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A variational principle for three-dimensional interactions between water waves and a floating rigid body with interior fluid motion

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A variational principle is given for the motion of a rigid body dynamically coupled to its interior fluid sloshing in three-dimensional rotating and translating coordinates. The fluid is assumed to be inviscid and incompressible. The Euler–Poincaré reduction framework of rigid body dynamics is adapted to derive the coupled partial differential equations for the angular momentum and linear momentum of the rigid body and for the motion of the interior fluid relative to the body coordinate system attached to the moving rigid body. The variational principle is extended to the problem of interactions between gravity-driven potential flow water waves and a freely floating rigid body dynamically coupled to its interior fluid motion in three dimensions.

Key words: variational methods, wave–structure interactions

1. Introduction

Studying the motion of a fluid in a stationary or forced vessel is a complicated problem both theoretically and experimentally. The works by Moiseyev & Rumyantsev (1968), Ibrahim (2005) and Faltinsen & Timokha (2009), and references therein, highlight the problems in these areas. The problem of dynamic coupling between rigid body motion and its interior fluid motion adds an additional layer of complexity to the sloshing problem because it allows for the potential enhancement or destabilisation of fluid dynamics due to the motion of the rigid body. Such problems involve theoretical and numerical difficulties of both the fluid mechanics and rigid body dynamics. Examples of where dynamic coupling is of interest are the sloshing of fluid in ships and on board spacecraft, transport of liquids by robots, motion planning for industrial control, motion of water waves in ocean wave energy converters and sloshing in automobile fuel tanks. As reported by Ramodanov & Sidorenko (2017) the pioneering contributions to the problem of dynamics of a rigid body with fluid-filled cavities are due to Stokes (1880) and Zhukovskii (1885). The first studies on the stability of steady rotations of a body containing an ellipsoidal cavity with an interior uniform-vorticity flow are due to Greenhill (1880) and Hough (1895). A number of further studies on the stability of the motion of a rigid body with cavities containing

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viscous fluid are done by Rumyantsev (1963), Moiseyev & Rumyantsev (1968) and Kostyuchenko, Shkalikov & Yurkin (1998). Chernousko (1965) considered the general problem of the motion of a viscous incompressible fluid in the cavity of a rigid body and of the motion of the rigid body itself. Using asymptotic methods, first presented by Zhukovskii (1948) for the motion of an ideal fluid, the hydrodynamic problem was reduced to the solution of three stationary linear boundary value problems. After solving the fluid problem, the reduced equations for the rigid body motion were solved using asymptotic analysis. Ramodanov & Sidorenko (2017) revisited the approach proposed by Chernousko (1972) based on the integral manifold theory (Strygin & Sobolev 1988) to model the dynamics of a rigid body with ellipsoidal cavity entirely filled with a highly viscous fluid. They studied the motion of a physical pendulum with a fluid-filled cavity on a rotating platform using the Euler angles to parameterise the rotation. Disser *et al.* (2016) studied inertial motions of the coupled system constituted by a rigid body containing a cavity entirely filled with a viscous liquid. In the cited works on the coupled fluid–body dynamics, the differential equations for the motion of the rigid body containing fluid are derived based on balance of linear momentum and angular momentum, or a reduced version of the Euler–Lagrange equations are presented using a parameterisation of the rotation tensor. Instead of using balance of linear momentum and angular momentum, or approximating the governing equations for the coupled fluid–body system using a parameterisation of the rotation tensor, one could apply a variational principle to derive the exact equations of motion and obtain a Hamiltonian formulation for the coupled system, which is particularly useful for numerical integration.

This paper concerns the variational principles and the derivation of the corresponding equations of motion for the coupled fluid–body dynamics, in three-dimensional rotating and translating coordinates, by using the Euler–Poincaré framework for the following two cases: (i) A rigid body container with interior inviscid fluid sloshing. (ii) The extension to a floating rigid body in potential-flow water waves with a sloshing fluid within the body. These two problems, which are rarely presented in the literature, are built on the existing variational formulations for interactions between potential water waves and an empty floating rigid body container.

Variational principles and Hamiltonian formulations for the classical water-wave and fluid sloshing problems are given by Luke (1967), Zakharov (1968), Broer (1974), Lukovsky (1976) and Miles (1976, 1977). Bokhove & Oliver (2006) derived the geometric link between the parcel Eulerian–Lagrangian formulation and the variational and Hamiltonian formulations for generalised two-dimensional vorticity streamfunction fluid dynamics, the rotating two-dimensional shallow-water equations and the rotating three-dimensional compressible Euler equations. Timokha (2016) used Clebsch potentials to generalise the Bateman–Luke variational formulation (Bateman 1932; Luke 1967) for the sloshing of an ideal incompressible fluid with rotational flows. Cotter & Bokhove (2010) derived a new water wave model from a constrained variational formulation which combines a depth-averaged vertical vorticity with depth-dependent potential flow. The irrotational water-wave model and the depth-averaged shallow-water equations are limiting forms of the given water-wave model. Miloh (1984) presented a variational principle for interactions between water waves and several bodies on or below a free surface which oscillate at a common frequency. Van Daalen, Van Groesen & Zandbergen (1993) derived a Hamiltonian formulation for surface waves in hydrodynamic interaction with freely floating bodies. Van Groesen & Andonowati (2017) presented a Boussinesq-type Hamiltonian formulation for wave–ship interactions.

Variational principles for the motion of a rigid body coupled to its interior fluid motion are given by Moiseyev & Rumyantsev (1968) and Lukovsky (2015) (and references therein). In chapter 3 of the work by Lukovsky (2015), the Bateman–Luke variational principle is used to develop a mathematical theory for interactions between a floating rigid body, containing tanks filled with liquids, and exterior ocean waves. Rumiantsev (1966) derived the Lagrangian equations of motion from the principle of least action in the Hamilton–Ostrogradskii form, for a rigid body with cavities, partially or completely filled with an ideal fluid possessing surface tension. Alemi Ardakani (2017) derived a coupled variational principle for the two-dimensional interactions between gravity-driven water waves and a rigid body dynamically coupled to its interior potential flow with uniform vorticity. Our main goal in the current paper is to develop new variational principles for interactions between three-dimensional potential water waves and a freely floating rigid body dynamically coupled to its interior inviscid and incompressible fluid sloshing. For this purpose, we need to apply calculus of variations in the proper rotation group of $\mathbb{R}^3$ using the Euler–Poincaré framework.

Holm, Marsden & Ratiu (1998*a*) derived the Euler–Poincaré equations, the Lagrangian analogue of the Lie–Poisson Hamiltonian equations, for the motion of a free rigid body and a heavy top. In the study of rigid body mechanics, the Lie group SO(3) is the configuration space and also the symmetry group of the Lagrangian functional which allows us to introduce a specific procedure for obtaining the reduced dynamics on the quotient space TSO(3)/SO(3) (Holm, Schmah & Stoica 2009). This procedure is called Euler–Poincaré reduction (Holm *et al.* 2009). The Euler–Poincaré reduction theorem, for rigid body dynamics and for dynamical systems with a broken symmetry such as the motion of a heavy top, is discussed in detail in chapter 7 of the work by Holm *et al.* (2009). The Euler–Poincaré equations for the motion of an ideal incompressible fluid, for a rotating stratified ideal incompressible fluid and for rotating shallow-water dynamics are given by Holm *et al.* (1998*a*), Holm, Marsden & Ratiu (1999). The Euler–Poincaré equations for the mean motion of ideal incompressible fluids with nonlinear dispersion in three dimensions, including rotation and stratification, are derived by Holm, Marsden & Ratiu (1998*b*). Gay-Balmaz, Marsden & Ratiu (2012) applied the Euler–Poincaré reduction theorem for semidirect products (Holm *et al.* 1998*a*) to develop the Lagrangian convective formulation of free boundary compressible hydrodynamics. Both the constrained variational principle and the equations of motion are presented. Lewis *et al.* (1986) determined the Poisson bracket structure for an incompressible fluid with a free boundary and showed that the equations for an ideal fluid having a free boundary with surface tension are Hamiltonian relative to this structure. Using the symmetry of particle relabelling, Mazer & Ratiu (1989) derived the non-canonical Poisson bracket in Eulerian representation as a reduction from the canonical bracket in Lagrangian representation. They extended the results of Lewis *et al.* (1986) to the case of ideal adiabatic self-gravitating flow with surface tension and obtained a Hamiltonian formulation of this problem.

Variational principles are useful in the formulation of the exact differential equations governing the motion of a fluid (Salmon 1983, 1988; Morrison 1998), for fluid–structure interactions (Alemi Ardakani 2017), in the derivation of approximate equations in geophysical fluid dynamics using asymptotic expansions of Lagrangian functionals (Oliver 2006, 2014), identifying conservation laws (Shepherd 1990), constructing variational symplectic numerical schemes (Marsden & West 2001) and developing approximate methods for multimodal expansion of solutions (Faltinsen

& Timokha 2009; Lukovsky 2015). The variational principles are connected with conservation of energy and the symplectic structure of the equations (Alemi Ardakani 2016).

The interests in this paper are threefold. Firstly, to derive a Lagrangian action functional for the coupled dynamics between a rigid body and its interior inviscid and incompressible fluid motion in three-dimensional rotating and translating coordinates. Secondly, to apply the Euler–Poincaré reduction framework to Hamilton's variational principle for the coupled fluid–vessel dynamics in order to derive the exact differential equations for the rotational and translational motion of the rigid body interacting with its interior fluid, and also to derive the partial differential equations governing the motion of the fluid. Thirdly, to extend the variational principle to the problem of three-dimensional interactions between gravity-driven water waves and a freely floating rigid body with interior fluid motion, and to derive the exact partial differential equations governing the nonlinear wave–structure–slosh interactions.

Gerrits & Veldman (2003) and Veldman *et al.* (2007) studied the problem of coupled liquid–solid dynamics for a liquid-filled spacecraft. They presented the coupled differential equations for the motion of a spacecraft containing fluid, describing the conservation of linear momentum and angular momentum, and for the motion of a viscous incompressible fluid sloshing on board a spacecraft relative to a moving (rotating and translating) reference frame attached to the spacecraft. The governing equations for the motion of the spacecraft containing fluid, neglecting the thruster-induced forces and torque on the spacecraft, are respectively

$$m\dot{\boldsymbol{q}} + \dot{\boldsymbol{\omega}} \times m\bar{\boldsymbol{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times m\bar{\boldsymbol{r}}) = -\int_V \rho \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\omega} \times \boldsymbol{u} - \boldsymbol{F} \right) \mathrm{d}V + m_s \boldsymbol{F}, \tag{1.1}$$

and

$$m\bar{\boldsymbol{r}} \times \dot{\boldsymbol{q}} + \boldsymbol{I}_t \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I}_t \boldsymbol{\omega} = -\int_V \rho \boldsymbol{r} \times \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\omega} \times \boldsymbol{u} - \boldsymbol{F} \right) \mathrm{d}V + m_s \bar{\boldsymbol{r}}_s \times \boldsymbol{F}, \tag{1.2}$$

where the integral is over the volume V of the fluid, $\boldsymbol{u} = (u, v, w)$ is the velocity of the fluid relative to the moving coordinate system attached to the spacecraft, $\boldsymbol{F}$ is the acceleration due to gravity, $\dot{\boldsymbol{q}}$ is the linear acceleration of the origin of the moving reference frame relative to an inertial reference frame, $\boldsymbol{\omega}$ is the angular velocity of the moving frame, $\dot{\boldsymbol{\omega}}$ is the angular acceleration of the moving frame, m is the total mass (body + fluid) of the spacecraft, $\boldsymbol{I}_t$ is the moment of inertia tensor of the coupled system (body + fluid) relative to the origin of the moving reference frame, $\boldsymbol{r}$ is the position of a liquid particle in the moving reference frame, $\bar{\boldsymbol{r}}$ is the centre of mass of the coupled system with respect to the moving reference frame, m_s and $\bar{\boldsymbol{r}}_s$ denote the mass and the centre of mass of the dry spacecraft relative to the moving reference frame, respectively. Using the divergence theorem, in (1.1) and (1.2) the force and torque that the fluid exerts on the boundary of the rigid body, via pressure and viscous effects, are replaced by $\mathrm{D}\boldsymbol{u}/\mathrm{D}t - \boldsymbol{F} + f$ and $\boldsymbol{r} \times (\mathrm{D}\boldsymbol{u}/\mathrm{D}t - \boldsymbol{F} + f)$, respectively, where f is the acceleration due to a virtual body force induced by the motion of the tank. It is shown in § 3 that equations (1.1) and (1.2) for the rigid body motion, neglecting viscosity of the fluid, can be recovered from a variational principle.

The paper starts with the derivation of an action functional for the coupled interactions between a rigid body, undergoing three-dimensional rotational and translational motions, and its interior fluid in § 2. In § 3, the detailed derivations of the Euler–Poincaré equations for the angular momentum and linear momentum of the rigid body are given. In § 4, the Euler–Poincaré equations of motion for the interior fluid of the rigid body are derived relative to the body coordinate system attached to the rotating–translating rigid body. In § 5, the proposed variational principle is extended to interactions between potential water waves and a floating rigid body containing inviscid fluid. The exact nonlinear hydrodynamic equations for the rigid body motion are derived. The paper ends with concluding remarks in § 6.

2. A Lagrangian functional for the coupled fluid–body dynamics

Consider a three-dimensional (3-D) rigid body which contains an inviscid and incompressible fluid, with free surface, undergoing three-dimensional rotational and translational motions. Using the calculus of variations, the Euler–Poincaré equations for the dynamic coupling between the rigid body motion and its interior fluid sloshing can be derived from a Lagrangian action functional. The derivation of this action functional is given below.

For the study of dynamic coupling between the rigid body motion and its interior rotating and translating fluid, three frames of reference are used. One spatial frame and two body frames. The spatial frame, which is fixed in space, has coordinates denoted by $\boldsymbol{X} = (X, Y, Z)$. The first body frame, which is placed at the centre of rotation of the moving body and used for the analysis of the rigid body motion, has coordinates denoted by $\boldsymbol{x}_b = (x_b, y_b, z_b)$. The second body frame, which is attached to the moving body and used for the analysis of the fluid motion inside the tank, has coordinates denoted by $\boldsymbol{x} = (x, y, z)$. The distance between the origin of the body frame $\boldsymbol{x}$ to the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$, is denoted by $\boldsymbol{d} = (d_1, d_2, d_3)$ which is a constant vector. So the position of a fluid particle relative to the body frame $\boldsymbol{x}_b$ is $\boldsymbol{x}_b = \boldsymbol{x} + \boldsymbol{d}$. The fluid–tank system has a uniform translation $\boldsymbol{q}(t) = (q_1(t), q_2(t), q_3(t))$ relative to the spatial frame $\boldsymbol{X}$, which is the vector from the origin of the spatial frame $\boldsymbol{X}$ to the origin of the body frame $\boldsymbol{x}_b$. The configuration of the fluid in a rotating–translating vessel is schematically shown in figure 1.

The Lagrangian action for the coupled fluid–vessel dynamics takes the form

$$\mathscr{L} = \int_{t_1}^{t_2} \left(\mathrm{KE}^{fluid} - \mathrm{PE}^{fluid} + \mathrm{KE}^{vessel} - \mathrm{PE}^{vessel} \right) \, \mathrm{d}t, \tag{2.1}$$

where KE^{fluid} is the kinetic energy of the fluid, KE^{vessel} is the kinetic energy of the vessel, PE^{fluid} is the potential energy of the fluid and PE^{vessel} is the potential energy of the vessel. The expressions are derived below. For the kinetic and potential energies, the velocity vector and the displacement vector should be relative to the spatial frame $\boldsymbol{X}$, but the analysis of the fluid motion is carried out in the body frame $\boldsymbol{x}$. The relationship between the two velocities is developed using the kinetic theory of rigid bodies (Murray, Lin & Sastry 1994; O'Reilly 2008).

