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A CLASS OF RIEMANNIAN MANIFOLDS SATISFYING R(X,Y)·R = 0

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1. Introduction

Let (M, g) be a Riemannian manifold and let R be its Riemannian curvature tensor. If (M, g) is a locally symmetric space, we have

$$(*) \qquad R(X,Y) \cdot R = 0 \qquad \text{for all tangent vectors } X, Y$$

where the endomorphism $R(X,Y)$ (i.e., the curvature transformation) operates on R as a derivation of the tensor algebra at each point of M. There is a question: Under what additional condition does this algebraic condition (*) on R imply that (M, g) is locally symmetric (i.e., $\nabla R = 0$)? A conjecture by K. Nomizu [5] is as follows: (*) implies $\nabla R = 0$ in the case where (M, g) is complete and irreducible, and $\dim M \geq 3$. He gave an affirmative answer in the case where (M, g) is a certain complete hypersurface in a Euclidean space ([5]).

With respect to this problem, K. Sekigawa and H. Takagi [8] proved that if (M, g) is a complete conformally flat Riemannian manifold with dim $M \geq 3$ and satisfies (*), then (M, g) is locally symmetric.

On the other hand, R.L. Bishop and B.O'Neill [1] constructed a wide class of Riemannian manifolds of negative curvature by warped product using convex functions. For two Riemannian manifolds B and F, a warped product is denoted by $B \times_f F$, where f is a positive C^∞-function on B. The purpose of this paper is to prove

THEOREM A. *Let (F, g) be a Riemannian manifold of constant curvature $K \leq 0$. Let E^n be an n-dimensional Euclidean space and let f be a positive C^∞-function on E^n. On a warped product $E^n \times_f F$, assume that*

(i) *the condition (*) is satisfied, and*

(ii) *the scalar curvature is constant.*

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Then $E^n \times {}_f F$ *is locally symmetric. The converse is clear.*

In theorem A, if $n \geq 2$, we see that $E^n \times {}_f F$ is not of constant curvature. If $n = 1$, we have

THEOREM B. *Let* (F, g) *be a Riemannian manifold of constant curvature* $K \leq 0$. *Let* E^1 *be a Euclidean* 1-*space and let* f *be a non-constant positive* C^∞-*function on* E^1. *Then* $E^1 \times {}_f F$ *satisfies the condition* (*) *if and only if* $E^1 \times {}_f F$ *is of constant curvature.*

Concerning theorem B, it is remarked that, as is stated in [1], p. 28, a hyperbolic m-space is expressed as $H^m = E^1 \times {}_f E^{m-1}$ for $f = e^t$ or $= E^1 \times {}_f H^{m-1}$ for $f = \cosh\ t$.

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2. The Riemannian curvature tensor of $E^n \times {}_f F$

Let (F, g) be a Riemannian manifold and let E^n be a Euclidean n-space. We consider the product manifold $E^n \times F$. For vector fields A, B, C, etc. on E^n, we denote vector fields $(A, 0)$, $(B, 0)$, $(C, 0)$, etc. on $E^n \times F$ by also A, B, C, etc. Likewise, for vector fields X, Y, etc. on F, we denote vector fields $(0, X)$, $(0, Y)$, etc. on $E^n \times F$ by X, Y, etc.

We denote the inner product of A and B on E^n by $\langle A, B \rangle$. Let f be a positive C^∞-function on E^n. Then the (Riemannian) inner product $\langle\ ,\ \rangle$ for $A + X$ and $B + Y$ on the warped product $E^n \times {}_f F$ at (a, x) is given by (cf. [1])

$$\langle A + X,\ B + Y \rangle_{(a,x)} = \langle A, B \rangle_{(a)} + f^2(a) g_x(X, Y). \tag{2.1}$$

