THE QUEER q-SCHUR SUPERALGEBRA

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We dedicate the paper to the memory of Professor James A. Green

Abstract

As a natural generalisation of q-Schur algebras associated with the Hecke algebra $\mathcal{H}_{r,R}$ (of the symmetric group), we introduce the queer q-Schur superalgebra associated with the Hecke–Clifford superalgebra $\mathcal{H}^{c}_{r,R}$, which, by definition, is the endomorphism algebra of the induced $\mathcal{H}^{c}_{r,R}$-module from certain q-permutation modules over $\mathcal{H}_{r,R}$. We will describe certain integral bases for these superalgebras in terms of matrices and will establish the base-change property for them. We will also identify the queer q-Schur superalgebras with the quantum queer Schur superalgebras investigated in the context of quantum queer supergroups and provide a constructible classification of their simple polynomial representations over a certain extension of the field $\mathbb{C}(v)$ of complex rational functions.

1. Introduction

In the determination of the polynomial representations of the complex general linear group $\mathrm{GL}_n(\mathbb{C})$, Issai Schur introduced certain finite-dimensional algebras to describe the homogeneous components of these representations and to set up a correspondence between the representations of $\mathrm{GL}_n(\mathbb{C})$ of a fixed homogeneous degree r and the representations of the symmetric group $\mathfrak{S}_r$ on r letters. This correspondence forms part of the well-known Schur–Weyl duality. Green publicised this algebraic approach in his 1980 monograph [18] and presented the theory 'in some ways very much in keeping with the present-day ideas on representations of algebraic groups'. In particular, he used the term *Schur algebras* for these finite-dimensional algebras and provided some standard combinatorial treatment for the structure and representations of Schur algebras.

Since the introduction of quantum groups in the mid 1980s, the family of Schur algebras has been enlarged to include q-Schur algebras, affine q-Schur algebras and

their various superanalogs. See, for example, [5, 6, 8, 13] and the references therein. Thus, the treatment in [18] has also been 'quantised' in various forms which provide applications to both structure and representations of the associated quantum linear groups, finite general linear groups, general linear p-adic groups, general linear supergroups and so on. In this paper, we will further generalise Green's standard treatment to a new class of Schur superalgebras, the queer q-Schur superalgebra.

Originally arising from the classification of complex simple (associative) superalgebras, there are two types of q-Schur superalgebras. The 'type M' q-Schur superalgebra links representations of quantum linear supergroups with those of Hecke algebras of type A, while the 'type Q' q-Schur superalgebra, or simply the queer q-Schur superalgebra, links the quantum queer supergroup with the Hecke–Clifford superalgebra. It is known that the former has various structures shared with the q-Schur algebra; see [10, 13, 15, 23], while the latter has been introduced in [14, 25] in the context of Schur–Weyl–Sergeev duality and investigated in [14] in terms of a Drinfeld–Jimbo-type presentation (cf. [15]).

In this paper, we will explore the algebraic properties of the queer q-Schur superalgebra. Thus, like the q-Schur algebra, we first introduce a queer q-Schur superalgebra as the endomorphism algebra of a direct sum of certain 'q-permutation supermodules' for the Hecke–Clifford superalgebra. We then construct its standard basis arising from analysing double cosets in the symmetric group. In particular, we use matrices to label the basis elements. In this way, we establish the base-change property for the queer q-Schur superalgebra. We then identify them in the generic case with the quotients of the quantum queer supergroups and classify their irreducible representations in this case.

Here is the layout of the paper. In Section 2, we review the definition of the Hecke–Clifford superalgebra $\mathcal{H}_{r,R}^{\mathsf{c}}$ over a commutative ring R and its standard basis. We then introduce certain (induced) q-permutation supermodules and define the queer q-Schur superalgebra $Q_q(n, r; R)$. In Section 3, we study the properties of q-permutation supermodules and investigate some new bases. In Section 4, we introduce some special elements in the Clifford superalgebras, which is the key to a basis for the endomorphism algebra of the induced q-trivial representation. An integral basis and the base-change property for the queer q-Schur superalgebra are given in Section 5. We then identify the queer q-Schur superalgebras as quotients of the quantum queer supergroup in Section 6. Using the identification together with a work by Jones and Nazarov [20], a complete set of irreducible polynomial (super)representations of the quantum queer supergroup is constructed in the last section.

Throughout the paper, let $\mathbb{Z}_2 = \{0, 1\}$. We will use a twofold meaning for $\mathbb{Z}_2$. We will regard $\mathbb{Z}_2$ as an abelian group when we use it to describe a superspace. However, for a matrix or an n-tuple with entries in $\mathbb{Z}_2$, we will regard it as a subset of $\mathbb{Z}$.

Let $\mathcal{A} = \mathbb{Z}[\boldsymbol{q}]$ and let R be a commutative ring which is also an $\mathcal{A}$-module by mapping $\boldsymbol{q}$ to $q \in R$. We assume that the characteristic of R is not equal to 2. If Q denotes an $\mathcal{A}$-algebra or an $\mathcal{A}$-module, then $Q_R = Q \otimes_{\mathcal{A}} R$ denotes the object obtained by base change to R. From Section 6 onwards, we will use the ring $\mathcal{Z} = \mathbb{Z}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ of Laurent polynomials in indeterminate $\boldsymbol{v}$, where $\boldsymbol{q} = \boldsymbol{v}^2$.

2. Hecke–Clifford superalgebras and queer q-Schur superalgebras

We first introduce an endomorphism algebra of an induced module from certain 'q-permutation modules' over the Hecke–Clifford superalgebra. We will prove in Section 6 that it is isomorphic to the quantum queer Schur superalgebra defined in [14] by studying the tensor space of the natural representation of the quantum queer supergroup $U_v(\mathfrak{q}_n)$.

Let R be a commutative ring of characteristic not equal to 2 and let $q \in R$. Let C_r denote the *Clifford superalgebra* over R generated by odd elements $c_1, \ldots, c_r$ subject to the relations

$$c_i^2 = -1, \quad c_i c_j = -c_j c_i, \quad 1 \le i \ne j \le r. \tag{2.1}$$

The *Hecke–Clifford superalgebra* $\mathcal{H}_{r,R}^{\mathrm{c}}$ is the associative superalgebra over the ring R with the even generators $T_1, \ldots, T_{r-1}$ and the odd generators $c_1, \ldots, c_r$ subject to (2.1) and the following additional relations: for $1 \le i, i' \le r-1$ and $1 \le j \le r$ with $|i - i'| > 1$ and $j \ne i, i+1$,

$$(T_i - q)(T_i + 1) = 0, \quad T_i T_{i'} = T_{i'} T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i c_j = c_j T_i, \quad T_i c_i = c_{i+1} T_i, \quad T_i c_{i+1} = c_i T_i - (q-1)(c_i - c_{i+1}). \tag{2.2}$$

Note that if q is invertible then T_i^{-1} exists and the last relation is obtained by taking the inverses of the last second relation.

For notational simplicity, if $R = \mathcal{A} := \mathbb{Z}[q]$, the integral polynomial ring in q, then we write $\mathcal{H}_r^{\mathrm{c}} = \mathcal{H}_{r,\mathcal{A}}^{\mathrm{c}}$.

Let $\mathfrak{S}_r$ be the symmetric group on $\{1, 2, \ldots, r\}$ which is generated by the simple transpositions $s_k = (k, k+1)$ for $1 \le k \le r-1$. The subalgebra $\mathcal{H}_{r,R}$ of $\mathcal{H}_{r,R}^{\mathrm{c}}$ generated by $T_1, \ldots, T_{r-1}$ is the (Iwahori–)Hecke algebra associated with $\mathfrak{S}_r$.

REMARK. Jones and Nazarov [20] introduced the notion of an affine Hecke–Clifford superalgebra which is generated by $T_1, \ldots, T_{r-1}, c_1, \ldots, c_r, X_1, \ldots, X_r$ subject to the relations (2.1), (2.2) and certain additional explicit relations which are not mentioned here. It is known from [20, Proposition 3.5] (see also [2, Remark 4.2]) that $\mathcal{H}_{r,R}^{\mathrm{c}}$ can naturally be viewed as a quotient superalgebra of the affine Hecke–Clifford superalgebra by the two-sided ideal generated by X_1.

We also note that the generator T_i used in [2] is equal to $v^{-1} T_i$ here, where $v^2 = q$ for some $v \in R$. Moreover, we choose $c_j^2 = -1$ in (2.1), following [22, 25, 26], whereas in [22, 25, 26] the authors take $C_j^2 = 1$. Thus, assuming that $\sqrt{-1} \in R$, we have $c_i = \sqrt{-1} C_i$.

Like the Hecke algebra $\mathcal{H}_{r,R}$, the symmetry of $\mathcal{H}_{r,R}^{\mathrm{c}}$ can be seen from the following (anti-)automorphisms of order 2.

LEMMA 2.1. *The R-algebra $\mathcal{H}_{r,R}^{\mathrm{c}}$ admits the following algebra involutions φ, ψ and anti-involutions τ, ι defined by:*

(1) $\varphi : T_i \mapsto -T_{r-i} + (q-1), \quad c_j \mapsto c_{r+1-j}$;
(2) $\psi : T_i \mapsto c_i T_i c_{i+1}, \quad c_j \mapsto c_j$;

(3) $\tau : T_i \mapsto -T_i + (q-1), \quad c_j \mapsto c_j;$

(4) $\iota : T_i \mapsto T_i - (q-1)c_i c_{i+1}, \quad c_j \mapsto c_j$

for all $1 \le i, j \le r$ and $i \ne r$.

PROOF. Note that the correspondence $T_i \mapsto -T_i + (q-1)$ in (3), denoted by $(\ )^{\#}$ in [8], defines an automorphism of the subalgebra $\mathcal{H}_{r,R}$. So (3) follows easily. (If $q^{-1} \in R$, then (3) is seen easily by the fact that $-T_i + (q-1) = -qT_i^{-1}$.) By directly checking the relations (2.1) and (2.2), there is an anti-involution

$$\gamma : T_i \longmapsto T_{r-i}, \quad c_j \longmapsto c_{r+1-j}.$$

Thus, compositing γ and τ gives (1).

Let $\dot{T}_i = c_i T_i c_{i+1}$. Then $\dot{T}_i, c_j$ generate $\mathcal{H}_{r,R}^{c}$ and satisfy all the relations (2.1) and (2.2) above with T_i replaced by $\dot{T}_i$. Hence, ψ in (2) is an automorphism of order 2. Finally, the map ι in (4) is the map ψ followed by the map τ defined relative to the generators $\dot{T}_i, c_j$, since, by the last relation in (2.2),

$$\iota(T_i) = T_i - (q-1)c_i c_{i+1} = -c_i T_i c_{i+1} + (q-1).$$

(Of course, by noting Remark 1, assertion (4) follows also from [2, (2.31)].) □

Let $T_{s_i} = T_i$ and $T_w = T_{s_{i_1}} \cdots T_{s_{i_k}}$, where $w = s_{i_1} \cdots s_{i_k} \in \mathfrak{S}_r$ is a reduced expression. Let $\le$ be the Bruhat order of $\mathfrak{S}_r$, that is, $\pi \le w$ if π is a subexpression of some reduced expression for w. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}_2^r$, set $c^{\alpha} = c_1^{\alpha_1} \cdots c_r^{\alpha_r}$. Observe that the symmetric group $\mathfrak{S}_r$ acts naturally on C_r by permuting the generators $c_1, \ldots, c_r$. We shall denote this action by $z \mapsto w \cdot z$ for $z \in C_r, w \in \mathfrak{S}_r$. Then

$$w \cdot c^{\alpha} = c_{w(1)}^{\alpha_1} \cdots c_{w(r)}^{\alpha_r} = \pm c^{\alpha \cdot w^{-1}},$$

where $\alpha \cdot w = (\alpha_{w(1)}, \ldots, \alpha_{w(r)})$, for $\alpha \in \mathbb{Z}_2^r, w \in \mathfrak{S}_r$, is the place permutation action. Moreover, by (2.2),

$$c_k^{\alpha_k} c_{k+1}^{\alpha_{k+1}} T_{s_k} = T_{s_k} c_k^{\alpha_k} c_k^{\alpha_{k+1}} + (q-1)(c_k^{\alpha_k + \alpha_{k+1}} - c_{k+1}^{\alpha_k} c_k^{\alpha_{k+1}}). \tag{2.3}$$

Thus, by induction on the length $\ell(w)$ of w, one proves the following formula: for any $\alpha \in \mathbb{Z}_2^r$ and $w \in \mathfrak{S}_r$,

$$c^{\alpha} T_w = T_w(w^{-1} \cdot c^{\alpha}) + \sum_{\pi < w, \beta \in \mathbb{Z}_2^r} a_{\pi,\beta}^{\alpha,w} T_\pi c^{\beta} = \pm T_w c^{\alpha \cdot w} + \sum_{\pi < w, \beta \in \mathbb{Z}_2^r} a_{\pi,\beta}^{\alpha,w} T_\pi c^{\beta} \tag{2.4}$$

for some $a_{\pi,\beta}^{\alpha,w} \in R$. Then, by the results [20, Proposition 2.1] and [2, Theorem 3.6] over fields, it is easy to deduce that the following holds.

LEMMA 2.2. *The superalgebra $\mathcal{H}_{r,R}^{c}$ is a free R-module. Moreover, both the sets $\mathscr{B} = \{T_w c^{\alpha} \mid w \in \mathfrak{S}_r, \alpha \in \mathbb{Z}_2^r\}$ and $\{c^{\alpha} T_w \mid w \in \mathfrak{S}_r, \alpha \in \mathbb{Z}_2^r\}$ form bases for $\mathcal{H}_{r,R}^{c}$.*

PROOF. By (2.4), it suffices to prove that $\mathscr{B}$ forms a basis. First, the set spans $\mathcal{H}_{r,R}^{\mathsf{c}}$ by (2.4). Suppose next that $R = \mathcal{A}$. By base change to the fraction field of $\mathcal{A}$ and [2, Theorem 3.6], we see that $\mathscr{B}$ is linearly independent and, hence, forms an $\mathcal{A}$-basis for $\mathcal{H}_r^{\mathsf{c}}$. For a general R, specialisation to R by sending $\boldsymbol{q}$ to q gives a basis $\{T_w c^\alpha \otimes 1 \mid w \in \mathfrak{S}_r, \alpha \in \mathbb{Z}_2^r\}$ for $\mathcal{H}_r^{\mathsf{c}} \otimes_{\mathcal{A}} R$. Now the presentation of $\mathcal{H}_{r,R}^{\mathsf{c}}$ gives an R-algebra epimorphism $f : \mathcal{H}_{r,R}^{\mathsf{c}} \to \mathcal{H}_r^{\mathsf{c}} \otimes_{\mathcal{A}} R$, sending T_i, c_j to $T_i \otimes 1, c_j \otimes 1$, respectively, and sending $\mathscr{B}$ to a basis. This forces that $\mathscr{B}$ itself must be a basis (and that f is an isomorphism). $\qquad\square$

COROLLARY 2.3. *Let R be a commutative ring which is an $\mathcal{A}$-module via the map $\mathcal{A} \to R, \boldsymbol{q} \mapsto q$.*

(1)   *There is an algebra isomorphism $\mathcal{H}_{r,R}^{\mathsf{c}} \cong \mathcal{H}_r^{\mathsf{c}} \otimes_{\mathcal{A}} R$.*

(2)   *As a module over the subalgebra $\mathcal{H}_{r,R}$, $\mathcal{H}_{r,R}^{\mathsf{c}}$ is $\mathcal{H}_{r,R}$-free with basis $\{c^\alpha \mid \alpha \in \mathbb{Z}_2^r\}$.*

An l-tuple $(\lambda_1, \ldots, \lambda_l)$ of nonnegative integers $\lambda_i \in \mathbb{N}$ is called a *composition* of r if $\sum_{i=1}^l \lambda_i = r$. For a composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of r, denote by $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_l}$ the standard Young subgroup of $\mathfrak{S}_r$ and let $\mathcal{D}_\lambda$ be the set of minimal length right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$. Let $\mathcal{H}_{\lambda,R}$ and $\mathcal{H}_{\lambda,R}^{\mathsf{c}}$ be the associated subalgebras of $\mathcal{H}_{r,R}$ and $\mathcal{H}_{r,R}^{\mathsf{c}}$, respectively. Then clearly $\mathcal{H}_{\lambda,R}^{\mathsf{c}} \cong \mathcal{H}_{\lambda,R} \otimes C_r$ as R-supermodules and moreover we have algebra isomorphisms

$$\mathcal{H}_{\lambda,R} \cong \mathcal{H}_{\lambda_1,R} \otimes \cdots \otimes \mathcal{H}_{\lambda_l,R}, \quad \mathcal{H}_{\lambda,R}^{\mathsf{c}} \cong \mathcal{H}_{\lambda_1,R}^{\mathsf{c}} \otimes \cdots \otimes \mathcal{H}_{\lambda_l,R}^{\mathsf{c}}. \tag{2.5}$$

We remind the reader that, for two superalgebras $\mathcal{S}$ and $\mathcal{T}$ over R, the (super)tensor product $\mathcal{S} \otimes \mathcal{T}$ is naturally a superalgebra, with multiplication defined by

$$(s \otimes t)(s' \otimes t') = (-1)^{\widehat{t \cdot s'}} (ss') \otimes (tt') \quad \text{for all } s, s' \in \mathcal{S}, t, t' \in \mathcal{T},$$

where t, s' are homogeneous. Here we used the notation $\widehat{a} = 0$ if a is even and $\widehat{a} = 1$ otherwise for a homogeneous element a in a superspace.

