorems are often given without proofs.

Thomas Pink had proposed a dramatic expansion of the conjectures of Manin–Mumford and André–Oort, that encompasses Zilber's conjecture alluded to above, as well as the Mordell–Lang conjecture (a theorem of Faltings). Chapter 8, by David Masser, presents a particular case of this Zilber–Pink conjecture: the "relative Manin–Mumford conjecture" for Abelian varieties.

This collection of papers will furnish the reader with both an introduction to the statements of Manin–Mumford/André–Oort type, and a preview of the techniques used in their solution following the strategy of Pila–Zannier. As this strategy combines results from number theory, algebraic geometry, and mathematical logic, the reading of this book will certainly be of great help to any reader who wishes to study the research papers.

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Three Papers of Maryanthe Malliaris and Saharon Shelah

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