We extend the function f on E^n to that on $E^n \times {}_f F$ by $f(a, x) = f(a)$. The Riemannian connections defined by $\langle\ ,\ \rangle$ on E^n and $E^n \times {}_f F$ are denoted by ∇^0 and ∇, respectively. The Riemannian connection defined by g on F is denoted by D. Then we have the identities (cf. Lemma 7.3, [1])

$$\nabla_A B = \nabla^0_A B, \tag{2.2}$$

$$\nabla_A X = \nabla_X A = (Af/f)X, \tag{2.3}$$

$$\nabla_X Y = D_X Y - (\langle X, Y \rangle / f) \text{ grad } f. \tag{2.4}$$

By (2.2) we identify ∇^0 with ∇ in the sequel. In (2.4) $\text{grad} f$ on E^n is identified with grad f on $E^n \times {}_f F$ and we have

$$\langle \text{grad } f, A \rangle = df(A) = Af.$$

The Riemannian curvature tensors by ∇ and D are denoted by R and S respectively. We use both notations $R(X,Y)$ and R_{XY}, etc.:

$$R(X,Y) = R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y], \quad \text{etc.}$$

Then, noticing that E^n is flat, we have (cf. Lemma 7.4, [1])

(2.5) $$R_{AB}C = 0,$$

(2.6) $$R_{AX}B = -(1/f)\langle\nabla_A \operatorname{grad} f, \ B\rangle X,$$

(2.7) $$R_{AB}X = R_{XY}A = 0,$$

(2.8) $$R_{AX}Y = R_{AY}X = (1/f)\langle X,Y\rangle \nabla_A \operatorname{grad} f,$$

(2.9) $$R_{XY}Z = S_{XY}Z - (\langle \operatorname{grad} f, \ \operatorname{grad} f\rangle/f^2)(\langle X,Z\rangle Y - \langle Y,Z\rangle X).$$

3. The condition (*)

From now on (§3 ∼ §8) we assume that (F,g) is of constatn curvature $K \leq 0$. Then we have

$$S_{XY}Z = K(g(X,Z)Y - g(Y,Z)X)$$
$$= (K/f^2)(\langle X,Z\rangle Y - \langle Y,Z\rangle X).$$

In this case, (2.9) is written as

(3.1) $$R_{XY}Z = P(\langle X,Z\rangle Y - \langle Y,Z\rangle X),$$

where we have put

(3.2) $$P = (K - \langle \operatorname{grad} f, \ \operatorname{grad} f\rangle)/f^2 \leq 0.$$

Now by definition we have

$$(R(X,Y)\cdot R)(Z,V)W = R_{XY}R_{ZV}W - R(R_{XY}Z,V)W - R(Z,R_{XY}V)W - R_{ZV}R_{XY}W$$

which vanishes by (3.1). Likewise, by (2.5) ∼ (2.8), (3.1), we have

$$(R(X,Y)\cdot R)(Z,A)W = 0,$$
$$(R(X,Y)\cdot R)(Z,B)A = 0,$$
$$(R(X,Y)\cdot R)(C,B)A = 0,$$

from which we have

$$(R(X,Y)\cdot R)(A,Z)W = -(R(X,Y)\cdot R)(Z,A)W = 0,$$

(3.3) $$(R(X,Y)\cdot R)(Z,W)A = -(R(X,Y)\cdot R)(A,Z)W - (R(X,Y)\cdot R)(W,A)Z = 0,$$

(3.4) $$(R(X,Y)\cdot R)(C,B)W = -(R(X,Y)\cdot R)(W,C)B - (R(X,Y)\cdot R)(B,W)C = 0.$$

Next, by similar calculations we have

$$(3.5)\qquad (R(X,A)\cdot R)(Z,V)W = (fP\nabla_A \operatorname{grad} f + \nabla_Q \operatorname{grad} f)(\langle V,W\rangle\langle X,Z\rangle - \langle Z,W\rangle\langle X,V\rangle)/f^2,$$

where we have put $Q = \nabla_A \operatorname{grad} f$.