Let $\boldsymbol{Q}(t) \in \mathrm{SO}(3)$ be a proper rotation in $\mathbb{R}^3$,

$$\boldsymbol{Q}^{\mathrm{T}} \boldsymbol{Q} = \boldsymbol{I} \quad \text{and} \quad \det(\boldsymbol{Q}) = 1. \tag{2.2}$$

Then the relation between the spatial displacement and the body displacement for a fluid particle is

$$\boldsymbol{X} = \boldsymbol{Q}(\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{q}. \tag{2.3}$$

To clarify, we write $\boldsymbol{x} = (x(a, b, c, t), y(a, b, c, t), z(a, b, c, t))$ as Cartesian coordinates of a fluid particle marked by Lagrangian labels $\boldsymbol{a} = (a, b, c)$ at time t. This formulation

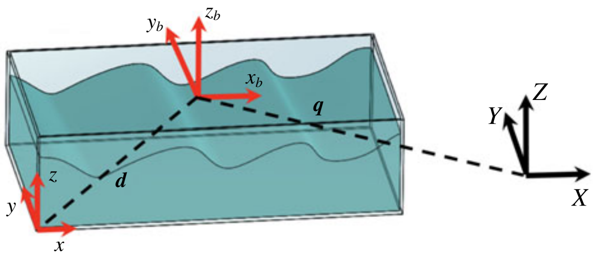

FIGURE 1. (Colour online) Schematic showing a configuration of the fixed coordinate system $\boldsymbol{X} = (X, Y, Z)$ relative to the moving coordinate systems $\boldsymbol{x}_b = (x_b, y_b, z_b)$ and $\boldsymbol{x} = (x, y, z)$, attached to the vessel. The distance between the origin of the spatial frame $\boldsymbol{X}$ and the point of rotation is denoted by $\boldsymbol{q}$. The distance from the origin of the body frame $\boldsymbol{x}$ to the point of rotation is denoted by $\boldsymbol{d}$.

is consistent with the theory of rigid body motion, where an arbitrary motion can be described by the pair $(\boldsymbol{Q}(t), \boldsymbol{q}(t))$.

The body angular velocity is a time-dependent vector

$$\boldsymbol{\Omega}(t) = (\Omega_1(t), \Omega_2(t), \Omega_3(t)), \tag{2.4}$$

relative to the body coordinate system $\boldsymbol{x}_b$ with entries determined from $\boldsymbol{Q}$ by

$$\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{Q}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} := \widehat{\boldsymbol{\Omega}}, \tag{2.5}$$

where $\widehat{\boldsymbol{\Omega}} \in \mathfrak{so}(3)$ is the Lie algebra of the group of rotations of $\mathbb{R}^3$, i.e. $\boldsymbol{Q}(t)$. The convention for the entries of the skew–symmetric matrix $\widehat{\boldsymbol{\Omega}}$ is such that

$$\text{Hat map:} \quad \widehat{\boldsymbol{\Omega}}\boldsymbol{r} = \boldsymbol{\Omega} \times \boldsymbol{r}, \quad \text{for any } \boldsymbol{r} \in \mathbb{R}^3, \quad \boldsymbol{\Omega} := (\Omega_1, \Omega_2, \Omega_3). \tag{2.6}$$

The body angular velocity is to be contrasted with the spatial angular velocity, the angular velocity viewed from the spatial frame, which is

$$\widehat{\boldsymbol{\Omega}}^{spatial} := \dot{\boldsymbol{Q}}\boldsymbol{Q}^{\mathrm{T}}. \tag{2.7}$$

As vectors the spatial and body angular velocities are related by $\boldsymbol{\Omega}^{spatial} = \boldsymbol{Q}\boldsymbol{\Omega}$. Either representation for the angular velocity can be used. But the body representation is the sensible choice leading to great simplification of the differential equations.

The relation between the body velocity and space velocity is

$$\dot{\boldsymbol{X}} = \boldsymbol{Q}(\dot{\boldsymbol{x}} + \boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d}) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}). \tag{2.8}$$

If $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ denotes the Eulerian velocity of a fluid particle relative to the body frame with $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$ the corresponding flow map, the fluid particle initially at position $\boldsymbol{a}$ is at position $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a}, t)$ at time t, then the Lagrangian velocity of the fluid particle is $\dot{\boldsymbol{x}}(\boldsymbol{a}, t) = \boldsymbol{u}(\boldsymbol{x}(\boldsymbol{a}, t), t)$, and the Lagrangian acceleration of the fluid particle is $\ddot{\boldsymbol{x}}(\boldsymbol{a}, t) = \mathrm{D}\boldsymbol{u}/\mathrm{D}t = \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla\boldsymbol{u}$. For an incompressible fluid the Jacobian J of the label-to-particle mapping $(a, b, c) \to (x, y, z)$ is the motion invariant, i.e. $\partial J/\partial t = 0$ which is the continuity equation $\nabla\cdot\boldsymbol{u}=0$ in the Lagrangian particle-path formulation. The Lagrangian labels (a, b, c) can be chosen such that

$$J=\frac{\partial(x, y, z)}{\partial(a, b, c)}=x_a(y_bz_c-y_cz_b)+x_b(y_cz_a-y_az_c)+x_c(y_az_b-y_bz_a)=1. \tag{2.9}$$

This means that at an initial time t_0 the Lagrangian labels (a, b, c) are physically possible coordinates, i.e. $(a, b, c)=(x_0, y_0, z_0)$. So in Eulerian coordinates (2.8) takes the form

$$\boldsymbol{U}=\boldsymbol{Q}(\boldsymbol{u}+\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}), \tag{2.10}$$

where $\boldsymbol{U}(\boldsymbol{X}, t)$ is the Eulerian velocity of a fluid particle in the spatial frame $\boldsymbol{X}$. Using this expression the kinetic energy of the fluid is

$$\begin{aligned}\mathrm{KE}^{fluid} &= \int_{\mathcal{V}}\frac{1}{2}\,\|\boldsymbol{U}\|^2\,\rho\,\mathrm{d}\mathcal{V}\\ &= \int_{\mathcal{V}}\left(\frac{1}{2}\,\|\boldsymbol{u}\|^2+\boldsymbol{u}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\right)+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\cdot(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))\right.\\ &\quad\left.+\frac{1}{2}\,\|\dot{\boldsymbol{q}}\|^2+\frac{1}{2}\,\|\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})\|^2\right)\rho\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{2.11}$$

where ρ is the density of the fluid which is considered to be inviscid and incompressible, and the integral is over the volume $(\mathcal{V})$ of the fluid inside the container. This expression can be simplified using the definition of the fluid mass moment of inertia, noting that

$$\begin{aligned}\int_{\mathcal{V}}\frac{1}{2}\,\|\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})\|^2\,\rho\,\mathrm{d}\boldsymbol{x} &= \int_{\mathcal{V}}\frac{1}{2}(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))\cdot(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))\rho\,\mathrm{d}\boldsymbol{x}\\ &= \int_{\mathcal{V}}\frac{1}{2}((\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})((\boldsymbol{x}+\boldsymbol{d})\cdot(\boldsymbol{x}+\boldsymbol{d}))-(\boldsymbol{\Omega}\cdot(\boldsymbol{x}+\boldsymbol{d}))^2)\rho\,\mathrm{d}\boldsymbol{x}\\ &= \boldsymbol{\Omega}\cdot\left(\left(\frac{1}{2}\int_{\mathcal{V}}\left(\|\boldsymbol{x}+\boldsymbol{d}\|^2\,\boldsymbol{I}-(\boldsymbol{x}+\boldsymbol{d})\otimes(\boldsymbol{x}+\boldsymbol{d})\right)\rho\,\mathrm{d}\boldsymbol{x}\right)\boldsymbol{\Omega}\right)\\ &= \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f\boldsymbol{\Omega},\end{aligned} \tag{2.12}$$

with

$$\boldsymbol{I}_f=\int_{\mathcal{V}}(\|\boldsymbol{x}+\boldsymbol{d}\|^2\boldsymbol{I}-(\boldsymbol{x}+\boldsymbol{d})\otimes(\boldsymbol{x}+\boldsymbol{d}))\rho\,\mathrm{d}\boldsymbol{x}, \tag{2.13}$$

where $\otimes$ denotes the tensor product, $\boldsymbol{I}$ is the 3×3 identity matrix and $\boldsymbol{I}_f$ is the mass moment of inertia of the fluid relative to the point of rotation, i.e. the origin of the body frame $\boldsymbol{x}_b$. So the kinetic energy of the fluid can be written as

$$\begin{aligned}\mathrm{KE}^{fluid} &= \int_{\mathcal{V}}\left(\frac{1}{2}\,\|\boldsymbol{u}\|^2+\boldsymbol{u}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\right)\right.\\ &\quad\left.+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\cdot(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))+\frac{1}{2}\,\|\dot{\boldsymbol{q}}\|^2\right)\rho\,\mathrm{d}\boldsymbol{x}+\frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f\boldsymbol{\Omega}.\end{aligned} \tag{2.14}$$

The potential energy of the fluid in (2.1) is

$$\mathrm{PE}^{fluid}=\int_{\mathcal{V}}\rho g\,(\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q})\cdot\hat{z}\,\mathrm{d}\boldsymbol{x}, \tag{2.15}$$

where $\hat{z}$ is the unit vector in the Z direction, and g is the acceleration due to gravity.

To derive the kinetic energy of the dry vessel note that the relation between the body velocity and the space velocity is

$$\dot{\boldsymbol{X}} = \dot{\boldsymbol{Q}}\boldsymbol{x}_b + \dot{\boldsymbol{q}} \quad \text{or} \quad \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{X}} = (\boldsymbol{\Omega} \times \boldsymbol{x}_b) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}. \tag{2.16a,b}$$

Then the kinetic energy of the vessel is

$$\begin{aligned}\mathrm{KE}^{vessel} &= \int_{\mathbb{V}} \frac{1}{2} \left\|\dot{\boldsymbol{X}}\right\|^2 \rho_v \,\mathrm{d}\mathbb{V} = \int_{\mathbb{V}} \frac{1}{2} \left\|\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{X}}\right\|^2 \rho_v \,\mathrm{d}\mathbb{V} \\ &= \int_{\mathbb{V}} \left(\frac{1}{2} \left\|\boldsymbol{\Omega} \times \boldsymbol{x}_b\right\|^2 + (\boldsymbol{\Omega} \times \boldsymbol{x}_b) \cdot \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}} + \frac{1}{2}\left\|\dot{\boldsymbol{q}}\right\|^2\right) \rho_v \,\mathrm{d}\mathbb{V},\end{aligned} \tag{2.17}$$

where ρ_v is the density of the vessel and the integral is over the volume $\mathbb{V}$ of the vessel. Taking into account that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{V}} \left\|\boldsymbol{\Omega} \times \boldsymbol{x}_b\right\|^2 \rho_v \,\mathrm{d}\mathbb{V} &= \frac{1}{2}\int_{\mathbb{V}} \left((\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\,(\boldsymbol{x}_b\cdot\boldsymbol{x}_b) - (\boldsymbol{\Omega}\cdot\boldsymbol{x}_b)^2\right) \rho_v \,\mathrm{d}\mathbb{V} \\ &= \boldsymbol{\Omega}\cdot\left(\left(\frac{1}{2}\int_{\mathbb{V}} \left(\|\boldsymbol{x}_b\|^2\boldsymbol{I} - \boldsymbol{x}_b\otimes\boldsymbol{x}_b\right)\rho_v \,\mathrm{d}\mathbb{V}\right)\boldsymbol{\Omega}\right) = \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega},\end{aligned} \tag{2.18}$$

with

$$\boldsymbol{I}_v = \int_{\mathbb{V}} (\|\boldsymbol{x}_b\|^2\boldsymbol{I} - \boldsymbol{x}_b\otimes\boldsymbol{x}_b)\rho_v \,\mathrm{d}\mathbb{V}, \tag{2.19}$$

and that

$$\int_{\mathbb{V}} (\boldsymbol{\Omega}\times\boldsymbol{x}_b)\cdot\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\rho_v \,\mathrm{d}\mathbb{V} \;=\; (\boldsymbol{\Omega}\times m_v\bar{\boldsymbol{x}}_v)\cdot\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}, \tag{2.20}$$

then the kinetic energy of the vessel simplifies to

$$\mathrm{KE}^{vessel} = \tfrac{1}{2}m_v\,\|\dot{\boldsymbol{q}}\|^2 + (\boldsymbol{\Omega}\times m_v\bar{\boldsymbol{x}}_v)\cdot\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}} + \tfrac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega}, \tag{2.21}$$

where $\boldsymbol{I}_v$ is the mass moment of inertia of the dry vessel relative to the point of rotation, m_v is the mass of the dry vessel and $\bar{\boldsymbol{x}}_v = (\bar{x}_v, \bar{y}_v, \bar{z}_v)$ is the centre of mass of the dry vessel relative to the body frame $\boldsymbol{x}_b$. The potential energy of the dry vessel in (2.1) is

$$\mathrm{PE}^{vessel} = m_v g(\boldsymbol{Q}\bar{\boldsymbol{x}}_v + \boldsymbol{q})\cdot\hat{z}. \tag{2.22}$$

Now substitution of (2.14), (2.15), (2.21) and (2.22) into the action integral (2.1) gives an explicit form of the Lagrangian functional for the dynamics of a rigid body coupled to its interior fluid motion

$$\begin{aligned}\mathscr{L}(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{u}) = \int_{t_1}^{t_2} &\left(\int_{\mathcal{V}}\left(\frac{1}{2}\|\boldsymbol{u}\|^2 + \boldsymbol{u}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\right)\right.\right. \\ &+ \left.\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\cdot(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})) + \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2 - g\,(\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q})\cdot\hat{z}\right)\rho\,\mathrm{d}\boldsymbol{x} + \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f\boldsymbol{\Omega} \\ &+ \left.\frac{1}{2}m_v\,\|\dot{\boldsymbol{q}}\|^2 + (\boldsymbol{\Omega}\times m_v\bar{\boldsymbol{x}}_v)\cdot\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega} - m_v g\,(\boldsymbol{Q}\bar{\boldsymbol{x}}_v + \boldsymbol{q})\cdot\hat{z}\right)\mathrm{d}t.\end{aligned} \tag{2.23}$$

Transformation of the Lagrangian functional (2.23) from the Eulerian setting to the Lagrangian particle-path setting gives

$$\begin{aligned}\mathscr{L}(\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{x},\dot{\boldsymbol{x}}) = \int_{t_1}^{t_2}\bigg(\int_{\mathcal{V}'}\bigg(\frac{1}{2}\|\dot{\boldsymbol{x}}\|^2+\dot{\boldsymbol{x}}\boldsymbol{\cdot}\big(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\big)\\ +\ \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\boldsymbol{\cdot}(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))+\frac{1}{2}\|\dot{\boldsymbol{q}}\|^2-g\,(\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q})\boldsymbol{\cdot}\hat{z}\bigg)\rho\,\mathrm{d}\boldsymbol{a}+\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{\cdot}\boldsymbol{I}_f\boldsymbol{\Omega}\\ +\ \frac{1}{2}m_v\|\dot{\boldsymbol{q}}\|^2+(\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v)\boldsymbol{\cdot}\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}+\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{\cdot}\boldsymbol{I}_v\boldsymbol{\Omega}-m_v g\,(\boldsymbol{Q}\overline{\boldsymbol{x}}_v+\boldsymbol{q})\boldsymbol{\cdot}\hat{z}\bigg)\,\mathrm{d}t,\end{aligned} \tag{2.24}$$

where the integral is over the volume $\mathcal{V}'$ of the reference space, and

$$\boldsymbol{I}_f=\int_{\mathcal{V}'}(\|\boldsymbol{x}+\boldsymbol{d}\|^2\boldsymbol{I}-(\boldsymbol{x}+\boldsymbol{d})\otimes(\boldsymbol{x}+\boldsymbol{d}))\rho\,\mathrm{d}\boldsymbol{a}. \tag{2.25}$$

Taking the first variations of the Lagrangian functional (2.24) with respect to $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ yields the Euler–Poincaré equations for the angular momentum and linear momentum of the rigid body containing fluid. Moreover, taking the first variation of the action integral (2.24) with respect to $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$, after the addition of a constraint term to this functional to enforce incompressibility of the fluid (see § 4), gives the Euler–Poincaré equation for the motion of the interior fluid relative to the body frame $\boldsymbol{x}$. In the action integrals (2.23) and (2.24), $\boldsymbol{q}(t)$ is relative to the spatial frame $\boldsymbol{X}$.