Denote by $\Lambda(n, r) \subset \mathbb{N}^n$ the set of compositions of r with n parts. Given $\lambda \in \Lambda(n, r)$, set

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w, \quad y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q^{-1})^{\ell(w)} T_w, \tag{2.6}$$

where $\ell(w)$ is the length of w. As a superanalog of the q-Schur algebra (cf. [8, Section 2.9]) or a quantum analog of the Schur superalgebra of type Q (cf. [2]), we introduce the *queer q-Schur superalgebra*

$$Q_q(n, r; R) = \text{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}} \left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \right). \tag{2.7}$$

In particular, we write

$$Q_q(n, r) := Q_q(n, r; \mathcal{A}) \quad \text{and} \quad Q_v(n, r) := Q_q(n, r; \mathcal{Z}), \tag{2.8}$$

where $\boldsymbol{q} = \boldsymbol{v}^2$.

Note that $\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,R}$ is a direct sum of q-permutation modules, which is known as the tensor space used in defining q-Schur algebras. Since $x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \cong x_\lambda \mathcal{H}_{r,R} \otimes_{\mathcal{H}_{r,R}} \mathcal{H}_{r,R}^{\mathsf{c}}$ as $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodules, which will be called a *q-permutation supermodule*, it is clear we have an $\mathcal{H}_{r,R}^{\mathsf{c}}$-module isomorphism

$$\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \cong \Big(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,R} \Big) \otimes_{\mathcal{H}_{r,R}} \mathcal{H}_{r,R}^{\mathsf{c}}, \tag{2.9}$$

which is an induced $\mathcal{H}_{r,R}^{\mathsf{c}}$-module from the tensor space. We will see in Section 6 that this module itself can be regarded as a tensor product of a free R-supermodule.

Given a right $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule M, we can twist the action with φ given in Lemma 2.1 to get a new $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule M^φ. Thus, as an R-module $M^\varphi = M$, the new $\mathcal{H}_{r,R}^{\mathsf{c}}$-action is defined by

$$m \cdot h = m\varphi(h) \quad \text{for all } h \in \mathcal{H}_{r,R}^{\mathsf{c}}, m \in M.$$

Clearly, $\mathcal{H}_{r,R}$ is stable under the involution φ. If M is a $\mathcal{H}_{r,R}$-module, we can define M^φ similarly. It is known that

$$(x_\lambda \mathcal{H}_{r,R})^\varphi \cong y_{\lambda^\circ} \mathcal{H}_{r,R} \cong y_\lambda \mathcal{H}_{r,R}$$

(see [8, (2.1)] or [6, Lemma 7.39]), where $\lambda^\circ = (\lambda_l, \lambda_{l-1}, \ldots, \lambda_1)$ if $\lambda = (\lambda_1, \ldots, \lambda_l)$. This implies that $(x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}})^\varphi \cong y_\lambda \mathcal{H}_{r,R}^{\mathsf{c}}$ due to the facts that $x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \cong x_\lambda \mathcal{H}_{r,R} \otimes_{\mathcal{H}_{r,R}} \mathcal{H}_{r,R}^{\mathsf{c}}$ and $y_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \cong y_\lambda \mathcal{H}_{r,R} \otimes_{\mathcal{H}_{r,R}} \mathcal{H}_{r,R}^{\mathsf{c}}$. Hence, we obtain a superanalog of [8, Theorem 2.24(i) and (ii)]

$$Q_q(n,r;R) \cong \operatorname{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}} \Big(\bigoplus_{\lambda \in \Lambda(n,r)} y_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \Big).$$

REMARK. We may also use the anti-involution τ in Lemma 2.1(3) to turn a left $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule to a right $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule or *vice versa*. More precisely, if M is a finite-dimensional left $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule, then we define a right $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule M^τ by setting $M^\tau = M$ as an R-module and making an $\mathcal{H}_{r,R}^{\mathsf{c}}$-action through τ:

$$m \cdot h = \tau(h)m, \quad \forall h \in \mathcal{H}_{r,R}^{\mathsf{c}}, m \in M.$$

In particular, if $M = \mathcal{H}_{r,R}^{\mathsf{c}} y_\lambda$, then

$$(\mathcal{H}_{r,R}^{\mathsf{c}} y_\lambda)^\tau \cong \tau(\mathcal{H}_{r,R}^{\mathsf{c}} y_\lambda) = \tau(y_\lambda) \mathcal{H}_{r,R}^{\mathsf{c}} = x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \tag{2.10}$$

by [8, (2.1)] again.

3. q-Permutation supermodules for $\mathcal{H}_{r,R}^{\mathsf{c}}$

The q-permutation supermodules $x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}}$ share certain nice properties with q-permutation modules $x_\lambda \mathcal{H}_{r,R}$ of the Hecke algebra $\mathcal{H}_{r,R}$, as we will see below.

Recall, for a composition λ of r, that $\mathcal{D}_\lambda$ is the set of minimal length right coset representatives of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_r$.

Lemma 3.1. *Let λ be a composition of r. Then:*

(1)   *the right $\mathcal{H}^{\mathsf{c}}_{r,R}$-supermodule $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R}$ (respectively, $y_\lambda \mathcal{H}^{\mathsf{c}}_{r,R}$) is R-free with basis*

$$\{x_\lambda T_d c^\alpha \mid d \in \mathcal{D}_\lambda, \alpha \in \mathbb{Z}_2^r\} \quad (\text{respectively, } \{y_\lambda T_d c^\alpha \mid d \in \mathcal{D}_\lambda, \alpha \in \mathbb{Z}_2^r\});$$

(2) $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} = \{h \in \mathcal{H}^{\mathsf{c}}_{r,R} \mid T_{s_k} h = qh, \forall s_k \in \mathfrak{S}_\lambda\};$

(3) $y_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} = \{h \in \mathcal{H}^{\mathsf{c}}_{r,R} \mid T_{s_k} h = -h, \forall s_k \in \mathfrak{S}_\lambda\}.$

Similar statements hold for the left $\mathcal{H}^{\mathsf{c}}_{r,R}$-supermodules $\mathcal{H}^{\mathsf{c}}_{r,R} x_\lambda$ and $\mathcal{H}^{\mathsf{c}}_{r,R} y_\lambda$.

PROOF. Part (1) is clear. We now prove part (2). Since $T_{s_k} x_\lambda = q x_\lambda$ for all $s_k \in \mathfrak{S}_\lambda$, we obtain $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \subseteq \{h \in \mathcal{H}^{\mathsf{c}}_{r,R} \mid T_{s_k} h = qh, \forall s_k \in \mathfrak{S}_\lambda\} =: \mathcal{E}$.

Conversely, given $h = \sum_{w \in \mathfrak{S}_r, \alpha \in \mathbb{Z}_2^r} h_{w,\alpha} T_w c^\alpha \in \mathcal{E}$ with $h_{w,\alpha} \in R$, write

$$h_\alpha = \sum_{w \in \mathfrak{S}_r} h_{w,\alpha} T_w c^\alpha = \Big(\sum_{w \in \mathfrak{S}_r} h_{w,\alpha} T_w\Big) c^\alpha.$$

Then $h_\alpha \in \mathcal{H}_{r,R} c^\alpha$ and h can be written as $h = \sum_{\alpha \in \mathbb{Z}_2^r} h_\alpha$. Assume that $T_{s_k} h = qh$ for all $s_k \in \mathfrak{S}_\lambda$. Then

$$\sum_{\alpha \in \mathbb{Z}_2^r} (T_{s_k} h_\alpha - qh_\alpha) = 0$$

for $s_k \in \mathfrak{S}_\lambda$. Observe that $T_{s_k} h_\alpha - qh_\alpha \in \mathcal{H}_{r,R} c^\alpha$ for each $\alpha \in \mathbb{Z}_2^r$. Then, by Corollary 2.3(2), one may deduce that $T_{s_k} h_\alpha = qh_\alpha$ for all $s_k \in \mathfrak{S}_\lambda$ and $\alpha \in \mathbb{Z}_2^r$. This implies that $T_{s_k}(\sum_{w \in \mathfrak{S}_r} h_{w,\alpha} T_w) = q(\sum_{w \in \mathfrak{S}_r} h_{w,\alpha} T_w)$ for all $s_k \in \mathfrak{S}_\lambda$ since c^α is invertible. Then, using the classical result for $\mathcal{H}_{r,R}$, we obtain $\sum_{w \in \mathfrak{S}_r} h_{w,\alpha} T_w \in x_\lambda \mathcal{H}_{r,R}$. This means that $h_\alpha \in x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R}$ for $\alpha \in \mathbb{Z}_2^r$. Therefore, part (2) is proved. The proof of part (3) is similar.

Applying the anti-involution τ in Lemma 2.1(3) to (1)–(3) gives the last assertion. □

Lemma 3.2. *The following isomorphism of R-modules holds for any composition λ, μ of r:*

$$\operatorname{Hom}_{\mathcal{H}^{\mathsf{c}}_{r,R}}(x_\mu \mathcal{H}^{\mathsf{c}}_{r,R}, x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R}) \cong x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \cap \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu.$$

PROOF. The proof is standard with the required isomorphism given by the map

$$\operatorname{Hom}_{\mathcal{H}^{\mathsf{c}}_{r,R}}(x_\mu \mathcal{H}^{\mathsf{c}}_{r,R}, x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R}) \longrightarrow x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \cap \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu, \quad f \longmapsto f(x_\mu). \qquad \square$$

We will prove that the R-modules $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \cap \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu$ are free for the special case $\lambda = \mu = (r)$ in Section 4 and for the general case in Section 5. We need some preparation. In the following, we will display a new basis for $\mathcal{H}^{\mathsf{c}}_{r,R}$ and $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R}$. We start with recalling several facts for Hecke algebras $\mathcal{H}_{r,R}$.

Given two compositions λ, μ of r, let $\mathcal{D}_{\lambda,\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$, where $\mathcal{D}_\mu^{-1} = \{d^{-1} \mid d \in \mathcal{D}_\mu\}$ is the set of minimal length left coset representatives of $\mathfrak{S}_\mu$ in $\mathfrak{S}_r$.

Lemma 3.3 ([3], [7, Lemma 1.6]). *Suppose that λ, μ are compositions of r.*

(1)   *The set $\mathcal{D}_{\lambda,\mu}$ is a system of $\mathfrak{S}_\lambda$–$\mathfrak{S}_\mu$ double coset representatives in $\mathfrak{S}_r$.*

(2)   *The element $d \in \mathcal{D}_{\lambda,\mu}$ is the unique element of minimal length in $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$.*

(3)   *If $d \in \mathcal{D}_{\lambda,\mu}$, there exists a composition $\nu(d)$ of r such that $\mathfrak{S}_{\nu(d)} = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$.*

(4)   *Each element $w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu$ can be written uniquely as $w = ud\sigma$ with $u \in \mathfrak{S}_\lambda$ and $\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$. Moreover, $\ell(w) = \ell(u) + \ell(d) + \ell(\sigma)$.*

Suppose that λ, μ are compositions of r. Observe that $\mathcal{D}_{\mu,\lambda} = \mathcal{D}_\mu \cap \mathcal{D}_\lambda^{-1} = \mathcal{D}_{\lambda,\mu}^{-1}$ is a system of $\mathfrak{S}_\mu$-$\mathfrak{S}_\lambda$ double coset representatives in $\mathfrak{S}_r$ and hence $d\mathfrak{S}_\mu d^{-1} \cap \mathfrak{S}_\lambda$ is a standard Young subgroup of $\mathfrak{S}_\lambda$ for each $d \in \mathcal{D}_{\lambda,\mu}$ by Lemma 3.3(3). Thus, $\nu(d^{-1})$ is well defined via $\mathfrak{S}_{\nu(d^{-1})} = d\mathfrak{S}_\mu d^{-1} \cap \mathfrak{S}_\lambda$ and moreover $\mathfrak{S}_{\nu(d^{-1})} = d\mathfrak{S}_{\nu(d)}d^{-1}$.

COROLLARY 3.4. *Suppose that λ, μ are compositions of r and $d \in \mathcal{D}_{\lambda,\mu}$.*

(1)   *For each $u \in \mathfrak{S}_\lambda$, the element ud can be written uniquely as $ud = u'd\tau$ with $u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda$ and $\tau \in \mathfrak{S}_{\nu(d)}$. Moreover, $\ell(ud) = \ell(u) + \ell(d) = \ell(u') + \ell(d) + \ell(\tau)$.*

(2)   *The coset $\mathfrak{S}_\lambda d = \{u'd\tau \mid u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda, \tau \in \mathfrak{S}_{\nu(d)}\}$.*

(3)   *Each element $w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu$ can be written uniquely as $w = u'd\pi$ with $u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda$ and $\pi \in \mathfrak{S}_\mu$.*

(4)   *For $d_1, d_2 \in \mathcal{D}_{\lambda,\mu}$ and $u_1 \in \mathcal{D}_{\nu(d_1^{-1})}^{-1} \cap \mathfrak{S}_\lambda, u_2 \in \mathcal{D}_{\nu(d_2^{-1})}^{-1} \cap \mathfrak{S}_\lambda$, if $u_1 \neq u_2$ or $d_1 \neq d_2$, then $u_1 d_1 \mathfrak{S}_\mu \cap u_2 d_2 \mathfrak{S}_\mu = \emptyset$.*

COROLLARY 3.5. *Suppose that λ, μ are compositions of r and $d \in \mathcal{D}_{\lambda,\mu}$. Then*

$$x_\lambda T_d = \left(\sum_{u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_{u'} \right) T_d \, x_{\nu(d)}.$$

It is known that $\mathcal{H}_{r,R}$ has the basis $\{T_w \mid w \in \mathfrak{S}_r\}$ satisfying the following:

$$T_w T_{s_i} = \begin{cases} T_{ws_i} & \text{if } \ell(ws_i) = \ell(w) + 1, \\ (q-1)T_w + qT_{ws_i} & \text{if } \ell(ws_i) = \ell(w) - 1. \end{cases} \tag{3.1}$$

Recall that $\leq$ is the Bruhat order of $\mathfrak{S}_r$. Then by (3.1) one can deduce that, for any $y, w \in \mathfrak{S}_r$, there is an element $y * w \in \mathfrak{S}_r$ such that $\ell(y * w) \leq \ell(y) + \ell(w)$ and

$$T_y T_w = \sum_{z \leq y*w} f_z^{y,w} T_z \tag{3.2}$$

for some $f_z^{y,w} \in R$ with $f_{y*w}^{y,w} \neq 0$ (see, for example, [6, Proposition 4.30]).

LEMMA 3.6. *Suppose that $\lambda, \mu \in \Lambda(n, r)$. Let $u \in \mathfrak{S}_\lambda, d \in \mathcal{D}_{\lambda,\mu}$, and $w \in \mathfrak{S}_\mu$. Then*

$$T_u T_d T_w = \sum_{y \leq ud*w} f_y^{ud,w} T_y \quad \text{for some } f_y^{ud,w} \in R.$$

In particular, $\ell(y) \leq \ell(u) + \ell(d) + \ell(w)$ whenever $f_y^{ud,w} \neq 0$.