$$(3.6)\qquad (R(X,A)\cdot R)(Z,B)W = (\langle fP\nabla_A \operatorname{grad} f, B\rangle + \langle\nabla_A \operatorname{grad} f,\ \nabla_B \operatorname{grad} f\rangle)(\langle X,W\rangle Z - \langle Z,W\rangle X)/f^2,$$

$$(3.7)\qquad (R(X,A)\cdot R)(Z,B)C = \langle X,Z\rangle(\langle\nabla_B \operatorname{grad} f, C\rangle\nabla_A \operatorname{grad} f - \langle\nabla_A \operatorname{grad} f, C\rangle\nabla_B \operatorname{grad} f)/f^2,$$

$$(3.8)\qquad (R(X,A)\cdot R)(C,B)G = (\langle\nabla_A \operatorname{grad} f, B\rangle\langle\nabla_C \operatorname{grad} f, G\rangle - \langle\nabla_A \operatorname{grad} f, C\rangle\langle\nabla_B \operatorname{grad} f, G\rangle)X/f^2.$$

Finally we have $R(A,B)\cdot R = 0$, since $R_{AB} = 0$.

LEMMA 3.1. *On $E^n \times {}_fF$, the condition* (*) *is equivalent to*

$$(3.9)\qquad fP\nabla_A \operatorname{grad} f + \nabla_Q \operatorname{grad} f = 0, \qquad Q = \nabla_A \operatorname{grad} f, \qquad \textit{and}$$

$$(3.10)\qquad \langle\nabla_B \operatorname{grad} f, C\rangle\nabla_A \operatorname{grad} f = \langle\nabla_A \operatorname{grad} f,\ C > \nabla_B \operatorname{grad} f.$$

Proof. $R(X,Y)\cdot R = 0$ and $R(A,B)\cdot R = 0$ hold always. If (*) holds, then (3.5) and (3.7) imply (3.9) and (3.10). Conversely, (3.5) and (3.9) imply $(R(X,A)\cdot R)(Z,V)W = 0$. Since

$$\langle\nabla_Q \operatorname{grad} f, B\rangle = \langle\nabla_B \operatorname{grad} f, Q\rangle = \langle\nabla_B \operatorname{grad} f,\ \nabla_A \operatorname{grad} f\rangle,$$

(3.6) and (3.9) imply $(R(X,A)\cdot R)(Z,B)W = 0$. (3.7) and (3.10) imply $(R(X,A)\cdot R)(Z,B)C = 0$. Similarly, (3.8) and (3.10), together with the fact that $\langle\nabla_A \operatorname{grad} f, B\rangle = \langle\nabla_B \operatorname{grad} f, A\rangle$, imply $(R(X,A)\cdot R)(C,B)G = 0$. Finally we have $(R(X,A)\cdot R)(Z,V)B = 0$ and $(R(X,A)\cdot R)(C,B)W = 0$ in the same way as (3.3) and (3.4).

4. The condition for $\nabla R = 0$

Using the identity

$$(\nabla_X R)(Y,Z)W = \nabla_X(R_{YZ}W) - R(\nabla_X Y, Z)W - R(Y,\nabla_X Z)W - R_{YZ}(\nabla_X W),$$

together with (2.3), (2.4) and (2.8), we get

$$(4.1)\qquad (\nabla_X R)(Y,Z)W =$$

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$$(\langle X,Y\rangle\langle Z,W\rangle-\langle X,Z\rangle\langle Y,W\rangle)(fP\operatorname{grad}f+\nabla_{\operatorname{grad}f}\operatorname{grad}f)/f^2,$$

where we have used $\nabla_X P = XP = 0$. Similarly we get

$$(\nabla_X R)(A,Y)W = ((\nabla_A \operatorname{grad} f)f + fPAf)(\langle Y,W\rangle X - \langle X,W\rangle Y)/f^2, \tag{4.2}$$

$$(\nabla_A R)(B,Y)W = \langle Y,W\rangle(f\nabla_A\nabla_B \operatorname{grad} f - f\nabla_T \operatorname{grad} f - Af\nabla_B \operatorname{grad} f)/f^2,\quad T=\nabla_A B, \tag{4.3}$$

$$(\nabla_X R)(Y,A)B = \langle X,Y\rangle(Bf\nabla_A \operatorname{grad} f - \langle\nabla_A \operatorname{grad} f, B\rangle \operatorname{grad} f)/f^2, \tag{4.4}$$

$$(\nabla_A R)(B,X)C = (Af\langle\nabla_B \operatorname{grad} f, C\rangle + f\langle\nabla_T \operatorname{grad} f, C\rangle - f\langle\nabla_A\nabla_B \operatorname{grad} f, C\rangle)X/f^2. \tag{4.5}$$

