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Uniqueness of standing waves for nonlinear Schrödinger equations

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For $m > 0$ and $p \in (1, (N+2)/(N-2))$, we show the uniqueness and a linearized non-degeneracy of solutions for the following problem:

$$\Delta u - |x|^m u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0.$$

1. Introduction and statement of main results

We are interested in the uniqueness and linearized non-degeneracy of solutions satisfying

$$\Delta u - |x|^m u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0. \tag{1.1}$$

Throughout this paper, we assume that $N \geqslant 2$, $m > 0$ and $p \in (1, (N+2)/(N-2))$ for $N \geqslant 3$ and $p \in (1, \infty)$ for $N = 2$. Equation (1.1) comes from the study of standing waves for the nonlinear Schrödinger equation

$$\mathrm{i}\hbar\psi_t = -\tfrac{1}{2}\hbar^2\Delta\psi + V(x)\psi - |\psi|^{p-1}\psi \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.2}$$

where $\hbar$ is the Planck constant and i is the imaginary unit. A solution $\psi(x,t)$ of (1.2) is called a standing wave if

$$\psi(x,t) = \exp\biggl(-\mathrm{i}\frac{E}{\hbar}t\biggr)u(x)$$

for some real-valued function u. Then the function u satisfies

$$\tfrac{1}{2}\hbar^2\Delta u - (V(x) - E)u + u^p = 0 \quad \text{in } \mathbb{R}^N.$$

We are interested in semi-classical standing waves. Thus, shifting E to 0 without loss of generality, we consider

$$\varepsilon^2\Delta u - V(x)u + u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0 \tag{1.3}$$

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for sufficiently small $\varepsilon > 0$. The case $\inf_{x\in\mathbb{R}^N} V(x) > 0$ has been studied extensively by many authors (see [2–4, 10] and references therein). On the other hand, for the case $\inf_{x\in\mathbb{R}^N} V(x) = 0$ some solutions contrasting with those for the case $\inf_{x\in\mathbb{R}^N} V(x) > 0$ have been obtained in [1–3]. In fact, for each isolated connected component L of $\{x \in \mathbb{R}^N \mid V(x) = 0\}$, there exists a solution u_ε of (1.3) such that

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty = 0, \qquad \liminf_{\varepsilon\to 0} \varepsilon^{-2/(p-1)}\|u_\varepsilon\|_\infty > 0$$

and that, for each $\delta > 0$, there exist some $C, c > 0$ satisfying

$$u_\varepsilon(x) \leqslant C \exp\biggl(-\frac{c}{\varepsilon}\operatorname{dist}(x, L)\biggr) \quad \text{if } \operatorname{dist}(x, L) > \delta.$$

In particular, if $L = \{x_0\}$ and $V(x) = |x - x_0|^m$ for small $|x - x_0|$,

$$\varepsilon^{-(2/(p-1))(m/(m+2))}u_\varepsilon(\varepsilon^{2/(m+2)}x + x_0)$$

converges to a least energy solution of (1.1) as $\varepsilon \to 0$. Furthermore, in [1] we glue together u_ε with other semi-classical standing waves of different energy scales. For this, we assume the uniqueness and linearized non-degeneracy of solutions of (1.1). For $m = 2$, the uniqueness of solutions of (1.1) was proved in [6]. In this paper, we will prove the uniqueness and linearized non-degeneracy of solutions of (1.1) for any $m > 0$.

In addition to the uniqueness of positive solutions, we are interested in a linearized non-degeneracy of the unique positive solution. To see a linearized non-degeneracy of the unique positive solution, we consider the uniqueness of solutions for the following problem, which is more general than (1.1):

$$\Delta u - (V(x) + \delta a(x))u + (1 + \delta b(x))u^p = 0, \quad u > 0 \text{ in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0. \tag{1.4}$$

Here, $\delta > 0$ and two functions a, b are radially symmetric functions with compact supports. To see a linearized non-degeneracy of the unique positive solution, we will require $a = bu_0^{p-1}$, where u_0 is a solution of (1.4) for $\delta = 0$. Then the u_0 is also a solution of (1.4) for $\delta > 0$. In view of the bifurcation theory, the uniqueness of solutions of (1.4) for $\delta > 0$ may imply a linearized non-degeneracy of (1.4) for $\delta = 0$. We will show the linearized non-degeneracy by a variational argument.

To state our main results, we define

$$\begin{aligned}
\beta &= \frac{2(N-1)(p-1)}{p+3} \in (0, 2),\\
L &= \frac{2(N-1)\{(N-2)p + N - 4\}}{(p+3)^2},\\
G(r) &= V(r)r^\beta + Lr^{-(2-\beta)},\\
H(r) &= r^{3-\beta}G'(r) = \beta r^2V(r) + r^3V'(r) - L(2-\beta).
\end{aligned}$$

We note that $L < 0$ for $N = 2$ and $L > 0$ for $N \geqslant 3$. We make the following conditions on V.