PROOF. Clearly, from Lemma 3.3(3) and (3.3), we have $T_u T_d = T_{ud}$. Then the lemma follows from (3.2). □

LEMMA 3.7. *For any given* $\lambda, \mu \in \Lambda(n,r)$, *the set*

$$\mathscr{B}' = \{T_u T_d c^\alpha T_\sigma \mid u \in \mathfrak{S}_\lambda, d \in \mathcal{D}_{\lambda,\mu}, \sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu, \alpha \in \mathbb{Z}_2^r\}$$

forms an R-basis for $\mathcal{H}_{r,R}^{\mathsf{c}}$.

PROOF. Let M be the R-submodule of $\mathcal{H}_{r,R}^{\mathsf{c}}$ spanned by the set $\mathscr{B}'$. Take an arbitrary $w \in \mathfrak{S}_r$ and $\alpha \in \mathbb{Z}_2^r$. We now prove that every $T_w c^\alpha \in M$ by induction on $\ell(w)$. Clearly, if $\ell(w) = 0$, then $T_w = 1$ and $c^\alpha \in M$. Assume now that $\ell(w) \geq 1$. By Lemma 3.3(1) and (4), there exist unique $d \in \mathcal{D}_{\lambda,\mu}$ and $u \in \mathfrak{S}_\lambda, \sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$ such that $w = ud\sigma$ and $\ell(w) = \ell(u) + \ell(d) + \ell(\sigma)$. Then by (3.1) we obtain $T_w = T_u T_d T_\sigma$. Hence, by (2.4) and Lemma 3.6,

$$T_w c^\alpha = T_u T_d (T_\sigma c^\alpha) = \pm T_u T_d \Big(c^{\alpha \cdot \sigma^{-1}} T_\sigma - \sum_{\pi < \sigma, \beta \in \mathbb{Z}_2^r} a_{\beta,\pi}^{\alpha \cdot \sigma^{-1}, \sigma} T_\pi c^\beta \Big)$$

$$= \pm T_u T_d c^{\alpha \cdot \sigma^{-1}} T_\sigma \mp \sum_{\pi < \sigma, \beta \in \mathbb{Z}_2^r} a_{\beta,\pi}^{\alpha \cdot \sigma^{-1}, \sigma} T_u T_d T_\pi c^\beta$$

$$= \pm T_u T_d c^{\alpha \cdot \sigma^{-1}} T_\sigma \mp \sum_{\pi < \sigma, \beta \in \mathbb{Z}_2^r} a_{\beta,\pi}^{\alpha \cdot \sigma^{-1}, \sigma} \sum_{u' \leq ud*\pi, \beta \in \mathbb{Z}_2^r} f_{u',\beta}^{ud,\pi} T_{u'} c^\beta, \qquad (3.3)$$

where u' satisfies $\ell(u') \leq \ell(u) + \ell(d) + \ell(\pi)$. Since $\pi < \sigma$, we have $\ell(u') \leq \ell(u) + \ell(d) + \ell(\pi) < \ell(u) + \ell(d) + \ell(\sigma) = \ell(w)$. Then, by induction on $\ell(w)$, each term $T_{u'} c^\beta$ on the right-hand side of (3.3) belongs to M and so does $T_w c^\alpha$ by (3.3). Hence, $M = \mathcal{H}_{r,R}^{\mathsf{c}}$ or, equivalently, $\mathcal{H}_{r,R}^{\mathsf{c}}$ is spanned by the set $\mathscr{B}'$ by Lemma 2.2. Finally, $|\mathscr{B}| = |\mathscr{B}'|$ forces that the set is linearly independent. $\square$

For $\lambda, \mu \in \Lambda(n,r)$, let $X_{\lambda,\mu} = (\bigcup_{d \in \mathcal{D}_{\lambda,\mu}} (\{d\} \times (\mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu))) \times \mathbb{Z}_2^r$.

COROLLARY 3.8. *For given* $\lambda, \mu \in \Lambda(n,r)$, *we may decompose* $\mathcal{H}_{r,R}^{\mathsf{c}}$ *into a direct sum of left* $\mathcal{H}_{\lambda,R}$-*modules or right* $\mathcal{H}_{\mu,R}^{\mathsf{c}}$-*modules:*

$$\mathcal{H}_{r,R}^{\mathsf{c}} = \bigoplus_{(d,\sigma,\alpha) \in X_{\lambda,\mu}} \mathcal{H}_{\lambda,R} T_d c^\alpha T_\sigma = \bigoplus_{d \in \mathcal{D}_{\lambda,\mu}} \bigoplus_{u \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_u T_d \mathcal{H}_{\mu,R}^{\mathsf{c}}.$$

In particular, the set

$$\{x_\lambda T_d c^\alpha T_\sigma \mid (d,\sigma,\alpha) \in X_{\lambda,\mu}\}$$
$$(\text{respectively, } \{T_u T_d c^\alpha x_\mu \mid d \in \mathcal{D}_{\lambda,\mu}, u \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda, \alpha \in \mathbb{Z}_2^r\})$$

forms a basis for $x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}}$ *(respectively,* $\mathcal{H}_{r,R}^{\mathsf{c}} x_\mu$*).*

PROOF. Clearly, $\mathcal{H}_{\lambda,R} T_d c^\alpha T_\sigma$ is spanned by $\{T_u T_d c^\alpha T_\sigma \mid u \in \mathfrak{S}_\lambda\}$. Since

$$\{T_u T_d c^\alpha T_\sigma \mid u \in \mathfrak{S}_\lambda, (d,\sigma,\alpha) \in X_{\lambda,\mu}\} = \dot{\bigcup_{(d,\sigma,\alpha) \in X_{\lambda,\mu}}} \{T_u T_d c^\alpha T_\sigma \mid u \in \mathfrak{S}_\lambda\}$$

is a disjoint union, the assertion follows from Lemma 3.7 and Corollary 3.4. $\square$

4. The elements $c_{q,i,j}$

For $r \geq 1$ and $1 \leq i < j \leq r$, we set

$$c_{q,i,j} = q^{j-i}c_i + q^{j-i-1}c_{i+1} + \cdots + qc_{j-1} + c_j, \quad c'_{q,i,j} = c_i + qc_{i+1} + \cdots + q^{j-i}c_j. \quad (4.1)$$

For simplicity, we write $c_{q,r} = c_{q,1,r}$ and $c'_{q,r} = c'_{q,1,r}$.

LEMMA 4.1. *Let $r \geq 1$. The following holds in $\mathcal{H}^c_{r,R}$:*

$$x_{(r)}c_{q,r} = c'_{q,r}x_{(r)} \quad and \quad y_{(r)}c'_{q,r} = c_{q,r}y_{(r)}.$$

PROOF. We only prove the first formula. The proof of the second one is similar. We apply induction on r. It is known that

$$x_{(r)} = x_{(r-1)}(1 + T_{r-1} + T_{r-1}T_{r-2} + \cdots + T_{r-1} \cdots T_1) \quad (4.2)$$

since $\{1, s_{r-1}, s_{r-1}s_{r-2}, \ldots, s_{r-1} \cdots s_2 s_1\} = \mathcal{D}_{(r-1,1)}$. Using (2.2), a direct calculation shows for $1 \leq k \leq r-1$ and $1 \leq l \leq r$,

$$T_{r-1} \cdots T_k c_l = \begin{cases} c_l T_{r-1} \cdots T_k & \text{if } 1 \leq l \leq k-1, \\ c_r T_{r-1} \cdots T_k & \text{if } l = k, \\ c_{l-1}T_{r-1} \cdots T_k - (q-1)T_{l-2} \cdots T_k(c_k - c_r)T_{r-1} \cdots T_l & \text{if } k+1 \leq l \leq r, \end{cases}$$

where we use the convention $T_{r-1} \cdots T_l = 1$ if $l = r$ and the formula in the last case is due to the following computation:

$$\begin{aligned}
T_{r-1} \cdots T_k c_l &= T_{r-1} \cdots T_l \cdot (c_{l-1}T_{l-1} - (q-1)(c_{l-1} - c_l)) \cdot T_{l-2} \cdots T_k \\
&= c_{l-1}T_{r-1} \cdots T_l T_{l-1} T_{l-2} \cdots T_k - (q-1)(T_{r-1} \cdots T_l) \\
&\quad \cdot (c_{l-1} - c_l) \cdot (T_{l-2} \cdots T_k) \\
&= c_{l-1}T_{r-1} \cdots T_k - (q-1)(c_{l-1} - c_r) \cdot (T_{r-1} \cdots T_l) \cdot (T_{l-2} \cdots T_k) \\
&= c_{l-1}T_{r-1} \cdots T_k - (q-1)(c_{l-1} - c_r) \cdot (T_{l-2} \cdots T_k) \cdot (T_{r-1} \cdots T_l) \\
&= c_{l-1}T_{r-1} \cdots T_k - (q-1)(T_{l-2} \cdots T_k) \cdot (c_k - c_r) \cdot (T_{r-1} \cdots T_l).
\end{aligned}$$

This implies that the following holds for $1 \leq k \leq r-1$ and $1 \leq l \leq r$:

$$\begin{aligned}
&x_{(r-1)}T_{r-1} \cdots T_k c_l \\
&= \begin{cases} x_{(r-1)}c_l T_{r-1} \cdots T_k & \text{if } 1 \leq l \leq k-1, \\ x_{(r-1)}c_r T_{r-1} \cdots T_k & \text{if } l = k, \\ x_{(r-1)}c_{l-1}T_{r-1} \cdots T_k - q^{l-k-1}(q-1)x_{(r-1)}(c_k - c_r)T_{r-1} \cdots T_l & \text{if } k+1 \leq l \leq r, \end{cases}
\end{aligned}$$

where the formula in the last case is due to the fact that $x_{(r-1)}T_{l-2} \cdots T_k = q^{l-k-1}x_{(r-1)}$.

Hence,

$$
\begin{aligned}
x_{(r-1)}T_{r-1}\cdots T_k c_{q,r} &= x_{(r-1)}T_{r-1}\cdots T_k (q^{r-k+1}c_{q,1,k-1} + q^{r-k}c_k + c_{q,k+1,r}) \\
&= x_{(r-1)}(q^{r-k+1}c_{q,1,k-1} + q^{r-k}c_r + c_{q,k,r-1})T_{r-1}\cdots T_k \\
&\quad - \sum_{l=k+1}^{r} q^{r-l}q^{l-k-1}(q-1)x_{(r-1)}(c_k - c_r)T_{r-1}\cdots T_l \\
&\qquad (\text{noting that } T_{r-1}\cdots T_l = 1 \text{ if } l = r) \\
&= x_{(r-1)}(q^{r-k+1}c_{q,1,k-1} + q^{r-k}c_r + c_{q,k,r-1})T_{r-1}\cdots T_k \\
&\quad - \sum_{l=k+1}^{r-1} (q-1)q^{r-k-1}x_{(r-1)}(c_k - c_r)T_{r-1}\cdots T_l - (q-1)q^{r-k-1}x_{(r-1)}(c_k - c_r).
\end{aligned}
$$

This together with (4.2) gives rise to

$$
\begin{aligned}
x_{(r)}c_{q,r} &= \Big(x_{(r-1)} + \sum_{k=1}^{r-1} x_{(r-1)}T_{r-1}\cdots T_k\Big)c_{q,r} \\
&= x_{(r-1)}c_{q,r} + \sum_{k=1}^{r-1} x_{(r-1)}(q^{r-k+1}c_{q,1,k-1} + q^{r-k}c_r + c_{q,k,r-1})T_{r-1}\cdots T_k \\
&\quad - \sum_{k=1}^{r-1}\sum_{l=k+1}^{r-1} (q-1)q^{r-k-1}x_{(r-1)}(c_k - c_r)T_{r-1}\cdots T_l \\
&\quad - \sum_{k=1}^{r-1} (q-1)q^{r-k-1}x_{(r-1)}(c_k - c_r). \tag{4.3}
\end{aligned}
$$

Combining the first term and the last sum in (4.3) yields

$$
\begin{aligned}
&x_{(r-1)}c_{q,r} - \sum_{k=1}^{r-1} x_{(r-1)}(q-1)q^{r-k-1}(c_k - c_r) \\
&= x_{(r-1)}\Big(\sum_{k=1}^{r} q^{r-k}c_k - \sum_{k=1}^{r-1}(q^{r-k} - q^{r-k-1})c_k + \sum_{k=1}^{r-1}(q^{r-k} - q^{r-k-1})c_r\Big) \\
&= x_{(r-1)}(q^{r-2}c_1 + q^{r-3}c_2 + \cdots + c_{r-1} + q^{r-1}c_r) = x_{(r-1)}(c_{q,1,r-1} + q^{r-1}c_r).
\end{aligned}
$$

Also, the second (double) sum in (4.3) can be rewritten as

$$
\begin{aligned}
&x_{(r-1)}\sum_{l=2}^{r-1}\Big(\sum_{k=1}^{l-1}(q-1)q^{r-k-1}(c_k - c_r)\Big)T_{r-1}\cdots T_l \\
&= x_{(r-1)}\sum_{l=2}^{r-1}((q^{r-l+1} - q^{r-l})c_{q,1,l-1} - (q^{r-1} - q^{r-l})c_r)T_{r-1}\cdots T_l.
\end{aligned}
$$

Hence, substituting into (4.3) yields

$$
x_{(r)}c_{q,r} = x_{(r-1)}(c_{q,1,r-1} + q^{r-1}c_r)\Big(1 + \sum_{k=1}^{r-1} T_{r-1}\cdots T_k\Big). \tag{4.4}
$$

Here the following calculation has been used for $2 \leq k \leq r - 1$:

$$(q^{r-k+1}c_{q,1,k-1} + q^{r-k}c_r + c_{q,k,r-1}) - ((q^{r-k+1} - q^{r-k})c_{q,1,k-1} - (q^{r-1} - q^{r-k})c_r)$$
$$= q^{r-k}c_{q,1,k-1} + c_{q,k,r-1} + q^{r-1}c^r = c_{q,1,r-1} + q^{r-1}c_r = c_{q,r-1} + q^{r-1}c_r,$$

which is also the first ($k = 1$) term of the first sum in (4.3). By induction on r,

$$x_{(r-1)}c_{q,1,r-1} = c'_{q,r-1}x_{(r-1)}$$

and, hence, one can deduce that

$$x_{(r-1)}(c_{q,1,r-1} + q^{r-1}c_r) = (c'_{q,r-1} + q^{r-1}c_r)x_{(r-1)} = c'_{q,r}x_{(r-1)}.$$

Thus, by (4.4) and (4.2),

$$x_{(r)}c_{q,r} = c'_{q,r}x_{(r-1)}(1 + T_{r-1} + \cdots + T_{r-1} \cdots T_1) = c'_{q,r}x_{(r)}.$$

This proves the lemma.   □

PROPOSITION 4.2. *For $r \geq 1$, the intersection $x_{(r)}\mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R}x_{(r)}$ is a free R-module with basis $\{x_{(r)}, x_{(r)}c_{q,r}\}$. A similar result holds with $x_{(r)}$ replaced by $y_{(r)}$ and $c_{q,r}$ by $c'_{q,r}$.*

PROOF. By Lemma 4.1, the set $\{x_{(r)}, x_{(r)}c_{q,r}\}$ is contained in the intersection $x_{(r)}\mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R}x_{(r)}$ and clearly it is linearly independent. It remains to show that $x_{(r)}\mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R}x_{(r)}$ is spanned by $\{x_{(r)}, x_{(r)}c_{q,r}\}$. Take an arbitrary $z \in x_{(r)}\mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R}x_{(r)}$. Since $z \in x_{(r)}\mathcal{H}^c_{r,R}$, by Lemma 3.1(1), we can write z as $z = \sum_{\alpha \in \mathbb{Z}_2^r} f_\alpha x_{(r)}c^\alpha$, with $f_\alpha \in R$. Moreover, since $z \in \mathcal{H}^c_{r,R}x_{(r)}$, by Lemma 3.1(2), $zT_{s_k} = qz, \quad \forall 1 \leq k \leq r - 1$.