LEMMA 4.1. *On $E^n \times_f F$, $\nabla R = 0$ if and only if*

$$fP\operatorname{grad} f + \nabla_{\operatorname{grad} f}\operatorname{grad} f = 0, \tag{4.6}$$

$$f\nabla_A\nabla_B \operatorname{grad} f - f\nabla_T \operatorname{grad} f - Af\nabla_B \operatorname{grad} f = 0,\quad T=\nabla_A B,\ \textit{and} \tag{4.7}$$

$$Bf\nabla_A \operatorname{grad} f - \langle\nabla_A \operatorname{grad} f, B\rangle \operatorname{grad} f = 0. \tag{4.8}$$

Proof. Necessity comes from (4.1), (4.3) and (4.4). Conversely, assume that (4.6) ~ (4.8) hold. Then, we have $(\nabla_X R)(Y,Z)W=0$ and $(\nabla_A R)(B,Y)W=0$ by (4.1) and (4.3). We take the inner products of A and both sides of (4.6) to get

$$\begin{aligned}0 &= fPAf + \langle\nabla_{\operatorname{grad} f}\operatorname{grad} f, A\rangle\\ &= fPAf + \langle\nabla_A \operatorname{grad} f,\ \operatorname{grad} f\rangle\\ &= fPAf + (\nabla_A \operatorname{grad} f)f.\end{aligned}$$

Therefore, we have $(\nabla_X R)(A,Y)W = 0$ by (4.2). Next we take the inner products of C and both sides of (4.7). Then we have $(\nabla_A R)(B,X)C = 0$ by (4.5). By (4.4) and (4.8) we have $(\nabla_X R)(Y,A)B = 0$. These, together with the first and second Bianchi identities, imply $(\nabla_X R)(Y,W)A = (\nabla_A R)(X,Y)W = (\nabla_A R)(Y,W)B = (\nabla_Y R)(A,B)W = (\nabla_X R)(A,B)C = (\nabla_A R)(B,C)X = 0$.
Finally, $(\nabla_A R)(B,C)G = 0$ follows from (2.5).

5. The scalar curvature

In this section, we obtain the expression of the scalar curvature. Let $(A_\alpha, X_i;\ \alpha = 1,\cdots,n;\ i = 1,\cdots,r = \dim F)$ be vector fields on some open set on $E^n \times F$ such that they make an orthonormal basis at each point of the open set. We denote by R_1 the Ricci curvature tensor. Then we have

$$R_1(Y, Z) = \sum_i \langle R(Y, X_i)Z, X_i\rangle + \sum_\alpha \langle R(Y, A_\alpha)Z, A_\alpha\rangle,$$

which is calculated by (2.8) and (3.1), and we get

$$\begin{aligned} R_1(Y, Z) &= P \sum_i \langle\langle Y, Z\rangle X_i - \langle X_i, Z\rangle Y, X_i\rangle \\ &\quad + \sum_\alpha \langle -(1/f)\langle Z, Y\rangle \nabla_{A_\alpha} \operatorname{grad} f, A_\alpha\rangle \\ &= [(r-1)P - (1/f)\sum_\alpha \langle \nabla_{A_\alpha} \operatorname{grad} f, A_\alpha\rangle]\langle Y, Z\rangle, \end{aligned}$$

where we have used

$$\begin{aligned} \sum_i \langle \langle X_i, Z\rangle Y, X_i\rangle &= \sum_i \langle Y, X_i\rangle \langle X_i, Z\rangle \\ &= \sum_i \langle \langle Y, X_i\rangle X_i, Z\rangle = \langle Y, Z\rangle. \end{aligned}$$

Similarly we have

$$\begin{aligned} R_1(B, C) &= \sum_i \langle R(B, X_i)C, X_i\rangle + \sum_\alpha \langle R(B, A_\alpha)C, A_\alpha\rangle \\ &= -(r/f)\langle \nabla_B \operatorname{grad} f, C\rangle. \end{aligned}$$

Therefore we get

$$\begin{aligned} \text{The scalar curvature} &= \sum_i R_1(X_i, X_i) + \sum_\alpha R_1(A_\alpha, A_\alpha) \\ &= r[(r-1)P - (2/f)\sum_\alpha \langle \nabla_{A_\alpha} \operatorname{grad} f, A_\alpha\rangle]. \end{aligned} \tag{5.1}$$

6. Two lemmas

LEMMA 6.1. *On* $E^n \times {}_fF$, (4.6) *is equivalent to* $P =$ constant.