(V0) a and b are radially symmetric non-negative functions with compact supports $a \in C^2$, $b \in C^3$, and $b = 1$ in a neighbourhood of 0.

(V1) $V(x) = V(|x|)$, $V \in C([0,\infty)) \cap C^2((0,\infty))$, $\liminf_{|x|\to\infty} V(x) > 0$ and $\lambda_1(-\Delta + V(x)) > 0$, where $\lambda_1(-\Delta + V(x))$ is the minimum value of the spectrum of $-\Delta + V(x)$.

(V2) one of the following hold:

(i) $\{r > 0 \mid G'(r) = 0,\ G(r) \geqslant 0\} = \emptyset$;

(ii) $\inf_{r>0} H(r) > 0$ for $N = 2$, and H has the unique simple zero in $(0,\infty)$ and $\limsup_{r\to 0} H(r) < 0$ for $N \geqslant 3$.

(V3) V is non-decreasing and non-constant on $(0,\infty)$.

A radial solution $u(x) = u(r)$, $r = |x|$ of (1.4) solves

$$\left.\begin{aligned} u_{rr} + \frac{N-1}{r}u_r - (V(r) + \delta a(r))u + (1 + \delta b(r))u^p = 0, \quad & u > 0 \text{ in } (0,\infty), \\ u_r(0) = 0, \quad & \\ \lim_{r\to\infty} u(r) = 0. \quad & \end{aligned}\right\} \tag{1.5}$$

Theorem 1.1. *Suppose that (V0), (V1) and (V2) hold. Then, for small $\delta \geqslant 0$, (1.5) has a unique solution.*

Definition 1.2. A solution w for $\delta = 0$ of (1.4) is *non-degenerate* in $\mathbb{H} \subset H^{1,2}(\mathbb{R}^N)$ if $\phi = 0$ is the unique solution in $\mathbb{H}$ of $\Delta\phi - V(x)\phi + pw^{p-1}\phi = 0$.

Theorem 1.3. *Suppose that (V1), (V2) and (V3) hold. Then, for $\delta = 0$, (1.4) has a unique solution which is non-degenerate in $H^{1,2}(\mathbb{R}^N)$.*

Remark 1.4. In fact, for $\delta = 0$, we can prove the uniqueness of radial solutions of (1.4) under (V1) and the following conditions weaker than (V2). One of the following holds:

(i) $\{r > 0 \mid G'(r) = 0,\ G(r) > 0\} = \emptyset$;

(ii) H is non-negative for all $r > 0$ and $H \not\equiv 0$ if $N = 2$, and the zero set of $H(r)$ is connected in $(0,\infty)$ and $\limsup_{r\to 0} H(r) < 0$ for $N \geqslant 3$.

We will give sufficient conditions for (V2) in § 4 (see propositions 4.1 and 4.2). In particular, we have the following.

Corollary 1.5. *Let $V(x) = |x|^m - \mu$, $m > 0$. Assume that $\lambda_1(-\Delta + |x|^m) > \mu$. Then, for $\delta = 0$, (1.4) has a unique solution which is non-degenerate in $H^{1,2}(\mathbb{R}^N)$.*

Remark 1.6. Let $\lambda(m,N) = \lambda_1(-\Delta + |x|^m)$. Then one can show that

$$\lim_{m\to 0} \lambda(m,N) = 1$$

and that $\lim_{m\to\infty} \lambda(m,N)$ exists and equals the square of the first zero of the Bessel function of order $\frac{1}{2}(N-2)$. It is also known that $\lambda(2,N) = N$.

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2. Proof of theorem 1.1

Under condition (V1), there exists a solution of (1.5) for small $\delta > 0$. To prove the uniqueness, we follow the approach used in [7, 8]. We use an argument similar to the proof of [7, theorem 0.1] for $N = 2$.

Let u be a solution of (1.5). For $r \geqslant 0$, let

$$K_\delta(r) \equiv 1 + \delta b(r) \quad \text{and} \quad V_\delta(r) \equiv V(r) + \delta a(r).$$

Note that

$$u'' + \frac{N-1}{r}u' - V_\delta(r)u + K_\delta(r)u^p = 0. \tag{2.1}$$

We define

$$\alpha = \frac{2(N-1)}{p+3}, \quad A_\delta(r) \equiv r^\alpha K_\delta^{1/(p+3)}(r), \quad v(r) = A_\delta(r)u(r).$$

Then v satisfies

$$B_\delta(r)v'' + \tfrac{1}{2}B_\delta'(r)v' - G_\delta(r)v + v^p = 0,$$

where

$$\begin{aligned} B_\delta(r) &\equiv r^\beta K_\delta^{-4/(p+3)}(r), \\ G_\delta(r) &\equiv B_\delta(r)\biggl[V_\delta(r) + \frac{L}{r^2} + \frac{(N-1)(p-1)}{(p+3)^2}\frac{K_\delta'(r)}{rK_\delta(r)} \\ &\qquad - \frac{p+4}{(p+3)^2}\frac{(K_\delta')^2(r)}{K_\delta^2(r)} + \frac{1}{p+3}\frac{K_\delta''(r)}{K_\delta(r)}\biggr]. \end{aligned}$$

