*Proceedings of the Royal Society of Edinburgh*, **146A**, 1091–1114, *2016*
DOI:10.1017/S0308210515000785

# Asymptotic behaviour of the lifespan of solutions for a semilinear heat equation in hyperbolic space

**Zhiyong Wang**
Department of Mathematics, Fujian Normal University, Fuzhou 350007, People's Republic of China (wangzhiyong236@163.com)

**Jingxue Yin***
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, People's Republic of China (yjx@scnu.edu.cn)

(MS received 30 December 2014; accepted 17 June 2015)

This paper is concerned with the asymptotic behaviour of the lifespan of solutions for a semilinear heat equation with initial datum $\lambda\phi(x)$ in hyperbolic space. The growth rates for both $\lambda \to 0$ and $\lambda \to \infty$ are determined.

*Keywords:* lifespan; hyperbolic space; asymptotic behaviour; heat equation

2010 *Mathematics subject classification:* Primary 35B44; 35K05; 35B40

## 1. Introduction

Consider the Cauchy problem

$$\left.\begin{aligned} u_t &= \Delta_{\mathbb{H}^N} u + \mathrm{e}^{\alpha t}|u|^{p-1}u, \quad && (x,t) \in \mathbb{H}^N \times (0, T_\lambda), \\ u(x,0) &= u_0(x) = \lambda\phi(x), && x \in \mathbb{H}^N, \end{aligned}\right\} \tag{1.1}$$

where $\Delta_{\mathbb{H}^N}$ is the Laplace–Beltrami operator on the $N$-dimensional hyperbolic space, $\alpha > 0$, $p > 1$, $\lambda$ is a positive parameter, $\phi$ is a non-negative bounded and continuous function in $\mathbb{H}^N$ that is not identically equal to zero, and $T_\lambda$ is the lifespan of the solution $u$.

Bandle *et al*. [3] recently established the global existence and blow-up profile for problem (1.1) with $\lambda = 1$ via the Fujita exponent

$$p_H^* = 1 + \frac{\alpha}{\lambda_0} \quad \text{with } \lambda_0 = \frac{(N-1)^2}{4}, \tag{1.2}$$

which is quite different to the case in Euclidean space $\mathbb{R}^N$. Indeed, they showed that if $1 < p < p_H^*$, then every non-trivial positive solution blows up in finite time, while if $p > p_H^*$, then the problem (1.1) possesses global solutions for small initial data. As for the critical exponent $p = p_H^*$, they proved that there exist non-trivial global positive solutions if $\alpha > \frac{2}{3}\lambda_0$. To our knowledge, this is the first work on the

*Corresponding author.

blow-up problem of heat equations on a manifold with negative sectional curvature. In a recent paper [16] we proved that there exist non-trivial global positive solutions for $p = p_H^*$ and $0 < \alpha \leqslant \frac{2}{3}\lambda_0$. It is worth mentioning that, from the conclusions in [3, 16], the problem on the hyperbolic space and the corresponding problem on bounded domains of $\mathbb{R}^N$ (see [13]) have a similar Fujita exponent, whereas the critical exponent in $\mathbb{H}^N$ is not a blow-up exponent but in $\mathbb{R}^N$ it is a blow-up one. Similar results on blow-up and global existence of solutions for parabolic equations or inequalities on more general Riemannian manifolds are established; see [12,15,18] and references therein.

Our interest lies in the asymptotic behaviour of the lifespan $T_\lambda$, i.e. the blow-up time of the solutions as $\lambda \to \infty$ and as $\lambda \to 0$, which is also different from the case in Euclidean space, namely,

$$\left.\begin{aligned} u_t &= \Delta u + \mathrm{e}^{\alpha t}|u|^{p-1}u, && (x,t) \in \mathbb{R}^N \times (0,T),\\ u(x,0) &= \lambda\phi(x), && x \in \mathbb{R}^N. \end{aligned}\right\} \tag{1.3}$$

In fact, for (1.3), as a direct consequence of the general discussion in [9] for $\alpha = 0$, the growth order of $T_\lambda$ as $\lambda \to 0$ can be estimated as

$$C_1 \frac{1}{\lambda^{p-1}} \leqslant T_\lambda \leqslant C_2 \frac{1}{\lambda^{p-1}},$$

where $C_1$, $C_2$ are positive constants, whereas Gui and Wang [6] showed that if $\lim_{|x|\to\infty} \phi(x) = A > 0$, then $\lim_{\lambda\to 0} \lambda^{p-1} T_\lambda = A^{1-p}/(p-1)$. For $\lambda \to \infty$, Gui and Wang also proved that

$$\lim_{\lambda\to\infty} \lambda^{p-1} T_\lambda = \frac{1}{p-1} \|\phi\|_{L^\infty(\mathbb{R}^N)}^{1-p},$$

provided that $\phi \gneqq 0$. Concerning equation (1.1) for $\alpha > 0$, we prefer to mention the corresponding problem in bounded domain for Euclidean space $\mathbb{R}^N$,

$$\left.\begin{aligned} u_t &= \Delta u + \mathrm{e}^{\alpha t}|u|^{p-1}u, && (x,t) \in \Omega \times (0,T),\\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,T),\\ u(x,0) &= \lambda\phi(x), && x \in \Omega, \end{aligned}\right\} \tag{1.4}$$

due to the similarity of the Fujita exponents in $\mathbb{H}^N$ and a bounded domain of $\mathbb{R}^N$. Payne and Phlippin [14, theorem 1] showed that

$$T_\lambda \leqslant C\lambda^{1-p}$$

for some constant $C > 0$.

The purpose of this paper is to study the asymptotic behaviour of the lifespan of solutions of (1.1) in the hyperbolic space as $\lambda \to 0$ and $\lambda \to \infty$ (for the definition of solutions see definition 2.1). We first consider the asymptotic behaviour as $\lambda \to 0$. Due to the presence of the Fujita exponent $p_H^*$, we should certainly distinguish the case in which $1 < p < p_H^*$ from the case in which $p \geqslant p_H^*$. However, we prefer first to give a general discussion that is valid for all $p$. We denote by $d(x,0)$ the hyperbolic distance between $x$ and 0 (see § 2.1). Let $B(\infty,\varepsilon)$ be the $\varepsilon$ neighbourhood of $\infty$, namely, $B(\infty,\varepsilon) = \{x \in \mathbb{H}^N \mid d(x,0) > 1/\varepsilon\}$.

THEOREM 1.1. *Let $p > 1$, let $\alpha > 0$ and let $\phi \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $\phi(x) \gneqq 0$. Suppose that there exists an $\varepsilon > 0$ such that*

$$\inf_{x \in B(\infty,\varepsilon)} \phi(x) > 0.$$

*Then*

$$\lim_{\lambda \to 0} \frac{T_\lambda}{\ln(1/\lambda)} = \frac{p-1}{\alpha}.$$

Here we have found an essential difference, namely, that the growth order is $\ln(1/\lambda)$, rather that $\lambda^{1-p}$ in the case for the Euclidean space. Remember that for $p$ in $(1, p_H^*)$ non-trivial solutions must blow up unconditionally, and so the following result is quite natural.

THEOREM 1.2. *Let $p > 1$, let $\alpha > 0$ and let $\phi \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $\phi(x) \gneqq 0$. If $1 < p < p_H^*$, then there exist $C_1, C_2 > 0$ such that*

$$C_1 \ln \frac{1}{\lambda} \leqslant T_\lambda \leqslant C_2 \ln \frac{1}{\lambda} \quad \text{as } \lambda \to 0.$$

Just as is known for the critical and supercritical cases, $p \geqslant p_H^*$ say, the solution will only blow up for large initial datum in a certain sense. However, the positive assumption of $u_0$ in $B(\infty, \varepsilon)$ can be weakened to a decay one. Indeed, we have the following delicate result.

THEOREM 1.3. *Let $p > 1$, let $\alpha > 0$ and let $\phi \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $\phi(x) \gneqq 0$. Let $p \geqslant p_H^*$ and let*

$$k^* = \frac{N-1}{2}\left(1 - \sqrt{\frac{p - p_H^*}{p-1}}\right).$$

*If for some $\varepsilon > 0$ and $0 < k < k^*$,*

$$\inf_{x \in B(\infty,\varepsilon)} \phi(x) \mathrm{e}^{kd(x,0)} > 0,$$

*then the solution $u(x,t;\lambda)$ blows up in finite time, and the lifespan of $u(x,t;\lambda)$ satisfies*

$$C_1 \ln \frac{1}{\lambda} \leqslant T_\lambda \leqslant C_2 \ln \frac{1}{\lambda} \quad \text{as } \lambda \to 0$$

*for some constants $C_1, C_2 > 0$.*

It is worth mentioning that the decay rate $k$ is optimal, since we may show that for $p > p_H^*$ and $k > k^*$ there exists a positive global solution of (1.1) with initial datum decaying more slowly than $\delta \mathrm{e}^{-kd(x,0)}$ in $B(\infty, \varepsilon)$ for some $\delta > 0$ (see appendix A).

Finally, we consider the asymptotic behaviour of the lifespan as $\lambda \to \infty$. We have a similar result to Gui and Wang's estimate of (1.3).