Proof. By (3.2) and (4.6) we have

$$(1/f)(K - \langle \operatorname{grad} f, \operatorname{grad} f\rangle) \operatorname{grad} f + \nabla_{\operatorname{grad} f} \operatorname{grad} f = 0.$$

Since this equation is considered as an equation on E^n, we introduce the natural coordinate symtem $(x^\alpha; \alpha = 1, \cdots, n)$ on E^n. Then the last equation is nothing but

$$\left(K - \sum_\alpha \frac{\partial f}{\partial x^\alpha}\frac{\partial f}{\partial x^\alpha}\right)\frac{\partial f}{\partial x^\beta} + f\sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}\frac{\partial f}{\partial x^\alpha} = 0,$$

which implies that each partial derivative of

$$P = \left[K - \sum_\alpha \left(\frac{\partial f}{\partial x^\alpha}\right)^2\right] \Big/ f^2 \tag{6.1}$$

vanishes. Thus, P is constant. The converse is clear.

LEMMA 6.2. *On $E^n \times {}_fF$, if the condition* (*) *is satisfied and the scalar curvature is constant, then P is constant.*

Proof. If f is constant, Lemma 6.2 is trivial. Therefore we assume that f is not constant. We put $A=\partial/\partial x^\alpha$, $B=\partial/\partial x^\beta$ and $C=\partial/\partial x^\gamma$, which are parallel on E^n. Then (3.9) and (3.10) are written as

$$fP\frac{\partial^2 f}{\partial x^\alpha \partial x^\delta}+\sum_\theta \frac{\partial^2 f}{\partial x^\theta \partial x^\delta}\,\frac{\partial^2 f}{\partial x^\theta \partial x^\alpha}=0, \tag{6.2}$$

$$\frac{\partial^2 f}{\partial x^\beta \partial x^\gamma}\,\frac{\partial^2 f}{\partial x^\alpha \partial x^\delta}=\frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma}\,\frac{\partial^2 f}{\partial x^\beta \partial x^\delta}. \tag{6.3}$$

Summing with respect to α and γ in (6.3), and substituting the result into (6.2), we have

$$\left(fP+\sum_\theta \frac{\partial^2 f}{\partial x^\theta \partial x^\theta}\right)\frac{\partial^2 f}{\partial x^\alpha \partial x^\delta}=0. \tag{6.4}$$

Define a subset Θ of E^n by

$$\Theta=\left\{x\in E^n\,;\left(\frac{\partial^2 f}{\partial x^\alpha \partial x^\delta}\right)(x)=0 \text{ for all } \alpha,\ \delta\right\}.$$

Let Θ_0 be a component of Θ. If Θ_0 contains an open set, f is of the form $f=a_\alpha x^\alpha+b$ on the interior of Θ_0 for some constant a_α, b (if the same letter appears as a subscript and as a superscript, we abbreviate $\sum$). Since f is positive and C^∞-differentiable, $\Psi=E^n-\Theta=E^n\cap\Theta^c$ can not be empty. Since Θ is closed, Ψ is a non-empty open set. On Ψ we have

$$fP+\sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\alpha}=0. \tag{6.5}$$

On the other hand, the scalar curvature is given by (5.1), which is also written as

$$\text{the scalar curvature}=r\left[(r-1)P-(2/f)\sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^\alpha}\right]. \tag{6.6}$$

By (6.5) and (6.6), we get

$$r(r+1)P=\text{the scalar curvature}=\text{constant}, \tag{6.7}$$

which shows that P is constant on Ψ.