Let

$$E_\delta(r;v) \equiv \tfrac{1}{2}B_\delta(r)(v')^2 - \tfrac{1}{2}G_\delta(r)v^2 + \frac{1}{p+1}v^{p+1}. \tag{2.2}$$

Then it follows that

$$E_\delta'(r;v) = -\tfrac{1}{2}G_\delta'(r)v^2. \tag{2.3}$$

Note that $K_\delta'(r) = 0$ in a neighbourhood of $\{0, \infty\}$, and that $G_0(r) = G(r)$. Since

$$2\alpha + \beta - 2 = \begin{cases} 2N - 4 - \dfrac{4(N-1)}{p+3} > N - 3 & \text{for } N \geqslant 3, \\ -\dfrac{4}{p+3} \in (-1, 0) & \text{for } N = 2, \end{cases}$$

it follows that

$$\begin{aligned} \lim_{r\to 0} G_\delta(r)v^2 &= \lim_{r\to 0} K_\delta^{-4/(p+3)}(Lr^{2\alpha+\beta-2}u^2(r) + V_\delta(r)r^{2\alpha+\beta}u^2(r)) \\ &= \begin{cases} -\infty, & N = 2, \\ 0, & N \geqslant 3. \end{cases} \end{aligned} \tag{2.4}$$

Hence, we see that

$$\lim_{r\to 0} E_\delta(r) = \begin{cases} \infty, & N = 2, \\ 0, & N \geqslant 3. \end{cases} \tag{2.5}$$

Note that

$$2\alpha + \beta = \frac{2(N-1)(p+1)}{p+3} < 2(N-1).$$

Then, from lemma A.1, we see that

$$\limsup_{r\to\infty} E_\delta(r) = 0. \tag{2.6}$$

We now claim that if $\delta \geqslant 0$ is small,

$$E_\delta(r) \geqslant 0 \quad \text{for all } r > 0 \text{ and } E_\delta \not\equiv 0. \tag{2.7}$$

Case (i) of (V2)

By (2.2) and (2.3), it follows that

$$\{r > 0 \mid E'_\delta(r) = 0,\ E_\delta(r) < 0\} \subset \{r > 0 \mid G'_\delta(r) = 0,\ G_\delta(r) > 0\} \equiv \mathcal{Z}.$$

Note that $G_\delta(r) = G(r)$ for large $r > 0$. Suppose that there exists $r_\delta \in \mathcal{Z} > 0$ for small $\delta > 0$. Then, from (V2) case (i), we see that $\{r_\delta\}$ is bounded. Then we may assume that $r_\delta \to r_0 \geqslant 0$ as $\delta \to 0$. Then, $G'(r_0) = 0$ and $G(r_0) \geqslant 0$. Since $\lim_{r\to 0} |G(r)| = \infty$, it follows that $r_0 > 0$. This is a contradiction. Thus, by (2.3), (2.5) and (2.6), we have $E_\delta \geqslant 0$ for all $r > 0$ if $\delta \geqslant 0$ is small as claimed.

Assume by contradiction that $E_\delta \equiv 0$ for small $\delta \geqslant 0$. Then G_δ is constant. This contradicts $\lim_{r\to 0} |G_\delta(r)| = \infty$. This proves (2.7) when (V2) case (i) holds.

Case (ii) of (V2)

Define $H_\delta(r) = r^{3-\beta} G'_\delta(r)$. Then, if $H_\delta \not\equiv 0$, $E_\delta \not\equiv 0$.

Let $N = 2$. Then, by (2.3), $E'_\delta(r) \leqslant 0$ if $\delta \geqslant 0$ is small. Hence, it follows from (2.5) and (2.6) that $E_\delta(r) \geqslant 0$.

Let $N \geqslant 3$ and $R \in (0, \infty)$ be the unique zero of H. From the fact that $\limsup_{r\to 0} H(r) < 0$, we see that H (and hence G') is negative in $(0, R)$ and positive in (R, ∞). Then, from (V0), we deduce that, for small $\delta > 0$, there exists $R_\delta > 0$ such that $\lim_{\delta\to 0} R_\delta = R$ and R_δ is a unique simple zero of H_δ. Moreover, $\limsup_{r\to 0} H_\delta(r) < 0$ for small $\delta > 0$. By (2.3), E_δ is non-decreasing in $(0, R_\delta)$ and non-increasing in (R_δ, ∞). Then, from (2.5) and (2.6), it follows that $E_\delta \gneqq 0$ for small $\delta \geqslant 0$. This proves (2.7) when case (ii) of (V2) holds.