THEOREM 1.4. *Let $\alpha > 0$ and let $p > 1$. Let $\phi \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $\phi(x) \gneqq 0$. Then there exists $\Lambda = \Lambda(p, \phi, \alpha, N) \geqslant 0$ such that $T_\lambda < \infty$ for $\lambda > \Lambda$, and*

$$\lim_{\lambda \to \infty} \lambda^{p-1} T_\lambda = \frac{1}{p-1} \|\phi\|_{L^\infty(\mathbb{H}^N)}^{-(p-1)}. \tag{1.5}$$

The paper is organized as follows. In § 2 we present basic properties of the hyperbolic space. Afterwards we introduce the heat kernel in $\mathbb{H}^N$ and give an *a priori* estimate for the lifespan, which will be used for the upper bound estimate on the lifespan as $\lambda \to 0$. In § 3 we consider asymptotic behaviour of the lifespan as $\lambda \to 0$, and show the proofs of theorems 1.1–1.3. In § 4 we present the proof of theorem 1.4. Finally, we give a remark on the sharpness of the blow-up criterion in theorem 1.3 in the appendix.

## 2. Preliminaries

In this section we summarize the mathematical background of our problem and give some basic estimates to be used in our proofs.

### 2.1. Basic properties of the hyperbolic space

There are several models for the hyperbolic space: the Klein model, the hyperboloid model, the Poincaré model, etc. (see [4]). We shall use the Poincaré model here, namely, the $N$-dimensional hyperbolic space $\mathbb{H}^N$ that is the unit ball in $\mathbb{R}^N$,

$$I = \{x \in \mathbb{R}^N \mid |x| < 1\},$$

equipped with the Riemannian metric

$$\mathrm{d}s_I^2 = \frac{4((\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^N)^2)}{(1-|x|^2)^2},$$

where $|\cdot|$ is the standard Euclidean norm. We denote the hyperbolic distance between the points $x$ and $y$ by $d(x, y)$, and denote the ball in $\mathbb{H}^N$ with centre $x$ and radius $\varepsilon$ by

$$B_{\mathbb{H}^N}(x, \varepsilon) := \{y \in \mathbb{H}^N \mid d(x, y) < \varepsilon\}.$$

We call a bijective map $T\colon \mathbb{H}^N \to \mathbb{H}^N$ an isometry if $d(T(x), T(y)) = d(x, y)$ for $x, y \in \mathbb{H}^N$. Let 0 denote $(0, 0, \dots, 0) \in I$. From [1] we know that there exists an isometric translation

$$T_y(x) = \frac{(1-|y|^2)(x-y) - |x-y|^2 y}{[x, y]^2} \tag{2.1}$$

that maps $y \in \mathbb{H}^N$ to 0. Here $[x, y]^2 = 1 + |x|^2|y|^2 - 2\sum_{i=1}^N x_i y_i$.

In the geodesic coordinates $(r, \theta)$ the Laplace–Beltrami operator $\Delta_{\mathbb{H}^N}$ can be written as

$$\Delta_{\mathbb{H}^N} u = \frac{1}{\sinh^{N-1} r}\frac{\partial}{\partial r}\left(\sinh^{N-1} r \frac{\partial u}{\partial r}\right) + \frac{1}{\sinh^2 r}\Delta_\theta u.$$

Hence, for a radial function $u(r)$,

$$\Delta_{\mathbb{H}^N} u(r) = u''(r) + (N-1)\frac{\cosh r}{\sinh r}u'(r). \tag{2.2}$$

The volume element of $\mathbb{H}^N$ can be written as

$$\mathrm{d}\mu = (\sinh r)^{N-1}\,\mathrm{d}r\,\mathrm{d}\theta. \tag{2.3}$$

### 2.2. The heat kernel in the hyperbolic space

The solution of the linear heat equation

$$u_t = \Delta_{\mathbb{H}^N} u, \quad (x,t) \in \mathbb{H}^N \times (0,\infty),$$
$$u(x,0) = \phi(x), \qquad x \in \mathbb{H}^N,$$

can be written as

$$u(x,t) = \int_{\mathbb{H}^N} g_N(x,y,t)\phi(y)\,\mathrm{d}\mu_y.$$

Here $g_N$ is the heat kernel in $\mathbb{H}^N$, which is a function of the distance of $x, y \in \mathbb{H}^N$ and the time $t$, that is,

$$g_N(x,y,t) = k_N(d(x,y),t). \tag{2.4}$$

In this paper we deal with the so-called mild solutions.

Definition 2.1. A function $u \in C(\mathbb{H}^N \times [0,\tau]) \cap L^\infty(\mathbb{H}^N \times (0,\tau))$ for any $\tau \in [0,T)$ is called a *mild solution* of problem (1.1) if

$$u(x,t;\lambda) = \lambda \int_{\mathbb{H}^N} g_N(x,y,t)\phi(y)\,\mathrm{d}\mu_y + \int_0^t \left( \int_{\mathbb{H}^N} g_N(x,y,t-s)\mathrm{e}^{\alpha s}|u|^{p-1}u(y,s;\lambda)\,\mathrm{d}\mu_y \right) \mathrm{d}s \tag{2.5}$$

with $(x,t) \in \mathbb{H}^N \times [0,T)$.

For the local well-posedness and regularity for mild solutions to problem (1.1) with initial datum $u_0 \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ we refer the reader to [3].

We summarize some elementary properties of the heat kernel in the following lemma.

Lemma 2.2 (Davies [5, ch. 5]). *Let $x, y, z \in \mathbb{H}^N$ and let $s, t > 0$. Then*

(i) $g_N(x,y,t) = g_N(y,x,t)$;

(ii) $g_N(Tx,Ty,t) = g_N(x,y,t)$, *where $T$ is an isometry in $\mathbb{H}^N$;*

(iii) *the semigroup property*

$$\int_{\mathbb{H}^N} g_N(x,y,t)g_N(y,z,s)\,\mathrm{d}\mu_y = g_N(x,z,s+t) \tag{2.6}$$

*is satisfied;*

(iv) *we have conservation of probability, namely,*

$$\int_{\mathbb{H}^N} g_N(x,y,t)\,\mathrm{d}\mu_y = 1 \quad \text{for } x \in \mathbb{H}^N \text{ and } t > 0. \tag{2.7}$$

Finally, we recall the following heat kernel estimate.

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

LEMMA 2.3 (Davies [5, theorem 5.7.2]). *Let $N \geqslant 2$ and let $\lambda_0 = (N-1)^2/4$. Then there exists $c_N > 0$ such that*

$$c_N^{-1} h_N(d(x,y),t) \leqslant g_N(x,y,t) \leqslant c_N h_N(d(x,y),t) \tag{2.8}$$

*for $t > 0$ and $x, y \in \mathbb{H}^N$, where*

$$h_N(d,t) = (4\pi t)^{-N/2}(1+d)(1+d+t)^{(N-3)/2} \exp\left\{-\lambda_0 t - \frac{N-1}{2}d - \frac{d^2}{4t}\right\}. \tag{2.9}$$

REMARK 2.4. Let $\bar{h}_N(d,t) = (4\pi t)^{-N/2}\mathrm{e}^{-d^2/4t}$ denote the heat kernel in $\mathbb{R}^N$. By (2.8) and (2.9) we see that if $d(x,y)$ and $t$ are small, then $h_N$ and $\bar{h}_N$ are similar; if $d$ or $t$ are large, then $h_N$ decays more quickly than $\bar{h}_N$.

### 2.3. An *a priori* estimate on the lifespan

In this section we shall prove an *a priori* linear estimate that will be used for proving the upper bound estimates on the lifespan as $\lambda \to 0$ in § 3.

First we show the following version of Jensen's inequality in the hyperbolic space.

LEMMA 2.5. *Let $p > 1$ and let $f \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $f \geqslant 0$. Then*

$$\int_{\mathbb{H}^N} g_N(x,y,t) f^p(y)\,\mathrm{d}\mu_y \geqslant \left(\int_{\mathbb{H}^N} g_N(x,y,t) f(y)\,\mathrm{d}\mu_y\right)^p \tag{2.10}$$

*for $x \in \mathbb{H}^N$ and $t > 0$.*

*Proof.* Fix $x \in \mathbb{H}^N$ and $t > 0$, and define the measure

$$\mathrm{d}\bar{\mu}_y = g_N(x,y,t)\,\mathrm{d}\mu_y$$

on $\mathbb{H}^N$. Since the conservation of probability (2.7) implies that

$$\int_{\mathbb{H}^N} \mathrm{d}\bar{\mu}_y = \int_{\mathbb{H}^N} g_N(x,y,t)\,\mathrm{d}\mu_y \equiv 1,$$

by Jensen's inequality [10, theorem 2.2] we obtain (2.10). □

We now prove the *a priori* estimate on the lifespan of the solution of (1.1). The proof is similar to that of the blow up of the non-trivial positive solutions (1.1) in $1 < p < p_H^*$ (see [3]). This kind of estimate was introduced by Weissler [17] for proving the blow up of the non-trivial positive solutions of (1.3) with $\alpha = 0$ in the critical case; see also [7, ch. 5]. For the convenience of the reader, we present it here.