Using a similar approach, the Lagrangian functional (2.23) can be derived for the case where $\boldsymbol{q}(t)$ is relative to the body frame $\boldsymbol{x}_b$, i.e. $\boldsymbol{q}_b(t)=\boldsymbol{Q}^{-1}\boldsymbol{q}$. The action integral for this case reads

$$\begin{aligned}\mathscr{L}\left(\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q}_b,\dot{\boldsymbol{q}}_b,\boldsymbol{u}\right)=\int_{t_1}^{t_2}\bigg(\int_{\mathcal{V}}\bigg(\frac{1}{2}\|\boldsymbol{u}+\dot{\boldsymbol{q}}_b\|^2+(\boldsymbol{u}+\dot{\boldsymbol{q}}_b)\boldsymbol{\cdot}\big(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}+\boldsymbol{q}_b)\big)\\ +\ (\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))\boldsymbol{\cdot}(\boldsymbol{\Omega}\times\boldsymbol{q}_b)+\frac{1}{2}\left\|\boldsymbol{\Omega}\times\boldsymbol{q}_b\right\|^2-g\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d}+\boldsymbol{q}_b)\boldsymbol{\cdot}\hat{z}\bigg)\rho\,\mathrm{d}\boldsymbol{x}\\ +\ \frac{1}{2}\boldsymbol{\Omega}\boldsymbol{\cdot}\boldsymbol{I}_f\boldsymbol{\Omega}+\frac{1}{2}m_v\|\dot{\boldsymbol{q}}_b\|^2+\dot{\boldsymbol{q}}_b\boldsymbol{\cdot}\big(\boldsymbol{\Omega}\times m_v\left(\overline{\boldsymbol{x}}_v+\boldsymbol{q}_b\right)\big)+\frac{1}{2}m_v\|\boldsymbol{\Omega}\times\boldsymbol{q}_b\|^2\\ +\ (\boldsymbol{\Omega}\times m_v\overline{\boldsymbol{x}}_v)\boldsymbol{\cdot}\left(\boldsymbol{\Omega}\times\boldsymbol{q}_b\right)+\frac{1}{2}\boldsymbol{\Omega}\boldsymbol{\cdot}\boldsymbol{I}_v\boldsymbol{\Omega}-m_v g\boldsymbol{Q}\left(\overline{\boldsymbol{x}}_v+\boldsymbol{q}_b\right)\boldsymbol{\cdot}\hat{z}\bigg)\,\mathrm{d}t,\end{aligned} \tag{2.26}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{q}_b$ are both relative to the body frame $\boldsymbol{x}_b$. Similarly, to obtain the equations of motion for the interior fluid, a constraint term should be added to the functional (2.26) to enforce incompressibility of the fluid (see § 4).

In the Lagrangian action (2.24) the rotation tensor $\boldsymbol{Q}(t)$ can be parameterised using the 3–2–1 Euler angles in three dimensions (O'Reilly 2008). This gives an expression for the body angular velocity $\widehat{\boldsymbol{\Omega}}=\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{Q}}$. After substituting for $\boldsymbol{\Omega}$ and $\boldsymbol{Q}$, in terms of the Euler angles and their derivatives, into the Lagrangian action, the Euler–Lagrange equations for the rotational and translational motion of the rigid body, containing fluid, can be provided by Hamilton's principle $\delta\mathscr{L}\left(\boldsymbol{Q},\dot{\boldsymbol{Q}},\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{x},\dot{\boldsymbol{x}}\right)=0$. But the problem in using the Euler angles is that the resulting Euler–Lagrange equations are unwieldy to work with directly to determine the motion of the rigid body as the differential equations become very lengthy. However, we can follow an alternative path, taking variations of the action functional (2.24) with respect to $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, applying similar calculus of variations used by Holm *et al.* (2009) for the Euler–Poincaré reduction of the non-free rigid body motion. See chapter 13 of Marsden & Ratiu (1999), and chapters 7 and 11 of Holm *et al.* (2009) for the background history and mathematics on the Euler–Poincaré reduction framework in Lagrangian mechanics. So our strategy is to adapt the Euler–Poincaré framework, first developed for rigid body dynamics, in order to derive the exact differential equations for the dynamic coupling between a rigid body and its interior fluid sloshing in three-dimensional rotating and translating coordinates.

In the presented derivations in this paper, the calculus of variations for the Lagrangian functional of the coupled (interior fluid + body) system (2.24) is done in the Lagrangian particle-path setting using the Euler–Poincaré framework, and then the equations of motion are transformed to Eulerian coordinates. Alternatively, we could take the variations of the Lagrangian functional (2.23) in Eulerian coordinates, after the addition of a constraint term to this functional to enforce incompressibility of the fluid, by using the so-called *Lin constraint* (Bretherton 1970; Oliver 2006) for the variations $\delta \boldsymbol{u}$ of the form

$$\delta \boldsymbol{u} = \dot{\boldsymbol{w}} + \nabla \boldsymbol{w}\boldsymbol{u} - \nabla \boldsymbol{u}\boldsymbol{w}, \tag{2.27}$$

where $\boldsymbol{w}$ is a vector-valued free variation.

3. The Euler–Poincaré equations for the rigid body motion

The Euler–Poincaré reduction theorem for the motion of a free rigid body and a heavy top is given by Holm *et al.* (1998*a*). Here, we apply this framework to dynamic coupling between rigid body motion and its interior inviscid and incompressible fluid motion in the Lagrangian particle-path formulation. In this section the detailed derivations of the Euler–Poincaré equations for the body angular velocity $\boldsymbol{\Omega}(t)$ and translational motion $\boldsymbol{q}(t)$ of the rigid body are given. The aim is to recover equations (1.1) and (1.2), which are in Eulerian coordinates, from a variational principle with precise definitions of dependent variables with respect to the spatial and body coordinate systems.

3.1. *The Euler–Poincaré equation for* $\boldsymbol{\Omega}(t)$

The equation of motion for the body angular velocity $\boldsymbol{\Omega}(t)$ is provided by Hamilton's variational principle:

$$\delta \mathscr{L}(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x}, \dot{\boldsymbol{x}}) = 0, \tag{3.1}$$

where the Lagrangian action $\mathscr{L}$ $(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x}, \dot{\boldsymbol{x}})$ is defined in (2.24) and the variations $\delta \boldsymbol{Q}$ are taken among paths $\boldsymbol{Q}(t) \in \mathrm{SO}(3)$, $t \in [t_1, t_2]$, with fixed endpoints, so that $\delta \boldsymbol{Q}(t_1) = \delta \boldsymbol{Q}(t_2) = \boldsymbol{0}$. The variations $\delta \boldsymbol{\Omega}$ are induced by the variations $\delta \boldsymbol{Q}$ via (Holm *et al.* 2009)

$$\delta \widehat{\boldsymbol{\Omega}} = \frac{\mathrm{d}\widehat{\boldsymbol{\Gamma}}}{\mathrm{d}t} + [\widehat{\boldsymbol{\Omega}}, \widehat{\boldsymbol{\Gamma}}] = \frac{\mathrm{d}\widehat{\boldsymbol{\Gamma}}}{\mathrm{d}t} + \widehat{\boldsymbol{\Omega}}\widehat{\boldsymbol{\Gamma}} - \widehat{\boldsymbol{\Gamma}}\widehat{\boldsymbol{\Omega}}, \tag{3.2}$$

where $\widehat{\boldsymbol{\Gamma}} \in \mathfrak{so}(3)$ is defined by

$$\widehat{\boldsymbol{\Gamma}} = \boldsymbol{Q}^{-1}\delta\boldsymbol{Q}. \tag{3.3}$$

Since $[\widehat{\boldsymbol{\Omega}}, \widehat{\boldsymbol{\Gamma}}] = \widehat{\boldsymbol{\Omega}\times\boldsymbol{\Gamma}}$, the equivalent vector representation of (3.2) is

$$\delta\boldsymbol{\Omega} = \dot{\boldsymbol{\Gamma}} + \boldsymbol{\Omega}\times\boldsymbol{\Gamma}. \tag{3.4}$$

Also it can be proved that (Marsden & Ratiu 1999; Holm *et al.* 2009)

$$\delta\boldsymbol{Q}^{-1} = -\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\boldsymbol{Q}^{-1} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\right) = -\boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1}. \tag{3.5a,b}$$

Now the Euler–Poincaré equation for $\boldsymbol{\Omega}(t)$ can be obtained by taking the first variation of the action integral $\mathscr{L}(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x}, \dot{\boldsymbol{x}})$ in (2.24) with respect to $\boldsymbol{\Omega}$ and $\boldsymbol{Q}$. Due to lengthy derivations, here we calculate the first variation of each term in (2.24) with respect to $\boldsymbol{\Omega}$ and $\boldsymbol{Q}$ separately. For the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ of the second term in the Lagrangian action (2.24), assuming that $\dot{\boldsymbol{q}}$, $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are constants, we have

$$\begin{aligned}
&\delta\int_{t_1}^{t_2}\int_{\mathcal{V}'}\langle\dot{\boldsymbol{x}}, \boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t\\
&\quad=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\langle\dot{\boldsymbol{x}}, \delta\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \delta\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t\\
&\quad=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left\langle\dot{\boldsymbol{x}}, \underbrace{(\dot{\boldsymbol{\Gamma}}+\boldsymbol{\Omega}\times\boldsymbol{\Gamma})}_{=\delta\boldsymbol{\Omega}}\times(\boldsymbol{x}+\boldsymbol{d})\underbrace{-\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\boldsymbol{Q}^{-1}}_{=\delta\boldsymbol{Q}^{-1}}\dot{\boldsymbol{q}}\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t\\
&\quad=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left\langle\dot{\boldsymbol{x}}, -(\boldsymbol{x}+\boldsymbol{d})\times\dot{\boldsymbol{\Gamma}} - (\boldsymbol{x}+\boldsymbol{d})\times(\boldsymbol{\Omega}\times\boldsymbol{\Gamma})\underbrace{-\widehat{\boldsymbol{\Gamma}}\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}}_{\text{using (3.3)}}\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t\\
&\quad=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left\langle\dot{\boldsymbol{x}}, -(\boldsymbol{x}+\boldsymbol{d})\times\dot{\boldsymbol{\Gamma}} - (\boldsymbol{x}+\boldsymbol{d})\times(\boldsymbol{\Omega}\times\boldsymbol{\Gamma})\underbrace{+\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\times\boldsymbol{\Gamma}}_{\substack{\text{using the}\\ \text{hat map}}}\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t\\
&\quad=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\big(\langle\dot{\boldsymbol{\Gamma}}, -\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})\rangle + \langle\boldsymbol{\Omega}\times\boldsymbol{\Gamma}, -\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})\rangle\\
&\qquad+\langle\boldsymbol{\Gamma}, \dot{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\rangle\big)\,\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t \rightarrow \left\{\begin{array}{l}\text{using the vector identity}\\ \boldsymbol{a}\boldsymbol{\cdot}(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{c}\boldsymbol{\cdot}(\boldsymbol{a}\times\boldsymbol{b})\end{array}\right.\\
&\quad=\int_{\mathcal{V}'}\big(\langle\boldsymbol{\Gamma}, -\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})\rangle|_{t_2} - \langle\boldsymbol{\Gamma}, -\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})\rangle|_{t_1}\big)\,\rho\,\mathrm{d}\boldsymbol{a}\\
&\qquad+\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left(\left\langle\boldsymbol{\Gamma}, \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d}))\right\rangle + \langle\boldsymbol{\Gamma}, ((\boldsymbol{x}+\boldsymbol{d})\times\dot{\boldsymbol{x}})\times\boldsymbol{\Omega}\rangle\right.\\
&\qquad\left.+\langle\boldsymbol{\Gamma}, \dot{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\rangle\right)\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t \quad\rightarrow \text{(integrating by parts)}\\
&\quad=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left\langle\boldsymbol{\Gamma}, \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})) + \boldsymbol{\Omega}\times(\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})) + \dot{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t,
\end{aligned} \tag{3.6}$$

where the terms $\langle\boldsymbol{\Gamma}, -\dot{\boldsymbol{x}}\times(\boldsymbol{x}+\boldsymbol{d})\rangle|_{t_{1,2}}$ vanish if we apply the inverse of the hat map to $\widehat{\boldsymbol{\Gamma}} = \boldsymbol{Q}^{-1}\delta\boldsymbol{Q}$, noting that $\delta\boldsymbol{Q}$ vanishes at the endpoints t_1 and t_2.

Now taking the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ of the third term in (2.24) using (3.3), (3.4) and (3.5*a*), assuming that $\dot{\boldsymbol{q}}$ and $\boldsymbol{x}$ are constants, gives

$$\delta\int_{t_1}^{t_2}\int_{\mathcal{V}'}\langle \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}},\, \boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left\langle \boldsymbol{\Gamma},\, \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\times(\boldsymbol{x}+\boldsymbol{d})\right)+(\boldsymbol{x}+\boldsymbol{d})\times\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\times\boldsymbol{\Omega}\right)\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t. \tag{3.7}$$

See appendix A for the proof of (3.7). To take the variation of the potential energy of the fluid in (2.24) with respect to $\boldsymbol{Q}$, assuming that $\boldsymbol{q}$ and $\boldsymbol{x}$ are constants, set

$$\boldsymbol{\Sigma}=\boldsymbol{Q}^{-1}\hat{z}, \tag{3.8}$$

which gives

$$\delta\boldsymbol{\Sigma}=\delta\boldsymbol{Q}^{-1}\hat{z}=-\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\boldsymbol{Q}^{-1}\hat{z}=-\widehat{\boldsymbol{\Gamma}}\boldsymbol{\Sigma}=\boldsymbol{\Sigma}\times\boldsymbol{\Gamma}, \tag{3.9}$$

and hence

$$\delta\int_{t_1}^{t_2}\int_{\mathcal{V}'}-g\,\underbrace{\langle\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{q},\hat{z}\rangle}_{\times\boldsymbol{Q}^{-1}}\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\delta\int_{t_1}^{t_2}\int_{\mathcal{V}'}-g\left\langle(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{-1}\boldsymbol{q},\,\underbrace{\boldsymbol{Q}^{-1}\hat{z}}_{=\boldsymbol{\Sigma}}\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}-g\left(\langle\delta\boldsymbol{Q}^{-1}\boldsymbol{q},\,\boldsymbol{\Sigma}\rangle+\langle(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{-1}\boldsymbol{q},\,\delta\boldsymbol{\Sigma}\rangle\right)\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}-g\left\langle\underbrace{-\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\boldsymbol{Q}^{-1}}_{=\delta\boldsymbol{Q}^{-1}}\boldsymbol{q},\,\boldsymbol{\Sigma}\right\rangle-g\left\langle(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{-1}\boldsymbol{q},\,\underbrace{\boldsymbol{\Sigma}\times\boldsymbol{\Gamma}}_{=\delta\boldsymbol{\Sigma}}\right\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}-g\left\langle\underbrace{-\widehat{\boldsymbol{\Gamma}}\boldsymbol{Q}^{-1}\boldsymbol{q}}_{\text{using (3.3)}},\,\boldsymbol{\Sigma}\right\rangle-g\langle(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{-1}\boldsymbol{q},\,\boldsymbol{\Sigma}\times\boldsymbol{\Gamma}\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}-g\left\langle\underbrace{\boldsymbol{Q}^{-1}\boldsymbol{q}\times\boldsymbol{\Gamma}}_{\substack{\text{using the}\\ \text{hat map}}},\,\boldsymbol{\Sigma}\right\rangle-g\langle(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{-1}\boldsymbol{q},\,\boldsymbol{\Sigma}\times\boldsymbol{\Gamma}\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\underbrace{-g(\langle\boldsymbol{\Gamma},\,\boldsymbol{\Sigma}\times\boldsymbol{Q}^{-1}\boldsymbol{q}\rangle+\langle\boldsymbol{\Gamma},\,((\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{Q}^{-1}\boldsymbol{q})\times\boldsymbol{\Sigma}\rangle)}_{\hookrightarrow\left\{\substack{\text{using the vector identity}\\ \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})}\right.}\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\int_{\mathcal{V}'}\langle\boldsymbol{\Gamma},\,-g(\boldsymbol{x}+\boldsymbol{d})\times\boldsymbol{\Sigma}\rangle\rho\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t. \tag{3.10}$$

Taking the variations $\delta\boldsymbol{\Omega}$ of the mass moment of inertia of the fluid in (2.24), assuming that $\boldsymbol{x}$ is constant, gives

$$\delta \int_{t_1}^{t_2} \left\langle \frac{1}{2}\boldsymbol{\Omega}, \boldsymbol{I}_f \boldsymbol{\Omega} \right\rangle \mathrm{d}t = \int_{t_1}^{t_2} \langle \delta\boldsymbol{\Omega}, \boldsymbol{I}_f \boldsymbol{\Omega}\rangle \, \mathrm{d}t \rightarrow \text{(noting that } \boldsymbol{I}_f \text{ is symmetric)}$$
$$= \int_{t_1}^{t_2} \langle \dot{\boldsymbol{\Gamma}} + (\boldsymbol{\Omega} \times \boldsymbol{\Gamma}), \boldsymbol{I}_f \boldsymbol{\Omega}\rangle \, \mathrm{d}t = \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I}_f \boldsymbol{\Omega}\right) + \boldsymbol{I}_f \boldsymbol{\Omega} \times \boldsymbol{\Omega} \right\rangle \mathrm{d}t, \quad (3.11)$$

where, when integrating by parts, we used the condition that the variations vanish at the endpoints in time.