Let us assume by contradiction that (2.1) has another positive solution $\tilde{u}$. We may assume that $\tilde{u}(0) > u(0)$. By lemma A.3, we can choose a $\tilde{u}$ which intersects u at most once. Then, by lemma A.2,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\tilde{u}(r)}{u(r)}\right) < 0, \quad r > 0. \tag{2.8}$$

Let $\tilde{v}(r) = A_\delta(r)\tilde{u}(r)$, $E_\delta(r) = E_\delta(r; v)$, $\tilde{E}_\delta(r) = E_\delta(r; \tilde{v})$ and

$$F_\delta(r) \equiv \tilde{E}_\delta(r) - \left(\frac{\tilde{v}}{v}\right)^2 E_\delta(r).$$

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It then follows from (2.8) that

$$\begin{aligned} F_\delta'(r) &= \tilde{E}_\delta' - \left(\frac{\tilde{v}}{v}\right)^2 E_\delta' - \left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\tilde{v}}{v}\right)^2\right\} E_\delta \\ &= -\tfrac{1}{2}G_\delta'\tilde{v}^2 + \tfrac{1}{2}G_\delta' v^2\left(\frac{\tilde{v}}{v}\right)^2 - \left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\tilde{v}}{v}\right)^2\right\} E_\delta \\ &= -\left\{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\tilde{v}}{v}\right)^2\right\} E_\delta \geqslant 0. \end{aligned} \tag{2.9}$$

This also implies that

$$0 < \left(\frac{\tilde{v}(r)}{v(r)}\right)^2 \leqslant \left(\frac{\tilde{v}(0)}{v(0)}\right)^2$$

for all $r \geqslant 0$. Then it follows that

$$\liminf_{r\to\infty} F_\delta(r) \leqslant 0. \tag{2.10}$$

On the other hand, note that

$$\begin{aligned} F_\delta(r) = A_\delta A_\delta' B_\delta\left\{\tilde{u}\tilde{u}' - \left(\frac{\tilde{u}}{u}\right)^2 uu'\right\} + \tfrac{1}{2}A_\delta^2 B_\delta\left\{(\tilde{u}')^2 - \left(\frac{\tilde{u}}{u}\right)^2(u')^2\right\} \\ + \frac{1}{p+1}\left\{(A_\delta\tilde{u})^{p+1} - (A_\delta u)^{p+1}\left(\frac{\tilde{u}}{u}\right)^2\right\} \end{aligned}$$

and that

$$\begin{gathered} A_\delta(r) = (1+\delta)^{1/(p+3)} r^\alpha, \quad B_\delta(r) = (1+\delta)^{-4/(p+3)} r^\beta \text{ for small } r > 0, \\ \beta + 2\alpha - 1 = 2N - 3 - \frac{4(N-1)}{p+3} > N - 2, \\ u'(0) = \tilde{u}'(0) = 0, \quad 0 < u(0) < \tilde{u}(0). \end{gathered}$$

Thus, we see that $F_\delta(r) \to 0$ as $r \to 0$. Then, from (2.9), (2.10), we see that $F_\delta \equiv 0$. By (2.8) and (2.9), it follows that $E_\delta \equiv 0$, which is a contradiction. This completes the proof of theorem 1.1.

3. Proof of theorem 1.3

We define

$$\|\phi\| = \left(\int_{\mathbb{R}^N} |\nabla\phi|^2 + V(x)\phi^2\,\mathrm{d}x\right)^{1/2},$$

and let X and X_{rad} be the completion of $C_0^\infty(\mathbb{R}^N)$ and $\{\phi \in C_0^\infty(\mathbb{R}^N) | \phi(x) = \phi(|x|)\}$ with respect to $\|\cdot\|$, respectively. We note that

$$\|\phi\|^2 \geqslant \lambda_1(-\Delta + V(x)) \int_{\mathbb{R}^N} \phi^2\,\mathrm{d}x$$

and $X \subset H^{1,2}(\mathbb{R}^N)$.

By the condition (V3), any solution of (1.4) with $\delta = 0$ is radially symmetric (see [5, 9]). Thus, it follows from theorem 1.1 that (1.4) with $\delta = 0$ has a unique solution. Let w be the unique solution of (1.5) with $\delta = 0$. Multiplying $-(r^{N-1}w_r)_r + r^{N-1}V(r)w = r^{N-1}w^p$ by w and integrating over $(0, r)$ yields

$$-r^{N-1}w'(r)w(r) + \int_0^r ((w')^2(s) + V(s)w^2(s))s^{N-1}\,\mathrm{d}s = \int_0^r w^{p+1}(s)s^{N-1}\,\mathrm{d}s.$$

Then we see that $w \in X_{\mathrm{rad}}$ by lemma A.1. We shall prove that w is non-degenerate in $H^{1,2}(\mathbb{R}^N)$. Assume by contradiction that there exists $\psi \in H^{1,2}(\mathbb{R}^N) \setminus \{0\}$ such that

$$\Delta\psi - V(x)\psi + pw^{p-1}\psi = 0, \quad \psi \not\equiv 0.$$

Then, by lemma A.5, $\psi(x) = \psi(|x|)$; hence, by lemma A.1, $\psi \in X_{\mathrm{rad}}$. It is obvious that $w \neq \psi$. Let $T \equiv \mathrm{span}\langle w, \psi\rangle$ be a two-dimensional subspace of X_{rad}. We take $R > 0$ to be sufficiently large such that

$$\alpha w + \beta\psi = 0 \in B_R(0), \quad \alpha, \beta \in \mathbb{R} \quad \Longrightarrow \quad \alpha = \beta = 0.$$