LEMMA 2.6. *Suppose that $u$ is a solution of the problem (1.1) with non-negative initial datum $u_0(x) \in L^\infty(\mathbb{H}^N) \cap C(\mathbb{H}^N)$ in $[0,T)$. Then*

$$\left(\int_{\mathbb{H}^N} g_N(0,y,t) u_0(y)\,\mathrm{d}\mu_y\right)^{1-p} \geqslant \frac{p-1}{\alpha}(\mathrm{e}^{\alpha t} - 1) \quad \text{for } 0 < t < T. \tag{2.11}$$

*In particular,*

$$T^* \leqslant \sup\{T > 0 \mid (2.11) \text{ holds}\},$$

*where $T^*$ is denoted by the lifespan of the solution $u$.*

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

*Proof.* Fix $0 < t < T$. Taking $\bar t \in [0,t]$, we have

$$u(x,\bar t) = \int_{\mathbb{H}^N} g_N(x,y,\bar t)u_0(y)\,\mathrm{d}\mu_y + \int_0^{\bar t}\int_{\mathbb{H}^N} g_N(x,y,\bar t - s)\mathrm{e}^{\alpha s}|u|^{p-1}u(y,s)\,\mathrm{d}\mu_y\,\mathrm{d}s. \tag{2.12}$$

Multiplying (2.12) by $g_N(x,0,t-\bar t)$ and integrating with respect to $x$ over $\mathbb{H}^N$, we obtain

$$\begin{aligned}\int_{\mathbb{H}^N} & g_N(x,0,t-\bar t)u(x,\bar t)\,\mathrm{d}\mu_x \\ &= \int_{\mathbb{H}^N} g_N(x,0,t-\bar t)\int_{\mathbb{H}^N} g_N(x,y,\bar t)u_0(y)\,\mathrm{d}\mu_y\,\mathrm{d}\mu_x \\ &\quad + \int_{\mathbb{H}^N} g_N(x,0,t-\bar t)\int_0^{\bar t}\int_{\mathbb{H}^N} g_N(x,y,\bar t - s)\mathrm{e}^{\alpha s}|u|^{p-1}u(y,s)\,\mathrm{d}\mu_y\,\mathrm{d}s\,\mathrm{d}\mu_x.\end{aligned}$$

By (2.6), lemma 2.5 and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{H}^N} & g_N(x,0,t-\bar t)u(x,\bar t)\,\mathrm{d}\mu_x \\ &= \int_{\mathbb{H}^N} g_N(y,0,t)u_0(y)\,\mathrm{d}\mu_y + \int_0^{\bar t}\int_{\mathbb{H}^N} g_N(0,y,t-s)\mathrm{e}^{\alpha s}|u|^{p-1}u(y,s)\,\mathrm{d}\mu_y\,\mathrm{d}s \\ &\geqslant \int_{\mathbb{H}^N} g_N(y,0,t)u_0(y)\,\mathrm{d}\mu_y + \int_0^{\bar t}\mathrm{e}^{\alpha s}\bigg(\int_{\mathbb{H}^N} g_N(0,y,t-s)u(y,s)\,\mathrm{d}\mu_y\bigg)^p\mathrm{d}s.\end{aligned} \tag{2.13}$$

Denoting the right-hand side of (2.13) by

$$G(\bar t) = \int_{\mathbb{H}^N} g_N(y,0,t)u_0(y)\,\mathrm{d}\mu_y + \int_0^{\bar t}\mathrm{e}^{\alpha s}\bigg(\int_{\mathbb{H}^N} g_N(0,y,t-s)u(y,s)\,\mathrm{d}\mu_y\bigg)^p\mathrm{d}s,$$

we have

$$G(0) = \int_{\mathbb{H}^N} g_N(y,0,t)u_0(y)\,\mathrm{d}\mu_y. \tag{2.14}$$

Differentiating $G(\bar t)$ with respect to $\bar t$, by (2.13) we obtain

$$G'(\bar t) = \mathrm{e}^{\alpha\bar t}\bigg(\int_{\mathbb{H}^N} g_N(y,0,t-\bar t)u(y,\bar t)\,\mathrm{d}\mu_y\bigg)^p \geqslant \mathrm{e}^{\alpha\bar t}G^p(\bar t),$$

that is,

$$G^{-p}(\bar t)G'(\bar t) \geqslant \mathrm{e}^{\alpha\bar t}.$$

Integrating with respect $\bar t$ over $[0,t]$, we obtain

$$\frac{1}{1-p}(G^{1-p}(t) - G^{1-p}(0)) \geqslant \frac{1}{\alpha}(\mathrm{e}^{\alpha t} - 1).$$

Thus,

$$G^{1-p}(0) \geqslant \frac{p-1}{\alpha}(\mathrm{e}^{\alpha t} - 1) \tag{2.15}$$

since $p > 1$. The lemma follows from (2.14) and (2.15) immediately. □

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

## 2.4. Notation

We conclude this section by introducing some notation. Let $F$ and $G$ be two non-negative functions defined on $X$. If there exists $C > 0$ such that $F(x) \leqslant CG(x)$ for $x \in X$, we write $F(x) \lesssim G(x)$ or $G(x) \gtrsim F(x)$. If $F(x) \lesssim G(x)$ and $G(x) \lesssim F(x)$, we write $F(x) \sim G(x)$. If $C$ depends on some variables or functions $a_1, a_2, \dots, a_k$ and $F < CG$, we write $F \lesssim_{a_1,a_2,\dots,a_k} G$. We define the notation $\gtrsim_{a_1,a_2,\dots,a_k}$ and $\sim_{a_1,a_2,\dots,a_k}$ similarly.

# 3. Lifespan estimates as $\lambda \to 0$

In this section we consider the lifespan as $\lambda \to 0$. The initial data will be small as $\lambda \to 0$, and the long time behaviour of the heat kernel and the time-weighted term will effect the blow-up time, so the results in this section will be significantly different from the results in the Euclidean space.

It is easy to get the following lower bound estimate on the lifespan as $\lambda \to 0$, which shows that $T_\lambda \to \infty$ as $\lambda \to 0$ or that $u(\cdot;\lambda)$ becomes a global solution.

Proposition 3.1. *If $p > 1$, then*

$$\liminf_{\lambda \to 0} \frac{T_\lambda}{\ln(1/\lambda)} \geqslant \frac{p-1}{\alpha}. \tag{3.1}$$

*Proof.* Consider the following ordinary differential equation (ODE) for $v(t;\lambda)$, where $\lambda > 0$ is a parameter:

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= \mathrm{e}^{\alpha t} v^p, \\ v(0;\lambda) &= \lambda \|\phi\|_{L^\infty(\mathbb{H}^N)}. \end{aligned}$$

By a direct calculation we obtain

$$v(t;\lambda) = \left[\frac{1-p}{\alpha}\left(\mathrm{e}^{\alpha t} - 1 + \frac{\alpha}{1-p}(\lambda\|\phi\|_{L^\infty(\mathbb{H}^N)})^{1-p}\right)\right]^{1/(1-p)}.$$

Thus, $v(t;\lambda)$ blows up at

$$\bar{T}_\lambda := \frac{1}{\alpha} \ln\left[1 + \frac{\alpha}{p-1}(\lambda\|\phi\|_{L^\infty(\mathbb{H}^N)})^{1-p}\right].$$

Noticing that $v$ is an upper solution of $u(x,t;\lambda)$, by the comparison principle in [3] we obtain $u(x,t;\lambda) \leqslant v(t;\lambda)$. Thus,

$$T_\lambda \geqslant \bar{T}_\lambda = \frac{1}{\alpha} \ln\left[1 + \frac{\alpha}{p-1}(\lambda\|\phi\|_{L^\infty})^{1-p}\right], \tag{3.2}$$

which implies (3.1) by L'Hospital's rule. □

To obtain the upper bound estimates on the lifespan as $\lambda \to 0$, we use the following *general strategy*.

Step 1. Prove linear estimates on the initial data.

Step 2. Use the results in step 1 and lemma 2.6 to derive the upper bound estimates.

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

### 3.1. A sharp lifespan estimate for the non-decaying initial datum

To prove theorem 1.1, by proposition 3.1, it is sufficient to show that

$$\limsup_{\lambda\to 0} \frac{T_\lambda}{\ln(1/\lambda)} \leqslant \frac{p-1}{\alpha}. \tag{3.3}$$

We shall prove (3.3) by the strategy stated at the beginning of this section.

We claim the following lemma, which will be proved later.

LEMMA 3.2. *Fix $\tau > 0$. Suppose that $\phi$ satisfies the assumptions in theorem 1.1. Then*

$$\inf_{x\in\mathbb{H}^N} \int_{\mathbb{H}^N} g_N(x,y,\tau)\phi(y)\,\mathrm{d}\mu_y > 0.$$

To prove (3.3), we need only supplement lemma 3.2 with the following lemma.

LEMMA 3.3. *Fix $\tau > 0$. Suppose that $\phi$ satisfies the assumptions in theorem 1.1. Then there exists $\varepsilon > 0$ such that*

$$\int_{\mathbb{H}^N} g_N(x,y,t)\phi(y)\,\mathrm{d}\mu_y \geqslant \varepsilon \quad \textit{for } t > \tau. \tag{3.4}$$

*Proof.* We fix $\tau$ in lemma 3.2 and

$$\varepsilon = \inf_{x\in\mathbb{H}^N} \int_{\mathbb{H}^N} g_N(x,y,\tau)\phi(y)\,\mathrm{d}\mu_y > 0.$$

Then, by (i), (iii) and (iv) in lemma 2.2, we have

$$\begin{aligned}
\int_{\mathbb{H}^N} g_N(x,y,t)\phi(y)\,\mathrm{d}\mu_y &= \int_{\mathbb{H}^N} \left(\int_{\mathbb{H}^N} g_N(x,z,t-\tau)g_N(y,z,\tau)\,\mathrm{d}\mu_z\right)\phi(y)\,\mathrm{d}\mu_y \\
&= \int_{\mathbb{H}^N} g_N(x,z,t-\tau)\left(\int_{\mathbb{H}^N} g_N(z,y,\tau)\phi(y)\,\mathrm{d}\mu_y\right)\mathrm{d}\mu_z \\
&\geqslant \varepsilon \int_{\mathbb{H}^N} g_N(x,z,t-\tau)\,\mathrm{d}\mu_z \\
&= \varepsilon \quad \text{for } t > \tau.
\end{aligned}$$

The proof is complete. □

We now give the proof of theorem 1.1.