For $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $1 \leq k \leq r - 1$, by (2.2) and (2.3),

$$c_j^{\alpha_j}T_{s_k} = T_{s_k}c_j^{\alpha_j} \quad \text{for } j \neq k, k+1,$$
$$c_{k+1}^{\alpha_{k+1}}T_{s_k} = T_{s_k}c_k^{\alpha_{k+1}}, \quad c_k^{\alpha_k}T_{s_k} = T_{s_k}c_{k+1}^{\alpha_k} + (q-1)(c_k^{\alpha_k} - c_{k+1}^{\alpha_k}),$$
$$c_k^{\alpha_k}c_{k+1}^{\alpha_{k+1}}T_{s_k} = T_{s_k}c_k^{\alpha_k}c_{k+1}^{\alpha_{k+1}} + (q-1)(c_k^{\alpha_k + \alpha_{k+1}} - c_{k+1}^{\alpha_k}c_k^{\alpha_{k+1}}).$$

Then, by the fact that $x_{(r)}T_{s_k} = qx_{(r)}$ for $1 \leq k \leq r - 1$,

$$x_{(r)}c^\alpha T_{s_k} = x_{(r)}T_{s_k}c_1^{\alpha_1} \cdots c_{k-1}^{\alpha_{k-1}}c_{k+1}^{\alpha_k}c_k^{\alpha_{k+1}}c_{k+2}^{\alpha_{k+2}} \cdots c_r^{\alpha_r}$$
$$+ (q-1)x_{(r)}c_1^{\alpha_1} \cdots c_{k-1}^{\alpha_{k-1}}(c_k^{\alpha_k + \alpha_{k+1}} - c_{k+1}^{\alpha_k}c_k^{\alpha_{k+1}})c_{k+2}^{\alpha_{k+2}} \cdots c_r^{\alpha_r}$$
$$= x_{(r)}c_1^{\alpha_1} \cdots c_{k-1}^{\alpha_{k-1}}c_k^{\alpha_k}c_{k+1}^{\alpha_{k+1}}c_{k+2}^{\alpha_{k+2}} \cdots c_r^{\alpha_r}$$
$$+ (q-1)x_{(r)}c_1^{\alpha_1} \cdots c_{k-1}^{\alpha_{k-1}}c_k^{\alpha_k + \alpha_{k+1}}c_{k+2}^{\alpha_{k+2}} \cdots c_r^{\alpha_r}.$$

This together with $zT_{s_k} = qz$ implies that

$$q \sum_{\alpha \in \mathbb{Z}_2^r} f_\alpha x_{(r)}c^\alpha = zT_{s_k} = \sum_{\alpha \in \mathbb{Z}_2^r} f_\alpha x_{(r)}c_1^{\alpha_1} \cdots c_{k-1}^{\alpha_{k-1}}c_k^{\alpha_k}c_{k+1}^{\alpha_{k+1}}c_{k+2}^{\alpha_{k+2}} \cdots c_r^{\alpha_r}$$
$$+ \sum_{\alpha \in \mathbb{Z}_2^r} (q-1)f_\alpha x_{(r)}c_1^{\alpha_1} \cdots c_{k-1}^{\alpha_{k-1}}c_k^{\alpha_k + \alpha_{k+1}}c_{k+2}^{\alpha_{k+2}} \cdots c_r^{\alpha_r}$$

for all $1 \leq k \leq r - 1$. Then by using Lemma 3.1(1) and by equating the coefficients of $x_{(r)}c^\alpha$ on both sides for each $\alpha \in \mathbb{Z}_2^r$,

$$
\begin{aligned}
qf_{(\alpha_1,\ldots,\alpha_{k-1},1,1,\alpha_{k+2},\ldots,\alpha_r)} &= -f_{(\alpha_1,\ldots,\alpha_{k-1},1,1,\alpha_{k+2},\ldots,\alpha_r)}, \\
qf_{(\alpha_1,\ldots,\alpha_{k-1},0,1,\alpha_{k+2},\ldots,\alpha_r)} &= f_{(\alpha_1,\ldots,\alpha_{k-1},1,0,\alpha_{k+2},\ldots,\alpha_r)}, \\
qf_{(\alpha_1,\ldots,\alpha_{k-1},0,0,\alpha_{k+2},\alpha_r)} &= f_{(\alpha_1,\ldots,\alpha_{k-1},0,0,\alpha_{k+2},\ldots,\alpha_r)} \\
&\quad + (q-1)(f_{(\alpha_1,\ldots,\alpha_{k-1},1,1,\alpha_{k+2},\ldots,\alpha_r)} + f_{(\alpha_1,\ldots,\alpha_{k-1},0,0,\alpha_{k+2},\ldots,\alpha_r)}), \\
qf_{(\alpha_1,\ldots,\alpha_{k-1},1,0,\alpha_{k+2},\ldots,\alpha_r)} &= f_{(\alpha_1,\ldots,\alpha_{k-1},0,1,\alpha_{k+2},\ldots,\alpha_r)} \\
&\quad + (q-1)(f_{(\alpha_1,\ldots,\alpha_{k-1},1,0,\alpha_{k+2},\ldots,\alpha_r)} + f_{(\alpha_1,\ldots,\alpha_{k-1},0,1,\alpha_{k+2},\ldots,\alpha_r)}),
\end{aligned}
\tag{4.5}
$$

where the relation (2.1) is used in the first and last equalities. Hence, since the characteristic of R is not equal to 2, (4.5) implies that

$$
f_{(\alpha_1,\ldots,\alpha_{k-1},1,0,\alpha_{k+1},\ldots,\alpha_r)} = qf_{(\alpha_1,\ldots,\alpha_{k-1},0,1,\alpha_{k+1},\ldots,\alpha_r)}, \quad f_{(\alpha_1,\ldots,\alpha_{k-1},1,1,\alpha_{k+1},\ldots,\alpha_r)} = 0 \tag{4.6}
$$

for all $\alpha_i \in \mathbb{Z}_2$ and $1 \leq k \leq r - 1$. Then one can deduce that

$$
\begin{cases}
f_{\epsilon_k} = q^{r-k} f_{\epsilon_r} & \text{for } 1 \leq k \leq r, \\
f_\alpha = 0 & \text{for } \alpha \in \mathbb{Z}_2^r \backslash \mathbf{0} \text{ and } \alpha \neq \epsilon_k, 1 \leq k \leq r,
\end{cases}
$$

where $\epsilon_k = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the kth position. This means that

$$
z = \sum_{\alpha \in \mathbb{Z}_2^r} f_\alpha x_{(r)} c^\alpha = f_{(0,\ldots,0)} x_{(r)} + f_{\epsilon_r} x_{(r)} (q^{r-1} c_1 + q^{r-2} c_2 + \cdots + c_r)
$$

$$
= f_{(0,\ldots,0)} x_{(r)} + f_{\epsilon_r} x_{(r)} c_{q,r}.
$$

Hence, z lies in the R-module spanned by $\{x_{(r)}, x_{(r)} c_{q,r}\}$ and the proposition is verified. □

Given $\lambda = (\lambda_1,\ldots,\lambda_N) \in \Lambda(N,r)$ and $\alpha \in \mathbb{Z}_2^N$ with $N \geq 1$ and $\alpha_k \leq \lambda_k$ for $1 \leq k \leq N$, we let $\widetilde{\lambda}_k = \lambda_1 + \cdots + \lambda_k$ for $1 \leq k \leq N$ and introduce the following elements in C_r:

$$
\begin{aligned}
c_\lambda^\alpha &= (c_{q,1,\widetilde{\lambda}_1})^{\alpha_1} (c_{q,\widetilde{\lambda}_1+1,\widetilde{\lambda}_2})^{\alpha_2} \cdots (c_{q,\widetilde{\lambda}_{N-1}+1,\widetilde{\lambda}_N})^{\alpha_N}, \\
(c_\lambda^\alpha)' &= (c'_{q,1,\widetilde{\lambda}_1})^{\alpha_1} (c'_{q,\widetilde{\lambda}_1+1,\widetilde{\lambda}_2})^{\alpha_2} \cdots (c'_{q,\widetilde{\lambda}_{N-1}+1,\widetilde{\lambda}_N})^{\alpha_N}.
\end{aligned}
\tag{4.7}
$$

For notational simplicity, we define for $\lambda, \mu \in \mathbb{N}^N$

$$
\lambda \leq \mu \iff \lambda_i \leq \mu_i \text{ for all } i = 1, 2, \ldots, N.
$$

COROLLARY 4.3. *Suppose that $\lambda = (\lambda_1,\ldots,\lambda_N) \in \Lambda(N,r)$. Then $x_\lambda \mathcal{H}_{\lambda,R}^c \cap \mathcal{H}_{\lambda,R}^c x_\lambda$ is a free R-module with basis $\{x_\lambda c_\lambda^\alpha = (c_\lambda^\alpha)' x_\lambda \mid \alpha \in \mathbb{Z}_2^N, \alpha \leq \lambda\}$. A similar result holds for the y_λ version.*

PROOF. Clearly, by Lemma 4.1 and (2.1), we have $x_\lambda c_\lambda^\alpha = (c_\lambda^\alpha)' x_\lambda$ for $\alpha \in \mathbb{Z}_2^N$ and the set is linearly independent. We now modify the proof of Proposition 4.2 to prove the assertion. Without loss of generality, we may assume that all $\lambda_i \neq 0$.

For $z = \sum_{\alpha \in \mathbb{Z}_2^r} f_\alpha x_\lambda c^\alpha \in x_\lambda \mathcal{H}_{\lambda,R}^c \cap \mathcal{H}_{\lambda,R}^c x_\lambda$, (4.6) continues to hold for all k such that $\widetilde{\lambda}_{i-1} + 1 \le k < \widetilde{\lambda}_i$ for all i (assuming that $\lambda_0 = 0$). Then we first observe that if there exist $\widetilde{\lambda}_{i-1} + 1 \le k < l \le \widetilde{\lambda}_i$ for some $1 \le i \le N$ such that $\alpha_k = \alpha_l = 1$, then $f_\alpha = 0$. Now it remains to consider those f_α for $\alpha = \mathbf{0}$ or for $\alpha \in \mathbb{Z}_2^r$ having the form $\alpha = \epsilon_{k_1} + \epsilon_{k_2} + \cdots + \epsilon_{k_t}$ for some $1 \le t \le N$, where there exist $1 \le i_1 < i_2 < \cdots < i_t \le N$ such that $\widetilde{\lambda}_{i_1 - 1} + 1 \le k_1 \le \widetilde{\lambda}_{i_1}, \widetilde{\lambda}_{i_2 - 1} + 1 \le k_2 \le \widetilde{\lambda}_{i_2}, \ldots, \widetilde{\lambda}_{i_t - 1} + 1 \le k_t \le \widetilde{\lambda}_{i_t}$. In the latter case, by (4.6), one can deduce that

$$f_\alpha = q^{\widetilde{\lambda}_{i_1} - k_1} q^{\widetilde{\lambda}_{i_2} - k_2} \cdots q^{\widetilde{\lambda}_{i_t} - k_t} f_{\epsilon_{\widetilde{\lambda}_{i_1}} + \epsilon_{\widetilde{\lambda}_{i_2}} + \cdots + \epsilon_{\widetilde{\lambda}_{i_t}}}.$$

Hence,

$$z = f_{\mathbf{0}} x_\lambda + \sum_{t=1}^{N} \sum_{1 \le i_1 < \cdots < i_t \le N} f_{\epsilon_{\widetilde{\lambda}_{i_1}} + \epsilon_{\widetilde{\lambda}_{i_2}} + \cdots + \epsilon_{\widetilde{\lambda}_{i_t}}} x_\lambda c_{q, \widetilde{\lambda}_{i_1 - 1} + 1, \widetilde{\lambda}_{i_1}} c_{q, \widetilde{\lambda}_{i_2 - 1} + 1, \widetilde{\lambda}_{i_2}} \cdots c_{q, \widetilde{\lambda}_{i_t - 1} + 1, \widetilde{\lambda}_{i_t}},$$

proving the spanning condition. $\square$

REMARK. (1) By (2.5), $\mathcal{H}_{\lambda,R}^c \cong \mathcal{H}_{\lambda_1,R}^c \otimes \cdots \otimes \mathcal{H}_{\lambda_N,R}^c$, where the isomorphism maps x_λ to $x_{(\lambda_1)} \otimes \cdots \otimes x_{(\lambda_N)}$. Hence, as free R-modules,

$$x_\lambda \mathcal{H}_{\lambda,R}^c \cong x_{(\lambda_1)} \mathcal{H}_{\lambda_1,R}^c \otimes \cdots \otimes x_{(\lambda_N)} \mathcal{H}_{\lambda_N,R}^c, \quad \mathcal{H}_{\lambda,R}^c x_\lambda \cong \mathcal{H}_{\lambda_1,R}^c x_{(\lambda_1)} \otimes \cdots \otimes \mathcal{H}_{\lambda_N,R}^c x_{(\lambda_N)}.$$

The corollary above implies the following isomorphism of free R-modules:

$$x_\lambda \mathcal{H}_{\lambda,R}^c \cap \mathcal{H}_{\lambda,R}^c x_\lambda \cong (x_{(\lambda_1)} \mathcal{H}_{\lambda_1,R}^c \cap \mathcal{H}_{\lambda_1,R}^c x_{(\lambda_1)}) \otimes \cdots \otimes (x_{(\lambda_N)} \mathcal{H}_{\lambda_N,R}^c \cap \mathcal{H}_{\lambda_N,R}^c x_{(\lambda_N)}).$$

(2) The elements $c_{q,i,j}$ have also been introduced in [24, Lemma 6.7]. They play an important role in (5.4) below for the construction of an integral basis. However, such a role was not mentioned in [24]. Instead, they play a key role in the classification of simple modules via the superalgebra Γ_λ (which is actually isomorphic to the endomorphism superalgebra $\mathrm{End}_{\mathcal{H}_{r,R}^c}(x_\lambda \mathcal{H}_{r,R}^c) \cong x_\lambda \mathcal{H}_{r,R}^c \cap \mathcal{H}_{r,R}^c x_\lambda$ by Corollary 4.3) defined in [24, Definition 6.8] (see [24, Theorems 6.14 and 6.32]).

5. Integral bases and the base-change property

We are now ready to prove, by constructing a basis, that the module $x_\lambda \mathcal{H}_{r,R}^c \cap \mathcal{H}_{r,R}^c x_\mu$ occurring in Lemma 3.2 is R-free. This will provide an integral R-basis for $\mathcal{Q}_q(n, r; R)$.

Let $M_n(\mathbb{N})$ be the set of $n \times n$ matrices $M = (m_{ij})$ and let

$$M_n(\mathbb{N})_r = \left\{ M \in M_n(\mathbb{N}) \,\middle|\, \sum m_{ij} = r \right\}.$$

Given $M \in M_n(\mathbb{N})$, define

$$\mathrm{ro}(M) = \left(\sum_j m_{1j}, \ldots, \sum_j m_{nj} \right), \quad \mathrm{co}(M) = \left(\sum_j m_{j1}, \ldots, \sum_j m_{jn} \right).$$

Then $\mathrm{ro}(M), \mathrm{co}(M) \in \Lambda(n, r)$ for any $M \in M_n(\mathbb{N})_r$. Given a composition $\lambda \in \Lambda(n, r)$ and $1 \le k \le n$, define the subsets $R_k^\lambda \subseteq \{1, \ldots, r\}$ as follows:

$$R_k^\lambda = \{\tilde{\lambda}_{k-1} + 1, \tilde{\lambda}_{k-1} + 2, \ldots, \tilde{\lambda}_{k-1} + \lambda_k\} \quad (\tilde{\lambda}_0 = 0, \tilde{\lambda}_i = \lambda_1 + \cdots + \lambda_i, 1 \le i \le n).$$

It is well known that double cosets of the symmetric group can be described in terms of matrices as follows. There is a bijection

$$\begin{aligned} \jmath : M_n(\mathbb{N})_r &\longrightarrow \mathfrak{J}_n(r) = \{(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(n, r), d \in \mathcal{D}_{\lambda,\mu}\} \\ M &\longmapsto (\mathrm{ro}(M), d_M, \mathrm{co}(M)), \end{aligned} \tag{5.1}$$

where $d_M \in \mathcal{D}_{\lambda,\mu}$ is the permutation satisfying $|R_i^{\mathrm{ro}(M)} \cap d_M(R_j^{\mathrm{co}(M)})| = m_{ij}$ for $1 \le i, j \le n$ with $M = (m_{ij})$. Moreover, the composition $\nu(d_M)$ (respectively $\nu(d_M^{-1})$), defined in Lemma 3.3(3), is obtained by reading the entries in M along columns (respectively rows), that is,

$$\begin{aligned} \nu(d_M) &= (m_{11}, \ldots, m_{n1}, m_{12}, \ldots, m_{n2}, \ldots, m_{1n}, \ldots, m_{nn}), \\ \nu(d_M^{-1}) &= (m_{11}, \ldots, m_{1n}, m_{21}, \ldots, m_{2n}, \ldots, m_{n1}, \ldots, m_{nn}). \end{aligned} \tag{5.2}$$