Choose $b \in C_0^\infty([0, \infty); [0, 1])$ such that $b(r) = 1$ for $r \in [0, R]$, $b(r) = 0$ for $r \in [R + 1, \infty)$ and let

$$K_\delta(r) \equiv 1 + \delta b(r), \quad V_\delta(r) = V(r) + \delta a(r) \equiv V(r) + \delta b(r)w^{p-1}(r), \quad r \geqslant 0,$$

with small $\delta > 0$. Then, we see that w is also a solution of (1.5) for all $\delta \geqslant 0$. We define an energy functional Γ_δ on X_{rad} as follows:

$$\Gamma_\delta(u) \equiv \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) + \delta a(x))u^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} (1 + \delta b(x))u^{p+1}\,\mathrm{d}x.$$

Then, the unique solution w of (1.5) corresponds to a critical point of the functional Γ_δ. It is standard to see that for small $\delta \geqslant 0$, there exists a minimizer of Γ_δ over a Nehari manifold M. Here M is defined by

$$M = \left\{u \in X_{\mathrm{rad}} \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) + \delta a(x))u^2 - (1 + \delta b(x))u^{p+1}\,\mathrm{d}x = 0\right\}.$$

The minimizer is a solution of (1.5). Thus, by theorem 1.1, the minimizer should be the solution w. Note that

$$\begin{aligned}\Gamma_\delta(u) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2\,\mathrm{d}x - \frac{1}{p+1}\int_{\mathbb{R}^N} u^{p+1}\,\mathrm{d}x \\ &\quad + \delta\int_{\mathbb{R}^N} \tfrac{1}{2}a(x)u^2 - \frac{1}{p+1}b(x)u^{p+1}\,\mathrm{d}x \\ &\equiv \Gamma(u) + \delta G(u),\end{aligned}$$

and that, for any $\varphi \in X_{\mathrm{rad}}$,

$$\begin{aligned}\Gamma_\delta''(w)(\varphi, \varphi) &= \Gamma''(w)(\varphi, \varphi) + \delta\int_{\mathbb{R}^N} a(x)\varphi^2 - pb(x)w^{p-1}\varphi^2\,\mathrm{d}x \\ &= \Gamma''(w)(\varphi, \varphi) + \delta(1 - p)\int_{\mathbb{R}^N} a(x)\varphi^2\,\mathrm{d}x.\end{aligned}$$

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Thus, we see that, for any $\varphi \in T \setminus \{0\}$, we have $\Gamma''_\delta(w)(\varphi,\varphi) < 0$. This contradicts our assertion that w is a minimizer of Γ_δ over the Nehari manifold M.

This completes the proof of theorem 1.3.

4. Proof of corollary 1.5

It is obvious that the function $V(x) = |x|^m - \mu$ with $m > 0$, $\lambda_1(-\Delta + |x|^m) > \mu$ satisfies (V1) and (V3). Thus, it suffices to show that V satisfies (V2).

Case 1 ($N = 2$). Since V is non-decreasing, the following proposition implies that $V(x) = |x|^m - \mu$, $m > 0$, satisfies case (i) of (V2).

Proposition 4.1. *Let $N = 2$. Assume that $V \in C([0,\infty)) \cap C^1((0,\infty))$ is non-decreasing. Then V satisfies case (i) of (V2).*

Proof. If $G'(r_*) = 0$ for some $r_* > 0$, then

$$V(r_*) = -\frac{r_* V'(r_*)}{\beta} - \frac{L(\beta-2)}{r_*^2\beta}.$$

Noting that $V'(r) \geqslant 0$ for all $r > 0$ and $L < 0$, we have

$$\begin{aligned} G(r_*) &= r_*^\beta\biggl\{-\frac{r_* V'(r_*)}{\beta} - \frac{L(\beta-2)}{r_*^2\beta}\biggr\} + Lr_*^{\beta-2} \\ &= \frac{2Lr_*^{\beta-2} - r_*^{\beta+1}V'(r_*)}{\beta} < 0. \end{aligned}$$

It follows that

$$\{r > 0 \mid G'(r) = 0,\ G(r) \geqslant 0\} = \emptyset,$$

namely, case (i) of (V2) holds. □

Case 2 ($N \geqslant 3$). The following proposition implies that case (ii) of (V2) holds for $V(x) = |x|^m - \mu$, $m > 0$.