*Proof of theorem 1.1.* Fix $\tau$ and $\varepsilon$ in lemma 3.3. By proposition 3.1 there exists $\Lambda > 0$ such that $T_\lambda > \tau$ for $\lambda < \Lambda$. Then, by lemma 2.6 and (3.4),

$$(\lambda\varepsilon)^{1-p} \geqslant \left(\int_{\mathbb{H}^N} g_N(x,y,t)\lambda\phi(y)\,\mathrm{d}\mu_y\right)^{1-p} \geqslant \frac{p-1}{\alpha}(\mathrm{e}^{\alpha t}-1)$$

for $\lambda < \Lambda$ and $t \in [\tau, T_\lambda)$. Thus,

$$(\lambda\varepsilon)^{1-p} \geqslant \frac{p-1}{\alpha}(\mathrm{e}^{\alpha T_\lambda}-1) \quad \text{for } \lambda < \Lambda. \tag{3.5}$$

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

Since $\lambda \to 0$ implies that $T_\lambda \to \infty$, by (3.5) and L'Hospital's rule we obtain

$$\limsup_{\lambda \to 0} \frac{T_\lambda}{\ln(1/\lambda)} \leqslant \frac{p-1}{\alpha},$$

that is, (3.3) holds. The theorem follows from (3.1) and (3.3). □

To prove lemma 3.2 we first give some lemmas.

Lemma 3.4. *Denote by $d = d(x,y)$ the distance between $x$ and $y$ in $\mathbb{H}^N$. Then*

$$\int_{\mathbb{H}^N} (1+d)(1+d+t)^{(N-3)/2} \exp\left\{-\frac{N-1}{2}d - \frac{d^2}{4t}\right\} \mathrm{d}\mu_y \sim t^{N/2} \mathrm{e}^{\lambda_0 t}$$

*for $x \in \mathbb{H}^N$ and $t > 0$.*

*Proof.* It is equivalent to prove that for $x \in \mathbb{H}^N$ and $t > 0$,

$$\int_{\mathbb{H}^N} t^{-N/2}(1+d)(1+d+t)^{(N-3)/2} \exp\left\{-\lambda_0 t - \frac{N-1}{2}d - \frac{d^2}{4t}\right\} \mathrm{d}\mu_y \sim 1. \quad (3.6)$$

By the kernel estimate in lemma 2.3, (3.6) is equivalent to

$$\int_{\mathbb{H}^N} g_N(x,y,t)\, \mathrm{d}\mu_y \sim 1,$$

which follows from the conservation of probability (2.7). This completes the proof. □

Lemma 3.5.
(i) *If there exist $m, M > 0$ such that*

$$0 < m < \alpha, \beta < M,$$

*then*

$$1 + d \sim_{m,M} \alpha + \beta d \quad \textit{for } d > 0.$$

(ii) *If there exist $m, M > 0$ such that*

$$0 < m < \alpha, \beta, \gamma < M,$$

*then*

$$1 + d + t \sim_{m,M} \alpha + \beta d + \gamma t \quad \textit{for } d > 0,\ t > 0.$$

(iii) *Let $t > 0$ and let $x, y \in \mathbb{H}^N$. If there exist $m, M > 0$ such that*

$$0 < m < \alpha, \beta, \tilde{\alpha}, \tilde{\beta}, \frac{\tilde{\gamma}}{\gamma} < M,$$

*then*

$$\int_{\mathbb{H}^N} (\alpha+\beta d)(\tilde{\alpha}+\tilde{\beta}d+\tilde{\gamma}t)^{(N-3)/2} \exp\left\{-\frac{N-1}{2}d - \frac{d^2}{\gamma t}\right\} \mathrm{d}\mu_y \sim_{m,M} (\gamma t)^{N/2} \mathrm{e}^{\lambda_0(\gamma t/4)},$$

*where we write $d = d(x,y)$.*

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

*Proof.* By

$$\alpha + \beta d \leqslant M(1+d) \quad \text{and} \quad 1 + d \leqslant \frac{1}{m}(\alpha + \beta d),$$

we obtain (i).

Since the proof of (ii) is similar to that of (i), we omit it.

By (i) and (ii), it follows that

$$\int_{\mathbb{H}^N} (\alpha + \beta d)(\tilde{\alpha} + \tilde{\beta} d + \tilde{\gamma} t)^{(N-3)/2} \mathrm{e}^{-(N-1)d/2 - d^2/\gamma t} \, \mathrm{d}\mu_y$$
$$\sim_{m,M} \int_{\mathbb{H}^N} (1+d)(1 + d + \tfrac{1}{4}\gamma t)^{(N-3)/2} \mathrm{e}^{-(N-1)d/2} \mathrm{e}^{-d^2/4(\gamma t/4)} \, \mathrm{d}\mu_y,$$

which together with lemma 3.4 implies (iii). □

LEMMA 3.6. *If $d(y,0) \leqslant 1$, then for every $x \in \mathbb{H}^N$ and $t > 0$,*

$$1 + d(x,y) \sim 1 + d(x,0),$$
$$1 + d(x,y) + t \sim 1 + d(x,0) + t.$$

*Proof.* The proof follows immediately from the triangle inequalities. □

LEMMA 3.7. *Fix $\tau > 0$. Let $N \geqslant 2$ and let $\phi \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $\phi(x) \gneqq 0$. If there exist $\varepsilon > 0$ and $\delta > 0$ such that $\phi(x) > \delta > 0$ for every $x \in B_{\mathbb{H}^N}(0,\varepsilon)$, then*

$$\int_{\mathbb{H}^N} g_N(x,y,t)\phi(y) \, \mathrm{d}\mu_y \gtrsim_{\delta,\varepsilon,\tau} \mathrm{e}^{-t\lambda_0/2} g_N(x,0,t/2) \quad \text{for } t \geqslant \tau.$$

REMARK 3.8. A similar estimate was obtained in [3, lemma 5.1]. The advantage of our form is that it is related to the lower bound and the hyperbolic heat kernel, which will be convenient when handling some integral estimates in our proofs.

*Proof.* By lemma 2.3 we have

$$\begin{aligned}
&\int_{\mathbb{H}^N} g_N(x,y,t)\phi(y) \, \mathrm{d}\mu_y \\
&\quad \geqslant \delta \int_{B_{\mathbb{H}^N}(0,\varepsilon)} g_N(x,y,t) \, \mathrm{d}\mu_y \\
&\quad \gtrsim_\delta \mathrm{e}^{-\lambda_0 t} t^{-N/2} \int_{B_{\mathbb{H}^N}(0,\varepsilon)} (1 + d(x,y))(1 + d(x,y) + t)^{(N-3)/2} \\
&\qquad \times \exp\left\{ -\frac{N-1}{2} d(x,y) - \frac{d(x,y)^2}{4t} \right\} \mathrm{d}\mu_y. \qquad (3.7)
\end{aligned}$$

We need to estimate the integral

$$\begin{aligned}
I := \int_{B_{\mathbb{H}^N}(0,\varepsilon)} &(1 + d(x,y))(1 + d(x,y) + t)^{(N-3)/2} \\
&\times \exp\left\{ -\frac{N-1}{2} d(x,y) - \frac{d(x,y)^2}{4t} \right\} \mathrm{d}\mu_y.
\end{aligned}$$

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

Without loss of generality we assume that $\varepsilon < 1$, otherwise we could perform the proof for $\epsilon = \min\{\varepsilon, 1\}$. By lemma 3.6 and the inequalities

$$-d(x,y) \geqslant -d(x,0) - d(y,0),$$
$$-d(x,y)^2 \geqslant -2d(x,0)^2 - 2d(y,0)^2,$$

we have

$$I \gtrsim (1+d(x,0))(1+d(x,0)+t)^{(N-3)/2} \exp\left\{-\frac{N-1}{2}d(x,0) - \frac{2d(x,0)^2}{4t}\right\} \times \int_{B_{\mathbb{H}^N}(0,\varepsilon)} \exp\left\{-\frac{N-1}{2}d(y,0) - \frac{2d(y,0)^2}{4t}\right\} \mathrm{d}\mu_y. \quad (3.8)$$

Since $t \geqslant \tau$, the last integral in (3.8) can be estimated by

$$\begin{aligned}\int_{B_{\mathbb{H}^N}(0,\varepsilon)} &\exp\left\{-\frac{N-1}{2}d(y,0) - \frac{2d(y,0)^2}{4t}\right\} \mathrm{d}\mu_y \\ &\geqslant \int_{B_{\mathbb{H}^N}(0,\varepsilon)} \exp\left\{-\frac{N-1}{2}d(y,0) - \frac{2d(y,0)^2}{4\tau}\right\} \mathrm{d}\mu_y \\ &\gtrsim_{\varepsilon,\tau} 1.\end{aligned}$$

Thus,

$$\mathrm{I} \gtrsim_{\varepsilon,\tau} (1+d(x,0))(1+d(x,0)+t)^{(N-3)/2} \exp\left\{-\frac{N-1}{2}d(x,0) - \frac{2d(x,0)^2}{4t}\right\}. \quad (3.9)$$