For the subset $\mathbb{Z}_2 = \{0, 1\}$ of $\mathbb{N}$, let $M_n(\mathbb{Z}_2)$ be the set of $n \times n$ matrices $B = (b_{ij})$ with $b_{ij} \in \mathbb{Z}_2$. Set

$$M_n(\mathbb{N}|\mathbb{Z}_2) := \{(A|B) \mid A \in M_n(\mathbb{N}), B \in M_n(\mathbb{Z}_2)\},$$
$$M_n(\mathbb{N}|\mathbb{Z}_2)_r := \{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2) \mid A + B \in M_n(\mathbb{N})_r\}.$$

Given $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_r$ with $A = (a_{ij}), B = (b_{ij})$, let $m_{ij} = a_{ij} + b_{ij}$ for $1 \le i, j \le n$ and then we have $A + B = (m_{ij}) \in M_n(\mathbb{N})_r$. Hence, by (5.1), we have the permutation $d_{A+B} \in \mathcal{D}_{\lambda,\mu}$ with $\lambda = \mathrm{ro}(A + B), \mu = \mathrm{co}(A + B)$ and, moreover,

$$\nu_{A|B} := \nu(d_{A+B}) = (m_{11}, \ldots, m_{n1}, m_{12}, \ldots, m_{n2}, \ldots, m_{1n}, \ldots, m_{nn}).$$

Let $\alpha_B = (b_{11}, \ldots, b_{n1}, \ldots, b_{1n}, \ldots, b_{nn}) \in \mathbb{Z}_2^{n^2}$. Then by (4.7) we are ready to introduce the following elements:

$$c_{A|B} = c_{\nu_{A|B}}^{\alpha_B} \in C_r, \tag{5.3}$$

$$T_{A|B} = x_\lambda T_{d_{A+B}} c_{A|B} \sum_{\sigma \in \mathcal{D}_{\nu_{A|B}} \cap \mathfrak{S}_\mu} T_\sigma. \tag{5.4}$$

Given $\lambda, \mu \in \Lambda(n, r)$ and $d \in \mathcal{D}_{\lambda,\mu}$, let

$$M_n(\mathbb{N})_{\lambda,\mu} := \{A \in M_n(\mathbb{N}) \mid \mathrm{ro}(A) = \lambda, \mathrm{co}(A) = \mu\},$$
$$M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu} := \{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2) \mid \mathrm{ro}(A + B) = \lambda, \mathrm{co}(A + B) = \mu\},$$
$$M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d := \{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu} \mid d_{A+B} = d\}.$$

Note that, by sending A to $(A|0)$, we may regard $M_n(\mathbb{N})_{\lambda,\mu}$ as a subset of $M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}$. Note also that

$$M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu} = \bigcup_{d \in \mathcal{D}_{\lambda,\mu}} M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d. \tag{5.5}$$

Lemma 5.1. *For $\lambda, \mu \in \Lambda(n,r)$, the set $\{T_{A|B} \mid (A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}\}$ is a linearly independent subset of $x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \cap \mathcal{H}_{r,R}^{\mathsf{c}} x_\mu$.*

Proof. We first prove that $T_{A|B} \in x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}} \cap \mathcal{H}_{r,R}^{\mathsf{c}} x_\mu$ for each $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}$. Clearly, $T_{A|B} \in x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}}$ by (5.4). Let $d = d_{A+B}$. Then, by (5.2),

$$\nu(d) = (m_{11}, \ldots, m_{n1}, m_{12}, \ldots, m_{n2}, \ldots, m_{1n}, \ldots, m_{nn}),$$

where $A + B = (m_{ij})$. Hence, by Corollary 4.3 and (5.3), $x_{\nu(d)} c_{A|B} = c'_{A|B} x_{\nu(d)}$, where $c'_{A|B} = (c^{\alpha_B}_{\nu_{A|B}})'$. On the other hand, the partition

$$\mathfrak{S}_\mu = \bigsqcup_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} \mathfrak{S}_{\nu(d)} \sigma \tag{5.6}$$

implies that $x_\mu = x_{\nu(d)} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma$. Hence, by (5.4) and Corollary 3.5, one can deduce that

$$T_{A|B} = \sum_{u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_{u'} T_d x_{\nu(d)} c_{A|B} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma$$

$$= \sum_{u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_{u'} T_d c'_{A|B} x_{\nu(d)} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma = \sum_{u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_{u'} T_d c'_{(A|B)} x_\mu,$$

proving that $T_{A|B} \in \mathcal{H}_{r,R}^{\mathsf{c}} x_\mu$.

We now prove that the set is linearly independent. Suppose that

$$\sum_{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}} f_{A|B} T_{A|B} = 0.$$

By the new basis $\mathscr{B}'$ on Lemma 3.7 and (5.5), we have, for every $d \in \mathcal{D}_{\lambda,\mu}$,

$$\sum_{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d} f_{A|B} T_{A|B} = 0. \tag{5.7}$$

Let w_λ^0 be the longest element in the Young subgroup $\mathfrak{S}_\lambda$. By (5.6) and the fact that $\ell(w\sigma) = \ell(w) + \ell(\sigma)$ for $w \in \mathfrak{S}_{\nu(d)}$ and $\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$, there exists a unique longest element σ_0 in $\mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$ such that $w_\mu^0 = w_{\nu(d)}^0 \sigma_0$. Suppose that $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d$. Then, by (2.6), Lemma 3.6, and (2.4),

$$T_{A|B} = T_{w_\lambda^0} T_d c_{A|B} T_{\sigma_0} + \sum_{\substack{u \in \mathfrak{S}_\lambda, \sigma \in \mathcal{D}_{\nu_{A|B}} \cap \mathfrak{S}_\mu \\ u < w_\lambda^0 \text{ or } \sigma < \sigma_0}} T_u T_d c_{A|B} T_\sigma$$

$$= T_{w_\lambda^0} T_d c_{A|B} T_{\sigma_0} + \mathfrak{l}_{A|B},$$

where $\mathfrak{l}_{A|B}$ is an R-linear combination of the elements $T_{ud} c_{A|B} T_\sigma$ with $u \in \mathfrak{S}_\lambda, \sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$ and $\ell(ud\sigma) < \ell(w_\lambda^0 d\sigma_0) = \ell(w_\lambda^0) + \ell(d) + \ell(\sigma_0)$. This together with (5.7) gives rise to

$$\sum_{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d} f_{A|B} T_{w_\lambda^0 d} c_{A|B} T_{\sigma_0} + \sum_{(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d} f_{A|B} \mathfrak{l}_{A|B} = 0.$$

By Lemma 3.7, one can deduce that

$$\sum_{(A|B)\in M_n(\mathbb{N}|\mathbb{Z}_2)^d_{\lambda,\mu}} f_{A|B} T_{w^0_\lambda d} c_{A|B} T_{\sigma_0} = 0. \tag{5.8}$$

Recall the bijection ȷ in (5.1). Let $M^d = (m^d_{ij}) \in M_n(\mathbb{N})$ be the unique matrix such that $\jmath(M^d) = (\lambda, d, \mu)$. Then $A + B = M^d$ for all $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)^d_{\lambda,\mu}$ and we have a bijection between the sets $M_n(\mathbb{N}|\mathbb{Z}_2)^d_{\lambda,\mu}$ and $\{B \in M_n(\mathbb{Z}_2) \mid \alpha_B \leq \nu(d)\}$, where $\nu(d) = (m^d_{11}, \dots, m^d_{n1}, \dots, m^d_{1n}, \dots, m^d_{nn})$ and α_B as defined in (5.3). Now it is easy to see that the set $\{c_{A|B} = c^{\alpha_B}_{\nu(d)} \mid B \in M_n(\mathbb{Z}_2), \alpha_B \leq \nu(d)\}$ is linearly independent in C_r. Therefore, by (5.8) and Lemma 3.7 again, we have $f_{A|B} = 0$ for all $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)^d_{\lambda,\mu}$. □

PROPOSITION 5.2. *Suppose that* $\lambda, \mu \in \Lambda(n, r)$. *Then the intersection* $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \cap \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu$ *is a free R-module with basis*

$$\{T_{A|B} \mid (A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}\}.$$

PROOF. By Lemma 5.1, it suffices to show that the set spans $x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \cap \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu$. Take an arbitrary $z \in x_\lambda \mathcal{H}^{\mathsf{c}}_{r,R} \cap \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu$. By Corollary 3.8, we may write

$$z = \sum_{(d,\sigma,\alpha)\in X_{\lambda,\mu}} a_{d,\sigma,\alpha} x_\lambda T_d c^\alpha T_\sigma \quad \text{for some } a_{d,\sigma,\alpha} \in R.$$

This together with Corollary 3.5 gives rise to

$$z = \sum_d \Big(\sum_{u' \in \mathcal{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_{u'} \Big) T_d \sum_{\sigma,\alpha} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha T_\sigma.$$

For each $d \in \mathcal{D}_{\lambda,\mu}$, write

$$z_d = \sum_{\sigma\in\mathcal{D}_{\nu(d)}\cap\mathfrak{S}_\mu, \alpha\in\mathbb{Z}^r_2} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha T_\sigma \in \mathcal{H}^{\mathsf{c}}_{\mu,R}. \tag{5.9}$$

Then

$$z = \sum_{d\in\mathcal{D}_{\lambda,\mu}} \sum_{u' \in \mathcal{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_{u'} T_d z_d. \tag{5.10}$$

Since $z \in \mathcal{H}^{\mathsf{c}}_{r,R} x_\mu$, by Lemma 3.1, we have $z T_{s_k} = q z$ for $s_k \in \mathfrak{S}_\mu$ and, hence,

$$\sum_{d\in\mathcal{D}_{\lambda,\mu}} \sum_{u' \in \mathcal{D}^{-1}_{\nu(d^{-1})} \cap \mathfrak{S}_\lambda} T_{u'} T_d (z_d T_{s_k} - q z_d) = 0.$$

Since $z_d, z_d T_{s_k} \in \mathcal{H}^c_{\mu,R}$ by (5.9) for $s_k \in \mathfrak{S}_\mu$, one can deduce that

$$T_{u'} T_d (z_d T_{s_k} - q z_d) \in T_{u'} T_d \mathcal{H}^c_{\mu,R}.$$

By the second direct sum decomposition in Corollary 3.8, we obtain $T_{u'} T_d (z_d T_{s_k} - q z_d) = 0$. Writing $z_d T_{s_k} - q z_d$ as a linear combination of $T_w c^\alpha$ with $w \in \mathfrak{S}_\mu, \alpha \in \mathbb{Z}^r_2$, linear

independence of $T_{u'}T_dT_wc^\alpha$ (Lemma 2.2) implies that $z_dT_{s_k} - qz_d = 0$ for $s_k \in \mathfrak{S}_\mu$ and $d \in \mathcal{D}_{\lambda,\mu}$. By Lemma 3.1 again, we have $z_d \in \mathcal{H}^\mathsf{c}_{r,R}x_\mu \cap \mathcal{H}^\mathsf{c}_{\mu,R} = \mathcal{H}^\mathsf{c}_{\mu,R}x_\mu$. Thus, we can write z_d for each $d \in \mathcal{D}_{\lambda,\mu}$ as

$$z_d = \sum_{\beta \in \mathbb{Z}_2^r} f_\beta^d c^\beta x_\mu = \sum_{\beta \in \mathbb{Z}_2^r} f_\beta^d c^\beta x_{\nu(d)} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma$$

for some $f_\beta^d \in R$ by (5.6). Hence, by (5.9),

$$\sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu, \alpha \in \mathbb{Z}_2^r} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha T_\sigma = z_d = \sum_{\beta \in \mathbb{Z}_2^r} f_\beta^d c^\beta x_{\nu(d)} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma.$$

This means that

$$\sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} \left(\sum_{\alpha \in \mathbb{Z}_2^r} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha - \sum_{\beta \in \mathbb{Z}_2^r} f_\beta^d c^\beta x_{\nu(d)} \right) T_\sigma = 0.$$

But the left-hand side belongs to $\mathcal{H}^\mathsf{c}_{\mu,R} = \bigoplus_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} \mathcal{H}^\mathsf{c}_{\nu(d),R}T_\sigma$, forcing that every summand is 0. Consequently,

$$\sum_{\alpha \in \mathbb{Z}_2^r} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha - \sum_{\beta \in \mathbb{Z}_2^r} f_\beta^d c^\beta x_{\nu(d)} = 0$$

for each $\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$. Therefore,

$$\sum_{\alpha \in \mathbb{Z}_2^r} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha = \sum_{\beta \in \mathbb{Z}_2^r} f_\beta^d c^\beta x_{\nu(d)} \in x_{\nu(d)} \mathcal{H}^c_{\nu(d),R} \cap \mathcal{H}^c_{\nu(d),R} x_{\nu(d)} \qquad (5.11)$$

for each $d \in \mathcal{D}_{\lambda,\mu}$ and $\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu$.

On the other hand, since $d \in \mathcal{D}_{\lambda,\mu}$, there exists a unique $M^d = (m_{ij}^d) \in M_n(\mathbb{N})$ such that $\jmath(M^d) = (\lambda, d, \mu)$ with $\nu(d) = (m_{11}^d, \ldots, m_{n1}^d, \ldots, m_{1n}^d, \ldots, m_{nn}^d)$. (Recall that $\nu(d)$ is defined by $\mathfrak{S}_{\nu(d)} = d^{-1}\mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$.) Then, by (5.11) and Corollary 4.3,

$$\sum_{\alpha \in \mathbb{Z}_2^r} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha = \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d x_{\nu(d)} c^{\alpha_B}_{\nu(d)} \qquad (5.12)$$

for some $g_B^d \in R$, where $B = (b_{ij}) \in M_n(\mathbb{Z}_2) \leq M^d$ means that $b_{ij} \leq m_{ij}^d$ for $1 \leq i, j \leq n$ and $\alpha_B = (b_{11}, \ldots, b_{n1}, \ldots, b_{1n}, \ldots, b_{nn}) \in \mathbb{Z}_2^{n^2}$.

For each $B \in M_n(\mathbb{Z}_2)$ with $B \leq M^d$, we let $A(d, B) = M^d - B \in M_n(\mathbb{Z}_2)$ and then $d = d_{A(d,B)+B}$ and $(A(d,B)|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\lambda,\mu}^d$. Then, by (5.3) and (5.12),

$$\sum_{\alpha \in \mathbb{Z}_2^r} a_{d,\sigma,\alpha} x_{\nu(d)} c^\alpha = \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d x_{\nu(d)} c_{A(d,B)|B}$$

and hence, by (5.9),

$$z_d = \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d x_{\nu(d)} c_{A(d,B)|B} T_\sigma.$$

Therefore, by (5.10), one can deduce that

$$
\begin{aligned}
z &= \sum_{d \in \mathcal{D}_{\lambda,\mu}} \sum_{u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_{u'} T_d \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d x_{\nu(d)} c_{A(d,B)|B} T_\sigma \\
&= \sum_{d \in \mathcal{D}_{\lambda,\mu}} \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d \sum_{u' \in \mathcal{D}_{\nu(d^{-1})}^{-1} \cap \mathfrak{S}_\lambda} T_{u'} T_d x_{\nu(d)} c_{A(d,B)|B} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma \\
&= \sum_{d \in \mathcal{D}_{\lambda,\mu}} \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d x_\lambda T_d c_{A(d,B)|B} \sum_{\sigma \in \mathcal{D}_{\nu(d)} \cap \mathfrak{S}_\mu} T_\sigma \\
&= \sum_{d \in \mathcal{D}_{\lambda,\mu}} \sum_{B \in M_n(\mathbb{Z}_2), B \leq M^d} g_B^d T_{A(d,B)|B}
\end{aligned}
$$

since $d = d_{A(d,B)+B}$. This proves the proposition. □

For $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_r$, define $\phi_{(A|B)} \in Q_q(n,r;R) = \mathrm{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}}(\bigoplus_{\mu \in \Lambda(n,r)} x_\mu \mathcal{H}_{r,R}^{\mathsf{c}})$ via

$$
\phi_{(A|B)}(x_\mu h) = \delta_{\mu, \mathrm{co}(A+B)} T_{A|B} h \tag{5.13}
$$

for $\mu \in \Lambda(n,r)$ and $h \in \mathcal{H}_{r,R}^{\mathsf{c}}$.