Similar calculations show that taking the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ of the remaining terms in (2.24) due to the kinetic and potential energies of the vessel, assuming that $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are constants, gives

$$\left.\begin{aligned}
\delta \int_{t_1}^{t_2} \left\langle \boldsymbol{\Omega} \times m_v \bar{\boldsymbol{x}}_v, \boldsymbol{Q}^{\mathrm{T}} \dot{\boldsymbol{q}} \right\rangle \mathrm{d}t &= \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -\frac{\mathrm{d}}{\mathrm{d}t}\left(m_v \bar{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right) \right\rangle \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, m_v \bar{\boldsymbol{x}}_v \times \left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times \boldsymbol{\Omega}\right) \right\rangle \mathrm{d}t, \\
\delta \int_{t_1}^{t_2} \left\langle \frac{1}{2}\boldsymbol{\Omega}, \boldsymbol{I}_v \boldsymbol{\Omega} \right\rangle \mathrm{d}t &= \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -\boldsymbol{I}_v \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_v \boldsymbol{\Omega} \times \boldsymbol{\Omega} \right\rangle \mathrm{d}t, \\
\delta \int_{t_1}^{t_2} \left\langle -m_v g \left(\boldsymbol{Q}\bar{\boldsymbol{x}}_v + \boldsymbol{q}\right), \hat{z} \right\rangle \mathrm{d}t &= \int_{t_1}^{t_2} \langle \boldsymbol{\Gamma}, -m_v g \bar{\boldsymbol{x}}_v \times \boldsymbol{\Sigma}\rangle \, \mathrm{d}t.
\end{aligned}\right\} \quad (3.12)$$

Now from (3.6), (3.7), (3.10), (3.11) and (3.12) it can be concluded that Hamilton's variational principle (3.1) for the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ reads

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \boldsymbol{\Gamma}, \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d})) + \boldsymbol{\Omega} \times (\dot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d})) + \dot{\boldsymbol{x}} \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \boldsymbol{\Gamma}, \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d})\right) + (\boldsymbol{x}+\boldsymbol{d}) \times \left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times \boldsymbol{\Omega}\right) \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \boldsymbol{\Gamma}, -g(\boldsymbol{x}+\boldsymbol{d}) \times \boldsymbol{\Sigma}\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t + \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I}_f \boldsymbol{\Omega}\right) + \boldsymbol{I}_f \boldsymbol{\Omega} \times \boldsymbol{\Omega} \right\rangle \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -\frac{\mathrm{d}}{\mathrm{d}t}\left(m_v \bar{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right) + m_v \bar{\boldsymbol{x}}_v \times \left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times \boldsymbol{\Omega}\right) \right\rangle \mathrm{d}t \\
&+ \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -\boldsymbol{I}_v \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_v \boldsymbol{\Omega} \times \boldsymbol{\Omega} \right\rangle \mathrm{d}t + \int_{t_1}^{t_2} \langle \boldsymbol{\Gamma}, -m_v g \bar{\boldsymbol{x}}_v \times \boldsymbol{\Sigma}\rangle \, \mathrm{d}t = \boldsymbol{0}. \quad (3.13)
\end{aligned}$$

Therefore, since the variational principle (3.13) holds for any curve $\boldsymbol{\Gamma}(t)$ in $\mathfrak{so}(3)$ such that $\boldsymbol{\Gamma}(t_1) = \boldsymbol{\Gamma}(t_2) = \boldsymbol{0}$, we find that the body angular velocity of the rigid body containing fluid is governed by the equation:

$$\begin{aligned}
&\int_{\mathcal{V}'} \left(\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d})) + \boldsymbol{\Omega} \times (\dot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d})) + \dot{\boldsymbol{x}} \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \right) \rho \, \mathrm{d}\boldsymbol{a} \\
&\quad + \int_{\mathcal{V}'} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d})\right) + (\boldsymbol{x}+\boldsymbol{d}) \times \left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times \boldsymbol{\Omega}\right) \right) \rho \, \mathrm{d}\boldsymbol{a} \\
&\quad - \int_{\mathcal{V}'} g(\boldsymbol{x}+\boldsymbol{d}) \times \boldsymbol{\Sigma} \rho \, \mathrm{d}\boldsymbol{a} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{I}_f \boldsymbol{\Omega}\right) + \boldsymbol{I}_f \boldsymbol{\Omega} \times \boldsymbol{\Omega} - \frac{\mathrm{d}}{\mathrm{d}t}\left(m_v \bar{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}\right) \\
&\quad + m_v \bar{\boldsymbol{x}}_v \times \left(\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \times \boldsymbol{\Omega}\right) - \boldsymbol{I}_v \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_v \boldsymbol{\Omega} \times \boldsymbol{\Omega} - m_v g \bar{\boldsymbol{x}}_v \times \boldsymbol{\Sigma} = \boldsymbol{0}, \quad (3.14)
\end{aligned}$$

which is the Euler–Poincaré equation for $\boldsymbol{\Omega}(t)$. Equation (3.14) after differentiating with respect to time and simplifying using (3.5*b*) and the hat map (2.6) reduces to

$$\int_{\mathcal{V}'} \left(\ddot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{\Omega} \times (\dot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d})) + \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d}) \right.$$
$$\left. - g(\boldsymbol{x}+\boldsymbol{d}) \times \boldsymbol{\Sigma}\right) \rho \, \mathrm{d}\boldsymbol{a} - \dot{\boldsymbol{I}}_f \boldsymbol{\Omega} - \left(\boldsymbol{I}_f + \boldsymbol{I}_v\right) \dot{\boldsymbol{\Omega}}$$
$$+ (\boldsymbol{I}_f + \boldsymbol{I}_v)\boldsymbol{\Omega} \times \boldsymbol{\Omega} - m_v \overline{\boldsymbol{x}}_v \times \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - m_v g \overline{\boldsymbol{x}}_v \times \boldsymbol{\Sigma} = \boldsymbol{0}. \tag{3.15}$$

Now set the mass moment of inertia of the coupled system (body + fluid) as

$$\boldsymbol{I}_t = \boldsymbol{I}_f + \boldsymbol{I}_v, \tag{3.16}$$

and note that

$$m_f \overline{\boldsymbol{x}}_f = \int_{\mathcal{V}} (\boldsymbol{x}+\boldsymbol{d}) \rho \, \mathrm{d}\boldsymbol{x} = \int_{\mathcal{V}'} (\boldsymbol{x}+\boldsymbol{d}) \rho \, \mathrm{d}\boldsymbol{a}, \tag{3.17}$$

where $\overline{\boldsymbol{x}}_f(t)$ is the centre of mass of the fluid relative to the body frame, and

$$m_f = \int_{\mathcal{V}} \rho \, \mathrm{d}\boldsymbol{x} = \int_{\mathcal{V}'} \rho \, \mathrm{d}\boldsymbol{a}, \tag{3.18}$$

is the mass of the fluid which is time independent. Also by setting

$$m = m_f + m_v, \tag{3.19}$$

which is the total mass of the coupled system, we have

$$m \overline{\boldsymbol{x}}(t) = m_f \overline{\boldsymbol{x}}_f + m_v \overline{\boldsymbol{x}}_v, \tag{3.20}$$

where $\overline{\boldsymbol{x}}$ is the centre of mass of the coupled system which is time dependent. Now the $\boldsymbol{\Omega}$-equation (3.15) simplifies to

$$\int_{\mathcal{V}'} \left(\ddot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{\Omega} \times (\dot{\boldsymbol{x}} \times (\boldsymbol{x}+\boldsymbol{d}))\right) \rho \, \mathrm{d}\boldsymbol{a}$$
$$- m\overline{\boldsymbol{x}} \times \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{I}}_f \boldsymbol{\Omega} - \boldsymbol{I}_t \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_t \boldsymbol{\Omega} \times \boldsymbol{\Omega} - mg\overline{\boldsymbol{x}} \times \boldsymbol{\Sigma} = \boldsymbol{0}. \tag{3.21}$$

Now, transforming this equation from the Lagrangian particle-path setting to Eulerian coordinates, replacing the Lagrangian variables $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ by their respective Eulerian quantities $\boldsymbol{u}$ and $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ respectively, the Euler–Poincaré equation (3.21) takes the form

$$\int_{\mathcal{V}} \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{\Omega} \times (\boldsymbol{u} \times (\boldsymbol{x}+\boldsymbol{d}))\right) \rho \, \mathrm{d}\boldsymbol{x}$$
$$- m\overline{\boldsymbol{x}} \times \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{I}}_f \boldsymbol{\Omega} - \boldsymbol{I}_t \dot{\boldsymbol{\Omega}} + \boldsymbol{I}_t \boldsymbol{\Omega} \times \boldsymbol{\Omega} - mg\overline{\boldsymbol{x}} \times \boldsymbol{\Sigma} = \boldsymbol{0}. \tag{3.22}$$

Note that $\ddot{\boldsymbol{q}}(t)$ in this equation is relative to the spatial frame $\boldsymbol{X}$.

The Euler–Poincaré equation (3.22) for $\boldsymbol{\Omega}(t)$ can be written in a form where $\boldsymbol{q}(t)$ is relative to the body frame $\boldsymbol{x}_b$, i.e. $\boldsymbol{q}_b(t) = \boldsymbol{Q}^{-1}\boldsymbol{q}$. It can be proved that

$$\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_b + \dot{\boldsymbol{\Omega}} \times \boldsymbol{q}_b + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{q}_b) + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{q}}_b. \tag{3.23}$$

Hence substitution of (3.23) into (3.22) gives

$$\begin{aligned}&\int_{\mathcal{V}}\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\times(\boldsymbol{x}+\boldsymbol{d})+\boldsymbol{\Omega}\times(\boldsymbol{u}\times(\boldsymbol{x}+\boldsymbol{d}))\right)\rho\,\mathrm{d}\boldsymbol{x}\\&\quad-m\bar{\boldsymbol{x}}\times\left(\ddot{\boldsymbol{q}}_b+\dot{\boldsymbol{\Omega}}\times\boldsymbol{q}_b+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{q}_b\right)+2\boldsymbol{\Omega}\times\dot{\boldsymbol{q}}_b\right)\\&\quad-\dot{\boldsymbol{I}}_f\boldsymbol{\Omega}-\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}+\boldsymbol{I}_t\boldsymbol{\Omega}\times\boldsymbol{\Omega}-mg\bar{\boldsymbol{x}}\times\boldsymbol{\Sigma}=\mathbf{0},\end{aligned}\tag{3.24}$$

which is the Euler–Poincaré equation for $\boldsymbol{\Omega}(t)$ with both $\boldsymbol{\Omega}$ and $\boldsymbol{q}_b$ relative to the body frame $\boldsymbol{x}_b$.

Gerrits & Veldman (2003) and Veldman *et al.* (2007) studied dynamics of a liquid-filled spacecraft and presented the equation governing the rotational motion of a rigid body containing fluid based on the balance of angular momentum. Their $\boldsymbol{\Omega}$-equation is equation (1.2). To compare the derived Euler–Poincaré equation for $\boldsymbol{\Omega}(t)$ with the equation for angular momentum of the rigid body (1.2) given by Gerrits & Veldman (2003) and Veldman *et al.* (2007), we rewrite equation (3.22) in the following form

$$\begin{aligned}&\int_{\mathcal{V}}-(\boldsymbol{x}+\boldsymbol{d})\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-mg\bar{\boldsymbol{x}}\times\boldsymbol{\Sigma}=m\bar{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}\\&\quad+\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}+\boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega}+\dot{\boldsymbol{I}}_f\boldsymbol{\Omega}+\int_{\mathcal{V}}\boldsymbol{u}\times((\boldsymbol{x}+\boldsymbol{d})\times\boldsymbol{\Omega})\rho\,\mathrm{d}\boldsymbol{x},\end{aligned}\tag{3.25}$$

using the vector identity

$$\boldsymbol{\Omega}\times(\boldsymbol{u}\times(\boldsymbol{x}+\boldsymbol{d}))+(\boldsymbol{x}+\boldsymbol{d})\times(\boldsymbol{\Omega}\times\boldsymbol{u})+\boldsymbol{u}\times((\boldsymbol{x}+\boldsymbol{d})\times\boldsymbol{\Omega})=\mathbf{0}.\tag{3.26}$$

Differentiating the mass moment of inertia of the fluid (2.13) with respect to time gives

$$\begin{aligned}\dot{\boldsymbol{I}}_f&=\int_{\mathcal{V}'}(2\dot{\boldsymbol{x}}\cdot(\boldsymbol{x}+\boldsymbol{d})\boldsymbol{I}-(\dot{\boldsymbol{x}}\otimes(\boldsymbol{x}+\boldsymbol{d})+(\boldsymbol{x}+\boldsymbol{d})\otimes\dot{\boldsymbol{x}}))\rho\,\mathrm{d}\boldsymbol{a}\\&=\int_{\mathcal{V}}(2\boldsymbol{u}\cdot(\boldsymbol{x}+\boldsymbol{d})\boldsymbol{I}-(\boldsymbol{u}\otimes(\boldsymbol{x}+\boldsymbol{d})+(\boldsymbol{x}+\boldsymbol{d})\otimes\boldsymbol{u}))\rho\,\mathrm{d}\boldsymbol{x},\end{aligned}\tag{3.27}$$

and so

$$\dot{\boldsymbol{I}}_f\boldsymbol{\Omega}=\int_{\mathcal{V}}(2\boldsymbol{u}\cdot(\boldsymbol{x}+\boldsymbol{d})\boldsymbol{I}-(\boldsymbol{u}\otimes(\boldsymbol{x}+\boldsymbol{d})+(\boldsymbol{x}+\boldsymbol{d})\otimes\boldsymbol{u}))\boldsymbol{\Omega}\rho\,\mathrm{d}\boldsymbol{x}.\tag{3.28}$$