By (3.7) and (3.9), we obtain

$$\int_{\mathbb{H}^N} g_N(x,y,t)\phi(y)\,\mathrm{d}\mu_y \gtrsim_{\delta,\varepsilon,\tau} \mathrm{e}^{-\lambda_0 t} t^{-N/2}(1+d(x,0))(1+d(x,0)+t)^{(N-3)/2} \times \exp\left\{-\frac{N-1}{2}d(x,0) - \frac{2d(x,0)^2}{4t}\right\}. \quad (3.10)$$

From lemma 3.5 and lemma 2.3 we get

$$\begin{aligned}\mathrm{e}^{-\lambda_0 t} t^{-N/2}&(1+d(x,0))(1+d(x,0)+t)^{(N-3)/2} \exp\left\{-\frac{N-1}{2}d(x,0) - \frac{2d(x,0)^2}{4t}\right\} \\ &\gtrsim \mathrm{e}^{-\lambda_0 t/2}\mathrm{e}^{-\lambda_0 t/2} t^{-N/2}(1+d(x,0))(1+d(x,0)+t/2)^{(N-3)/2} \\ &\quad\times \exp\left\{-\frac{N-1}{2}d(x,0) - \frac{d(x,0)^2}{4(t/2)}\right\} \\ &\gtrsim \mathrm{e}^{-\lambda_0 t/2} g_N(x,0,t/2). \qquad (3.11)\end{aligned}$$

Lemma 3.7 follows from (3.10) and (3.11). □

We have the following corollary, which is the key ingredient for proving lemma 3.2.

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

Corollary 3.9. *Fix $\tau > 0$. Let $N \geqslant 2$ and let $\phi \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $\phi(x) \geqslant 0$. If there exist $\varepsilon > 0$ and $\delta > 0$ such that $\phi(x) > \delta > 0$ for every $x \in B_{\mathbb{H}^N}(x_0, \varepsilon)$, then*

$$\int_{\mathbb{H}^N} g_N(x, y, t)\phi(y)\,\mathrm{d}\mu_y \gtrsim_{\delta,\varepsilon,\tau} \mathrm{e}^{-\lambda_0 t/2} g_N(x, x_0, t/2) \quad \textit{for } t \geqslant \tau. \tag{3.12}$$

*Proof.* Let $T_{x_0} : \mathbb{H}^N \to \mathbb{H}^N$ be the isometry that maps $x_0$ to 0 (see (2.1)). Denote the inverse of $T_{x_0}$ by $T_{x_0}^{-1}$. Let

$$\bar{\phi}(x) = \phi(T_{x_0}^{-1} x) \quad \text{for } x \in \mathbb{H}^N.$$

Then $\bar{\phi}$ satisfies lemma 3.7. Hence,

$$\begin{aligned}\int_{\mathbb{H}^N} g_N(T_{x_0} x, y, t)\phi(T_{x_0}^{-1} y)\,\mathrm{d}\mu_y &= \int_{\mathbb{H}^N} g_N(T_{x_0} x, y, t)\bar{\phi}(y)\,\mathrm{d}\mu_y \\ &\gtrsim_{\delta,\varepsilon,\tau} \mathrm{e}^{-\lambda_0 t/2} g_N(T_{x_0} x, 0, t/2) \quad \text{for } t \geqslant \tau. \end{aligned} \tag{3.13}$$

From the change of variables $T_{x_0}^{-1} y = z$ in (3.13) we obtain

$$\begin{aligned}\int_{\mathbb{H}^N} g_N(T_{x_0} x, T_{x_0} z, t)\phi(z)\,\mathrm{d}\mu_z &\gtrsim_{\delta,\varepsilon,\tau} \mathrm{e}^{-\lambda_0 t/2} g_N(T_{x_0} x, 0, t/2) \\ &= \mathrm{e}^{-\lambda_0 t/2} g_N(T_{x_0} x, T_{x_0}(T_{x_0}^{-1} 0), t/2) \quad \text{for } t \geqslant \tau,\end{aligned}$$

which together with lemma 2.2(ii) implies that

$$\begin{aligned}\int_{\mathbb{H}^N} g_N(x, z, t)\phi(z)\,\mathrm{d}\mu_z &\gtrsim_{\delta,\varepsilon,\tau} \mathrm{e}^{-\lambda_0 t/2} g_N(x, T_{x_0}^{-1} 0, t/2) \\ &= \mathrm{e}^{-\lambda_0 t/2} g_N(x, x_0, t/2) \quad \text{for } t \geqslant \tau.\end{aligned}$$

The proof is complete. □

We now turn to the proof of lemma 3.2.

*Proof of lemma 3.2.* Without loss of generality we assume that

$$\phi(x) = \begin{cases} 0, & d(x, 0) < R, \\ 1, & d(x, 0) > R + 1, \end{cases}$$

where $R$ is a large positive number. Since $\phi \in L^\infty(\mathbb{H}^N)$ and $\phi(x) \gneqq 0$, the function

$$\varPhi(x) := \int_{\mathbb{H}^N} g_N(x, y, \tau)\phi(y)\,\mathrm{d}\mu_y$$

is positive and continuous on $\mathbb{H}^N$. Thus,

$$\inf_{x \in B_{\mathbb{H}^N}(0, R+2)} \varPhi(x) > 0. \tag{3.14}$$

For $x \in \mathbb{H}^N \setminus B_{\mathbb{H}^N}(0, R + 2)$, taking $t = \tau$ in corollary 3.9 and by (2.4), we have

$$\int_{\mathbb{H}^N} g_N(x, y, \tau)\phi(y)\,\mathrm{d}\mu_y \gtrsim \mathrm{e}^{-(\tau/2)\lambda_0} g_N(x, x, \tau/2) = \mathrm{e}^{-(\tau/2)\lambda_0} g_N(0, 0, \tau/2) \gtrsim 1. \tag{3.15}$$

The lemma follows from (3.14) and (3.15) immediately. □

### 3.2. The lifespan estimates without non-decaying assumption

In this section we present lifespan estimates for general initial data as $\lambda \to 0$.

To prove theorem 1.2 and theorem 1.3, by proposition 3.1 we need only show that

$$\limsup_{\lambda\to 0} \frac{T_\lambda}{\ln(1/\lambda)} \leqslant \infty. \tag{3.16}$$

3.2.1. *The lifespan estimates in the subcritical case*

In the subcritical case we have $1 < p < p_H^* = 1 + \alpha/\lambda_0$, that is,

$$\lambda_0 + \frac{\alpha}{1-p} < 0. \tag{3.17}$$

By lemma 2.6 we can give the proof of theorem 1.2.

*Proof of theorem 1.2.* By proposition 3.1, we note that there exists $\Lambda > 0$ such that $T_\lambda > 1$ for $\lambda < \Lambda$. Since $\phi(x) \gneqq 0$, there exist $x_0 \in \mathbb{H}^N$, $\varepsilon > 0$ and $\delta > 0$ such that

$$\phi(x) > \delta \quad \text{for } x \in B_{\mathbb{H}^N}(x_0, \varepsilon).$$

Taking $x = 0$ and $\tau = 1$ in corollary 3.9, we have

$$\int_{\mathbb{H}^N} g_N(0, y, t)\phi(y)\,\mathrm{d}\mu_y \gtrsim_{\delta,\varepsilon} \mathrm{e}^{-\lambda_0 t/2} g_N(0, x_0, t/2) \quad \text{for } t \geqslant 1. \tag{3.18}$$

By (3.18) and lemma 2.6, it follows that

$$(\lambda \mathrm{e}^{-\lambda_0 t/2} g_N(0, x_0, t/2))^{1-p} \gtrsim_{\varepsilon,\delta} \frac{p-1}{\alpha}(\mathrm{e}^{\alpha t} - 1) \gtrsim_{\varepsilon,\delta,p,\alpha} \mathrm{e}^{\alpha t} \tag{3.19}$$

for $1 < t < T_\lambda$ and $\lambda < \Lambda$. Since $T_\lambda > 1$, the kernel estimate (2.8) implies that

$$\begin{aligned}
\lambda \mathrm{e}^{-(T_\lambda/2)\lambda_0} g_N\left(0, x_0, \frac{T_\lambda}{2}\right) &\gtrsim \lambda \mathrm{e}^{-(T_\lambda/2)\lambda_0} \left(\frac{T_\lambda}{2}\right)^{-N/2} (1+d)\left(1 + d + \frac{T_\lambda}{2}\right)^{(N-3)/2} \\
&\qquad \times \exp\left\{-\lambda_0 \frac{T_\lambda}{2} - \frac{N-1}{2} d - \frac{d^2}{4(T_\lambda/2)}\right\} \\
&\gtrsim_d \lambda \mathrm{e}^{-(T_\lambda/2)\lambda_0} \left(\frac{T_\lambda}{2}\right)^{-N/2} \left(\frac{T_\lambda}{2}\right)^{(N-3)/2} \mathrm{e}^{-(T_\lambda/2)\lambda_0} \\
&\gtrsim_d \lambda \mathrm{e}^{-T_\lambda \lambda_0} T_\lambda^{-3/2},
\end{aligned} \tag{3.20}$$

where we write $d = d(0, x_0)$. Equations (3.19) and (3.20) imply that

$$(\lambda \mathrm{e}^{-T_\lambda \lambda_0} (T_\lambda)^{-3/2})^{1-p} \gtrsim_{d,\varepsilon,\delta,p,\alpha} \mathrm{e}^{\alpha T_\lambda},$$

that is,

$$T_\lambda^{3/2} \exp\left(\lambda_0 + \frac{\alpha}{1-p}\right) T_\lambda \gtrsim_{d,\varepsilon,\delta,p,\alpha} \lambda \quad \text{for } \lambda < \Lambda. \tag{3.21}$$

Since $1 < p < p_H^*$ implies that $\lambda_0 + \alpha/(1-p) < 0$, by (3.21) we obtain

$$\exp\left\{\frac{1}{2}\left(\lambda_0 + \frac{\alpha}{1-p}\right) T_\lambda\right\} \gtrsim_{d,\varepsilon,\delta,p,\alpha} \lambda \quad \text{for } \lambda > \Lambda. \tag{3.22}$$

Equation (3.16) follows from (3.17) and (3.22). □

3.2.2. *The lifespan problem in the supercritical and critical cases*

First we recall an elementary lemma that will be used to derive the linear estimate on the initial datum.