THEOREM 5.3. *Let R be a commutative ring of characteristic not equal to 2. Then the algebra $Q_q(n,r;R)$ is a free R-module with a basis given by the set*

$$
\{\phi_{(A|B)} \mid (A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_r\}.
$$

PROOF. By (2.7), one can deduce that

$$
Q_q(n,r;R) = \bigoplus_{\lambda,\mu \in \Lambda(n,r)} \mathrm{Hom}_{\mathcal{H}_{r,R}^{\mathsf{c}}}(x_\mu \mathcal{H}_{r,R}^{\mathsf{c}}, x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}}).
$$

Then, by Lemma 3.2 and Proposition 5.2, the set

$$
\{\phi_{(A|B)} \mid \mathrm{ro}(A+B) = \lambda, \mathrm{co}(A+B) = \mu\}
$$

is an R-basis for $\mathrm{Hom}_{\mathcal{H}_{r,R}^{\mathsf{c}}}(x_\mu \mathcal{H}_{r,R}^{\mathsf{c}}, x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}})$ for each pair $\lambda, \mu \in \Lambda(n,r)$. Therefore, the set

$$
\{\phi_{(A|B)} \mid (A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_r\}
$$

forms an R-basis for $Q_q(n,r;R)$. □

By Theorem 5.3, we have the following base-change property for $Q_q(n,r;R)$.

COROLLARY 5.4. *Maintain the assumption on R as above. Suppose that R is an $\mathcal{A}$-algebra via $\boldsymbol{q} \mapsto q$. Then*

$$
Q_q(n,r;R) \cong Q_{\boldsymbol{q}}(n,r)_R := Q_{\boldsymbol{q}}(n,r) \otimes_{\mathcal{A}} R.
$$

6. Identification with the quotients of the quantum queer supergroup

In this section, we shall show that the queer q-Schur superalgebra $Q_v(n,r)$ coincides with the quantum queer Schur superalgebra constructed in [14]. In particular, they are homomorphic images of the quantum queer supergroup $U_v(\mathfrak{q}_n)$.

Let $q = v^2$ for some $v \in R$. Let $V(n|n)_R = V_0 \oplus V_1$ be a free R-supermodule with basis $e_1, \ldots, e_n$ for V_0 and basis $e_{-1}, \ldots, e_{-n}$ for V_1. With the fixed ordered basis $\{e_1, \ldots, e_n, e_{-1}, \ldots, e_{-n}\}$, we often identify the R-algebra $\text{End}_R(V(n|n)_R)$ of all R-linear maps on $V(n|n)_R$ with the $2n \times 2n$ matrix algebra $M_{2n}(R)$ over R and, hence, $\text{End}_R(V(n|n)_R^{\otimes r})$ with $M_{2n}(R)^{\otimes r}$.

Let

$$I(n|n) = \{1, 2, \ldots, n, -1, -2, \ldots, -n\},$$
$$I(n|n, r) = \{\underline{i} = (i_1, \ldots, i_r) \mid i_k \in I(n|n), 1 \le k \le r\}.$$

Then $\mathfrak{S}_r$ acts on $I(n|n, r)$ by the place permutation

$$\underline{i}s_k = (i_1, \ldots, i_{k-1}, i_{k+1}, i_k, i_{k+2}, \ldots, i_r).$$

For $\underline{i} = (i_1, \ldots, i_r) \in I(n|n, r)$, set $e_{\underline{i}} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$; then the set $\{e_{\underline{i}} \mid \underline{i} \in I(n|n, r)\}$ forms a basis for $V(n|n)_R^{\otimes r}$.

Denote by $E_{i,j}$ for $i, j \in I(n|n)$ the standard elementary matrix with the (i, j)th entry being 1 and 0 elsewhere. Then $\{E_{i,j} \mid i, j \in I(n|n)\}$ can be viewed as the standard basis for $\text{End}_R(V(n|n)_R)$, that is, $E_{i,j}(e_k) = \delta_{j,k} e_i$. Following [25], we set

$$\Omega = \sum_{1 \le a \le n} (E_{-a,a} - E_{a,-a}),$$
$$T = \sum_{i,j \in I(n|n)} (-1)^{\widehat{j}} E_{i,j} \otimes E_{j,i}, \tag{6.1}$$
$$S = v \sum_{i \le j \in I(n|n)} S_{i,j} \otimes E_{i,j} \in \text{End}_R(V(n|n)_R^{\otimes 2}),$$

where $\widehat{j} = \widehat{e_j} = 0$ if $j > 0$ and $\widehat{j} = \widehat{e_j} = 1$ if $j < 0$, and $S_{i,j}$ for $i \le j$ are defined as follows:

$$S_{a,a} = 1 + (v - 1)(E_{a,a} + E_{-a,-a}), \quad 1 \le a \le n,$$
$$S_{-a,-a} = 1 + (v^{-1} - 1)(E_{a,a} + E_{-a,-a}), \quad 1 \le a \le n,$$
$$S_{b,a} = (v - v^{-1})(E_{a,b} + E_{-a,-b}), \quad 1 \le b < a \le n,$$
$$S_{-b,-a} = -(v - v^{-1})(E_{a,b} + E_{-a,-b}), \quad 1 \le a < b \le n,$$
$$S_{-b,a} = -(v - v^{-1})(E_{-a,b} + E_{a,-b}), \quad 1 \le a, b \le n.$$

To endomorphisms $A \in \text{End}_R(V(n|n)_R)$ and $Z = \sum_t X_t \otimes Y_t \in \text{End}_R(V(n|n)_R)^{\otimes 2} = \text{End}_R(V(n|n)_R^{\otimes 2})$, we associate the following elements in $\text{End}_R(V(n|n)_R^{\otimes r})$:

$$A^{(k)} = \text{id}^{\otimes(k-1)} \otimes A \otimes \text{id}^{\otimes(r-k)}, \quad 1 \le k \le r,$$
$$Z^{(j,k)} = \sum_t X_t^{(j)} Y_t^{(k)}, \quad 1 \le j \ne k \le r.$$

Let $\bar{S} = TS$. It follows from [25, Theorems 5.2 and 5.3] that there exists a *left* $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule structure on $V(n|n)_R^{\otimes r}$ given by

$$\Psi_r : \mathcal{H}_{r,R}^{\mathsf{c}} \longrightarrow \text{End}_R(V(n|n)_R^{\otimes r}), \quad T_i \longmapsto \bar{S}^{(i,i+1)}, \quad c_j \longmapsto \Omega^{(j)} \tag{6.2}$$

for all $1 \le i \le r-1, 1 \le j \le r$. Here we remark that the even generators $T_1, \ldots, T_{r-1}$ for $\mathcal{H}_{r,R}^{\mathsf{c}}$ are related to the even generators $t_1, \ldots, t_{r-1}$ in [25] via $T_i = vt_i$ for $1 \le i \le r-1$. This above action has been explicitly worked out in [27, Lemma 3.1] via (6.1).

LEMMA 6.1. *The $\mathcal{H}^c_{r,R}$-supermodule structure on $V(n|n)^{\otimes r}_R$ is given by the following formulas: for $\underline{i} = (i_1, \ldots, i_r) \in I(n|n, r)$ and $1 \le k \le r - 1$, $1 \le j \le r$,*

$$c_j \cdot e_{\underline{i}} = (-1)^{\hat{e}_{i_1} + \cdots + \hat{e}_{i_j}} e_{i_1} \otimes \cdots \otimes e_{i_{j-1}} \otimes e_{-i_j} \otimes e_{i_{j+1}} \otimes \cdots \otimes e_{i_r},$$

$$T_k \cdot e_{\underline{i}} = \begin{cases} v^2 e_{\underline{i}s_k} + (v^2 - 1)e_{\underline{i}_k^-} & \text{if } i_k = i_{k+1} \ge 1, \\ -e_{\underline{i}s_k} & \text{if } i_k = i_{k+1} \le -1, \\ e_{\underline{i}s_k} & \text{if } i_k = -i_{k+1} \ge 1, \\ v^2 e_{\underline{i}s_k} + (v^2 - 1)e_{\underline{i}} & \text{if } i_k = -i_{k+1} \le -1, \\ v e_{\underline{i}s_k} + (v^2 - 1)e_{\underline{i}_k^-} + (v^2 - 1)e_{\underline{i}} & \text{if } |i_k| < |i_{k+1}| \text{ and } i_{k+1} \ge 1, \\ (-1)^{\hat{i}_k} v e_{\underline{i}s_k} & \text{if } |i_k| < |i_{k+1}| \text{ and } i_{k+1} \le -1, \\ v e_{\underline{i}s_k} + (-1)^{\hat{i}_{k+1}} (v^2 - 1)e_{\underline{i}_k^-} & \text{if } |i_{k+1}| < |i_k| \text{ and } i_k \ge 1, \\ (-1)^{\hat{i}_{k+1}} e_{\underline{i}s_k} + (v^2 - 1)e_{\underline{i}} & \text{if } |i_{k+1}| < |i_k| \text{ and } i_k \le -1, \end{cases}$$

where $\underline{i}_k^- = (i_1, \ldots, i_{k-1}, -i_k, -i_{k+1}, i_{k+2}, \ldots, i_r)$.

For each $\underline{i} \in I(n|n, r)$, define $\text{wt}(\underline{i}) = \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r)$ by setting

$$\lambda_a = |\{k \mid a = |i_k|, 1 \le k \le r\}|, \quad \forall 1 \le a \le n.$$

For each $\lambda \in \Lambda(n, r)$, define $\underline{i}_\lambda$ by

$$\underline{i}_\lambda = (\underbrace{-1, \ldots, -1}_{\lambda_1}, \underbrace{-2, \ldots, -2}_{\lambda_2}, \ldots, \underbrace{-n, \ldots, -n}_{\lambda_n}).$$

Recall from (2.6) the elements y_λ.

COROLLARY 6.2. *The following holds as left $\mathcal{H}^c_{r,R}$-supermodules:*

$$V(n|n)^{\otimes r}_R \cong \bigoplus_{\lambda \in \Lambda(n,r)} \mathcal{H}^c_{r,R} y_\lambda.$$

PROOF. For each $\lambda \in \Lambda(n, r)$, let $(V(n|n)^{\otimes r}_R)_\lambda$ be the R-submodule of $V(n|n)^{\otimes r}_R$ spanned by the elements $e_{\underline{i}}$ with $\underline{i} \in I(n|n, r)$ and $\text{wt}(\underline{i}) = \lambda$. Then, by Lemma 6.1, $(V(n|n)^{\otimes r}_R)_\lambda$ is stable under the action of $\mathcal{H}^c_{r,R}$ and, moreover, $V(n|n)^{\otimes r}_R$ can be decomposed as

$$V(n|n)^{\otimes r}_R = \bigoplus_{\lambda \in \Lambda(n,r)} (V(n|n)^{\otimes r}_R)_\lambda. \tag{6.3}$$

Clearly, $e_{\underline{i}_\lambda} \in (V(n|n)^{\otimes r}_R)_\lambda$. For each $w \in \mathfrak{S}_r$, we have $\underline{i}_\lambda w = (i_{w(1)}, \ldots, i_{w(r)})$, where we write $\underline{i}_\lambda = (i_1, \ldots, i_r)$. Then we can easily deduce that

$$\{e_{\underline{i}} \mid \underline{i} \in I(n|n, r), \text{wt}(\underline{i}) = \lambda, i_k \le -1, 1 \le k \le r\} = \{e_{\underline{i}_\lambda w} \mid w \in \mathfrak{S}_r\} = \{e_{\underline{i}_\lambda d} \mid d \in \mathcal{D}_\lambda\}.$$

Moreover, by Lemma 6.1, one can deduce that $\mathcal{H}_{r,R} e_{\underline{i}_\lambda} = R\text{-span}\{e_{\underline{i}_\lambda d} \mid d \in \mathcal{D}_\lambda\}$. Meanwhile, by Frobenius reciprocity, there exists an $\mathcal{H}^c_{r,R}$-homomorphism

$$\mathcal{H}^c_{r,R} \otimes_{\mathcal{H}_{r,R}} \mathcal{H}_{r,R} e_{\underline{i}_\lambda} \to \mathcal{H}^c_{r,R} e_{\underline{i}_\lambda} = (V(n|n)^{\otimes r}_R)_\lambda, \quad c^\alpha \otimes e_{\underline{i}_\lambda d} \mapsto c^\alpha \cdot e_{\underline{i}_\lambda d} \tag{6.4}$$

which is bijective due to the fact that $\{c^\alpha \cdot e_{i_\lambda d} \mid d \in \mathcal{D}_\lambda, \alpha \in \mathbb{Z}_2^r\}$ is a basis for $(V(n|n)_R^{\otimes r})_\lambda$. Observe that the tensor product $V(0|n)_R^{\otimes r}$ of the odd R-submodule $V(0|n)_R$ of $V(n|n)_R$ admits a basis $\{e_{\underline{i}} \mid \underline{i} \in I(n|n, r), i_k \leq -1, 1 \leq k \leq r\}$, which can be identified with $\bigoplus_{\lambda \in \Lambda(n,r)} \mathcal{H}_{r,R} e_{\underline{i}_\lambda}$. Then, by Lemma 6.1 and [9, Lemma 2.1], there is an $\mathcal{H}_{r,R}$-module isomorphism

$$\mathcal{H}_{r,R} e_{\underline{i}_\lambda} \cong \mathcal{H}_{r,R} y_\lambda. \tag{6.5}$$

Taking a direct sum over $\Lambda(n, r)$ and induction to $\mathcal{H}_{r,R}^{\mathsf{c}}$ and noting (6.3) and (6.4) gives the required isomorphism. $\qquad \square$

Recall the anti-involution τ defined in Lemma 2.1(3) and the twist functor in Remark in Section 2. In particular, $(V(n|n)_R^{\otimes r})^\tau$ affords a right $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodule. By (2.10), we immediately have the following result.

COROLLARY 6.3. *There is an isomorphism of right $\mathcal{H}_{r,R}^{\mathsf{c}}$-supermodules:*

$$(V(n|n)_R^{\otimes r})^\tau \cong \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}},$$

which induces an algebra isomorphism

$$\mathrm{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}}(V(n|n)_R^{\otimes r}) \cong \mathrm{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}}((V(n|n)_R^{\otimes r})^\tau) \cong \mathrm{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}}\bigg(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,R}^{\mathsf{c}}\bigg) = Q_{v^2}(n, r; R).$$

Hence, if $R = \mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, the endomorphism superalgebra $\mathrm{End}_{\mathcal{H}_{r,R}^{\mathsf{c}}}(V(n|n)_R^{\otimes r})$ is the quantum queer Schur superalgebra $Q_v(n, r)$ considered in [14]. We remark that in [14], we used the indeterminate q instead of v and the notation $Qq(n, r)$ denotes the queer q-Schur superalgebra over the rational function field $\mathbb{C}(q)$. The isomorphism above shows that our notation is consistent as, by (2.8), we used the same notation to denote the right-hand side $Q_{v^2}(n, r; \mathcal{Z})$; see (2.8).

Olshanski [25] introduced the quantum deformation $U_v(\mathfrak{q}_n)$ over $\mathbb{C}(v)$ of the universal enveloping algebra of the queer Lie superalgebra $\mathfrak{q}(n)$ and defined a superalgebra homomorphism

$$\Phi_r : U_v(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathbb{C}(v)}(V(n|n)_{\mathbb{C}(v)}^{\otimes r}).$$

We refer the reader to [25] for the details of the definitions of $U_v(\mathfrak{q}(n))$ and Φ_r. The following result is known as the double centraliser property and forms the first part of the quantum analog of Schur–Weyl–Sergeev duality for $U_v(\mathfrak{q}_n)$ and $\mathcal{H}_{r,\mathbb{C}(v)}^{\mathsf{c}}$, which can be regarded a super analog of the duality established by Jimbo in [19] for Hecke algebra associated with symmetric groups. Recall also the algebra homomorphism Ψ_r defined in (6.2).

PROPOSITION 6.4 [25, Theorem 5.3]. *The algebras $\Phi_r(U_v(\mathfrak{q}_n))$ and $\Psi_r(\mathcal{H}_{r,\mathbb{C}(v)}^{\mathsf{c}})$ form mutual centralisers in $\mathrm{End}_{\mathbb{C}(v)}(V(n|n)_{\mathbb{C}(v)}^{\otimes r})$. In particular, Φ_r induces an algebra epimorphism*

$$\Phi_r : U_v(\mathfrak{q}_n) \longrightarrow \mathrm{End}_{\mathcal{H}_{r,\mathbb{C}(v)}^{\mathsf{c}}}(V(n|n)_{\mathbb{C}(v)}^{\otimes r}) = Q_v(n, r)_{\mathbb{C}(v)}.$$

Thus, by [14, Theorem 9.2], the queer q-Schur superalgebra $Q_v(n, r)_{\mathbb{C}(v)}$ has a presentation with even generators $K_i^{\pm 1}, E_j, F_j$ and odd generators $K_{\bar{i}}, E_{\bar{j}}, F_{\bar{j}}$, for $1 \leq i \leq n$ and $1 \leq j \leq n - 1$, subject to the relations (QQ1)–(QQ9) given in [14].