The summation of the last two terms on the right-hand side of (3.25) gives

$$\begin{aligned}&\int_{\mathcal{V}}((2\boldsymbol{u}\cdot(\boldsymbol{x}+\boldsymbol{d})\boldsymbol{I}-(\boldsymbol{u}\otimes(\boldsymbol{x}+\boldsymbol{d})+(\boldsymbol{x}+\boldsymbol{d})\otimes\boldsymbol{u}))\,\boldsymbol{\Omega}\\&\quad+\boldsymbol{u}\times((\boldsymbol{x}+\boldsymbol{d})\times\boldsymbol{\Omega}))\,\rho\,\mathrm{d}\boldsymbol{x}=\int_{\mathcal{V}}(\boldsymbol{x}+\boldsymbol{d})\times(\boldsymbol{\Omega}\times\boldsymbol{u})\,\rho\,\mathrm{d}\boldsymbol{x}.\end{aligned}\tag{3.29}$$

Substitution of (3.29) into (3.25) gives

$$\begin{aligned}&\int_{\mathcal{V}}-(\boldsymbol{x}+\boldsymbol{d})\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-mg\bar{\boldsymbol{x}}\times\boldsymbol{\Sigma}\\&\quad=m\bar{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}+\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}+\boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega},\end{aligned}\tag{3.30}$$

which recovers equation (1.2), neglecting the torque due to viscosity of the fluid in (1.2). However, it should be noted that in the works by Gerrits & Veldman (2003) and Veldman *et al.* (2007) it is stated that $\dot{\boldsymbol{q}}$ in (1.2) is the acceleration of the moving origin with respect to an inertial reference frame, but our calculations show precisely that $\dot{\boldsymbol{q}}$ is actually $\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}$ which is relative to the body frame (see (3.23)). Also it should be noted that the terms involving $\boldsymbol{F}$ in (1.2) are gathered in (3.30) as $-mg\bar{\boldsymbol{x}}\times\boldsymbol{\Sigma}$ with a precise definition for $\boldsymbol{\Sigma}$ in the body coordinate system. Using (3.23) equation (3.30) can be written in terms of $\boldsymbol{q}_b$ as

$$\int_{\mathcal{V}} -(\boldsymbol{x}+\boldsymbol{d})\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-mg\bar{\boldsymbol{x}}\times\boldsymbol{\Sigma}$$
$$=m\bar{\boldsymbol{x}}\times(\ddot{\boldsymbol{q}}_b+\dot{\boldsymbol{\Omega}}\times\boldsymbol{q}_b+\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{q}_b)+2\boldsymbol{\Omega}\times\dot{\boldsymbol{q}}_b)+\boldsymbol{I}_t\dot{\boldsymbol{\Omega}}+\boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega}. \quad (3.31)$$

3.2. *The Euler–Poincaré equation for $\boldsymbol{q}(t)$*

The governing equation for the translational motion of the rigid body $\boldsymbol{q}(t)$ is provided by Hamilton's variational principle (3.1) by taking the variations $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ of the Lagrangian action (2.24) with fixed endpoints $\delta\boldsymbol{q}(t_1)=\delta\boldsymbol{q}(t_2)=\boldsymbol{0}$, and assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are constants. Applying similar calculations to the calculus of variations presented in § 3.1, it can be proved that Hamilton's principle leads to

$$\int_{\mathcal{V}'}\left(-\ddot{\boldsymbol{x}}-\boldsymbol{\Omega}\times\dot{\boldsymbol{x}}-\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))-\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-g\boldsymbol{\Sigma}\right)\rho\,\mathrm{d}\boldsymbol{a}$$
$$-m_v\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Omega}\times m_v\bar{\boldsymbol{x}}_v)-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m_v\bar{\boldsymbol{x}}_v)-m_vg\boldsymbol{\Sigma}=\boldsymbol{0}, \quad (3.32)$$

which is the Euler–Poincaré equation for the translational motion $\boldsymbol{q}(t)$ of the rigid body in the spatial frame $\boldsymbol{X}$. This equation, after differentiating with respect to time and applying (3.17), (3.18) and (3.20), simplifies to

$$\int_{\mathcal{V}'}(-\ddot{\boldsymbol{x}}-2\boldsymbol{\Omega}\times\dot{\boldsymbol{x}})\,\rho\,\mathrm{d}\boldsymbol{a}-m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\dot{\boldsymbol{\Omega}}\times m\bar{\boldsymbol{x}}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m\bar{\boldsymbol{x}})-mg\boldsymbol{\Sigma}=\boldsymbol{0}. \quad (3.33)$$

Transforming this equation from the Lagrangian particle-path setting to Eulerian coordinates, replacing the Lagrangian variables $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ by their respective Eulerian quantities $\boldsymbol{u}$ and $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ respectively, the Euler–Poincaré equation (3.33) reduces to

$$\int_{\mathcal{V}}\left(-\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}-2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-\dot{\boldsymbol{\Omega}}\times m\bar{\boldsymbol{x}}$$
$$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m\bar{\boldsymbol{x}})-mg\boldsymbol{\Sigma}=\boldsymbol{0}. \quad (3.34)$$

The $\boldsymbol{q}$-equation (3.34) can be written in terms of the translational acceleration in the body frame $\ddot{\boldsymbol{q}}_b$ using (3.23) as

$$\int_{\mathcal{V}}\left(-\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}-2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-m\left(\ddot{\boldsymbol{q}}_b+\dot{\boldsymbol{\Omega}}\times\boldsymbol{q}_b+\boldsymbol{\Omega}\times\left(\boldsymbol{\Omega}\times\boldsymbol{q}_b\right)+2\boldsymbol{\Omega}\times\dot{\boldsymbol{q}}_b\right)$$
$$-\dot{\boldsymbol{\Omega}}\times m\bar{\boldsymbol{x}}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m\bar{\boldsymbol{x}})-mg\boldsymbol{\Sigma}=\boldsymbol{0}, \quad (3.35)$$

or in the form

$$\int_{\mathcal{V}}\left(-\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}-2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\rho\,\mathrm{d}\boldsymbol{x}-m\left(\ddot{\boldsymbol{q}}_b+2\boldsymbol{\Omega}\times\dot{\boldsymbol{q}}_b\right)-m\dot{\boldsymbol{\Omega}}\times\left(\bar{\boldsymbol{x}}+\boldsymbol{q}_b\right)$$
$$-m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\bar{\boldsymbol{x}}+\boldsymbol{q}_b))-mg\boldsymbol{\Sigma}=\boldsymbol{0}. \quad (3.36)$$

The final form of the Euler–Poincaré equation for the linear momentum of the rigid body is (3.34) in the spatial frame $\boldsymbol{X}$ or (3.36) in the body frame $\boldsymbol{x}_b$.

The Euler–Poincaré equation (3.34) for $\boldsymbol{q}(t)$ recovers equation (1.1) for linear momentum of the rigid body, neglecting the force due to the viscosity of the fluid in (1.1), given by Gerrits & Veldman (2003) and Veldman *et al.* (2007). However, it should be noted that in the works by Gerrits & Veldman (2003) and Veldman *et al.* (2007) it is stated that $\dot{\boldsymbol{q}}$ in (1.1) is the acceleration of the moving origin with respect to an inertial reference frame, but from our calculations it is obvious that $\dot{\boldsymbol{q}}$ is actually $\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}$ which is relative to the body frame. The terms involving $\boldsymbol{F}$ in (1.1) are gathered in (3.34) as $-mg\boldsymbol{\Sigma}$ with $\boldsymbol{\Sigma}$ defined in (3.8).

3.3. *Reconstruction of* $\boldsymbol{Q}(t) \in SO(3)$

The Euler–Poincaré equations (3.30) and (3.34) determine the body angular velocity $\boldsymbol{\Omega}(t)$ and translational motion $\boldsymbol{q}(t)$ of the rigid body, respectively. The tangent vectors $\dot{\boldsymbol{Q}}(t) \in \mathrm{TSO}(3)$ along the integral curve in the rotation group $\boldsymbol{Q}(t) \in \mathrm{SO}(3)$ may be retrieved via the reconstruction formula (Holm *et al.* 2009)

$$\dot{\boldsymbol{Q}} = \boldsymbol{Q}\widehat{\boldsymbol{\Omega}}. \tag{3.37}$$

The solution of (3.37) yields the integral curve $\boldsymbol{Q}(t) \in \mathrm{SO}(3)$ for the orientation of the rigid body.

Finally, differentiating the constraint equation $\boldsymbol{\Sigma}(t) = \boldsymbol{Q}^{-1}(t)\hat{z}$ gives

$$\dot{\boldsymbol{\Sigma}}(t) = \boldsymbol{\Sigma}(t) \times \boldsymbol{\Omega}(t) \quad \text{with } \boldsymbol{\Sigma}(0) = \boldsymbol{Q}^{-1}(0)\hat{z}. \tag{3.38}$$

So the evolutionary system for the rigid body motion (3.30) and (3.34) is completed by (3.37) and (3.38).

4. The Euler–Poincaré equation for the interior fluid motion

In Eulerian coordinates, the interior fluid occupies the region $\mathcal{V}$

$$0 \leqslant x \leqslant L_1, \quad 0 \leqslant y \leqslant L_2, \quad 0 \leqslant z \leqslant h(x, y, t), \tag{4.1}$$

where L_1 and L_2 are given positive constants, and $z = h(x, y, t)$ is the position of the free surface. The boundary conditions are

$$\left.\begin{array}{lll} u = 0 & \text{at } x = 0 & \text{and} \quad x = L_1, \\ v = 0 & \text{at } y = 0 & \text{and} \quad y = L_2, \\ w = 0 & \text{at } z = 0, & \end{array}\right\} \tag{4.2}$$

which are the no-flow boundary condition on the rigid walls, and at the free surface, the kinematic and dynamic boundary conditions are respectively

$$w = h_t + uh_x + vh_y \quad \text{and} \quad p = 0 \quad \text{at } z = h(x, y, t), \tag{4.3a,b}$$

where the surface tension is neglected in the boundary condition for the pressure p.

The Euler–Poincaré equation for the position $\boldsymbol{x}(t)$ of fluid particles in the body frame $\boldsymbol{x}$ can be provided by Hamilton's variational principle (3.1) by taking the variations $\delta\boldsymbol{x}$ and $\delta\dot{\boldsymbol{x}}$ of the Lagrangian action (2.24), in Lagrangian coordinates, with fixed endpoints $\delta\boldsymbol{x}(t_1) = \delta\boldsymbol{x}(t_2) = \boldsymbol{0}$, assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are constants. However, a constraint term should be added to the Lagrangian functional (2.24) to enforce incompressibility of the fluid $\nabla \cdot \boldsymbol{u} = 0$. The action functional (2.24) is modified to

$$\mathscr{L}_t = \mathscr{L}(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x}, \dot{\boldsymbol{x}}) + \mathscr{L}_c \quad \text{with} \quad \mathscr{L}_c = \int_{t_1}^{t_2} \int_{\mathcal{V}'} p(\boldsymbol{a}, t)(J-1)\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t, \tag{4.4}$$

where $p(\boldsymbol{a}, t)$ is the pressure field of the interior fluid and $\mathscr{L}_c$ enforces incompressibility of the fluid in the Lagrangian particle-path formulation. Hence

$$\delta\mathscr{L}_t = \delta\mathscr{L}(\boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x}, \dot{\boldsymbol{x}}) + \delta\mathscr{L}_c = \delta\mathscr{L} + \delta \int_{t_1}^{t_2} \int_{\mathcal{V}'} p(\boldsymbol{a}, t)(J-1)\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t = 0, \tag{4.5}$$

where the integral is over the volume $\mathcal{V}'$ of the reference space

$$0 \leqslant a \leqslant L_1, \quad 0 \leqslant b \leqslant L_2, \quad 0 \leqslant c \leqslant L_3, \tag{4.6}$$

where L_1, L_2 and L_3 are given positive constants. The free surface $z = h(x, y, t)$ is the map of the boundary surface $\{(a, b, c) : c = L_3\}$.

After some calculations, presented in appendix B, it is proved that taking the variations $\delta\boldsymbol{x}$ and $\delta\dot{\boldsymbol{x}}$ of the first component of the Lagrangian functional $\mathscr{L}_t$, in the Lagrangian particle-path setting, leads to

$$\begin{aligned} \delta\mathscr{L} = \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \delta\boldsymbol{x}, -\ddot{\boldsymbol{x}} - \dot{\boldsymbol{\Omega}} \times (\boldsymbol{x} + \boldsymbol{d}) - 2\boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} \\ - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{x} + \boldsymbol{d})) - g\boldsymbol{\Sigma} \rangle\, \rho\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t. \end{aligned} \tag{4.7}$$

Taking the variations δp and $\delta\boldsymbol{x}$ of the constraint component of $\mathscr{L}_t$ in (4.4*b*) gives

$$\begin{aligned} \delta\mathscr{L}_c = {} & \delta \int_{t_1}^{t_2} \int_{\mathcal{V}'} p(J-1)\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathcal{V}'} \delta p(J-1)\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t \\ & + \int_{t_1}^{t_2} \int_{\mathcal{V}'} p(\delta x_a(y_b z_c - y_c z_b) + \delta x_b(y_c z_a - y_a z_c) + \delta x_c(y_a z_b - y_b z_a))\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t \\ & + \int_{t_1}^{t_2} \int_{\mathcal{V}'} p(\delta y_a(x_c z_b - x_b z_c) + \delta y_b(x_a z_c - x_c z_a) + \delta y_c(x_b z_a - x_a z_b))\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t \\ & + \int_{t_1}^{t_2} \int_{\mathcal{V}'} p(\delta z_a(x_b y_c - x_c y_b) + \delta z_b(x_c y_a - x_a y_c) + \delta z_c(x_a y_b - x_b y_a))\, \mathrm{d}\boldsymbol{a}\, \mathrm{d}t. \end{aligned} \tag{4.8}$$

After integrating by parts and imposing the no-flow boundary conditions in the Lagrangian particle-path setting

$$\left. \begin{array}{lll} x_b = x_c = 0 & \text{at } a = 0 & \text{and} \quad a = L_1, \\ y_a = y_c = 0 & \text{at } b = 0 & \text{and} \quad b = L_2, \\ z_a = z_b = 0 & \text{at } c = 0, & \end{array} \right\} \tag{4.9}$$

and the free-surface pressure boundary condition $p = 0$ at $c = L_3$, $\delta\mathscr{L}_c$ in (4.8) reduces to

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$$\delta\mathscr{L}_c = \int_{t_1}^{t_2}\int_{\mathcal{V}'} \delta p(J-1)\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t$$

$$+\int_{t_1}^{t_2}\int_{\mathcal{V}'}\left\langle \delta\boldsymbol{x}, \overbrace{\begin{bmatrix}(y_cz_b-y_bz_c)p_a+(y_az_c-y_cz_a)p_b+(y_bz_a-y_az_b)p_c\\(x_bz_c-x_cz_b)p_a+(x_cz_a-x_az_c)p_b+(x_az_b-x_bz_a)p_c\\(x_cy_b-x_by_c)p_a+(x_ay_c-x_cy_a)p_b+(x_by_a-x_ay_b)p_c\end{bmatrix}}^{=-\nabla p \text{ in Eulerian coordinates } \leftarrow \rho}\right\rangle \mathrm{d}\boldsymbol{a}\,\mathrm{d}t. \tag{4.10}$$

Therefore, since $\delta\boldsymbol{x}$ and δp are arbitrary, from the variations (4.7) and (4.10) and Hamilton's principle (4.5) we conclude that

$$\ddot{\boldsymbol{x}}+\frac{1}{\rho}\begin{bmatrix}(y_bz_c-y_cz_b)p_a+(y_cz_a-y_az_c)p_b+(y_az_b-y_bz_a)p_c\\(x_cz_b-x_bz_c)p_a+(x_az_c-x_cz_a)p_b+(x_bz_a-x_az_b)p_c\\(x_by_c-x_cy_b)p_a+(x_cy_a-x_ay_c)p_b+(x_ay_b-x_by_a)p_c\end{bmatrix}$$
$$=-\dot{\boldsymbol{\Omega}}\times(\boldsymbol{x}+\boldsymbol{d})-2\boldsymbol{\Omega}\times\dot{\boldsymbol{x}}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))-\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-g\boldsymbol{\Sigma},\quad J-1=0. \tag{4.11}$$

Now, transforming the equations of motion (4.11) from Lagrangian coordinates to Eulerian coordinates we obtain

$$\left.\begin{aligned}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\frac{1}{\rho}\nabla p=-\dot{\boldsymbol{\Omega}}\times(\boldsymbol{x}+\boldsymbol{d})-2\boldsymbol{\Omega}\times\boldsymbol{u}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))-\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}-g\boldsymbol{\Sigma},\\ \nabla\cdot\boldsymbol{u}=0,\end{aligned}\right\} \tag{4.12}$$

which are the Euler–Poincaré equations for the interior fluid motion relative to the body frame $\boldsymbol{x}$. The Euler equations in rotating coordinates are presented by Alemi Ardakani & Bridges (2011). Here we have recovered these equations using a variational principle in the Lagrangian particle-path formulation.