LEMMA 3.10. *If $r > 0$ and $t > 0$, then*

$$(r+t)^{(N-3)/2} \gtrsim \min\{r^{(N-3)/2}, t^{(N-3)/2}\}, \quad N \geqslant 2.$$

*Proof.* For $N \geqslant 3$ we have $(N-3)/2 \geqslant 0$, which implies that

$$2(r+t)^{(N-3)/2} \geqslant t^{(N-3)/2} + r^{(N-3)/2}. \tag{3.23}$$

For $N = 2$ we have $(N-3)/2 < 0$. Without loss of generality we assume that $0 < r \leqslant t$. Then

$$\min\{r^{(N-3)/2}, t^{(N-3)/2}\} = t^{(N-3)/2} \leqslant \left(\frac{r+t}{2}\right)^{(N-3)/2}. \tag{3.24}$$

The lemma follows from (3.23) and (3.24). □

The key ingredient of our argument is the following linear estimate.

LEMMA 3.11. *Suppose that $p$, $k$, $\alpha$ and $\phi$ satisfy the assumptions of theorem 1.3. Then*

$$\int_{\mathbb{H}^N} g_N(y,0,t)\phi(y)\,\mathrm{d}\mu_y \gtrsim \mathrm{e}^{-\lambda_0 t + \gamma^2 t} \quad \text{for } t > \frac{R}{2\gamma}, \tag{3.25}$$

*where $\gamma = \sqrt{\lambda_0} - k$.*

*Proof.* Let $r = d(y,0)$. By the heat kernel estimate and (2.3), the volume element in hyperbolic space, we have

$$\int_{\mathbb{H}^N} g_N(y,0,t)\phi(y)\,\mathrm{d}\mu_y \gtrsim \int_R^\infty t^{-N/2}(1+r)(1+r+t)^{(N-3)/2} \times \exp\left\{-\lambda_0 t - \frac{N-1}{2}r - \frac{r^2}{4t}\right\}\mathrm{e}^{-kr}(\sinh r)^{N-1}\,\mathrm{d}r.$$

Since $(r+1) \sim r$ and $\sinh r \sim \mathrm{e}^r$ for $r > R$, we have

$$\int_{\mathbb{H}^N} g_N(y,0,t)\phi(y)\,\mathrm{d}\mu_y \gtrsim t^{-N/2}\mathrm{e}^{-\lambda_0 t}\int_R^\infty r(r+t)^{(N-3)/2}\exp\left\{-\frac{r^2}{4t} + \left(\frac{N-1}{2} - k\right)r\right\}\mathrm{d}r. \tag{3.26}$$

Let

$$\mathrm{I} = \int_R^\infty r r^{(N-3)/2}\exp\left\{-\frac{r^2}{4t} + \left(\frac{N-1}{2} - k\right)r\right\}\mathrm{d}r,$$

$$\mathrm{II} = \int_R^\infty r t^{(N-3)/2}\exp\left\{-\frac{r^2}{4t} + \left(\frac{N-1}{2} - k\right)r\right\}\mathrm{d}r.$$

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

By lemma 3.10 and (3.26) we obtain

$$\int_{\mathbb{H}^N} g_N(y,0,t)\phi(y)\,\mathrm{d}\mu_y \gtrsim t^{-N/2}\mathrm{e}^{-\lambda_0 t}\min\{\mathrm{I},\mathrm{II}\}. \tag{3.27}$$

Next we shall estimate I and II. By the assumptions of this lemma we have

$$\gamma > 0.$$

From the change of variables $r/2\sqrt{t} - \gamma\sqrt{t} = s$ we have

$$\begin{aligned}\mathrm{I} &= \int_{R/2\sqrt{t}-\gamma\sqrt{t}}^{\infty} [(s+\gamma\sqrt{t})2\sqrt{t}]^{(N-1)/2}\mathrm{e}^{-s^2+\gamma^2 t}(2\sqrt{t})\,\mathrm{d}s \\ &\gtrsim \mathrm{e}^{\gamma^2 t} t^{(N+1)/4}\int_{R/2\sqrt{t}-\gamma\sqrt{t}}^{\infty} \mathrm{e}^{-s^2}(s+\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s.\end{aligned}$$

By $\gamma > 0$, we obtain

$$\frac{R}{2\sqrt{t}} - \gamma\sqrt{t} \leqslant 0 \quad \text{for } t > \frac{R}{2\gamma}.$$

Thus,

$$\int_{R/2\sqrt{t}-\gamma\sqrt{t}}^{\infty} \mathrm{e}^{-s^2}(s+\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s > \int_0^{\infty} \mathrm{e}^{-s^2}(s+\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s \quad \text{for } t > \frac{R}{2\gamma}.$$

Hence,

$$\mathrm{I} \gtrsim \mathrm{e}^{\gamma^2 t} t^{(N+1)/4}\int_0^{\infty} \mathrm{e}^{-s^2}(s+\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s \quad \text{for } t > \frac{R}{2\gamma}. \tag{3.28}$$

Now we estimate $\int_0^\infty \mathrm{e}^{-s^2}(s+\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s$. Noting that $N-1>0$, we have

$$\int_0^{\infty} \mathrm{e}^{-s^2}(s+\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s \gtrsim \int_0^{\infty} \mathrm{e}^{-s^2}(s)^{(N-1)/2}\,\mathrm{d}s + \int_0^{\infty} \mathrm{e}^{-s^2}(\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s. \tag{3.29}$$

Since

$$\int_0^{\infty} \mathrm{e}^{-s^2}(s)^{(N-1)/2}\,\mathrm{d}s \gtrsim 1 \quad \text{and} \quad \int_0^{\infty} \mathrm{e}^{-s^2}(\gamma\sqrt{t})^{(N-1)/2}\,\mathrm{d}s \gtrsim t^{(N-1)/4},$$

by (3.28) and (3.29) we get

$$\mathrm{I} \gtrsim t^{N/2}\mathrm{e}^{\gamma^2 t} \quad \text{for } t > \frac{R}{2\gamma}. \tag{3.30}$$

In a similar way we have

$$\mathrm{II} \gtrsim t^{N/2}\mathrm{e}^{\gamma^2 t} \quad \text{for } t > \frac{R}{2\gamma}. \tag{3.31}$$

The lemma follows from (3.27), (3.30) and (3.31). □

We now give the proof of theorem 1.3.

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

*Proof of theorem 1.3.* Let $\gamma = \sqrt{\lambda_0} - l$. First we prove the blow-up criterion. Since $p > 1$, by lemma 2.6 and lemma 3.11 we obtain

$$(\mathrm{e}^{-\lambda_0 t+\gamma^2 t})^{1-p} \gtrsim \frac{p-1}{\alpha}(\mathrm{e}^{\alpha t} - 1) \quad \text{for } \frac{R}{2\gamma} < t < T_\lambda. \tag{3.32}$$

If $u$ is a global solution, then (3.32) implies that

$$(1-p)(-\lambda_0 + \gamma^2) \geqslant \alpha.$$

Thus,

$$\gamma < \sqrt{\lambda_0 - \frac{\alpha}{p-1}},$$

which together with $\gamma = \sqrt{\lambda_0} - k$ implies that

$$k > \sqrt{\lambda_0} - \sqrt{\lambda_0 - \frac{\alpha}{p-1}} = k^*,$$

which contradicts the assumption that $k < k^*$. Thus, the solution $u(x,t)$ must blow up in finite time.

Now we estimate the upper bound of the lifespan. By lemma 2.6 and the decay estimate (3.25), we obtain

$$(\lambda \mathrm{e}^{-\lambda_0 t+\gamma^2 t})^{1-p} \gtrsim \frac{p-1}{\alpha}(\mathrm{e}^{\alpha t} - 1) \quad \text{for } \frac{R}{2\gamma} < t < T_\lambda, \tag{3.33}$$

where $\gamma = \sqrt{\lambda_0} - k$. For $\lambda \to 0$, by the lower estimate on the lifespan (see proposition 3.1), we have

$$T_\lambda \to \infty.$$

Thus, taking $t = T_\lambda$ in (3.33), by $\gamma > \sqrt{\lambda_0 - \alpha/(p-1)}$ we get

$$T_\lambda \lesssim \ln\frac{1}{\lambda} \quad \text{as } \lambda \to 0.$$

Now (3.16) follows, and this completes the proof. □

## 4. The lifespan estimates as $\lambda \to \infty$

The purpose of this section is devoted to the lifespan estimate as $\lambda \to \infty$. We shall use Kaplan's eigenfunction method (see [8]) to prove theorem 1.4, which was used by Gui and Wang [6] to study the lifespan problem of (1.3).

First we prove a comparison lemma for ODEs, which will be used for the upper bound estimate on the lifespan.