Proposition 4.2. *Let $N \geqslant 3$. Assume that $V \in C([0,\infty)) \cap C^2((0,\infty))$, and that V and $rV'(r)$ are non-decreasing and $\lim_{r\to\infty} V(r) = \infty$. Then V satisfies case (ii) of (V2).*

Proof. Since $\lim_{r\to 0} rV'(r) \in [0,\infty)$ and $L > 0$, $\lim_{r\to 0} H(r) = -L(2-\beta) < 0$. By the assumptions, a function

$$\frac{H(r)}{r^2} = \beta V(r) + rV'(r) - \frac{L(2-\beta)}{r^2}$$

is strictly increasing. Note that $\lim_{r\to\infty} H(r)/r^2 = \infty$ and $\lim_{r\to 0} H(r)/r^2 = -\infty$. Thus, H has a unique zero. Note that

$$\frac{\mathrm{d}}{\mathrm{d}r}\biggl(\frac{H(r)}{r^2}\biggr) = \beta V'(r) + (rV')'(r) + 2\frac{L(2-\beta)}{r^3}.$$

Since $\beta, V', (rV')' \geqslant 0$ and $2L(2-\beta)/r^3 > 0$ for $r > 0$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}r}\biggl(\frac{H(r)}{r^2}\biggr) > 0 \quad \text{for all } r > 0.$$

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Then, for a unique zero $r_0 > 0$ of H,

$$0 < \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{H(r)}{r^2}\right)\bigg|_{r=r_0} = \frac{-2H(r_0) + r_0 H'(r_0)}{(r_0)^3} = \frac{H'(r_0)}{(r_0)^2}.$$

Thus, we see that $H'(r_0) > 0$. Hence, case (ii) of (V2) holds. □

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Appendix A.

LEMMA A.1. *Let u be a solution of*

$$u_{rr} + \frac{N-1}{r}u_r - V(r)u + K(r)u^p = 0, \quad u(r) > 0 \textit{ in } r \in (0,\infty), \quad \lim_{r\to\infty} u(r) = 0,$$

where $V \in C((0,\infty))$, $\liminf_{r\to\infty} V(r) \equiv c \in (0,\infty]$ and $K \in C((0,\infty))$, $0 \leqslant K \leqslant c_1 < \infty$. Then, for any $\gamma \in (0, \sqrt{c})$, there exists a constant $C > 0$ such that

$$u(r) \leqslant C\exp(-\gamma r), \quad r \geqslant 1.$$

Moreover, for sufficiently large $r > 0$, $u_r(r) < 0$ and

$$\lim_{r\to\infty} r^{N-1}u_r(r) = \liminf_{r\to\infty} r^{N-1}V(r)u(r) = 0.$$

Proof. The decay property $u(r) \leqslant C\exp(-\gamma r)$ comes from standard comparison principles. Since $\lim_{r\to\infty} u(r) = 0$ and $u(r) > 0$ for $r > 0$, there exists r_0 such that $V(r)u(r) - K(r)(u(r))^p > 0$ for $r > r_0$. Note that

$$(r^{N-1}u_r)_r = r^{N-1}(V(r)u(r) - K(r)(u(r))^p).$$

Then, integrating the above equation over $[r, R]$, we see that

$$R^{N-1}u_r(R) - r^{N-1}u_r(r) = \int_r^R s^{N-1}(V(s)u(s) - K(s)(u(s))^p)\,\mathrm{d}s. \tag{A 1}$$

Suppose that for some $r_1 > r_0$, $u_r(r_1) \geqslant 0$. Since $\lim_{r\to\infty} u(r) = 0$ and $u(r) > 0$ for $r > 0$, there exists $r_2 > r_1$ such that $u_r(r_2) \leqslant 0$. Then, from (A 1), we see that

$$0 \geqslant (r_2)^{N-1}u_r(r_2) - (r_1)^{N-1}u_r(r_1) = \int_{r_1}^{r_2} s^{N-1}(V(s)u(s) - K(s)(u(s))^p)\,\mathrm{d}s > 0;$$

this is a contradiction. Thus, we see that $u_r(r) < 0$ for $r > r_0$. Then, since $(r^{N-1}u_r)_r = r^{N-1}(V(r)u(r) - K(r)(u(r))^p) > 0$ for $r > r_0$, it follows that

$$\lim_{r\to\infty} r^{N-1}u_r(r) \equiv m$$

exists and $m \leqslant 0$. Suppose that $m < 0$. Then, $r^{N-1}u_r(r) < \frac{1}{2}m$ for sufficiently large $r > 0$. This contradicts the exponential decay of u; it thus follows that $m = 0$. Then, it follows from (A 1) that

$$-r^{N-1}u_r = \int_r^\infty s^{N-1}(V(s)u(s) - K(s)(u(s))^p)\,\mathrm{d}s < \infty.$$

This implies that $\liminf_{r\to\infty} r^{N-1}V(r)u(r) = 0$, which completes the proof. □

Lemma A.2. *Let u_1, u_2 be two distinct solutions of*

$$u_{rr} + \frac{N-1}{r}u_r - V(r)u + K(r)f(u) = 0, \quad u > 0 \text{ in } (0,\infty),$$
$$u_r(0) = 0, \qquad \lim_{r\to\infty} u(r) = 0,$$

where $V \in C([0,\infty))$, $\liminf_{r\to\infty} V(r) > 0$, $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $f(s)/s$ is monotonically increasing, $\lim_{s\to 0} f(s)/s = 0$ and $K \in C((0,\infty))$, $0 < c_1 \leqslant K \leqslant c_2 < \infty$.