The momentum equation in (4.12) can be written in terms of the translational acceleration in the body frame $\ddot{\boldsymbol{q}}_b$ as

$$\begin{aligned}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\frac{1}{\rho}\nabla p &= -\dot{\boldsymbol{\Omega}}\times(\boldsymbol{x}+\boldsymbol{d}+\boldsymbol{q}_b)-2\boldsymbol{\Omega}\times(\boldsymbol{u}+\dot{\boldsymbol{q}}_b)\\ &\quad -\boldsymbol{\Omega}\times\big(\boldsymbol{\Omega}\times\big(\boldsymbol{x}+\boldsymbol{d}+\boldsymbol{q}_b\big)\big)-\ddot{\boldsymbol{q}}_b-g\boldsymbol{\Sigma}.\end{aligned} \tag{4.13}$$

Having developed a variational principle for rigid body dynamics with interior fluid motion in three-dimensional rotating and translating coordinates, this variational principle can be extended to the problem of interactions between water waves and a floating rigid body with interior inviscid fluid motion.

5. A variational principle for interactions between water waves and a floating rigid body containing fluid

Van Daalen *et al.* (1993) extended Luke's variational principle for the classical water-wave problem to the hydrodynamic interaction with a freely floating empty rigid body, i.e. without interior fluid, in three dimensions. However, they did not present the exact differential equations for the rigid body motion, due to the approximation used for the angular velocity in the kinetic energy of the rigid body. Alemi Ardakani

(2017) derived a coupled variational principle for the two-dimensional interactions between ocean waves and a freely floating rigid body with interior fluid sloshing with uniform vorticity. The complete set of equations of motion for the exterior water waves, the exact hydrodynamic equations for the planar rigid body motion and the full set of equations of motion for the interior potential flow of the body are derived.

The classical water-wave problem in three dimensions is described by the equations

$$\left.\begin{aligned} \Delta\Phi := \Phi_{XX} + \Phi_{YY} + \Phi_{ZZ} = 0 \quad &\text{for} \quad -H(X,Y) < Z < \eta(X,Y,t), \\ \Phi_t + \tfrac{1}{2}\nabla\Phi\cdot\nabla\Phi + gZ = 0 \quad &\text{on} \quad Z = \eta(X,Y,t), \\ \Phi_Z = \eta_t + \Phi_X\eta_X + \Phi_Y\eta_Y \quad &\text{on} \quad Z = \eta(X,Y,t), \\ \Phi_Z + \Phi_X H_X + \Phi_Y H_Y = 0 \quad &\text{on} \quad Z = -H(X,Y), \end{aligned}\right\} \tag{5.1}$$

where (X, Y, Z) is the spatial coordinate system, $\Phi(X, Y, Z, t)$ is the velocity potential of an irrotational fluid lying between $Z = -H(X, Y)$ and $Z = \eta(X, Y, t)$ with the gravity acceleration g acting in the negative Z direction. In the horizontal directions X and Y, the fluid domain is cut off by a cylindrical vertical surface $\mathcal{S}$ of infinite radius which extends from the bottom to the free surface. Then an extension of Luke's variational principle, for 3-D water waves, as reported by Van Daalen *et al.* (1993) reads

$$\delta\mathscr{L}_w(\Phi,\eta) = \delta\int_{t_1}^{t_2}\int_{V(t)} -\rho\left(\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gZ\right)\mathrm{d}V\,\mathrm{d}t = 0, \tag{5.2}$$

where the Bernoulli pressure, playing the role of the Lagrangian density, is integrated over the transient fluid domain $V(t)$, with variations in $\Phi(X, Y, Z, t)$ and $\eta(X, Y, t)$ subject to the restrictions $\delta\Phi = 0$ at the endpoints of the time interval, t_1 and t_2. In (5.2) the gradient vector field is denoted by ∇, and ρ is the water density. The variational principle (5.2) recovers the complete set of equations of motion for the water-wave problem described by (5.1).

In §§ 3 and 4, a variational principle is developed for dynamic coupling between rigid body motion and its interior inviscid and incompressible fluid sloshing in three-dimensional rotating and translating coordinates. This variational principle can be extended to the problem of 3-D water waves in hydrodynamic interaction with a freely floating rigid body containing fluid by the addition of the extended version of Luke's variational principle (5.2) to Hamilton's variational principle (4.5) as

$$\begin{aligned} \delta\mathcal{L}(\Phi,\eta,\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{x},\dot{\boldsymbol{x}},p) &= \delta\int_{t_1}^{t_2}\int_{V(t)} -\rho\left(\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gZ\right)\mathrm{d}V\,\mathrm{d}t \\ &+ \delta\int_{t_1}^{t_2}\left(\int_{\mathcal{V}'}\left(\frac{1}{2}\|\dot{\boldsymbol{x}}\|^2 + \dot{\boldsymbol{x}}\cdot\left(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\right) + \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}\cdot(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d}))\right.\right. \\ &\left. + \frac{1}{2}\|\dot{\boldsymbol{q}}\|^2 - g(\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{q})\cdot\hat{\boldsymbol{z}}\right)\rho\,\mathrm{d}\boldsymbol{a} + \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_f\boldsymbol{\Omega} \\ &\left. + \frac{1}{2}m_v\|\dot{\boldsymbol{q}}\|^2 + (\boldsymbol{\Omega}\times m_v\bar{\boldsymbol{x}}_v)\cdot\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{I}_v\boldsymbol{\Omega} - m_v g(\boldsymbol{Q}\bar{\boldsymbol{x}}_v + \boldsymbol{q})\cdot\hat{\boldsymbol{z}}\right)\mathrm{d}t \\ &+ \delta\int_{t_1}^{t_2}\int_{\mathcal{V}'} p(\boldsymbol{a},t)(J-1)\,\mathrm{d}\boldsymbol{a}\,\mathrm{d}t = 0, \end{aligned} \tag{5.3}$$

where $V(t)$ consists of a fluid bounded by the impermeable bottom S_b defined by the equation $Z = -H(X, Y)$, the free surface S_η defined by the equation $Z = \eta(X, Y, t)$, the

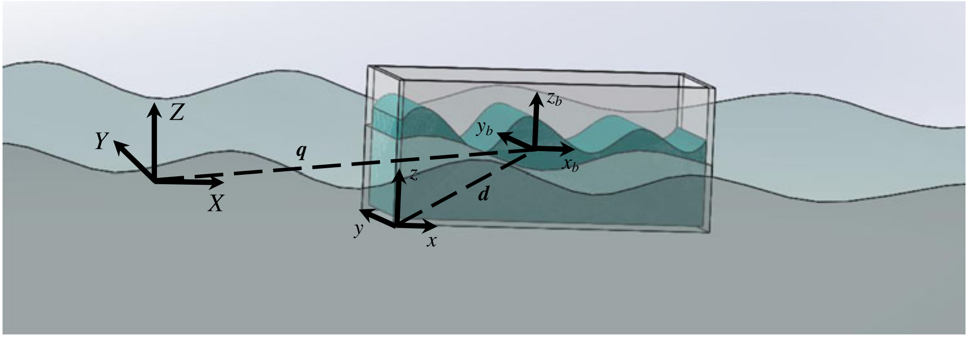

FIGURE 2. (Colour online) Schematic showing a floating structure containing fluid in hydrodynamic interaction with exterior ocean waves.

vertical surface $\mathcal{S}$ and the wetted surface S_w of the rigid body interacting with exterior water waves. The configuration of the fluid in a rotating–translating floating structure interacting with exterior water waves is schematically shown in figure 2.

In order to take the variations $\delta\Phi$ and $\delta\eta$ in (5.3), the variational Reynold's transport theorem should be used, since the domain of integration V is time dependent. The background mathematics on the variational analogue of Reynold's transport theorem can be found in Flanders (1973), Daniliuk (1976) and Gagarina, van der Vegt & Bokhove (2013). Then, according to the usual procedure in the calculus of variations, the variational principle (5.3) for the variations $\delta\Phi$, $\delta\eta$, $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\boldsymbol{x}$, $\delta\dot{\boldsymbol{x}}$ and δp becomes

$$
\begin{aligned}
\delta\mathcal{L}\left(\Phi, \eta, \boldsymbol{\Omega}, \boldsymbol{Q}, \boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{x}, \dot{\boldsymbol{x}}, p\right) &= \int_{t_1}^{t_2}\int_{S_\eta} -\left.\left(\Phi_t + \frac{1}{2}\nabla\Phi\boldsymbol{\cdot}\nabla\Phi + gZ\right)\right|^{Z=\eta} \rho\,\delta\eta\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t \\
&+ \int_{t_1}^{t_2}\int_{S_w} P(X, Y, Z, t)(\delta\boldsymbol{X}_w\boldsymbol{\cdot}\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t - \int_{t_1}^{t_2}\int_{V(t)} (\delta\Phi_t + \nabla\Phi\boldsymbol{\cdot}\nabla\delta\Phi)\rho\,\mathrm{d}V\,\mathrm{d}t \\
&+ \int_{t_1}^{t_2}\int_{\mathcal{V}} \left\langle \boldsymbol{\Gamma}, -(\boldsymbol{x}+\boldsymbol{d})\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\right\rangle \rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+ \int_{t_1}^{t_2} \left\langle \boldsymbol{\Gamma}, -m\overline{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \boldsymbol{I}_t\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega} - mg\overline{\boldsymbol{x}}\times\boldsymbol{\Sigma}\right\rangle \mathrm{d}t \\
&+ \int_{t_1}^{t_2}\int_{\mathcal{V}} \left\langle \boldsymbol{Q}^{-1}\delta\boldsymbol{q}, -\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - 2\boldsymbol{\Omega}\times\boldsymbol{u}\right\rangle \rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+ \int_{t_1}^{t_2} \left\langle \boldsymbol{Q}^{-1}\delta\boldsymbol{q}, -m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{\Omega}}\times m\overline{\boldsymbol{x}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m\overline{\boldsymbol{x}}) - mg\boldsymbol{\Sigma}\right\rangle \mathrm{d}t \\
&+ \int_{t_1}^{t_2}\int_{\mathcal{V}} \left\langle \delta\boldsymbol{x}, -\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \rho^{-1}\nabla p - \dot{\boldsymbol{\Omega}}\times(\boldsymbol{x}+\boldsymbol{d}) - 2\boldsymbol{\Omega}\times\boldsymbol{u} - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}\right. \\
&\left. - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})) - g\boldsymbol{\Sigma}\right\rangle \rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{\mathcal{V}} \delta p\,\nabla\boldsymbol{\cdot}\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0, \qquad (5.4)
\end{aligned}
$$

where the results of §§ 3 and 4 are applied in taking the variations of the second and third components of the variational principle (5.3). Note that in (5.4) after taking the variations $\delta\boldsymbol{\Omega}$, $\delta\boldsymbol{Q}$, $\delta\boldsymbol{q}$, $\delta\dot{\boldsymbol{q}}$, $\delta\boldsymbol{x}$, $\delta\dot{\boldsymbol{x}}$ and δp of the coupled (interior fluid + body) system in the Lagrangian particle-path setting, the results are transformed to Eulerian coordinates. In (5.4), $\boldsymbol{X}_w$ denotes the position of a point on the wetted body surface S_w relative to the spatial frame $\boldsymbol{X}$, $\boldsymbol{n}$ is the unit normal vector along $\partial V \supset S_w$ in the spatial frame, $\ell = (1+\eta_X^2+\eta_Y^2)^{1/2}$ giving $\mathrm{d}S = \ell\,\mathrm{d}X\mathrm{d}Y$, and P is the pressure field of the exterior water waves defined by

$$P(X, Y, Z, t) = -\rho\left(\Phi_t + \tfrac{1}{2}\nabla\Phi\boldsymbol{\cdot}\nabla\Phi + gZ\right) \quad \text{on } S_w. \tag{5.5}$$

These variations are subject to the restrictions that they vanish at the endpoints of the time interval. Moreover, the variations in η and Φ vanish on the vertical boundary at infinity, i.e. on $\mathcal{S}$.

The change in $\boldsymbol{X}_w$ due to the variations in $\boldsymbol{Q}$ and $\boldsymbol{q}$ is given by

$$\delta\boldsymbol{X}_w = \delta\boldsymbol{Q}\boldsymbol{x}_w + \delta\boldsymbol{q}, \tag{5.6}$$

where $\boldsymbol{x}_w$ is the position of a point on the wetted rigid body surface S_w relative to the body frame $\boldsymbol{x}_b$. Using the variations (5.6), the second integral on the right-hand side of (5.4) simplifies to

$$\begin{aligned}
\int_{t_1}^{t_2}\int_{S_w} P\langle\delta\boldsymbol{X}_w, \boldsymbol{n}\rangle\,\mathrm{d}S\,\mathrm{d}t &= \int_{t_1}^{t_2}\int_{S_w} P\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{X}_w, \boldsymbol{Q}^{-1}\boldsymbol{n}\rangle\,\mathrm{d}S\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{S_w} P\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{Q}\boldsymbol{x}_w + \boldsymbol{Q}^{-1}\delta\boldsymbol{q}, \boldsymbol{Q}^{-1}\boldsymbol{n}\rangle\,\mathrm{d}S\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{S_w} P\langle\widehat{\boldsymbol{\Gamma}}\boldsymbol{x}_w + \boldsymbol{Q}^{-1}\delta\boldsymbol{q}, \boldsymbol{Q}^{-1}\boldsymbol{n}\rangle\,\mathrm{d}S\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{S_w} P\langle\boldsymbol{\Gamma}\times\boldsymbol{x}_w + \boldsymbol{Q}^{-1}\delta\boldsymbol{q}, \boldsymbol{Q}^{-1}\boldsymbol{n}\rangle\,\mathrm{d}S\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{S_w} P\langle\boldsymbol{\Gamma}, -\boldsymbol{Q}^{-1}\boldsymbol{n}\times\boldsymbol{x}_w\rangle + P\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q}, \boldsymbol{Q}^{-1}\boldsymbol{n}\rangle\,\mathrm{d}S\,\mathrm{d}t\\
&= \int_{t_1}^{t_2}\int_{S_w} P\langle\boldsymbol{\Gamma}, \boldsymbol{x}_w\times\boldsymbol{n}_b\rangle + P\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q}, \boldsymbol{n}_b\rangle\,\mathrm{d}S\,\mathrm{d}t,
\end{aligned} \tag{5.7}$$

where

$$\boldsymbol{n}_b = \boldsymbol{Q}^{-1}\boldsymbol{n}, \tag{5.8}$$

is the unit normal vector along S_w in the body frame $\boldsymbol{x}_b$. Taking into account the motion of $V(t)$, we may write