LEMMA 4.1. *Let $v_1, v_2 \in C(\mathbb{R})$. Assume that $v_1(\xi) > v_2(\xi) \geqslant 0$ for every $\xi \in [\xi_0, \infty)$. If $x, y \in C^1([a,b])$ satisfy*

$$\begin{aligned} x'(t) &\geqslant v_1(x(t)), \quad a \leqslant t \leqslant b, \\ y'(t) &= v_2(y(t)), \quad a \leqslant t \leqslant b, \\ x(a) &= y(a) \geqslant \xi_0, \end{aligned}$$

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

*then*

$$x(t) > y(t) \quad \text{for } a < t < b.$$

*Proof.* The proof is similar to that of a comparison lemma in [2, § 2.7]. We define

$$\bar{t} = \sup\{t \in (a,b) \mid x(s) > y(s),\ a < s < t\}.$$

We claim that $\bar{t} = b$. Indeed, by the definition of a derivative and

$$x(a) = y(a), \qquad x'(a) \geqslant v_1(x(a)) > v_2(x(a)) = v_2(y(a)) = y'(a),$$

it follows that there exists $\tau > 0$ such that

$$x(t) > y(t) \quad \text{for } a < t < a + \tau.$$

Hence, the set $\{t > a \mid x(s) > y(s),\ a < s < t\}$ is not empty. Then $\bar{t} > a$. By the assumptions $v_2(\xi) \geqslant 0$ for $\xi \geqslant \xi_0$ and $y(a) \geqslant \xi_0$, we see that $y(t)$ is non-decreasing in $[a, \bar{t}\,]$. Thus, by the definition of $\bar{t}$,

$$x(t) > y(t) \geqslant y(0) \geqslant \xi_0 \quad \text{for } a < t < \bar{t}. \tag{4.1}$$

If $\bar{t} < b$, then by the continuity of $x$ and $y$, and by (4.1), we obtain

$$x(\bar{t}\,) = y(\bar{t}\,) \geqslant \xi_0 \tag{4.2}$$

and

$$x'(\bar{t}\,) \geqslant v_1(x(\bar{t}\,)) > v_2(x(\bar{t}\,)) = v_2(y(\bar{t}\,)) = y'(\bar{t}\,). \tag{4.3}$$

Therefore, by the definition of a derivative, (4.2) and (4.3), there exists $\sigma > 0$ such that

$$x(t) < y(t) \quad \text{for } \bar{t} - \sigma < t < \bar{t},$$

which contradicts the definition of $\bar{t}$. Thus, $\bar{t} = b$ and the lemma is proved. □

Now we are in a position to prove theorem 1.4.

*Proof of theorem 1.4.* We shall estimate the lower bound and the upper bound of the lifespan.

First, we prove that

$$\liminf_{\lambda\to\infty} \frac{T_\lambda}{(\lambda\|\phi\|_{L^\infty(\mathbb{H}^N)})^{1-p}/(p-1)} \geqslant 1. \tag{4.4}$$

Recalling (3.2) in the proof of proposition 3.1, we see that

$$T_\lambda \geqslant \frac{1}{\alpha} \ln\left[1 + \frac{\alpha}{p-1}(\lambda\|\phi\|_{L^\infty})^{1-p}\right].$$

Then (4.4) follows by L'Hospital's rule.

Next, we show that

$$\limsup_{\lambda\to\infty} \frac{T_\lambda}{(\lambda\|\phi\|_{L^\infty(\mathbb{H}^N)})^{1-p}/(p-1)} \leqslant 1 \tag{4.5}$$

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

by Kaplan's eigenfunction method, introduced in [8]. Take $x_0 \in \mathbb{H}^N$ such that $\phi(x_0) \gneqq 0$. Denote $k_\varepsilon$ as the first eigenvalue of $-\Delta_{\mathbb{H}^N}$ in the ball $B_{\mathbb{H}^N}(x_0, \varepsilon)$ and let $\rho_\varepsilon \geqslant 0$ be the corresponding eigenfunction, normalized so that

$$\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \rho_\varepsilon(y)\,\mathrm{d}\mu_y = 1.$$

Let $T_\lambda$ be the lifespan of the solution $u(x, t; \lambda)$ and define

$$w_{\varepsilon,\lambda}(t) = \int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} u(y, t; \lambda)\rho_\varepsilon(y)\,\mathrm{d}\mu_y \quad \text{for } t \in (0, T_\lambda).$$

By (1.1) and the proof of lemma 2.5,

$$\frac{\mathrm{d}w_{\varepsilon,\lambda}}{\mathrm{d}t} \geqslant -k_\varepsilon w_{\varepsilon,\lambda} + \mathrm{e}^{\alpha t} w_{\varepsilon,\lambda}^p > -k_\varepsilon w_{\varepsilon,\lambda} + w_{\varepsilon,\lambda}^p,$$

since $\alpha \geqslant 0$. Take $\delta \in (0, 1)$ as a parameter. Consider the system of $w_{\varepsilon,\lambda}(t)$ and $v(t; \delta)$:

$$\left.\begin{aligned} \frac{\mathrm{d}w_{\varepsilon,\lambda}}{\mathrm{d}t} &\geqslant -k_\varepsilon w_{\varepsilon,\lambda} + w_{\varepsilon,\lambda}^p, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= (1-\delta)v^p, \\ v(0;\delta) = w_{\varepsilon,\lambda}(0) &= \lambda \int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y. \end{aligned}\right\} \tag{4.6}$$

By a direct calculation, we see that if

$$\lambda \geqslant \Lambda := \left(\frac{k_\varepsilon}{\delta}\right)^{1/(p-1)} \bigg/ \int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y,$$

then (4.6) satisfies the assumptions of lemma 4.1. Thus,

$$w_{\varepsilon,\lambda}(t) \geqslant v(t;\delta) \quad \text{for } 0 < t < T_\lambda.$$

Since the blow-up time of $v(t; \delta)$ is

$$\begin{aligned} \underline{T_\delta} &:= \frac{1}{(p-1)(1-\delta)} (w_{\varepsilon,\lambda}(0))^{1-p} \\ &= \frac{1}{(p-1)(1-\delta)} \left(\lambda \int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y\right)^{1-p}, \end{aligned}$$

we obtain that the solution of (1.1) blows up in finite time and

$$T_\lambda \leqslant \underline{T_\delta} = \frac{1}{(p-1)(1-\delta)} \left(\lambda \int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y\right)^{1-p} \quad \text{for } \lambda > \Lambda.$$

Thus,

$$\limsup_{\lambda\to\infty} \frac{T_\lambda}{\lambda^{1-p}} \leqslant \frac{1}{(p-1)(1-\delta)} \left(\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y\right)^{1-p}.$$

Since $0 < \delta < 1$ is arbitrary, it follows that

$$\limsup_{\lambda\to\infty} \frac{T_\lambda}{\lambda^{1-p}} \leqslant \frac{1}{p-1}\left(\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y\right)^{1-p}. \tag{4.7}$$

By the continuity of $\phi$, $\rho_\varepsilon \geqslant 0$ and $\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \rho_\varepsilon = 1$, we have

$$\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y = \phi(x_\varepsilon)\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \rho_\varepsilon(y)\,\mathrm{d}\mu_y = \phi(x_\varepsilon)$$

for some $x_\varepsilon \in B_{\mathbb{H}^N}(x_0,\varepsilon)$. Letting $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_{B_{\mathbb{H}^N}(x_0,\varepsilon)} \phi(y)\rho_\varepsilon(y)\,\mathrm{d}\mu_y = \phi(x_0),$$

since $\phi(x)$ is continuous. Thus, by (4.7) and $p > 1$,

$$\limsup_{\lambda\to\infty} \frac{T_\lambda}{\lambda^{1-p}} \leqslant \frac{1}{p-1}(\phi(x_0))^{1-p}.$$

Hence,

$$\limsup_{\lambda\to\infty} \frac{T_\lambda}{\lambda^{1-p}} \leqslant \inf_{\substack{x\in\mathbb{H}^N,\\ \phi(x)\neq 0}} \frac{1}{p-1}(\phi(x))^{1-p} = \frac{1}{p-1}(\|\phi\|_{L^\infty(\mathbb{H}^N)})^{1-p},$$

and (4.5) follows.

The proof is complete by (4.4) and (4.5). □

## Appendix A. Remark on the blow-up criterion in the supercritical case

The following proposition reveals that there exist global solutions with initial datum with a slower decay rate than that satisfying the blow-up conditions in theorem 1.3.

Proposition A.1. *If $p > p_H^*$ and*

$$k > k^* := \frac{N-1}{2}\left(1 - \sqrt{\frac{p-p_H^*}{p-1}}\right),$$

*then there exists $u_0 \in C(\mathbb{H}^N) \cap L^\infty(\mathbb{H}^N)$ with $u_0 > 0$ and*

$$\liminf_{d(x,0)\to\infty} u_0(x)\mathrm{e}^{kd(x,0)} > 0$$

*such that the solution $u(x,t)$ of (1.1) with the initial datum $u_0$ is global.*

Remark A.2. In the $p > p*$ case we do not know whether there exists a global solution with initial datum having the decay rate $\mathrm{e}^{-k^* d(x,0)}$. In the critical case the existence of the global solution with the decay rate $\mathrm{e}^{-k^* d(x,0)}$ follows from [11, theorem 1.4 and lemma 3.4] for $1 < p < 2n/(n-2) - 1$ if $n \geqslant 3$ and $p > 1$ if $n = 2$.