(i) *If*

$$0 < u_1(r) < u_2(r), \qquad 0 \leqslant r < \sigma,$$
$$u_1(r) > u_2(r) > 0, \quad \sigma < r < \infty,$$

for some $\sigma > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_1}{u_2}\right) > 0$$

for all $r > 0$.

(ii) *If*

$$u_1(r) < u_2(r), \quad 0 \leqslant r < \sigma,$$

for some $\sigma > 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_1}{u_2}\right) > 0$$

for all $r \in (0,\sigma)$.

Proof. This is an extension of [7, lemma 1.2].

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_1}{u_2}\right) = \frac{(r^{N-1}u_{1,r})u_2 - (r^{N-1}u_{2,r})u_1}{r^{N-1}u_2^2}.$$

Let $g = g(r)$ be the numerator of the right-hand side of the above equation. Then $\lim_{r\to\infty} g(r) = 0 = g(0)$ by lemma A.1. Using

$$\begin{aligned} g_r(r) &= (r^{N-1}u_{1,r})_r u_2 - (r^{N-1}u_{2,r})_r u_1 \\ &= r^{N-1}\{(V(r)u_1 - K(r)f(u_1))u_2 - (V(r)u_2 - K(r)f(u_2))u_1\} \\ &= r^{N-1}K(r)u_1u_2\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right), \end{aligned}$$

we easily see the consequences in both cases (i) and (ii). □

LEMMA A.3. *Let u_1, u_2 be two distinct solutions of*

$$\left.\begin{aligned} u_{rr}+\frac{N-1}{r}u_r-V(r)u+K(r)u^p=0,\quad u>0\ \textit{in}\ (0,\infty),\\ u_r(0)=0,\\ \lim_{r\to\infty}u(r)=0, \end{aligned}\right\} \tag{A 2}$$

where $V\in C([0,\infty))$, $\liminf_{r\to\infty}V(r)>0$ and $K\in C([0,\infty))$, $0<c_1\leqslant K\leqslant c_2<\infty$. Assume that $u_1(0)<u_2(0)$. Then there exists a solution u_3 of (A 2) such that

$$u_3(0)\geqslant u_2(0),$$
$$\#\{r>0\mid u_1(r)=u_3(r)\}\leqslant 1.$$

Proof. This is an extension of [7, proposition 1.1].

Let $u(r;\alpha)$, $\alpha>0$ be the solution of an initial-value problem

$$u_{rr}+\frac{N-1}{r}u_r-V(r)u+K(r)u^p=0 \text{ in } r\in(0,\infty),\quad u(0)=\alpha,\quad u_r(0)=0.$$

Let $n(\alpha)=\#\{r>0\mid u_1(r)=u(r;\alpha)\}$ for $\alpha\geqslant u_2(0)$. We have only to consider the case $n(u_2(0))\geqslant 2$. Then, for $\alpha>u_2(0)$ sufficiently close to $u_2(0)$, $n(\alpha)\geqslant 2$. Let

$$\alpha_*=\sup\{\alpha>u_2(0)\mid n(\tilde\alpha)\geqslant 2 \text{ for all } \tilde\alpha\in(u_2(0),\alpha)\}.$$

For $\alpha\in(u_2(0),\alpha_*)$, let $\sigma_1(\alpha)<\sigma_2(\alpha)$ be the first and second intersection points of u_1 and $u(\cdot\,,\alpha)$. We claim that $u(\cdot\,,\alpha)>0$ in $(0,\sigma_2(\alpha))$ for $\alpha\in(u_2(0),\alpha_*)$. Moreover, we claim that $\alpha_*<\infty$.

The first claim is proved by the uniqueness of the initial-value problem of ODE. The second claim is proved by the first claim and a rescaling argument. Indeed, assume by contradiction that $\alpha_*=\infty$. Then, for all large α, $0<u_1<u(\cdot\,;\alpha)$ in $[0,\sigma_1(\alpha))$, $0<u(\cdot\,;\alpha)<u_1$ in $(\sigma_1(\alpha),\sigma_2(\alpha))$ and $u(\sigma_2(\alpha);\alpha)=u_1(\sigma_2(\alpha))$. On the other hand, $v(r;\alpha)=\alpha^{-1}u(r/\alpha^{(p-1)/2};\alpha)$ solves

$$v_{rr}+\frac{N-1}{r}v_r-\frac{1}{\alpha^{p-1}}V\left(\frac{r}{\alpha^{(p-1)/2}}\right)v+K\left(\frac{r}{\alpha^{(p-1)/2}}\right)v^p=0\quad\text{in } r\in(0,\infty),$$
$$v(0)=1,\quad v_r(0)=0.$$