On Θ_0, if $a_\alpha=0$ for all $\alpha=1,\cdots,n$, then P is constant on Θ_0 too. So we assume that at least one of a_α is not zero. Then, by (6.1) and $K\leq 0$, we get

$$P = (K - \textstyle\sum_\alpha a_\alpha^2)/(a_\beta x^\beta + b)^2 < 0. \tag{6.8}$$

We easily see that the function P on Θ_0 given by (6.8) can not be C^∞-differentiably extended to P on $\Theta_0 \cup \Psi$ so that P is constant on Ψ. Therefore Θ can not contain any open set where f is not constant. Hence, we have (6.7) on E^n.

7. Proof of Theorem A

Since $E^n \times {}_f F$ satisfies the condition (*) and the scalar curvature is constant, P is constant by Lemma 6.2. By Lemma 6.1 we see that (4.6) is equivalent to (6.1) with $P =$ constant. Now we solve (6.1) and show that the solution f satisfies (4.7) and (4.8). Then $E^n \times {}_f F$ is locally symmetric by Lemma 4.1. (6.1) is

$$K - \textstyle\sum_\alpha \left(\dfrac{\partial f}{\partial x^\alpha}\right)^2 - P f^2 = 0. \tag{7.1}$$

We solve the last partial differential equation by Lagrange-Charpit method. First we put

$$p_\alpha = \frac{\partial f}{\partial x^\alpha}, \quad \alpha = 1, \cdots, n.$$

Then the characteristic differential equations of (7.1) are

$$\begin{aligned} \frac{dx^1}{-2p_1} = \frac{dx^2}{-2p_2} = \cdots &= \frac{dx^n}{-2p_n} \\ &= \frac{df}{-2(p_1)^2 - \cdots - 2(p_n)^2} \\ &= \frac{-dp_1}{-2fPp_1} = \cdots = \frac{-dp_n}{-2fPp_n}. \end{aligned} \tag{7.2}$$

If f is constant, Theorem A is trivial. Hence, we assume that f is not constant. Then at least one of $p_1, \cdots, p_n$ does not vanish. So we assume $p_1 \neq 0$ (locally, if necessary) and furthermore we can assume that p_1 is positive. In this case, the last $(n-1)$ equatoins of (7.2) give the first integrals

$$p_\alpha = s_\alpha p_1, \quad \alpha = 2, \cdots, n,$$

where s_α are constants. Then (7.1) is

$$K - (p_1)^2(1 + s_2^2 + \cdots + s_n^2) - P f^2 = 0.$$

If we put $s_1 = 1$, we have

$$p_1 = \left[\frac{K - Pf^2}{\sum_\alpha s_\alpha^2}\right]^{1/2},$$

$$df = p_\alpha dx^\alpha = p_1 s_\alpha dx^\alpha.$$

Then we get

$$\frac{df}{[K - Pf^2]^{1/2}} = \frac{d(s_\beta x^\beta)}{[\sum s_\alpha^2]^{1/2}}. \tag{7.3}$$

By putting $[K - Pf^2]^{1/2} = \sqrt{-P}f + y$, we have

$$f = \frac{K - y^2}{2\sqrt{-P}y}, \tag{7.4}$$

$$\frac{-(K + y^2)dy/(2\sqrt{-P}y^2)}{(K + y^2)/2y} = \frac{d(s_\beta x^\beta)}{[\sum_\alpha s_\alpha^2]^{1/2}}. \tag{7.5}$$

Therefore we have

$$y = b\exp[-(-P/\sum s_\alpha^2)^{1/2}(s_\beta x^\beta)]. \tag{7.6}$$

If we put $[-P/\sum s_\alpha^2]^{1/2}s_\beta = c_\beta$, then, by (7.4) and (7.6), we have

$$f = \frac{1}{2\sqrt{-P}}\left[\frac{K}{b}\exp(c_\beta x^\beta) - b\exp(-c_\beta x^\beta)\right], \tag{7.7}$$

which is a solution of (7.1). Consequently, we see that f satisfies (4.7) and (4.8), which are written as

$$f\frac{\partial^3 f}{\partial x^\alpha \partial x^\beta \partial x^\gamma} - \frac{\partial f}{\partial x^\alpha}\frac{\partial^2 f}{\partial x^\beta \partial x^\gamma} = 0,$$

$$\frac{\partial f}{\partial x^\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\gamma} - \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}\frac{\partial f}{\partial x^\gamma} = 0.$$