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_1}^{t_2}\int_{V(t)}\delta\Phi\,\rho\,\mathrm{d}V\,\mathrm{d}t = &-\int_{t_1}^{t_2}\int_{S_\eta}\eta_t\delta\Phi\Big|^{Z=\eta}\rho\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t\\
&-\int_{t_1}^{t_2}\int_{S_w}(\dot{\boldsymbol{X}}_w\boldsymbol{\cdot}\boldsymbol{n})\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t - \int_{t_1}^{t_2}\int_{V(t)}\delta\Phi_t\rho\,\mathrm{d}V\,\mathrm{d}t.
\end{aligned} \tag{5.9}$$

This is the same as the variational Reynold's transport theorem but with variational derivatives replaced by time derivatives. Noting that the left-hand side vanishes due to the restriction $\delta\Phi = 0$ at times $t = t_1$ and $t = t_2$, this expression simplifies to

$$-\int_{t_1}^{t_2}\int_{V(t)}\delta\Phi_t\rho\,\mathrm{d}V\,\mathrm{d}t = \int_{t_1}^{t_2}\int_{S_\eta}\eta_t\delta\Phi\Big|^{Z=\eta}\rho\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{S_w}(\dot{\boldsymbol{X}}_w\boldsymbol{\cdot}\boldsymbol{n})\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t. \tag{5.10}$$

With Green's first identity, we may write

$$\int_{t_1}^{t_2}\int_{V(t)} \nabla\Phi\cdot\nabla\delta\Phi\,\rho\,\mathrm{d}V\mathrm{d}t = -\int_{t_1}^{t_2}\int_{V(t)} \Delta\Phi\delta\Phi\,\rho\,\mathrm{d}V\mathrm{d}t + \int_{t_1}^{t_2}\int_{\partial V} (\nabla\Phi\cdot\boldsymbol{n})\,\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t$$

$$= -\int_{t_1}^{t_2}\int_{V(t)} \Delta\Phi\delta\Phi\,\rho\,\mathrm{d}V\mathrm{d}t + \int_{t_1}^{t_2}\int_{S_\eta} (-\eta_X\Phi_X - \eta_Y\Phi_Y + \Phi_Z)\,\delta\Phi\Big|^{Z=\eta}\rho\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{S_b} (\Phi_X H_X + \Phi_Y H_Y + \Phi_Z)\delta\Phi\Big|_{Z=-H}\rho\,\mathrm{d}S\mathrm{d}t + \int_{t_1}^{t_2}\int_{S_w} \frac{\partial\Phi}{\partial\boldsymbol{n}}\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t. \tag{5.11}$$

Now, using the expressions (5.7), (5.10) and (5.11), the variational principle (5.4) simplifies to

$$\delta\mathcal{L}(\Phi,\eta,\boldsymbol{\Omega},\boldsymbol{Q},\boldsymbol{q},\boldsymbol{x},p) = \int_{t_1}^{t_2}\int_{S_\eta} -\left(\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gZ\right)\Big|^{Z=\eta}\rho\delta\eta\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{S_w} P(X,Y,Z,t)\,\langle\boldsymbol{\Gamma},\boldsymbol{x}_w\times\boldsymbol{n}_b\rangle + P(X,Y,Z,t)\left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q},\boldsymbol{n}_b\right\rangle\,\mathrm{d}S\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{S_\eta} (\eta_t + \eta_X\Phi_X + \eta_Y\Phi_Y - \Phi_Z)\delta\Phi\Big|^{Z=\eta}\rho\ell^{-1}\,\mathrm{d}S\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{S_w} \left(\dot{\boldsymbol{X}}_w\cdot\boldsymbol{n} - \frac{\partial\Phi}{\partial\boldsymbol{n}}\right)\delta\Phi\,\rho\,\mathrm{d}S\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{V(t)} \Delta\Phi\delta\Phi\,\rho\,\mathrm{d}V\,\mathrm{d}t$$

$$- \int_{t_1}^{t_2}\int_{S_b} (\Phi_X H_X + \Phi_Y H_Y + \Phi_Z)\delta\Phi\Big|_{Z=-H}\rho\,\mathrm{d}S\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{\mathcal{V}} \left\langle\boldsymbol{\Gamma}, -(\boldsymbol{x}+\boldsymbol{d})\times\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{u}\right)\right\rangle\rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \left\langle\boldsymbol{\Gamma}, -m\bar{\boldsymbol{x}}\times\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \boldsymbol{I}_t\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega}\times\boldsymbol{I}_t\boldsymbol{\Omega} - mg\bar{\boldsymbol{x}}\times\boldsymbol{\Sigma}\right\rangle\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{\mathcal{V}} \left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q}, -\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - 2\boldsymbol{\Omega}\times\boldsymbol{u}\right\rangle\rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2} \left\langle\boldsymbol{Q}^{-1}\delta\boldsymbol{q}, -m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{\Omega}}\times m\bar{\boldsymbol{x}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times m\bar{\boldsymbol{x}}) - mg\boldsymbol{\Sigma}\right\rangle\,\mathrm{d}t$$

$$+ \int_{t_1}^{t_2}\int_{\mathcal{V}} \left\langle\delta\boldsymbol{x}, -\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \rho^{-1}\nabla p - \dot{\boldsymbol{\Omega}}\times(\boldsymbol{x}+\boldsymbol{d}) - 2\boldsymbol{\Omega}\times\boldsymbol{u} - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}\right.$$

$$\left. - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{d})) - g\boldsymbol{\Sigma}\right\rangle\rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{\mathcal{V}} \delta p\nabla\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = 0. \tag{5.12}$$

From (5.12), we conclude that invariance of $\mathcal{L}$ with respect to a variation in the free-surface elevation η yields the dynamic free-surface boundary condition in (5.1), invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ yields the field equation in (5.1) in the domain $V(t)$, invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ at $Z=-H(X,Y)$ gives the bottom boundary condition in (5.1), invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ at $Z=\eta(X,Y,t)$

gives the kinematic free-surface boundary condition in (5.1) and invariance of $\mathcal{L}$ with respect to a variation in the velocity potential Φ on S_w gives the contact condition on the wetted surface of the rigid body,

$$\frac{\partial \Phi}{\partial \boldsymbol{n}} = \dot{\boldsymbol{X}}_w \cdot \boldsymbol{n} \quad \text{on } S_w. \tag{5.13}$$

Invariance of $\mathcal{L}$ with respect to $\boldsymbol{\Gamma}$ gives the hydrodynamic equation of motion for the rotational motion $\boldsymbol{\Omega}(t)$ of the floating rigid body interacting with exterior water waves and dynamically coupled to its interior fluid motion

$$\int_{\mathcal{V}} -(\boldsymbol{x}+\boldsymbol{d}) \times \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} \right) \rho \, \mathrm{d}\boldsymbol{x} - mg\bar{\boldsymbol{x}} \times \boldsymbol{\Sigma} - m\bar{\boldsymbol{x}} \times \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} \\ - \boldsymbol{I}_t\dot{\boldsymbol{\Omega}} - \boldsymbol{\Omega} \times \boldsymbol{I}_t\boldsymbol{\Omega} + \int_{S_w} P(X, Y, Z, t)(\boldsymbol{x}_w \times \boldsymbol{n}_b) \, \mathrm{d}S = \boldsymbol{0}. \tag{5.14}$$

Invariance of $\mathcal{L}$ with respect to $\boldsymbol{Q}^{-1}\delta\boldsymbol{q}$ gives the hydrodynamic equation of motion for the translational motion $\boldsymbol{q}(t)$ of the floating rigid body containing fluid and interacting with exterior water waves

$$\int_{\mathcal{V}} \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} \right) \rho \, \mathrm{d}\boldsymbol{x} + m\boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Omega}} \times m\bar{\boldsymbol{x}} \\ + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times m\bar{\boldsymbol{x}}) + mg\boldsymbol{\Sigma} - \int_{S_w} P(X, Y, Z, t)\boldsymbol{n}_b \, \mathrm{d}S = \boldsymbol{0}. \tag{5.15}$$

Finally, the invariance of $\mathcal{L}$ with respect to $\delta\boldsymbol{x}$ and δp gives the Euler equations and continuity equation for the interior inviscid and incompressible fluid of the rigid body in (4.12), respectively. The terms including the pressure field $P(X, Y, Z, t)$ in the hydrodynamic equations of motion (5.14) and (5.15) are the moments and forces respectively acting on the rigid body due to interactions with exterior water waves.

In summary, the equations of motion for the exterior water waves in $V(t)$ are (5.1) with the contact boundary condition (5.13). The equations of motion for the interior fluid of the rigid body are (4.12) which are dynamically coupled to the hydrodynamic equations of motion for the floating rigid body (5.14) and (5.15). The evolutionary system for the rigid body motion is completed by (3.37) and (3.38).

6. Concluding remarks

The paper is devoted to the derivation of a variational principle for dynamic coupling between a rigid body undergoing three-dimensional rotational and translational motions and its interior inviscid and incompressible fluid motion. The Euler–Poincaré reduction framework of rigid body dynamics is used to derive the exact differential equations for the linear momentum and angular momentum of the rigid body containing fluid, and the Euler equations for the motion of the interior fluid of the body relative to the rotating–translating coordinate system attached to the moving rigid body. The variational principle is extended to the problem of three-dimensional interactions between gravity-driven water waves and a freely floating rigid body dynamically coupled to its interior fluid motion. The exact nonlinear hydrodynamic equations of motion for the angular momentum and linear momentum of the floating body are derived.

The presented variational principle (4.5) for the coupled fluid–body dynamics and the variational principle (5.3) for wave–structure–slosh interactions can be a starting point for further analytical and numerical analysis of the dynamics of a liquid-filled spacecraft, fluid sloshing dynamics in moving tanks, a freely floating ship with fluid-filled tanks in hydrodynamic interaction with exterior water waves and the dynamics of floating structures such as ducted wave energy converters (Leybourne *et al.* 2014) and offshore wind turbines (Calderer *et al.* 2018) interacting with ocean waves. The proposed variational principle (4.5) and the corresponding Euler–Poincaré equations can be reduced for mathematical modelling of a suspended container with interior fluid sloshing undergoing pendular oscillations and constrained to rotate on the surface of sphere.

Variational principles are useful in constructing structure-preserving numerical schemes. Instead of discretising the Euler–Lagrange or Euler–Poincaré equations, a discretisation of the Lagrangian formulation allows for the derivation of integrators that are symplectic, preserve energy over long-time integration and respect a discrete version of Kelvin–Noether theorem which holds for solutions of the Euler–Poincaré equations. Gagarina *et al.* (2014) developed a variational finite element method for nonlinear free-surface gravity water waves using the potential-flow approximation for an inviscid and incompressible fluid with an irrotational velocity field. Their formulation stems from Luke's and Miles' variational principle (Luke 1967; Miles 1977) together with a space-plus-time approach for finite element discretisations that are continuous in space and discontinuous in time. Pavlov *et al.* (2011) developed variational Lie group integrators for incompressible perfect fluids based on the Hamilton–d'Alembert's principle, which are structure-preserving, exhibit long-time energy behaviour and give rise to a discrete form of Kelvin's circulation theorem. Desbrun *et al.* (2014) proposed structure-preserving Euler–Poincaré variational discretisations for rotating Euler equations and 2-D stratified flow in the Boussinesq approximation, based on a finite-dimensional approximation of the group of volume-preserving diffeomorphisms. A direction of great interest is to extend the variational symplectic methods of Gagarina *et al.* (2014, 2016) and Kalogirou & Bokhove (2016), for the exterior potential water waves, and the geometric structure-preserving discretisations of continuum theories by Gawlik *et al.* (2011), Pavlov *et al.* (2011) and Desbrun *et al.* (2014), for the interior rotating Euler equations, to develop hybrid numerical discretisations for the proposed variational principle (5.3) for 3-D interactions between exterior surface waves and a floating structure with interior fluid sloshing.

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Appendix A. Proof of equation (3.7)

For the variations $\delta\boldsymbol{\Omega}$ and $\delta\boldsymbol{Q}$ of the third term in the action functional (2.24) we have

$$
\begin{aligned}
\delta & \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \boldsymbol{Q}^{\mathrm{T}} \dot{\boldsymbol{q}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left(\left\langle \delta \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle + \left\langle \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}}, \delta \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \right) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left(\left\langle -\boldsymbol{Q}^{-1} \delta \boldsymbol{Q} \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \right. \\
&\quad \left. + \left\langle \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}}, \left(\dot{\boldsymbol{\Gamma}} + (\boldsymbol{\Omega} \times \boldsymbol{\Gamma}) \right) \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \right) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} (\langle -\widehat{\boldsymbol{\Gamma}} \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \rangle \\
&\quad + \langle \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}}, (\dot{\boldsymbol{\Gamma}} + (\boldsymbol{\Omega} \times \boldsymbol{\Gamma})) \times (\boldsymbol{x}+\boldsymbol{d}) \rangle) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left(\left\langle \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times \boldsymbol{\Gamma}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle + \left\langle \dot{\boldsymbol{\Gamma}}, -\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \right. \\
&\quad \left. + \left\langle \boldsymbol{\Omega} \times \boldsymbol{\Gamma}, -\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \right) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left(\left\langle \boldsymbol{\Gamma}, (\boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d})) \times \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right\rangle - \left\langle \boldsymbol{\Gamma}, -\frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d}) \right) \right\rangle \right. \\
&\quad \left. + \left\langle \boldsymbol{\Gamma}, -(\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d})) \times \boldsymbol{\Omega} \right\rangle \right) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left(\left\langle \boldsymbol{\Gamma}, \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d}) \right) + (\boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d})) \times \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right\rangle \right. \\
&\quad \left. + \left\langle \boldsymbol{\Gamma}, \left((\boldsymbol{x}+\boldsymbol{d}) \times \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right) \times \boldsymbol{\Omega} \right\rangle \right) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \boldsymbol{\Gamma}, \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times (\boldsymbol{x}+\boldsymbol{d})) + (\boldsymbol{x}+\boldsymbol{d}) \times (\boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \times \boldsymbol{\Omega}) \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t, \qquad \text{(A 1)}
\end{aligned}
$$

where the Jacobi identity is used in the last line. In integration by parts we used that $\boldsymbol{\Gamma}(t)$ vanishes at the endpoints, i.e. $\boldsymbol{\Gamma}(t_1) = \boldsymbol{\Gamma}(t_2) = \boldsymbol{0}$.