As $\alpha\to\infty$, $v(r;\alpha)$ converges in $C^1_{\mathrm{loc}}([0,\infty))$ to the solution w of

$$w_{rr}+\frac{N-1}{r}w_r+K(0)w^p=0 \text{ in } r\in(0,\infty),$$
$$w(0)=1,\quad w_r(0)=0.$$

It follows that if α is large, there exists some $r\in(\sigma_1(\alpha),\sigma_2(\alpha))$ such that $u(r;\alpha)=0$. This is a contradiction and shows that $\alpha_*<\infty$.

Now we observe that one of the following holds:

(i) $\lim_{\alpha\nearrow\alpha_*}\sigma_1(\alpha)=\lim_{\alpha\nearrow\alpha_*}\sigma_2(\alpha)=\infty$, or

(ii) $\lim_{\alpha\nearrow\alpha_*}\sigma_1(\alpha)<\infty$, $\lim_{\alpha\nearrow\alpha_*}\sigma_2(\alpha)=\infty$.

Moreover, $n(\alpha_*) = 0$ in case (i), $n(\alpha_*) = 1$ in case (ii). Finally, from lemma A.2, we see that $u(\cdot, \alpha_*)$ is a solution of (A 2). □

LEMMA A.4 (Kabeya and Tanaka [7, lemma 2.3]). *Let w be a solution of*

$$u_{rr} + \frac{N-1}{r} u_r - V(r)u + u^p = 0, \quad u > 0, \quad r \in (0, \infty),$$
$$u_r(0) = 0, \qquad \lim_{r \to \infty} u(r) = 0,$$

where $V \in C([0, \infty))$, $\liminf_{r\to\infty} V(r) > 0$ and $V(r_1) \leqslant V(r_2)$ for $r_1 \leqslant r_2$. Then $w'(r) < 0$ for $r > 0$.

Proof. This comes from the standard moving-plane method (see [5]). □

LEMMA A.5. *Assume $V(x) = V(|x|)$, $V(r) \in C^1((0, \infty)) \cap C([0, \infty))$, and $V' \geqslant 0$, $V' \not\equiv 0$. Let w be a solution of (1.5) and let $\psi \in H^1$ satisfy*

$$\Delta\psi - V(r)\psi + pw^{p-1}\psi = 0.$$

Then $\psi(x) = \psi(|x|)$.

Proof. Suppose that ψ is not radially symmetric. Then, we may assume that

$$\psi(x_1, x_2, \dots, x_N) \neq \psi(-x_1, x_2, \dots, x_N).$$

Define

$$\phi(x_1, x_2, \dots, x_N) = \psi(x_1, x_2, \dots, x_N) - \psi(-x_1, x_2, \dots, x_N).$$

Let Ω be a connected component of $\{x = (x_1, x_2, \dots, x_N) \mid \phi(x) > 0, x_1 > 0\}$ and $\Omega_\epsilon \equiv \{x \in \Omega \mid \phi(x) > \epsilon\}$. By Sard's theorem, there exists $\epsilon_m > 0$ with $\lim_{m\to\infty} \epsilon_m = 0$ such that $\{\epsilon_m\}_{m=1}^{\infty}$ are regular values of ϕ. Note that

$$\Delta\phi - V(r)\phi + pw^{p-1}\phi = 0$$

and

$$\Delta \frac{\partial w}{\partial x_1} - V(r)\frac{\partial w}{\partial x_1} + pw^{p-1}\frac{\partial w}{\partial x_1} = \frac{\partial V}{\partial x_1} w. \tag{A 3}$$

Multiplying both sides of (A 3) by ϕ and integrating by parts on Ω_{ϵ_m}, we see that

$$\int_{\partial\Omega_{\epsilon_m}} \frac{\partial^2 w}{\partial x_1 \partial\nu}\phi - \int_{\partial\Omega_{\epsilon_m}} \frac{\partial\phi}{\partial\nu}\frac{\partial w}{\partial x_1} = \int_{\Omega_{\epsilon_m}} \frac{\partial V}{\partial x_1} w\phi,$$

where ν is an outward unit vector normal to $\partial\Omega_{\epsilon_m}$. Note that

$$\frac{\partial w}{\partial x_1} < 0 \quad \text{for } x_1 > 0.$$

Taking $m \to \infty$, we deduce that $\partial\Omega = \{(x_1, x_2, \dots, x_N) \mid x_1 = 0\}$. This again implies that

$$\int_{\{x \mid x_1 > 0\}} \frac{\partial V}{\partial x_1} w\phi = 0.$$

This is a contradiction since

$$\frac{\partial V}{\partial x_1} = V_r \frac{x_1}{r} \gneq 0 \quad \text{for } x_1 > 0.$$

This completes the proof. □

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