8. Proof of Theorem B

Assume that $E^1 \times {}_f F$ satisfies condition (*). Then (3.9) or (6.2) is written as

$$\left(fP + \frac{d^2 f}{dx^2}\right)\frac{d^2 f}{dx^2} = 0, \tag{8.1}$$

where x is the natural coordinate system of E^1. Similarly as in §6, we define Θ and Ψ. Then on Ψ, by (6.1) and (8.1), we have

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(8.2)
$$\left[K - \left(\frac{df}{dx}\right)^2\right] + f\,\frac{d^2 f}{dx^2} = 0,$$

which implies that the derivative of $P = [K - (df/dx)^2]/f^2$ is zero so that P is constant on each component of Ψ. On the other hand, on an open interval contained in Θ, f is of the form $f = cx + d$ for some constants c and d. Since P is C^∞-differentiable, Θ can not contain any open interval where f is not constant. Thus, P is constant on E^1. Since f is non-constant, P is a negative constant. Now we have

$$K - \left(\frac{df}{dx}\right)^2 - Pf^2 = 0,$$

whose solution f is

$$f = \frac{1}{2\sqrt{-P}}\left[\frac{K}{b}\exp\sqrt{-P}\,x - b\exp(\sqrt{-P}\,x)\right]$$

where $b < 0$ is a constant. Then we have

$$\nabla_A \operatorname{grad} f = -fPA,$$

and hence, (2.6), (2.8) are expressed as

(8.3)
$$\begin{aligned} R_{AX}B &= (-1/f) < -PfA,\ B > X \\ &= P(\langle A, B\rangle X - \langle X, B\rangle A), \end{aligned}$$

(8.4)
$$\begin{aligned} R_{AX}Y &= (1/f)\langle X, Y\rangle(-PfA) \\ &= P(\langle A, Y\rangle X - \langle X, Y\rangle A). \end{aligned}$$

Thus, (8.3), (2.7), (8.4) and (3.1) show that $E^1 \times {}_f F$ is of constant curvature $P < 0$.

9. Remarks

(i) If (F, g) is a complete Riemannian manifold, then $E^n \times {}_f F$ is also a complete Riemannian manifold (cf. Lemma 7.2, [1]).

(ii) Assume that (F, g) is of constant curvature $K < 0$. If $(\partial^2 f/\partial x^\alpha \partial x^\beta)$ is non-singular at some point of E^n and n is sufficiently small with respect to $r = \dim F$ (for example, $n = 2$), then $E^n \times {}_f F$ is irreducible. In fact, by a result due to D. Montgomery and H. Samelson [3] we see that there is no proper subgroup of the orthogonal group $0(n + r)$ of order greater than $(n + r - 1)(n + r - 2)/2$, provided $n + r \neq 4$. On the other hand, the holonomy algebra is generated by (cf. [4])

$$R_{AX},\ R_{XY},\ \cdots,\ \text{etc.}$$

which are given by (2.5) ~ (2.8), (3.1). And under the circumstance stated above the restricted homogeneous holonomy group at the point is $SO(n+r)$.

(iii) It is an open question if one can get complete solutions of non-linear partial differential equations (6.1), (6.2) and (6.3) (i.e., the condition (*) on $E^n \times {}_fF$, $n \geqq 2$). If one can get the complete solutions, then one sees whether the assumption on the scalar curvature is necessary or not in Theorem A.

(iv) The condition (*) is expressed in local coordinates as

$$\nabla_r \nabla_s R^h{}_{ijk} - \nabla_s \nabla_r R^h{}_{ijk} = 0.$$

In [6], K. Nomizu and H. Ozeki showed that if $\nabla\nabla R = 0$ (more generally, $\nabla^k R = 0$ for some k) on a (complete) Riemannian manifold, then $\nabla R = 0$.

(v) Studies concerning $R(X,Y)\cdot R$ were made also by A. Lichnerowich [2], p. 11, P. J. Ryan [7], K. Sekigawa and S. Tanno [9], J. Simons [10], S. Tanno and T. Takahashi [11], etc.

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