REMARK. It would be natural to expect that one may use the integral basis $\{\phi_{(A|B)}\}_{(A|B)}$ given in Theorem 5.3 to describe the images of the generators $K_i^{\pm 1}, E_j, F_j, K_{\bar{i}}, E_{\bar{j}}, F_{\bar{j}}$ and, thus, to obtain a realisation of $U_v(\mathfrak{q}_n)$ similar to the ones given in [1, 10].

7. Irreducible $Q_v(n, r)_{\mathbb{K}}$-supermodules

In this section, we shall give a construction of all irreducible $Q_v(n, r)_{\mathbb{K}}$-supermodules over a certain field extension $\mathbb{K}$ of $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. But we first look at a few general facts for the superalgebra $Q_q(n, r)$ over $\mathcal{A} = \mathbb{Z}[q]$.

Let $\omega = (1^r)$. In the case $n \geq r$, we can view ω as an element in $\Lambda(n, r)$ and define

$$e_\omega = \phi_{(A_\omega | B_\omega)} \in Q_q(n, r),$$

where $A_\omega = \mathrm{diag}(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0)$ and $B_\omega = 0$. Then, by (5.4) and (5.13),

$$e_\omega(x_\lambda h) = \delta_{\omega, \lambda} x_\lambda h, \quad e_\omega^2 = e_\omega \tag{7.1}$$

for $\lambda \in \Lambda(n, r), h \in \mathcal{H}_r^c$.

Recall from [8, Theorem 2.24] that the q-Schur algebra $\mathcal{S}_q(n, r)$ is defined as

$$\mathcal{S}_q(n, r) := \mathrm{End}_{\mathcal{H}_r}\left(\bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_r \right).$$

Let

$$\mathcal{T}(n, r) = \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_r \quad \text{and} \quad \mathcal{T}^c(n, r) = \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H}_r^c.$$

Since $\mathcal{T}^c(n, r) = \mathcal{T}(n, r) \otimes_{\mathcal{H}_r} \mathcal{H}_r^c$ (see (2.9)), there is a natural algebra embedding

$$\epsilon : \mathcal{S}_q(n, r) \longrightarrow Q_q(n, r), \quad f \longmapsto f \otimes 1.$$

Equivalently, the q-Schur algebra $\mathcal{S}_q(n, r)$ can be identified as a subalgebra of $Q_q(n, r)$ via the following way:

$$\mathcal{S}_q(n, r) = \{\phi \in Q_q(n, r) \mid \phi(x_\lambda) \in \mathcal{H}_r, \forall \lambda \in \Lambda(n, r)\}. \tag{7.2}$$

On the other hand, if $n \geq r$, the evaluation map gives algebra isomorphisms

$$e_\omega Q_q(n, r) e_\omega \cong \mathcal{H}_r^c \quad \text{and} \quad e_\omega \mathcal{S}_q(n, r) e_\omega \cong \mathcal{H}_r.$$

We will identify them in the sequel. In particular, $Q_q(n, r) e_\omega$ is a $Q_q(n, r)$-$\mathcal{H}_r^c$-bisupermodule.

LEMMA 7.1. *If $n \geq r$, then there is a $Q_q(n, r)$-$\mathcal{H}_r^c$-bisupermodule isomorphism*

$$Q_q(n, r)e_\omega \cong \mathcal{T}^c(n, r).$$

Moreover, restriction gives an $\mathcal{S}_q(n, r)$-$\mathcal{H}_r^c$-bisupermodule isomorphism

$$Q_q(n, r)e_\omega \cong \mathcal{S}_q(n, r)e_\omega \otimes_{\mathcal{H}_r} \mathcal{H}_r^c.$$

PROOF. By (7.1) and (7.2), $e_\omega \in \mathcal{S}_q(n, r)$ and $x_\omega = 1$. Hence, the evaluation map gives an $\mathcal{S}_q(n, r)$-$\mathcal{H}_r$-bimodule isomorphism

$$\mathcal{S}_q(n, r)e_\omega = \operatorname{Hom}_{\mathcal{H}_r}(x_\omega \mathcal{H}_r, \mathcal{T}(n, r)) \cong \mathcal{T}(n, r)$$

and a $Q_q(n, r)$-$\mathcal{H}_r^c$-bisupermodule isomorphism

$$Q_q(n, r)e_\omega = \operatorname{Hom}_{\mathcal{H}_r^c}(x_\omega \mathcal{H}_r^c, \mathcal{T}^c(n, r)) \cong \mathcal{T}^c(n, r).$$

Thus, the restriction gives an $\mathcal{S}_q(n, r)$-$\mathcal{H}_r^c$-bisupermodule isomorphism

$$Q_q(n, r)e_\omega \cong \mathcal{T}(n, r) \otimes_{\mathcal{H}_r} \mathcal{H}_r^c \cong \mathcal{S}_q(n, r)e_\omega \otimes_{\mathcal{H}_r} \mathcal{H}_r^c.$$

This proves the lemma.   □

Assume that $\mathbb{F}$ is a field which is a $\mathcal{Z}$-algebra such that the image of q is not a root of unity. Thus, both $\mathcal{H}_{r,\mathbb{F}}$ and $\mathcal{S}_q(n, r)_{\mathbb{F}} = \mathcal{S}_q(n, r) \otimes_{\mathcal{Z}} \mathbb{F}$ are semisimple $\mathbb{F}$-algebras. The following result generalises the category equivalence between $\mathcal{S}_q(n, r)_{\mathbb{F}}$-mod and $\mathcal{H}_{r,\mathbb{F}}$-mod in the case $n \geq r$ and the proof is somewhat standard; see, for example, [5, Theorem 4.1.3].

PROPOSITION 7.2. *Assume that $n \geq r$. The superalgebras $Q_q(n, r)_{\mathbb{F}}$ and $\mathcal{H}_{r,\mathbb{F}}^c$ are Morita equivalent.*

PROOF. We consider the two functors generally defined in [18, Section 6.2]. Firstly, the $Q_q(n, r)_{\mathbb{F}}$-$\mathcal{H}_{r,\mathbb{F}}^c$-bisupermodule $\mathcal{T}^c(n, r)_{\mathbb{F}}$ induces a functor

$$\mathcal{F} : \mathcal{H}_{r,\mathbb{F}}^c\text{-smod} \longrightarrow Q_q(n, r)_{\mathbb{F}}\text{-smod}, \quad L \longmapsto \mathcal{T}^c(n, r)_{\mathbb{F}} \otimes_{\mathcal{H}_{r,\mathbb{F}}^c} L. \qquad (7.3)$$

Secondly, there exists the functor

$$\mathcal{G} : Q_q(n, r)_{\mathbb{F}}\text{-smod} \to \mathcal{H}_{r,\mathbb{F}}^c\text{-smod}, \quad M \mapsto e_\omega M.$$

It is clear (cf. [18, (6.2d)]) that $\mathcal{G} \circ \mathcal{F} \cong \operatorname{id}_{\mathcal{H}_{r,\mathbb{F}}^c\text{-smod}}$. So, it suffices to prove that $\mathcal{F} \circ \mathcal{G} \cong \operatorname{id}_{Q_q(n, r)_{\mathbb{F}}\text{-smod}}$.

It is known from the case for the q-Schur algebra that, for any $\mathcal{S}_q(n, r)_{\mathbb{F}}$-module M, there is a left $\mathcal{S}_q(n, r)_{\mathbb{F}}$-module isomorphism

$$\psi : \mathcal{S}_q(n, r)_{\mathbb{F}} e_\omega \otimes_{\mathcal{H}_{r,\mathbb{F}}} e_\omega M \cong M$$

defined by $\psi(x \otimes m) = xm$ for $x \in \mathcal{S}_q(n, r)_{\mathbb{F}} e_\omega$ and $m \in e_\omega M$. This together with Lemma 7.1 gives rise to a left $\mathcal{S}_q(n, r)_{\mathbb{F}}$-module isomorphism

$$\Psi : Q_q(n, r)_{\mathbb{F}} e_\omega \otimes_{\mathcal{H}_{r,\mathbb{F}}^c} e_\omega M = \mathcal{S}_q(n, r)_{\mathbb{F}} e_\omega \otimes_{\mathcal{H}_{r,\mathbb{F}}} e_\omega M \cong M.$$

We now claim that Ψ is a $Q_q(n,r)_{\mathbb{F}}$-supermodule isomorphism. Indeed, considering the basis $\{\phi_{(A|B)} \mid (A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)\}$ for $Q_q(n,r)_{\mathbb{F}}$,

$$Q_q(n,r)_{\mathbb{F}} e_\omega = \operatorname{Hom}_{\mathcal{H}_{r,\mathbb{F}}^{\mathsf{c}}}\Big(x_\omega \mathcal{H}_{r,\mathbb{F}}^{\mathsf{c}}, \bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,\mathbb{F}}^{\mathsf{c}}\Big) = \bigoplus_{\lambda \in \Lambda(n,r)} \phi_{(A(\omega,\lambda)|0)} \mathcal{H}_{r,\mathbb{F}}^{\mathsf{c}},$$

where $A(\omega,\lambda) = (a_{ij}^{\omega,\lambda})$ with $a_{ij}^{\omega,\lambda} = 1$ if $\lambda_1 + \cdots + \lambda_{i-1} + 1 \le j \le \lambda_1 + \cdots + \lambda_{i-1} + \lambda_i$ and $a_{ij}^{\omega,\lambda} = 0$ otherwise. Clearly, $\phi_{(A(\omega,\lambda)|0)}(x_\omega) = x_\lambda \in \mathcal{H}_{r,\mathbb{F}}$ for $\lambda \in \Lambda(n,r)$. Therefore, $\phi_{(A(\omega,\lambda)|0)} \in S_q(n,r)_{\mathbb{F}} e_\omega$ by (7.2). This implies that

$$\Psi(\phi_{(A(\omega,\lambda)|0)} h \otimes m) = \psi(\phi_{(A(\omega,\lambda)|0)} \otimes hm) = \phi_{(A(\omega,\lambda)|0)} hm$$

for any $h \in \mathcal{H}_{r,\mathbb{F}}^{\mathsf{c}}, m \in e_\omega M$. Hence, $\Psi(x \otimes m) = xm$ for $x \in Q_q(n,r)_{\mathbb{F}} e_\omega, m \in e_\omega M$. This means that Ψ is a $Q_q(n,r)_{\mathbb{F}}$-supermodule homomorphism. □

We are now ready to look at the classification of irreducible representations of the superalgebra $Q_v(n,r)_{\mathbb{K}}$, where

$$\mathbb{K} := \mathbb{C}(v)(\sqrt{[\![2]\!]}, \sqrt{[\![3]\!]}, \ldots, \sqrt{[\![r]\!]})$$

is the field extension of $\mathbb{C}(v)$ with $[\![k]\!] = (q^k - q^{-k})/(q - q^{-1})$ for $k \in \mathbb{N}$ ($q = v^2$). It is known from [20, Proposition 2.2] (cf. [21]) that $\mathcal{H}_{r,\mathbb{C}(v)}^{\mathsf{c}}$ is a semisimple superalgebra. Thus, $Q_v(n,r)_{\mathbb{C}(v)}$ is semisimple. The following result will imply that $\mathbb{K}$ is a splitting field of $Q_v(n,r)_{\mathbb{C}(v)}$.

A partition ξ of r is said to be *strict* if its nonzero parts are distinct. Denote by $\mathcal{SP}(r)$ the set of strict partitions ξ of r. For $\xi \in \mathcal{SP}(r)$, denote by $l(\xi)$ the length of ξ and let

$$\delta(\xi) = \begin{cases} 0 & \text{if } l(\xi) \text{ is even,} \\ 1 & \text{if } l(\xi) \text{ is odd.} \end{cases}$$

An irreducible supermodule X over an $\mathbb{F}$-superalgebra $\mathscr{A}$ is called split irreducible if $X \otimes_{\mathbb{F}} \mathbb{E}$ is irreducible over $\mathscr{A}_{\mathbb{E}} = \mathscr{A} \otimes_{\mathbb{F}} \mathbb{E}$ for any field extension $\mathbb{E} \supseteq \mathbb{F}$.

PROPOSITION 7.3 [20, Corollary 6.8]. *For each $\xi \in \mathcal{SP}(r)$, there exists a left simple $\mathcal{H}_{r,\mathbb{K}}^{\mathsf{c}}$-supermodule U^ξ such that $\{U^\xi \mid \xi \in \mathcal{SP}(r)\}$ forms a complete set of nonisomorphic irreducible left $\mathcal{H}_{r,\mathbb{K}}^{\mathsf{c}}$-supermodules. Moreover, every U^ξ is split irreducible and U^ξ is of type $\mathtt{M}$ (respectively, $\mathtt{Q}$) if $\delta(\xi) = 0$ (respectively, $\delta(\xi) = 1$).*

If we take $\mathbb{F} = \mathbb{K}$ in (7.3), then Propositions 7.2 and 7.3 imply the following immediately.

COROLLARY 7.4. *If $n \ge r$, then the set $\{\mathcal{T}^{\mathsf{c}}(n,r)_{\mathbb{K}} \otimes_{\mathcal{H}_{r,\mathbb{K}}^{\mathsf{c}}} U^\xi \mid \xi \in \mathcal{SP}(r)\}$ is a complete set of nonisomorphic irreducible $Q_v(n,r)_{\mathbb{K}}$-supermodules.*

For positive integers n, r, let

$$\mathcal{SP}(n,r) = \{\xi \in \mathcal{SP}(r) \mid \mathcal{T}^{\mathsf{c}}(n,r)_{\mathbb{K}} \otimes_{\mathcal{H}_{r,\mathbb{K}}^{\mathsf{c}}} U^\xi \ne 0\}.$$

Clearly, we have from the above that $\mathcal{SP}(n,r) = \mathcal{SP}(r)$ for all $n \ge r$.

Recall from the last section that there is a commuting action of $U_v(\mathfrak{q}_n)_\mathbb{K}$ and $\mathcal{H}_{r,\mathbb{K}}^c$ on $V(n|n)_\mathbb{K}^{\otimes r}$. For $\xi \in \mathcal{SP}(r)$, let

$$V(\xi) = \mathrm{Hom}_{\mathcal{H}_{r,\mathbb{K}}^c}(U^\xi, V(n|n)_\mathbb{K}^{\otimes r}).$$

This is a $U_v(\mathfrak{q}_n)_\mathbb{K}$-module. If $V(\xi) \neq 0$, then U^ξ is a direct summand of $V(n|n)_\mathbb{K}^{\otimes r}$. Hence, it is a simple $Q_v(n, r)_\mathbb{K}$-supermodule and, hence, a simple $U_v(\mathfrak{q}_n)_\mathbb{K}$-supermodule.

LEMMA 7.5. *Let $\xi \in \mathcal{SP}(r)$. If $n < r$ and $l(\xi) > n$, then $V(\xi) = 0$. In particular,*

$$\mathcal{SP}(n, r) = \{\xi \in \mathcal{SP}(r) \mid l(\xi) \leq n\}.$$

PROOF. By (6.3) and (6.5), we have that $V(n|n)_\mathbb{K}^{\otimes r}$ is isomorphic to a direct sum of $\mathcal{H}_{r,\mathbb{K}}^c y_\lambda$ with $\lambda \in \Lambda(n, r)$. By [4, Lemma 3.50], there exist some nonnegative integers $k_{\xi,\lambda}$ with $k_{\xi,\xi} \neq 0$ such that (Though the result there is for the specialisation $(\mathcal{H}_r^c)_\mathbb{C}$ of $\mathcal{H}_r^c$ at $q = 1$. However, by the category equivalence between the module categories, the proof there carries over.)

$$\mathcal{H}_{r,\mathbb{K}}^c y_\lambda \cong \bigoplus_{\xi \in \mathcal{SP}(r), \xi \trianglerighteq \lambda} k_{\xi,\lambda} U^\xi,$$

where $\trianglerighteq$ is the dominance order. Since $\xi \trianglerighteq \lambda$ implies that $l(\xi) \leq l(\lambda) = n$, it follows that $V(\xi) = 0$ if $l(\xi) > n$, proving the first assertion. To prove the last assertion, we write $U^\xi = \mathcal{H}_{r,\mathbb{K}}^c \epsilon$ for some idempotent ϵ. Then, as a left $Q_v(n, r)_\mathbb{K}$-module,

$$V(\xi) \cong \epsilon V(n|n)_\mathbb{K}^{\otimes r} \cong (V(n|n)_\mathbb{K}^{\otimes r})^\tau \epsilon \cong \mathcal{T}^c(n, r)_\mathbb{K} \otimes_{\mathcal{H}_{r,\mathbb{K}}^c} U^\xi.$$

The last assertion follows from the fact that $k_{\xi,\xi} \neq 0$. □

We now use Green's theorem [18, (6.2g)] to get the following classification theorem.