We will use the comparison principle to prove proposition A.1. A similar argument was used in [3] to prove the existence of global solutions in the supercritical case, while the decay rate of the upper solutions in their proof is $e^{-\lambda_0 d(x,0)}$, which does not satisfy the condition in proposition A.1.

The key ingredient of our proof is a transformation, which translates problem (1.1) to a parabolic equation that does not explicitly contain the time $t$. To be more precise, suppose that $u(x,t)$ is a solution of (1.1). Let

$$v(x,t) = e^{\beta t} u(x,t),$$

where $\beta = \alpha/(p-1)$. Then $v(x,t)$ satisfies

$$\left.\begin{aligned} v_t &= \Delta_{\mathbb{H}^N} v + \beta v + |v|^{p-1} v, \quad (x,t) \in \mathbb{H}^N \times (0,T), \\ v(x,0) &= v_0(x) = u_0(x), \qquad\qquad x \in \mathbb{H}^N. \end{aligned}\right\} \tag{A 1}$$

By (2.2) it is easily seen that $\bar{u}(x,t) = e^{-\beta t}\bar{v}(x)$ is an upper solution of problem (1.1) *if and only if* $\bar{v}(x)$ satisfies

$$\Delta_{\mathbb{H}^N} \bar{v} + \beta \bar{v} + |\bar{v}|^{p-1}\bar{v} \leqslant 0 \quad \text{for } x \in \mathbb{H}^N.$$

We use the linear ground state, which was introduced in [3], to construct upper solutions.

Definition A.3. We say that $w$ is a *ground state* if $w$ is a positive classical solution of

$$\Delta_{\mathbb{H}^N} w + \kappa w = 0 \quad \text{in } \mathbb{H}^N \ (\kappa \in \mathbb{R}).$$

It is easily seen that a radial ground state satisfies

$$\begin{gathered} w'' + (N-1)\coth\rho\, w' + \kappa w = 0, \quad \rho \in [0,\infty), \\ w(0) > 0, \quad w'(0) = 0. \end{gathered}$$

The next lemma gives the condition on the existence and asymptotic behaviour of the ground state.

Lemma A.4 (Bandle *et al*. [3, proposition A.1 and lemma A.1]).

(i) *There exists a ground state if and only if*

$$\kappa \leqslant \lambda_0.$$

(ii) *For any $\kappa \leqslant \lambda_0$ and $c > 0$ there exists a unique radial ground state $w$ such that $w(0) = c$. There holds*

$$\lim_{\rho\to\infty} w(\rho) e^{-\nu\rho} = k$$

*for some $k > 0$, where*

$$\nu := \sqrt{\lambda_0 - \kappa} - \sqrt{\lambda_0}.$$

*In particular, if $\kappa > 0$, then*

$$\lim_{\rho\to\infty} w(\rho) = 0.$$

https://doi.org/10.1017/S0308210515000785 Published online by Cambridge University Press

Remark A.5. We remark that for $\kappa > 0$ the ground state $w(\rho)$ is strictly decreasing for $\rho > 0$. In fact, it is easily seen that, by (2.2),

$$(\sinh^{N-1} \rho w'(\rho))' = -\kappa w(\rho) < 0.$$

Thus, $\sinh^{N-1} \rho w'(\rho) < \sinh^{N-1}(0) w'(0) = 0$, which implies that $w'(\rho) < 0$ for $\rho > 0$.

We are now in a position to give the proof of proposition A.1.

*Proof of proposition A.1.* Since $p > p_H^*$ and

$$k^* = \frac{N-1}{2}\left(1 - \sqrt{\frac{p - p_H^*}{p-1}}\right),$$

by the comparison principle of (1.1) it is sufficient to show that the proposition holds for $k^* < k < (N-1)/2 = \sqrt{\lambda_0}$. We use two steps to construct an upper solution of (A 1).

Step 1. There exists a radial ground state with decay rate $e^{-kd(x,0)}$. Let $w$ satisfy

$$w'' + (N-1)\coth \rho w' + \kappa w = 0, \quad \rho \in [0, \infty),$$

where

$$\kappa = \lambda_0 - (\sqrt{\lambda_0} - k)^2 \leqslant \lambda_0,$$

which together with lemma A.4 implies that there exists a ground state with the decay rate $e^{-kd(x,0)}$.

Step 2. We can construct an upper solution with decay rate $e^{-kd(x,0)}$ by scaling $w$. Since

$$k > k^* = \sqrt{\lambda_0} - \sqrt{\lambda_0 - \frac{\alpha}{p-1}},$$

we have

$$\kappa = \lambda_0 - (\sqrt{\lambda_0} - k)^2 > \frac{\alpha}{p-1}.$$

We shall use the linear term to control the increase of the nonlinear term. More precisely, letting

$$\eta = \left(\frac{w(0)^{p-1}}{\kappa - \alpha/(p-1)}\right)^{1/(p-1)},$$

by remark A.5 we have

$$\frac{1}{\eta}\left(\kappa - \frac{\alpha}{p-1}\right) w(\rho) \geqslant \left(\frac{w(\rho)}{\eta}\right)^p \quad \text{for } \rho > 0. \tag{A 2}$$

Then (A 2) holds. Let

$$\bar{v}(\rho, t) = \frac{1}{\eta} w(\rho) \quad \text{for } \rho \geqslant 0.$$

By (A 2), we have

$$\begin{aligned} 0 &= \bar{v}'' + (N-1)\coth\rho\bar{v}' + \kappa\bar{v} \\ &= \bar{v}'' + (N-1)\coth\rho\bar{v}' + \frac{\alpha}{p-1}\bar{v} + \left(\kappa - \frac{\alpha}{p-1}\right)\bar{v} \\ &\geqslant \bar{v}'' + (N-1)\coth\rho\bar{v}' + \frac{\alpha}{p-1}\bar{v} + \bar{v}^p. \end{aligned}$$

Thus, $\bar{v}$ is an upper solution of (A 1) (take $\beta = \alpha/(p-1)$ in (A 1)).

Let $\bar{u}(\rho,t) = \mathrm{e}^{-\beta t}\bar{v}(\rho,t)$. Then $\bar{u}$ is a global upper solution of (1.1). This completes the proof. □

## Acknowledgements

J.Y. was supported in part by the NSFC (Grant no. 11371153) and the Specialized Research Fund for the Doctoral Program of the High Educational Department of China. Z.W. was supported by the NSFC (Grant no. 11526052) of China.

## References

1 L. V. Ahlfors. Möbius transformations in several dimensions. Ordway Professorship Lectures in Mathematics, University of Minnesota, School of Mathematics, Minneapolis, MN, 1981.

2 V. I. Arnol′d. *Ordinary differential equations* (transl. from Russian and ed. R. A. Silverman) (Cambridge, MA: MIT Press, 1973).

3 C. Bandle, M. Assunta Pozio and A. Tesei. The Fujita exponent for the Cauchy problem in the hyperbolic space. *J. Diff. Eqns* **251** (2011), 2143–2163.

4 J. W. Cannon, W. J. Floyd, R. Kenyon and W. R. Parry. Hyperbolic geometry. In *Flavors of geometry*, Mathematical Sciences Research Institute Publications, vol. 31, pp. 59–115 (Cambridge University Press, 1997).

5 E. B. Davies. *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92 (Cambridge University Press, 1989).

6 C. Gui and X. Wang. Life spans of solutions of the Cauchy problem for a semilinear heat equation. *J. Diff. Eqns* **115** (1995), 166–172.

7 B. Hu. *Blow-up theories for semilinear parabolic equations*, Lecture Notes in Mathematics, vol. 2018 (Springer, 2011).

8 S. Kaplan. On the growth of solutions of quasi-linear parabolic equations. *Commun. Pure Appl. Math.* **16** (1963), 305–330.

9 T.-Y. Lee and W.-M. Ni. Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem. *Trans. Am. Math. Soc.* **333** (1992), 365–378.

10 E. H. Lieb and M. Loss. *Analysis*, 2nd edn, Graduate Studies in Mathematics, vol. 14 (Providence, RI: American Mathematical Society, 2001).

11 G. Mancini and K. Sandeep. On a semilinear elliptic equation in $\mathbb{H}^n$. *Annali Scuola Norm. Sup. Pisa* **7** (2008), 635–671.

12 P. Mastrolia, D. D. Monticelli and F. Punzo. Nonexistence of solutions to parabolic differential inequalities with a potential on riemannian manifolds. *Math. Annalen* (2016), DOI: 10.1007/s00208-016-1393-2.

13 F. Merle. Limit of the solution of a nonlinear Schrödinger equation at blow-up time. *J. Funct. Analysis* **84** (1989), 201–214.

14 L. E. Payne and G. A. Philippin. Blow-up phenomena in parabolic problems with time dependent coefficients under Dirichlet boundary conditions. *Proc. Am. Math. Soc.* **141** (2013), 2309–2318.

15 F. Punzo. Blow-up of solutions to semilinear parabolic equations on Riemannian manifolds with negative sectional curvature. *J. Math. Analysis Applic.* **387** (2012), 815–827.

16 Z. Wang and J. Yin. A note on semilinear heat equation in hyperbolic space. *J. Diff. Eqns* **256** (2014), 1151–1156.

17 F. B. Weissler. Existence and nonexistence of global solutions for a semilinear heat equation. *Israel J. Math.* **38** (1981), 29–40.

18 Q. S. Zhang. Blow-up results for nonlinear parabolic equations on manifolds. *Duke Math. J.* **97** (1999), 515–539.