Appendix B. Derivation of equation (4.7)

For the variations $\delta \dot{\boldsymbol{x}}$ of the first term in (2.24) we have

$$
\delta \int_{t_1}^{t_2} \int_{\mathcal{V}'} \frac{1}{2} \langle \dot{\boldsymbol{x}}, \dot{\boldsymbol{x}} \rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \delta \boldsymbol{x}, -\ddot{\boldsymbol{x}} \rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t. \qquad \text{(B 1)}
$$

For the variations $\delta \boldsymbol{x}$ and $\delta \dot{\boldsymbol{x}}$ of the second term in (2.24), assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$ and $\dot{\boldsymbol{q}}$ are constants, we have

$$
\begin{aligned}
\delta & \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \dot{\boldsymbol{x}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{\mathrm{T}} \dot{\boldsymbol{q}} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left(\left\langle \delta \dot{\boldsymbol{x}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right\rangle + \langle \dot{\boldsymbol{x}}, \boldsymbol{\Omega} \times \delta \boldsymbol{x} \rangle \right) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \delta \boldsymbol{x}, -\frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{-1} \dot{\boldsymbol{q}} \right) + \dot{\boldsymbol{x}} \times \boldsymbol{\Omega} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t
\end{aligned}
$$

$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \delta \boldsymbol{x}, -\left(\dot{\boldsymbol{\Omega}} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - \boldsymbol{Q}^{-1}\dot{\boldsymbol{Q}}\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} + \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}\right) + \dot{\boldsymbol{x}} \times \boldsymbol{\Omega} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t$$

$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \delta \boldsymbol{x}, -(\dot{\boldsymbol{\Omega}} \times (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{\Omega} \times \dot{\boldsymbol{x}} - \widehat{\boldsymbol{\Omega}}\boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} + \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}}) + \dot{\boldsymbol{x}} \times \boldsymbol{\Omega} \rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t$$

$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \delta \boldsymbol{x}, -\dot{\boldsymbol{\Omega}} \times (\boldsymbol{x}+\boldsymbol{d}) - 2\boldsymbol{\Omega} \times \dot{\boldsymbol{x}} + \boldsymbol{\Omega} \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} - \boldsymbol{Q}^{-1}\ddot{\boldsymbol{q}} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t, \tag{B 2}$$

where, when integrating by parts, we used the condition that the variations vanish at the endpoints in time. Similarly, for the variations $\delta \boldsymbol{x}$ of the third term and also the potential energy of the fluid in (2.24), assuming that $\boldsymbol{\Omega}$, $\boldsymbol{Q}$, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ are constants, we have respectively

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{q}}, \boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d}) \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}}, \boldsymbol{\Omega} \times \delta\boldsymbol{x} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t$$
$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \left\langle \delta\boldsymbol{x}, -\boldsymbol{\Omega} \times \boldsymbol{Q}^{-1}\dot{\boldsymbol{q}} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t, \tag{B 3}$$

and

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{V}'} -g \left\langle \boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{q}, \hat{z} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t$$
$$= \delta \int_{t_1}^{t_2} \int_{\mathcal{V}'} -g \left\langle (\boldsymbol{x}+\boldsymbol{d}) + \boldsymbol{Q}^{-1}\boldsymbol{q}, \boldsymbol{\Sigma} \right\rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t = \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \delta\boldsymbol{x}, -g\boldsymbol{\Sigma} \rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t. \tag{B 4}$$

Taking the variations $\delta \boldsymbol{x}$ of the mass moment of inertia of the fluid in (2.24), assuming that $\boldsymbol{\Omega}$ is constant, gives

$$\delta \int_{t_1}^{t_2} \frac{1}{2} \langle \boldsymbol{\Omega}, \boldsymbol{I}_f \boldsymbol{\Omega} \rangle \, \mathrm{d}t = \int_{t_1}^{t_2} \frac{1}{2} \langle \boldsymbol{\Omega}, \delta \boldsymbol{I}_f \boldsymbol{\Omega} \rangle \, \mathrm{d}t$$
$$= \int_{t_1}^{t_2} \left(\left\langle \boldsymbol{\Omega}, \left(\int_{\mathcal{V}'} \langle \delta\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{d} \rangle \boldsymbol{I} \rho \, \mathrm{d}\boldsymbol{a} \right) \boldsymbol{\Omega} \right\rangle \right.$$
$$\left. - \left\langle \boldsymbol{\Omega}, \left(\int_{\mathcal{V}'} \frac{1}{2} (\delta\boldsymbol{x} \otimes (\boldsymbol{x}+\boldsymbol{d}) + (\boldsymbol{x}+\boldsymbol{d}) \otimes \delta\boldsymbol{x}) \rho \, \mathrm{d}\boldsymbol{a} \right) \boldsymbol{\Omega} \right\rangle \right) \mathrm{d}t$$
$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} (\langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle \langle \delta\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{d} \rangle - \langle \boldsymbol{\Omega}, \delta\boldsymbol{x} \rangle \langle \boldsymbol{\Omega}, \boldsymbol{x}+\boldsymbol{d} \rangle) \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t$$
$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \delta\boldsymbol{x}, \langle \boldsymbol{\Omega}, \boldsymbol{\Omega} \rangle (\boldsymbol{x}+\boldsymbol{d}) - \langle \boldsymbol{\Omega}, \boldsymbol{x}+\boldsymbol{d} \rangle \boldsymbol{\Omega} \rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t$$
$$= \int_{t_1}^{t_2} \int_{\mathcal{V}'} \langle \delta\boldsymbol{x}, -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\boldsymbol{x}+\boldsymbol{d})) \rangle \rho \, \mathrm{d}\boldsymbol{a} \, \mathrm{d}t, \tag{B 5}$$

where the vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{c} \rangle \boldsymbol{b} - \langle \boldsymbol{a}, \boldsymbol{b} \rangle \boldsymbol{c}$ is applied in the last line. Now from (B 1), (B 2), (B 3), (B 4) and (B 5), it can be concluded that $\delta \mathscr{L}$, for the variations $\delta \boldsymbol{x}$ and $\delta \dot{\boldsymbol{x}}$, takes the form (4.7).

REFERENCES

ALEMI ARDAKANI, H. 2016 A symplectic integrator for dynamic coupling between nonlinear vessel motion with variable cross-section and bottom topography and interior shallow-water sloshing. *J. Fluids Struct.* **65**, 30–43.

ALEMI ARDAKANI, H. 2017 A coupled variational principle for 2D interactions between water waves and a rigid body containing fluid. *J. Fluid Mech.* **827**, R2 1–12.

ALEMI ARDAKANI, H. & BRIDGES, T. J. 2011 Shallow-water sloshing in vessels undergoing prescribed rigid-body motion in three dimensions. *J. Fluid Mech.* **667**, 474–519.

BATEMAN, H. 1932 *Partial Differential Equations of Mathematical Physics*. Cambridge University Press.

BOKHOVE, O. & OLIVER, M. 2006 Parcel Eulerian–Lagrangian fluid dynamics of rotating geophysical flows. *Proc. R. Soc.* A **462**, 2575–2592.

BRETHERTON, F. P. 1970 A note on Hamilton's principle for perfect fluids. *J. Fluid Mech.* **44**, 19–31.

BROER, L. J. F. 1974 On the Hamiltonian theory for surface waves. *Appl. Sci. Res.* **29**, 430–446.

CALDERER, A., GUO, X., SHEN, L. & SOTIROPOULOS, F. 2018 Fluid–structure interaction simulation of floating structures interacting with complex, large-scale ocean waves and atmospheric turbulence with application to floating offshore wind turbines. *J. Comput. Phys.* **355**, 144–175.

CHERNOUSKO, F. L. 1965 Motion of a rigid body with cavities filled with viscous fluid at small Reynolds numbers. *USSR Comput. Maths Math. Phys.* **5**, 99–127.

CHERNOUSKO, F. L. 1972 *Motion of a Rigid Body with Cavities Containing a Viscous Fluid.* NASA Technical Translations.

COTTER, C. & BOKHOVE, O. 2010 Variational water-wave model with accurate dispersion and vertical vorticity. *J. Engng Maths* **67**, 33–54.

VAN DAALEN, E. F. G., VAN GROESEN, E. & ZANDBERGEN, P. J. 1993 A Hamiltonian formulation for nonlinear wave-body interactions. In *Proceedings of the Eight International Workshop on Water Waves and Floating Bodies, Canada*, pp. 159–163. IWWWFB.

DANILIUK, I. I. 1976 On integral functionals with a variable domain of integration. In *Proceedings of the Steklov Institute of Mathematics*, vol. 118, pp. 1–44. American Mathematical Society.

DESBRUN, M., GAWLIK, E. S., GAY-BALMAZ, F. & ZEITLIN, V. 2014 Variational discretization for rotating stratified fluids. *J. Discrete Continuous Dyn. Syst.* **34**, 477–509.

DISSER, K., GALDI, G. P., MAZZONE, G. & ZUNINO, P. 2016 Intertial motions of a rigid body with a cavity filled with a viscous liquid. *Arch. Rat. Mech. Anal.* **221**, 487–526.

FALTINSEN, O. M & TIMOKHA, A. N. 2009 *Sloshing*. Cambridge University Press.

FLANDERS, H. 1973 Differentiation under the integral sign. *Am. Math. Mon.* **80**, 615–627.

GAGARINA, E., AMBATI, V. R., NURIJANYAN, S., VAN DER VEGT, J. J. W. & BOKHOVE, O. 2016 On variational and symplectic time integrators for Hamiltonian systems. *J. Comput. Phys.* **306**, 370–389.

GAGARINA, E., AMBATI, V. R., VAN DER VEGT, J. J. W. & BOKHOVE, O. 2014 Variational space–time (dis)continuous Galerkin method for nonlinear free surface water waves. *J. Comput. Phys.* **275**, 459–483.

GAGARINA, E., VAN DER VEGT, J. & BOKHOVE, O. 2013 Horizontal circulation and jumps in Hamiltonian wave models. *Nonlinear Process. Geophys.* **20**, 483–500.

GAWLIK, E. S., MULLEN, P., PAVLOV, D., MARSDEN, J. E. & DESBRUN, M. 2011 Geometric, variational discretization of continuum theories. *Physica* D **240**, 1724–1760.

GAY-BALMAZ, F., MARSDEN, J. E. & RATIU, T. S. 2012 Reduced variational formulations in free boundary continuum mechanics. *J. Nonlinear Sci.* **22**, 463–497.

GERRITS, J. & VELDMAN, A. E. P. 2003 Dynamics of liquid-filled spacecraft. *J. Engng Maths* **45**, 21–38.

GREENHILL, A. G. 1880 On the general motion of a liquid ellipsoid. *Proc. Camb. Phil. Soc.* **4**, 4–14.

VAN GROESEN, E. & ANDONOWATI 2017 Hamiltonian Boussinesq formulation of wave–ship interactions. *Appl. Math. Model.* **42**, 133–144.

HOLM, D. D., MARSDEN, J. E. & RATIU, T. S. 1998*a* The Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. Maths* **137**, 1–81.

HOLM, D. D., MARSDEN, J. E. & RATIU, T. S. 1998*b* The Euler–Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.* **80**, 4173–4176.

HOLM, D. D., MARSDEN, J. E. & RATIU, T. S. 1999 The Euler–Poincaré equations in geophysical fluid dynamics. arXiv:chao-dyn/9903035.

HOLM, D. D., SCHMAH, T. & STOICA, C. 2009 *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*. Oxford University Press.

HOUGH, S. S. 1895 The oscillations of a rotating ellipsoidal shell containing fluid. *Phil. Trans. R. Soc. Lond.* A **186**, 469–506.

IBRAHIM, R. A. 2005 *Liquid Sloshing Dynamics*. Cambridge University Press.

KALOGIROU, A. & BOKHOVE, O. 2016 Mathematical and numerical modelling of wave impact on wave-energy buoys. In *Proceedings of the International Conference on Offshore Mechanics and Arctic Engineering*, p. 8. The American Society of Mechanical Engineers.

KOSTYUCHENKO, A. G., SHKALIKOV, A. A. & YURKIN, M. Y. 1998 On the stability of a top with a cavity filled with a viscous fluid. *Funct. Anal. Appl.* **32**, 100–113.

LEWIS, D., MARSDEN, J. E., MONTGOMERY, R. & RATIU, T. S. 1986 The Hamiltonian structure for dynamic free boundary problems. *Physica* D **18**, 391–404.

LEYBOURNE, M., BATTEN, W. M. J., BAHAJ, A. S., MINNS, N. & O'NIANS, J. 2014 Preliminary design of the OWEL wave energy converter pre-commercial demonstrator. *Renewable Energy* **61**, 51–56.

LUKE, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

LUKOVSKY, I. A. 1976 Variational method in the nonlinear problems of the dynamics of a limited liquid volume with free surface. In *Oscillations of Elastic Constructions with Liquid*, pp. 260–264. Volna (in Russian).

LUKOVSKY, I. A. 2015 *Nonlinear Dynamics: Mathematical Models for Rigid Bodies with a Liquid*. De Gruyter.

MARSDEN, J. E. & RATIU, T. S. 1999 *Introduction to Mechanics and Symmetry*. Springer.

MARSDEN, J. E. & WEST, M. 2001 Discrete mechanics and variational integrators. *Acta Numerica* **10**, 357–514.

MAZER, A. & RATIU, T. S. 1989 Hamiltonian formulation of adiabatic free-boundary Euler flows. *J. Geom. Phys.* **6**, 271–291.

MILES, J. W. 1976 Nonlinear surface waves in closed basins. *J. Fluid Mech.* **75**, 419–448.

MILES, J. W. 1977 On Hamilton's principle for surface waves. *J. Fluid Mech.* **83**, 153–158.

MILOH, T. 1984 Hamilton's principle, Lagrange's method, and ship motion theory. *J. Ship Res.* **28**, 229–237.

MOISEYEV, N. N. & RUMYANTSEV, V. V. 1968 *Dynamic Stability of Bodies Containing Fluid*. Springer.

MORRISON, P. J. 1998 Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.* **70**, 467–521.

MURRAY, R. M., LIN, Z. X. & SASTRY, S. S. 1994 *A Mathematical Introduction to Robotic Manipulation*. CRC Press.

OLIVER, M. 2014 A variational derivation of the geostrophic momentum approximation. *J. Fluid Mech.* **751**, R2 1–10.

OLIVER, M. 2006 Variational asymptotics for rotating shallow water near geostrophy: a transformational approach. *J. Fluid Mech.* **551**, 197–234.

O'REILLY, O. M. 2008 *Intermediate Dynamics for Engineers: a Unified Treatment of Newton–Euler and Lagrangian Mechanics*. Cambridge University Press.

PAVLOV, D., MULLEN, P., TONG, Y, KANSO, E., MARSDEN, J. E. & DESBRUN, M. 2011 Sructure-preserving discretization of incompressible fluids. *Physica* D **240**, 443–458.

RAMODANOV, S. M. & SIDORENKO, V. V. 2017 Dynamics of a rigid body with an ellipsoidal cavity filled with viscous fluid. *Intl J. Non-Linear Mech.* **95**, 42–46.

RUMIANTSEV, V. V. 1966 On the theory of motion of rigid bodies with fluid-filled cavities. *J. Appl. Math. Mech.* **30**, 57–77.

RUMYANTSEV, V. V. 1963 Lyapunov's method in the study of the stability of rigid bodies with fluid-filled cavities. *Izv. Akad. Nauk SSSR (Series Mekh. Mashinostr.)* **6**, 119–140.

SALMON, R. 1983 Practical use of Hamilton's principle. *J. Fluid Mech.* **132**, 431–444.

SALMON, R. 1988 Hamiltonian fluid mechanics. *Annu. Rev. Fluid Mech.* **20**, 225–256.

SHEPHERD, T. G. 1990 Symmetries, conservation laws, and Hamiltonian structure in geophysical fluid dynamics. *Adv. Geophys.* **32**, 287–338.

STOKES, G. G. 1880 *Mathematical and Physical Papers*, vol. 1. Cambridge University Press.

STRYGIN, V. V. & SOBOLEV, V. A. 1988 *Separation of Motions By the Integral Manifolds Method*. Nauka (in Russian).

TIMOKHA, A. N. 2016 The Bateman-Luke variational formalism in a sloshing with rotational flows. *Dopov. Nac. Akad. Nauk Ukr.* **4**, 30–34.

VELDMAN, A. E. P., GERRITS, J., LUPPES, R., HELDER, J. A. & VREEBURG, J. P. B. 2007 The numerical simulation of liquid sloshing on board spacecraft. *J. Comput. Phys.* **224**, 82–99.

ZAKHAROV, V. E. 1968 Stability of periodic waves of finite-amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9**, 190–194.

ZHUKOVSKII, N. Y. 1885 On the motion of a rigid body with cavities filled with a homogeneous liquid drop. *Zh. Fiz.-Khim. Obs. Phys.* **17**, 81–113.

ZHUKOVSKII, N. E. 1948 Motion of a rigid body having a cavity filled with fluid. *Collected Works*, vol. 1, pp. 31–152. Gostekhizdat.