THEOREM 7.6. *For positive integers n, r, the set*

$$\{\mathcal{T}^c(n, r)_\mathbb{K} \otimes_{\mathcal{H}_{r,\mathbb{K}}^c} U^\xi \cong V(\xi) \mid \xi \in \mathcal{SP}(n, r)\}$$

forms a complete set of nonisomorphic irreducible $Q_v(n, r)_\mathbb{K}$-supermodules.

PROOF. The case for $n \geq r$ is seen above. We now assume that $n < r$. Consider the natural embedding

$$\Lambda(n, r) \hookrightarrow \Lambda(r, r), \quad \lambda \mapsto \overline{\lambda} = (\lambda_1, \dots, \lambda_n, 0, \dots, 0).$$

For $\lambda \in \Lambda(n, r)$, let $D(\lambda) = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, 0, \dots, 0) \in M_r(\mathbb{N})$ and set

$$e = \sum_{\lambda \in \Lambda(n,r)} \phi_{(D(\lambda)|0)}.$$

Then

$$Q_v(n, r)_\mathbb{K} \cong \mathrm{End}_{\mathcal{H}_{r,\mathbb{K}}^c}\left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}_{r,\mathbb{K}}^c\right) = e\,\mathrm{End}_{\mathcal{H}_{r,\mathbb{K}}^c}\left(\bigoplus_{\mu \in \Lambda(r,r)} x_\mu \mathcal{H}_{r,\mathbb{K}}^c\right)e = eQ_v(r, r)_\mathbb{K}e. \quad (7.4)$$

Meanwhile,

$$e(\mathcal{T}^{\mathrm{c}}(r,r)_{\mathbb{K}} \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi}) = e\mathcal{T}^{\mathrm{c}}(r,r)_{\mathbb{K}} \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi} = e\Big(\bigoplus_{\mu \in \Lambda(r,r)} x_{\mu} \mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}\Big) \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi}$$

$$= \Big(\bigoplus_{\lambda \in \Lambda(n,r)} x_{\lambda} \mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}\Big) \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi} = \mathcal{T}^{\mathrm{c}}(n,r)_{\mathbb{K}} \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi}.$$

Since $\{\mathcal{T}^{\mathrm{c}}(r,r)_{\mathbb{K}} \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi} \mid \xi \in \mathcal{SP}_r\}$ is a complete set of nonisomorphic simple $Q_\nu(r,r)_{\mathbb{K}}$-supermodules, (7.4) together with [18, (6.2g)] implies that the set

$$\{\mathcal{T}^{\mathrm{c}}(n,r)_{\mathbb{K}} \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi} \mid \xi \in \mathcal{SP}_r\} \setminus \{0\} = \{\mathcal{T}^{\mathrm{c}}(n,r)_{\mathbb{K}} \otimes_{\mathcal{H}^{\mathrm{c}}_{r,\mathbb{K}}} U^{\xi} \mid \xi \in \mathcal{SP}(n,r)\}$$

is a complete set of nonisomorphic simple $Q_\nu(n,r)_{\mathbb{K}}$-supermodules. □

REMARK. (1) By regarding the category $\bigoplus_{r \geq 0} Q_\nu(n,r)_{\mathbb{K}}$−**smod** as a full subcategory of the category of finite-dimensional $U_\nu(\mathfrak{q}_n)$-supermodules, we recover the category $O^{\geq 0}_{\mathrm{int}}$ of tensor modules (the counterpart of polynomial representations studied by Green [18]) investigated in [17].

(2) We remark that, for the q-Schur superalgebras $\mathcal{S}_\nu(m|n,r)_{\mathbb{F}}$ of type M with $m + n \geq r$, their irreducible representations at (odd) roots of unity have been classified in [11], while a nonconstructible classification of irreducible $Q_\nu(n,r)_{\mathbb{F}}$-supermodules is obtained in [24, Theorem 6.32] by a generalised cellular structure [12, 16]. We remark that Green's codeterminant basis was a first such basis for the Schur algebra. Moreover, unlike the situation in [11], the link between representations of $Q_\nu(n,r)_{\mathbb{F}}$ and the quantum queer supergroup has not yet been established, since we do not know if the surjective map Φ_r given in Theorem 6.4 can be extended to the roots of unity case.

Acknowledgement

Appendix A. Comparison with Mori's basis

In [24, Proposition 6.10], Mori also obtained a basis for $x_\mu \mathcal{H}^{\mathrm{c}}_{r,R} \cap \mathcal{H}^{\mathrm{c}}_{r,R} x_\lambda$ for $\lambda, \mu \in \Lambda(n,r)$, using a different approach via the combinatorics of circled tableaux. In terms of our notation, we may describe Mori's basis as follows. First, denote the basis elements given in Corollary 3.8 by m_T, where T runs over all circled row-standard tableaux. Then use a combinatorial relation between a circled row-semistandard tableau S and certain circled row-standard tableaux T to define m_S as a linear combination of those m_T involved. We will see below that, if S gives the matrix pair $(A|B) \in M_n(\mathbb{N}|\mathbb{Z}_2)_{\mu,\lambda}$, then $m_\mathsf{S} = T_{A|B}$. In other words, Mori's definition is simply to write the element $T_{A|B}$ defined in (5.4) as a linear combination of the basis given in Corollary 3.8 for $\mathcal{H}^c_{r,R} x_\lambda$. We now establish the above-described relationship.

A Young diagram of a composition λ is defined by $Y(\lambda) = \{(i, j)|i \geq 1, 1 \leq j \leq \lambda_i\}$. A tableau of shape λ is a function $\mathsf{T} : Y(\lambda) \to \{1, 2, \ldots\}$. The weight of a tableau T is a composition $\mu = (\mu_1, \mu_2, \ldots)$ whose ith component is $\mu_i = |\mathsf{T}^{-1}(i)|$. The tableau T is said to be *row-semistandard* if the entries in each row of T are weakly increasing. We denote by $\mathrm{Tab}_{\lambda;\mu}$ the set of row-semistandard tableaux of shape λ and weight μ. It is known (cf. [24, Section 5.2]) that the set $\mathrm{Tab}_{\lambda;\mu}$ is in bijection with the set $\mathcal{D}_{\mu,\lambda}$ and hence in bijection with the set $M_n(\mathbb{N})_{\mu,\lambda}$. More precisely, given $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, let $M(\mathsf{S}) = (m_{ij})$ with m_{ij} being the number of i's in the jth row of S. Then clearly $M(\mathsf{S}) \in M_n(\mathbb{N})_{\mu,\lambda}$ and, by (5.1), we obtain a double coset representative $d_\mathsf{S} := d_{M(\mathsf{S})} \in \mathcal{D}_{\mu,\lambda}$. Conversely, it is easy to see that the matrix $M \in M_n(\mathbb{N})_{\mu,\lambda}$ uniquely determines a row-semistandard tableau S.

For each $\mathsf{S} \in \mathrm{Tab}_{\lambda;\mu}$, let $\mathrm{Tab}_\mathsf{S} = \{\mathsf{T} \in \mathrm{Tab}_{\lambda;(1^r)} \mid \mathsf{T}_\mu = \mathsf{S}\}$, where T_μ is the row-semistandard tableau obtained from T by replacing its entries $1, 2, \ldots, \mu_1$ by 1, $\mu_1 + 1, \ldots, \mu_1 + \mu_2$ by 2 and so forth. It is known from the discussion in [24, Section 5.2] that for each $\mathsf{T} \in \mathrm{Tab}_\mathsf{S}$,

$$d_\mathsf{T} = ud_\mathsf{S} \quad \text{and} \quad |\mathcal{D}^{-1}_{\nu(d_\mathsf{S}^{-1})} \cap \mathfrak{S}_\mu| = |\mathrm{Tab}_\mathsf{S}|, \tag{A.1}$$

where $u \in \mathcal{D}^{-1}_{\nu(d_\mathsf{S}^{-1})} \cap \mathfrak{S}_\mu$.

Following [24], we introduce the notion of circled tableau. A circled tableau of shape λ is a map $\mathsf{S} : Y(\lambda) \to \{1, 2, \ldots\} \sqcup \{\textcircled{1}, \textcircled{2}, \ldots\}$. From a circled tableau S we obtain its underlying ordinary tableau $\mathsf{S}^\times$ by removing circles from numbers. The weight of a circled tableau is defined as that of the underlying tableau. Denote by $\mathrm{Tab}^c_{\lambda;\mu}$ the set of circled tableaux S such that $\mathsf{S}^\times \in \mathrm{Tab}_{\lambda;\mu}$ and circled numbers must be placed at the rightmost of a bar $\boxed{i\ i\ \cdots\ i}$ in a row for every i.

For $\mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}$, define $M^c(\mathsf{S}) = (A|B) \in M(\mathbb{N}|\mathbb{Z}_2)_{\mu,\lambda}$ by letting $B = (b_{ij})$ with $b_{ij} = 0$ if the bar containing i in the jth row of S has the form $\boxed{i\ i\ \cdots\ i}$ and $b_{ij} = 1$ if the bar containing i in the jth row of S has the form $\boxed{i\ i\ \cdots\ \textcircled{i}}$, and $A = M(\mathsf{S}^\times) - B$. For example, for the circled tableau $\mathsf{S} = \boxed{\begin{array}{ccc} 1 & \textcircled{1} & 2 & \textcircled{3} \\ 1 & 5 & \textcircled{5} \\ \textcircled{4} \end{array}}$, we have $M^c(\mathsf{S}) = (A|B)$ with

$$M(\mathsf{S}^\times) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Conversely, given $(A|B) \in M(\mathbb{N}|\mathbb{Z}_2)_{\mu,\lambda}$, we have $A + B \in M(\mathbb{N})_{\mu,\lambda}$ and hence it uniquely determines an uncircled tableau in $\mathrm{Tab}_{\lambda;\mu}$ and the matrix B determines the places of the circled numbers. Hence, we obtain the following result.

LEMMA A.1. *For $\lambda, \mu \in \Lambda(n, r)$, the map $M^c : \mathrm{Tab}^c_{\lambda;\mu} \to M(\mathbb{N}|\mathbb{Z}_2)_{\mu,\lambda}$ is a bijection.*

In particular, we may use the set $\mathrm{Tab}^c_{\lambda;(1^r)}$ to label the basis elements for $\mathcal{H}^c_{r,R} x_\lambda$ given in Corollary 3.8 by setting, for $\mathsf{T} \in \mathrm{Tab}^c_{\lambda;(1^r)}$,

$$m_\mathsf{T} = T_{d_{\mathsf{T}^\times}} c^{\alpha_\mathsf{T}} x_\lambda \quad \text{(see [24, Section 6.4]),} \tag{A.2}$$

where $\alpha_\mathsf{T} = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{Z}_2^r$ with $\alpha_k = 1$ if the kth entry is circled and $\alpha_k = 0$ otherwise according to the top-to-bottom reading order.

By extending this definition to $\mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}$, Mori further defined the element $m_\mathsf{S} \in x_\mu \mathcal{H}^c_{r,R} \cap \mathcal{H}^c_{r,R} x_\lambda$. Suppose that the circled bars in S are $\boxed{i_1\ i_1 \cdots ①}$, $\boxed{i_2\ i_2 \cdots ②}$, ..., $\boxed{i_k\ i_k \cdots ⓚ}$ with lengths $r_1, r_2, \ldots, r_k$. By distributing

$$\boxed{i\ i \cdots ①} \mapsto \boxed{①\ i \cdots i} + q\,\boxed{i\ ① \cdots i} + \cdots + q^{r-1}\,\boxed{i\ i \cdots ①}$$

for every $i = i_t$ and $r = r_t$, $1 \le t \le k$, we first make a formal linear combination $\sum_{0 \le l_t \le r_t - 1, 1 \le t \le k} q^{l_1 + l_2 + \cdots + l_k} \mathsf{R}$ of circled row-semistandard tableaux R, where $\mathsf{R} = \mathsf{R}(l_1, l_2, \ldots, l_k)$ are the circled tableaux obtained from S by replacing the circled bars $\boxed{i_1\ i_1 \cdots ①}$, $\boxed{i_2\ i_2 \cdots ②}$, ..., $\boxed{i_k\ i_k \cdots ⓚ}$ by $\boxed{i_1 \cdots ① \cdots i_1}$, $\boxed{i_2 \cdots ② \cdots i_2}$, ..., $\boxed{i_k \cdots ⓚ \cdots i_k}$, in which the circled numbers are placed in the (l_1+1)th, (l_2+1)th, ..., (l_k+1)th places. Then define

$$m_\mathsf{S} = \sum_{0 \le l_t \le r_t - 1, 1 \le t \le k} q^{l_1 + l_2 + \cdots + l_k} \sum_{\mathsf{T} \in \Gamma(\mathsf{R})} m_\mathsf{T}, \tag{A.3}$$

where $\Gamma(\mathsf{R})$ consists of all $\mathsf{T} \in \mathrm{Tab}^c_{\lambda;(1^r)}$ such that $\mathsf{T}^\times \in \mathrm{Tab}_{\mathsf{S}^\times}$ and its positions of circles are the same as that of $\mathsf{R} = \mathsf{R}(l_1, l_2, \ldots, l_k)$.

PROPOSITION A.2. *For $\mathsf{S} \in \mathrm{Tab}^c_{\lambda;\mu}$, if $M^c(\mathsf{S}) = (A|B)$, then $m_\mathsf{S} = T_{A|B}$.*

PROOF. Observe that, for each $\mathsf{R} = \mathsf{R}(l_1, l_2, \ldots, l_k)$, $\Gamma(\mathsf{R})$ is a disjoint union of

$$\Gamma(\mathsf{R}, \mathsf{U}) := \{\mathsf{T} \in \mathrm{Tab}^c_{\lambda;(1^r)} \mid \mathsf{T}^\times = \mathsf{U}, \mathsf{T} \text{ and } \mathsf{R} \text{ have the same positions of circles}\}$$

for all $\mathsf{U} \in \mathrm{Tab}_{\mathsf{S}^\times}$. Thus, by (A.2), (A.1), (5.3), and (5.4), (A.3) becomes

$$\begin{aligned}
m_\mathsf{S} &= \sum_{0 \le l_t \le r_t - 1, 1 \le t \le k} q^{l_1 + l_2 + \cdots + l_k} \sum_{\mathsf{U} \in \mathrm{Tab}_{\mathsf{S}^\times}} \sum_{\mathsf{T} \in \Gamma(\mathsf{R}(l_1,l_2,\ldots,l_k),\mathsf{U})} T_{d_{\mathsf{T}^\times}} c^{\alpha_\mathsf{T}} x_\lambda \\
&= \sum_{\mathsf{U} \in \mathrm{Tab}_{\mathsf{S}^\times}} T_{d_\mathsf{U}} \sum_{0 \le l_t \le r_t - 1, 1 \le t \le k} q^{l_1 + l_2 + \cdots + l_k} \sum_{\mathsf{T} \in \Gamma(\mathsf{R}(l_1,l_2,\ldots,l_k),\mathsf{U})} c^{\alpha_\mathsf{T}} x_\lambda \\
&= \sum_{u \in \mathcal{D}^{-1}_{\nu(d^{-1}_{\mathsf{S}^\times})} \cap \mathfrak{S}_\mu} T_u T_{d_{\mathsf{S}^\times}} c'_{A|B} x_\lambda \\
&= T_{A|B},
\end{aligned}$$

as desired.   □

REMARK. With this identification and by Proposition 5.2, we now can remove the condition that $1 + q$ is not a zero divisor in R, stated in [24, Proposition 6.10]. Of course, the standard assumption that the characteristic of R is not equal to 2 in the super theory will always be maintained.

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